Gauss–Berezin integral operators, spinors over orthosymplectic supergroups, and Lagrangian super-Grassmannians

Yuri A. Neretin[1]

We obtain explicit formulas for the spinor representation ρ of the real orthosymplectic supergroup $\mathrm{OSp}(2p|2q,\mathbb{R})$ by integral 'Gauss–Berezin' operators. Next, we extend ρ to a complex domain and get a representation of a larger semigroup, which is a counterpart of Olshanski subsemigroups in semisimple Lie groups. Further, we show that ρ can be extended to an operator-valued function on a certain domain in the Lagrangian super-Grassmannian (graphs of elements of the supergroup $\mathrm{OSp}(2p|2q,\mathbb{C})$ are Lagrangian super-subspaces) and show that this function is a 'representation' in the following sense: we consider Lagrangian subspaces as linear relations and composition of two Lagrangian relations in general position corresponds to a product of Gauss–Berezin operators[2].

1 Introduction

In the present paper, we consider algebras, superalgebras, functional spaces over complex numbers $\mathbb{C}$ and in few cases over real numbers $\mathbb{R}$. The transposition of matrices is denoted by $A \mapsto A^t$. The symbol 1_n denotes the unit matrix of size n.

1.1. Orthosymplectic spinors. Let $x_1, \dots, x_p$ be real variables, $\xi_1, \dots, \xi_q$ be Grassmann variables, $\xi_i\xi_j = -\xi_j\xi_i$ for all i, j (in particular, $\xi_i^2 = 0$). We consider differential operators

$$1, \quad x_k x_l, \quad x_k \frac{\partial}{\partial x_l}, \quad \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_l}, \quad \xi_m \xi_n, \quad \xi_m \frac{\partial}{\partial \xi_n}, \quad \frac{\partial}{\partial \xi_m}\frac{\partial}{\partial \xi_n}, \tag{1.1}$$

$$x_k \xi_m, \quad x_k \frac{\partial}{\partial \xi_m}, \quad \xi_m \frac{\partial}{\partial x_l}, \quad \frac{\partial}{\partial x_k}\frac{\partial}{\partial \xi_m}, \tag{1.2}$$

where $1 \leqslant k, l \leqslant p$, $1 \leqslant m, n \leqslant q$, acting in the space of polynomials in $x_1, \dots, x_p, \xi_1, \dots, \xi_q$. Denote by $\mathcal{R}$ the space of all *complex* linear combinations of such operators. We say, that a parity of a nonzero monomial of degree 2 in x_k, $\frac{\partial}{\partial \xi_l}$, ξ_m, $\frac{\partial}{\partial \xi_n}$ is $\overline{1}$ if it contains either one ξ_k or one $\frac{\partial}{\partial \xi_l}$. Otherwise parity is $\overline{0}$. So monomials (1.1) have parity $\overline{0}$ and (1.2) parity $\overline{1}$. We also say that a nonzero linear combination of monomials of parity $\overline{0}$ (resp. $\overline{1}$) has parity $\overline{0}$ (resp. $\overline{1}$). Let u, $v \in \mathcal{R}$ have parities $p(u)$, $p(v)$. We define the supercommutator of operators u, v by

$$[u, v]_s := uv - (-1)^{p(u)p(v)} vu \tag{1.3}$$

[1]Supported by MSHE "Priority 2030" strategic academic leadership program.

[2]This paper is an extended variant of preprint `https://arxiv.org/abs/0707.0570v3` and a strongly revised version of my earlier preprint [51] exploring other realization of spinors.

and extend this operation to the whole $\mathcal{R}$ by bilinearity.

It is easy to see that the space $\mathcal{R}$ is closed with respect to the supercommutator, and therefore we get a Lie superalgebra, it is isomorphic to a direct sum of the orthosymplectic Lie superalgebra $\mathfrak{osp}(2p|2q)$ and a trivial one-dimensional Lie algebra $\mathbb{C}$.

Let us recall a definition of $\mathfrak{osp}(2p|2q)$. Denote the block $(p+p)\times(p+p)$-matrix $\begin{pmatrix}0&1_p\\-1_p&0\end{pmatrix}$ by J and the block $(q+q)\times(q+q)$-matrix $\begin{pmatrix}0&1_q\\1_q&0\end{pmatrix}$ by I. The orthosymplectic Lie superalgebra $\mathfrak{osp}(2p|2q)$ consists of complex block $(2p+2q)\times(2p+2q)$-matrices $\begin{pmatrix}A&B\\C&D\end{pmatrix}$ satisfying the condition[3]

$$\begin{pmatrix}J&0\\0&I\end{pmatrix}\begin{pmatrix}A&B\\C&D\end{pmatrix}+\begin{pmatrix}A^t&C^t\\-B^t&D^t\end{pmatrix}\begin{pmatrix}J&0\\0&I\end{pmatrix}=0.$$

We say that parity of matrices of the form $\begin{pmatrix}*&0\\0&*\end{pmatrix}$ is $\overline{0}$, parity of $\begin{pmatrix}0&*\\ *&0\end{pmatrix}$ is $\overline{1}$, and define a supercommutator by formula (1.3).

