SYSTEM LIFETIME WITH N WORKING ELEMENTS AND REPAIRING DEVICE AND ITS ASYMPTOTIC BEHAVIOUR UNDER CONDITION OF THE ELEMENT'S FAST REPAIR

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Annotation. In this paper we consider the system with n identical elements and one repairing device. In each time moment only one element works while the rest ones stay in reserve. The distribution of element working and repairing period are general. We obtain the asymptotic distribution of the system lifetime under the condition of element fast repair.

Key words. Reliability theory, system lifetime, asymptotic behaviour.

1 Introduction

In this paper we consider the following problem from mathematical reliability theory. We suppose that there are n identical working elements in system and one repairing device. At any time moment, there is only one element working, called the main one, while the rest ones are staying in cold reserve. Also only one element could be restoring at every time moment, and if there are another broken elements, they stay in a queue. When the main element breaks, any reserve element immediately starts working and broken element immediately begins its restoration. Repairing device is supposed to be absolutely reliable. The system crashes when all n working elements break down.

Here we suppose that element operating time and its recovery time have general distribution.

In this paper we perform relations determining the system lifetime distribution and results for such working time distribution under the condition of element fast repair.

2 Model Description

For the system described above, we introduce the following notation: η – the element operating time is distributed according to the law $G(t)$; ξ – element recovery time is distributed according to the law $F(t)$. We assume that, both functions are absolutely continuous and have all moments that are finite. Let $b = E\eta < \infty$. We suppose that all random variables, which define the system, are mutually independent.

Our aim is to find the distribution of the system's lifetime and to study its asymptotic behaviour under conditions of element fast repair, i.e.

$$P(\xi > \eta) = \int_0^\infty (1 - F(t))\ dG(t) = \varepsilon \to 0$$

with some change of some ξ distribution characteristic. Denote by τ_j - the time period before the system breaks down, if there are $j,\ j = 0, 1, 2, ..., n-1$ broken elements by the moment when system started working. Also let $m(0)$ be the number of broken elements when the first main element started working. $\zeta(\eta)$ – the number of restored elements during η – the first main element operation period.

3 System lifetime distribution and asymptotic

Let express the system lifetime as the sum of the the first main element operating time and the remaining lifetime of the system.

For n=2 considering problem is solved and the results can be found in [1],[2] .That is why, further we assume that $n > 2$:

$$\tau_0 = \eta + \tau_1,$$

$$\tau_1 = (\eta + \tau_1)I(\zeta(\eta) = 1) + (\eta + \tau_2)I(\zeta(\eta) = 2),$$

$$\tau_j = (\eta + \tau_1)I(\zeta(\eta) = j) + \sum_{k=0}^{j-1}(\eta + \tau_{j+1-k})I(\zeta(\eta) = k), \quad 1 < j < n-1,$$

$$\tau_{n-1} = (\eta + \tau_{n-1})I(\zeta(\eta) = n-1) + \sum_{k=1}^{n-2}(\eta + \tau_{j+1-k})I(\zeta(\eta) = k) + \eta I(\zeta(\eta) = 0)$$

Here $I(A)$ is an indicator of event A.

Let's find the probability of the event $\{\zeta(\eta) = j \mid m(0) = j; \eta\}$. Let

$$K_j(x) = P(\xi_1 + ... + \xi_j \le x \mid \xi_1 + ... + \xi_{j+1} > x)$$

– the probability that exactly j elements will be repaired during x – time period. Then

$$K_j(x) = \int_0^x P(\xi_{j+1} > x-y)\; P(\xi_1+...+\xi_j \in dy) = \int_0^x (1-F(x-y))\; dF^{*(j)}(y)$$

Here $F^{*(j)}(y)$ – j-fold convolution of the function $F(y)$ with itself.

Then, if we take the Laplace-Stieltjes transform from the left and right side of the stochastic equations above, we will get:

$$\varphi_0(s) = g(s)\varphi_1(s)$$

$$\varphi_1(s) = (g(s) - g_0(s))\varphi_1(s) + \varphi_2(s)g_0(s), \tag{1}$$

$$\varphi_j(s) = \left(g(s) - \sum_{k=0}^{j-1} g_k(s)\right)\varphi_1(s) + \sum_{k=0}^{j-1} g_k(s)\varphi_{j+1-k}(s), \quad 1 < j < n-1,$$

$$\varphi_1(s) = \left(g(s) - \sum_{k=0}^{n-2} g_k(s)\right)\varphi_1(s) + \sum_{k=0}^{n-2} g_k(s)\varphi_{n-k}(s) + g_0(s)$$

Here:

$$\varphi_j(s) = Ee^{-s\tau_j}, \quad g(s) = \int_0^\infty e^{-sx}\, dG(x), \quad g_0(s) = \int_0^\infty e^{-sx}\,(1-F(t))\; dG(t),$$

$$g_j(s) = \int_0^\infty e^{-sx} \int_0^x (1-F(x-y))\; dF^{*(j)}(y)\; dG(x), \quad j = 1, 2, ..., n-2$$

We want to prove the following convergence:

$$\varphi_j(\varepsilon^{n-1}s) \to \frac{1}{1+bs} \tag{2}$$

when $\varepsilon \to 0$.

This is equivalent to the fact that the system lifetime distribution converges to exponential distribution with parameter b when $\varepsilon \to 0$.

We will use the concept of asymptotic expansion and asymptotic equivalence to prof (2) [3].

