# Comment on the paper "Direct and inverse problems for a third order self-adjoint differential operator with periodic boundary conditions and nonlocal potential"

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**Abstract.** This comment shows that the main results of [Y. Liu and J. Yan, Results Math. (2024) 79:21] immediately follow from the abstract results of [O. Dobosevich and R. Hryniv, Integr. Equ. Oper. Theory (2021) 93:18].

**Keywords:** inverse spectral problem; one-rank perturbation; self-adjoint operator.

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The main uniqueness results of [1] can be easily deduced from [2] as follows.

Let $A$ be an unbounded self-adjoint operator in a Hilbert space $H$, satisfying the following assumptions:

(A1) The operator $A$ has simple discrete spectrum $\{\lambda_n\}_{n \in I}$, where $I = \mathbb{Z}$ or, if $A$ is bounded below, one can put $I = \mathbb{N}$.

(A2) $\inf\limits_{n \in I} |\lambda_{n+1} - \lambda_n| =: d > 0$.

Let $B = A + \alpha \langle \cdot, v \rangle v$, where $\alpha \in \mathbb{R}$, $v \in H$, $\|v\| = 1$, and $\langle \cdot, \cdot \rangle$ denotes the scalar product in $H$. Denote by $\{u_n\}_{n \in I}$ the normalized eigenfunctions of $A$ and by $v_n := \langle v, u_n \rangle$, $n \in I$, the Fourier coefficients of $v$. Set

$$I_0 := \{n \in I \colon v_n = 0\}, \quad I_1 := \{n \in I \colon v_n \neq 0\}.$$

The spectrum of $B$ has the form $\sigma_0(B) \cup \sigma_1(B)$, where $\sigma_0(B) := \{\lambda_n \colon n \in I_0\}$ and $\sigma_1(B)$ are the zeros of the characteristic function

$$F(\lambda) = \alpha \sum_{n \in I_1} \frac{|v_n|^2}{\lambda_n - \lambda} + 1. \tag{1}$$

This formula easily follows from (2.5) in [2]. The zeros of $F(\lambda)$ are real, simple, and interlace with $\{\lambda_n\}_{n \in I_1}$ as mentioned on p.15 of [2].

Theorem 3.1 in [2] asserts that the eigenvalues of $B$ can be numbered as $\{\mu_n\}_{n \in I}$ so that

$$\sum_{n \in I} |\mu_n - \lambda_n| < \infty.$$

Lemma 4.5 in [2] implies that

$$F(\lambda) = \prod_{n \in I} \frac{\mu_n - \lambda}{\lambda_n - \lambda}. \tag{2}$$

Consequently, given $\{\lambda_n\}_{n\in I}$ and $\{\mu_n\}_{n\in I}$, one can construct the function $F(\lambda)$ by (2) and find the residues $\alpha|v_n|^2 = -\operatorname*{Res}_{\lambda=\lambda_n} F(\lambda)$, $n \in I_1$. This proves Lemma 5.4 in [1] if we consider $H = L^2_{\mathbb{C}}(0,1)$ and $A = iy'''$ with periodic boundary conditions. Note that the proofs of Theorems 5.5, 5.6 and Corollary 5.7 in [1] rely only on Lemma 5.4. Thus, we can easily deduce these results from (1) and (2) without derivation of complicated formulas for characteristic functions, resolvents, etc. Moreover, the abstract approach of [2] readily implies similar results for any self-adjoint operator $A$ satisfying the assumptions (A1) and (A2). The Ambarzumian-type theorem (Theorem 5.1 in [1]) also holds for any such operator, since its proof uses only the structure of the spectrum $\sigma_0(B)\cup\sigma_1(B)$ of a one-rank perturbation $B$ but not the specific form of the operator $A$.

# References

[1] Liu, Y.; Yan, J. Direct and inverse problems for a third order self-adjoint differential operator with periodic boundary conditions and nonlocal potential, Results Math. (2024), 79:21.

[2] Dobosevych, O.; Hryniv, R. Direct and inverse spectral problems for rank-one perturbations of self-adjoint operators, Integr. Equ. Oper. Theory (2021) 93:18.