If $p=0$, then we get the usual spinor representation of the orthogonal Lie algebra $\mathfrak{o}(2q,\mathbb{C})$ in the Grassmann algebra consisting of 'functions' in variables ξ_1, ..., ξ_q. If $q=0$, then we get a representation of symplectic Lie algebra $\mathfrak{sp}(2q)$ (symplectic spinors or oscillator representation). Spinors and symplectic spinors are distinguished objects of representation theory. Orthosymplectic spinors were considered in numerous works, for instance, Berezin [8] (with the construction mentioned above), Serov [65] (where the spinor representations of the orthosymplectic supergroups $\mathrm{OSp}(2p|r)$ were obtained), and [3], [4], [14], [15], [16], [21], [24], [36], [41], [54].

Remark on notation. In notation $\mathfrak{osp}(2p|2q)$ for the orthosymplectic Lie superalgebra, I firstly write $2p$ corresponding to the 'human' (real or complex) variables and the symplectic Lie algebra $\mathfrak{sp}(2p)$; $2q$ corresponds to Grassmann variables and the orthogonal Lie algebra $\mathfrak{o}(2q)$. In literature, our $\mathfrak{osp}(2p|2q)$ can be denoted by $\mathfrak{osp}(2q|2p)$ or $\mathfrak{spo}(2p|2q)$. ⊠

In this paper we write explicit formula for representation of the corresponding global object, which is larger than the real supergroup $\mathrm{OSp}(2p|2q)$. We do not assume that the reader is familiar with super-mathematics and discuss orthosymplectic spinors as a topic of analysis and as a story about integral operators. The text is self-closed, de facto we use a minimal version of language of super-algebra and super-analysis[4] and follow DeWitt's [18] way — to consider linear spaces (modules) over Grassmann algebra with infinite number of

[3] J is a canonical form of a non-degenerate skew-symmetric matrix, I is a canonical form of a nondegenerate complex symmetric matrix of an even order. For our aims, I is more convenient than the unit matrix, which is more natural for general theory.

[4] Lie superalgebras can have a life of their own, without supergroups, supermanifolds, superintegration, etc., see [32], [11], Chapter 1. We prefer to discuss supergroups — Lie superalgebras are actually present only in Section 10

generators[5].

1.2. Berezin formulas. Apparently, first elements of a strange analogy between the spinor representation of the orthogonal groups and the oscillator representation of symplectic groups[6] were observed by K. O. Friedrichs in the early 1950s, see [22]. He considered spinors over symplectic groups $\mathrm{Sp}(2n,\mathbb{R})$ as a kind of a self-obvious object (obtained by an application of the Stone-von Neumann theorem) and initiated a discussion about their extension to the case $n=\infty$, see some historical comments in [52].

In the beginning of 1960s, Feliks Berezin obtained explicit formulas [5], [6] for both the representations. We briefly recall his results. First of all, let us realize the *real* symplectic group $\mathrm{Sp}(2n,\mathbb{R})$ as the group of *complex* $(n+n)\times(n+n)$ matrices

$$g=\begin{pmatrix}\Phi&\Psi\\ \overline{\Psi}&\overline{\Phi}\end{pmatrix} \tag{1.4}$$

satisfying the condition

$$g\begin{pmatrix}0&1\\-1&0\end{pmatrix}g^t=\begin{pmatrix}0&1\\-1&0\end{pmatrix}. \tag{1.5}$$

Similarly, we realize the *real* orthogonal group $\mathrm{O}(2n,\mathbb{R})$ as the group of *complex* matrices

$$g=\begin{pmatrix}\Phi&\Psi\\ -\overline{\Psi}&\overline{\Phi}\end{pmatrix} \tag{1.6}$$

satisfying

$$g\begin{pmatrix}0&1\\1&0\end{pmatrix}g^t=\begin{pmatrix}0&1\\1&0\end{pmatrix}. \tag{1.7}$$