So then we will write approximation series for functions $g(\varepsilon^{n-1}s), g_j(\varepsilon^{n-1}s)$ $j = 0, 1, 2, ...,$ when $\varepsilon \to 0$:

$$g(\varepsilon^{n-1}s) \sim 1 - \varepsilon^{n-1}bs$$

$$g_0(\varepsilon^{n-1}s) \sim \int_0^\infty (1-F(x))\; dG(x)$$

$$- \varepsilon^{n-1}s \int_0^\infty x(1-F(x))\ dG(x) =$$
$$= \varepsilon \ - \ \varepsilon^{n-1}s\ \gamma_0$$

$$g_j(\varepsilon^{n-1}s) \sim \int_0^\infty \int_0^x (1-F(x-y))\ dF^{*(j)}(y)\ dG(x) -$$
$$- \varepsilon^{n-1}s \int_0^\infty x \int_0^x (1-F(x-y))\ dF^{*(j)}(y)\ dG(x) =$$
$$= g_j(0) - \varepsilon^{n-1}s\ \gamma_j \quad j = 1, 2, ...$$

Then, using the Hölder's inequality, we will get:

$$\gamma_0 = \int_0^\infty x(1-F(x))G'(x)\ dx \le$$
$$\le \left(\int_0^\infty x^2\ dG(x)\right)^{\frac{1}{2}} \cdot \left(\int_0^\infty (1-F(x))\ dG(x)\right)^{\frac{1}{2}} \le$$
$$\le Const \cdot \sqrt{\varepsilon}$$

Thus $\gamma_0 \to 0$ when $\varepsilon \to 0$. Then for $j = 1, 2, ...$ we have:

$$g_j(0) = \int_0^\infty \int_0^x (1-F(x-y))\ dF^{*(j)}(y)\ dG(x) \le$$

$$\le \int_0^\infty (1-F(x))\ dG(x) \cdot \int_0^\infty\ dF^{*(j)}(y) = \varepsilon$$

And also

$$\gamma_j = \int_0^\infty x \left(\int_0^x (1-F(x-y))\ dF^{*(j)}(y)\right)\ dG(x) \le$$

$$\le \int_0^\infty \left(\int_0^\infty x\ (1-F(x-y))\ dG(x)\right)\ dF^{*(j)}(y) \le$$

According to results for γ_0:

$$\le Const \cdot \sqrt{\varepsilon} \cdot \int_0^\infty dF^{*(j)}(y) = Const \cdot \sqrt{\varepsilon}$$

So, we have confirmed that

$$\begin{aligned} g_j(0) &\to 0, \qquad j = 1, 2, ... \\ \gamma_j &\to 0, \qquad j = 0, 1, 2, ... \end{aligned} \tag{3}$$

when $\varepsilon \to 0$.

Then, let's notice, that from (1) we can get:

$$\frac{\varphi_0(\varepsilon^{n-1}s)}{\varphi_1(\varepsilon^{n-1}s)} = g(\varepsilon^{n-1}s) \sim 1 - \varepsilon^{n-1}sb.$$

So, $\varphi_0(\varepsilon^{n-1}s) \sim \varphi_1(\varepsilon^{n-1}s)$.

From the next equation from (1), deviding the both parts by φ_1 we will get:

$$1 \sim 1 - \varepsilon^{n-1}sb - \varepsilon + \varepsilon^{n-1}s\gamma_0 + \frac{\varphi_2}{\varphi_1} \cdot \left(\varepsilon - \varepsilon^{n-1}s\gamma_0\right)$$

Here we have omitted the dependence of the argument $\varepsilon^{n-1}s$ due to brevity. We will do the same sometimes below. Then, using (3) and leaving the slowest decreasing terms with ε in the last expression, we get $\varphi_2 \sim \varphi_1$ and $\varphi_2 \sim \varphi_1 \cdot (1 + \varepsilon^{n-2}sb)$.

For $2 < j < n-1$ again, deviding the both parts of j-th equation in (1) by φ_1, using (3) and leaving the slowest decreasing terms with ε, we have following:

$$1 + \varepsilon^{n-j}sb \sim 1 - \varepsilon + g_1(0) \cdot \varepsilon^{n-j}sb + \frac{\varphi_{j+1}}{\varphi_1} \cdot (\varepsilon - \varepsilon^{n-1}s\gamma_0)$$

$$\frac{\varphi_{j+1}}{\varphi_1} \sim 1 + \varepsilon^{n-j-1}sb$$

So, we got, that all $\varphi_k \sim \varphi_1 \cdot (1 + \varepsilon^{n-k}sb)$, $k = 0, 2, 3, \cdots, n-1$. Particularly, $\varphi_k(\varepsilon^{n-1}s) \sim \varphi_1(\varepsilon^{n-1}s)$, $k = 0, 2, 3, \cdots, n-1$, when $\varepsilon \to 0$.

And finally, we divide the last equation in system (1) by φ_1, and, using (3), leaving the slowest decreasing terms with ε, we get:

$$\varphi_1 \cdot (1 + \varepsilon \cdot sb) \sim \varphi_1 \cdot (1 - \varepsilon + \varepsilon sb \cdot g_1(0)) + (\varepsilon - \varepsilon^{n-1}s\gamma_0)$$

$$\varphi_1 \cdot \varepsilon \cdot (1 + sb - sb \cdot g_1(0)) \sim \varepsilon \cdot (1 - \varepsilon^{n-2}s\gamma_0)$$

So then, (2) is proved.

References

[1] Gnedenko B.V., Belyaev Yu.K., Soloviev A.D. "Mathematical methods in the theory of reliability" *M.:Nauka*, 1965.

[2] Barlow R.E., Proschan F., "Mathematical theory of reliability" *Jh. Wiley & Sons, Inc. NY London Sydney*, 1968.

[3] Arkhipov, G. I. "Lectures on mathematical analysis" *M:Drofa* 2004.