The oscillator (spinor) representation of $\mathrm{Sp}(2n,\mathbb{R})$ (see (1.4), (1.5)) is realized by the following integral operators $W(\cdot)$

$$W\begin{pmatrix}\Phi&\Psi\\ \overline{\Psi}&\overline{\Phi}\end{pmatrix}f(z)=\pm(\det\Phi)^{-1/2}\times$$
$$\times\int_{\mathbb{C}^n}\exp\Bigl\{\frac12\begin{pmatrix}z&\overline{u}\end{pmatrix}\begin{pmatrix}\overline{\Psi}\Phi^{-1}&(\Phi^t)^{-1}\\ \Phi^{-1}&-\Phi^{-1}\Psi\end{pmatrix}\begin{pmatrix}z^t\\ \overline{u}^t\end{pmatrix}\Bigr\}f(u)e^{-|u|^2}\,du\,d\overline{u} \tag{1.8}$$

in the space of holomorphic functions on $\mathbb{C}^n$. Here the symbol ^{t} denotes the transposition of matrices; $z=\begin{pmatrix}z_1&\dots&z_n\end{pmatrix}$, $u=\begin{pmatrix}u_1&\dots&u_n\end{pmatrix}$ are row vectors,

$$\begin{pmatrix}z&\overline{u}\end{pmatrix}:=\begin{pmatrix}z_1&\dots&z_n&\overline{u}_1&\dots&\overline{u}_n\end{pmatrix}$$

[5]Different authors have different points of view to a formalization of 'super-analysis'. DeWitt's book was an object a justified critisism in [60], [44]. Our work is far from subleties of analysis on supermanifolds, translation of our results to the more common functorial language is more-or-less automatical.

[6]'*Spinor representation*' of orthogonal group is a common term (the representation was discovered by Élie Cartan [13], 1913), the term '*oscillator representation*' has several synonyms, namely, the Weil representation, the Shale–Weil representation, the Segal–Shale–Weil representation, the harmonic representation, the metaplectic representation, the symplectic spinors. The term 'oscillator representation' was proposed by Irving Segal (who was the first who described this representation [61], 1959). For further references, see [50], [53].

also is a row vector, and the expression in the curly brackets is a product of a row vector, matrix, and a column vector (i.e., the whole expression is a scalar). We write $W(\cdot)$ in honor of A.Weil.

On the other hand, Berezin obtained formulas for the spinor representation of the group $\mathrm{O}(2n,\mathbb{R})$ (we realize $\mathrm{O}(2n,\mathbb{R})$ by matrices (1.6), (1.7)). Exactly, he wrote 'integral operators' of the form

$$\operatorname{spin}\begin{pmatrix}\Phi & \Psi\\ \overline{\Psi} & \overline{\Phi}\end{pmatrix} f(\xi) = \pm(\det\Phi)^{1/2}\times \\ \times\int \exp\Bigl\{\frac{1}{2}\begin{pmatrix}\xi & \overline{\eta}\end{pmatrix}\begin{pmatrix}-\overline{\Psi}\Phi^{-1} & (\Phi^t)^{-1}\\ -\Phi^{-1} & -\Phi^{-1}\Psi\end{pmatrix}\begin{pmatrix}\xi^t\\ \overline{\eta}^t\end{pmatrix}\Bigr\} f(\eta)e^{-\eta\overline{\eta}^t}\,d\eta\,d\overline{\eta}, \quad (1.9)$$

here $\xi = \begin{pmatrix}\xi_1 & \dots & \xi_n\end{pmatrix}$, $\eta = \begin{pmatrix}\eta_1 & \dots & \eta_n\end{pmatrix}$ are row-matrices; ξ_j, η_j, $\overline{\eta}_j$ are anti-commuting variables.[7,8]. The integral in the right-hand side is the Berezin integral, see Section 2.

In fact, Berezin in his book [6][9] and the subsequent work [7] conjectured that there is the analysis of Grassmann variables parallel to the usual analysis (and these works contain important elements of this analysis). The book includes parallel exposition of the bosonic Fock space (analysis in infinite number of complex variables) and the fermionic Fock space (Grassmann analysis in infinite number of variables), this parallel seemed mysterious. However, the strangest elements of this analogy were formulas (1.8) and (1.9). This pushed him at the end of 60s – beginning of 70s to the invention of the 'super-analysis', which mixes even (complex or real) variables and odd (Grassmann) variables, see [34], Sect. 5.

Apparently, the first case of joining of classical and Grassmann analysis was the paper by Berezin and G. I. Kats [9], 1970, where formal Lie supergroups corresponding to Lie superalgebras were introduced. At least in 1965 Lie superalgebras appeared in algebraic topology (see Milnor, J. Moore [43], this work was one of starting points for Berezin and Kats). Starting 1971-73 Lie superalgebras and supersymmetries were used in the quantum field theory (e.g., [26], [69], [70], [58], 'super' originates from this literature).

1.3. Aims of this paper. Orthosymplectic spinors. We wish to unite formulas (1.8)–(1.9) and to write explicitly the representation of the su-

[7]Actually, in both the cases Berezin considered $n=\infty$.

[8]Such a formula makes sense only for an open dense subset in $\mathrm{SO}(2n,\mathbb{R})\subset \mathrm{O}(2n,\mathbb{R})$; this produces some difficulties below. Spinors over infinite-dimensional symplectic and orthogonal groups were independently introduced by Shale and Stinespring [66], [67]. However, the starting point of super-analysis and standpoint of the present work were Berezin's formulas (1.8), (1.9).

[9]On 'intellectual history' of this book, its origins and influence, see [52].

pergroup[10] $\mathrm{OSp}(2p|2q,\mathbb{R})$, the operators of the representation have the form

$$T(g)f(z,\xi)=\iint\exp\left\{\frac12\begin{pmatrix}z&\xi&\overline{u}&\overline{\eta}\end{pmatrix}\cdot\mathfrak{R}(g)\cdot\begin{pmatrix}z^t\\ \xi^t\\ \overline{u}^t\\ \overline{\eta}^t\end{pmatrix}\right\}\times \\ \times f(u,\eta)\,e^{-u\overline{u}^t-\eta\overline{\eta}^t}\,du\,d\overline{u}\,d\eta\,d\overline{\eta}, \quad (1.10)$$

where g ranges in the supergroup $\mathrm{OSp}(2p|2q,\mathbb{R})$ and $\mathfrak{R}(g)$ is a certain block matrix of size $(p+q+p+q)$ composed of elements a supercommutative algebra $\mathcal{A}$ (we prefer to think that $\mathcal{A}$ is the Grassmann algebra with infinite number of generators $\mathfrak{a}_j$), see Section 5.

1.4. Aims of this paper. Gauss–Berezin integral operators. In [45], [48], [47] it was shown that spinor and oscillator representations are actually representations of categories. We obtain a (non perfect) super-counterpart of these constructions. Let us explain this in more detail.

First, let us consider the 'bosonic' case. Consider a symmetric $(n+n)\times(n+n)$-matrix,

$$S=\begin{pmatrix}A&B\\B^t&C\end{pmatrix}.$$

Consider a *Gaussian integral operator*[11]

$$\mathfrak{B}_+\begin{bmatrix}A&B\\B^t&C\end{bmatrix}f(z)= \\ =\int_{\mathbb{C}^n}\exp\left\{\frac12\begin{pmatrix}z&\overline{u}\end{pmatrix}\begin{pmatrix}A&B\\B^t&C\end{pmatrix}\begin{pmatrix}z^t\\ \overline{u}^t\end{pmatrix}\right\}f(u)e^{-|u|^2}du\,d\overline{u}. \quad (1.11)$$

These operators are more general than (1.8), it can be shown (this was observed by G. I. Olshanski) that the symmetric matrices $\begin{pmatrix}\overline{\Psi}\Phi^{-1}&(\Phi^t)^{-1}\\ \Phi^{-1}&-\Phi^{-1}\Psi\end{pmatrix}$ in Berezin's formula (1.8) are unitary.

It can be readily checked that bounded Gaussian operators form a semigroup, which includes the group $\mathrm{Sp}(2n,\mathbb{R})$; its algebraic structure is described below in Section 3.

On the other hand, we can introduce *Berezin operators* that are fermionic counterparts of Gaussian operators. Namely, consider a skew-symmetric matrix

[10]Thus, this paper returns to the initial point of super-analysis. It seems strange that formula (1.10) was not written by Berezin himself. The author obtained it after reading the posthumous uncompleted book [8] of Berezin; in my opinion, it contains traces of attempts to do this (see also [10]). In the present paper, we use tools that were unknown to Berezin. A straightforward extension of [6] leads to cumbersome calculations. Certainly, these difficulties were surmountable. In any case, the present paper is a kind of a 'lost chapter' of the Berezin's book [6] and my book [50].

[11]We use the symbol $\mathfrak{B}$ in honour of Berezin.

$\begin{pmatrix} A & B \\ -B^t & C \end{pmatrix}$ and the integral operator

$$\mathfrak{B}_- \begin{bmatrix} A & B \\ -B^t & C \end{bmatrix} f(\xi) = \\ = \int \exp\Bigl\{ \frac{1}{2} \begin{pmatrix} \xi & \overline{\eta} \end{pmatrix} \begin{pmatrix} A & B \\ -B^t & C \end{pmatrix} \begin{pmatrix} \xi^t \\ \overline{\eta}^t \end{pmatrix} \Bigr\} f(u) e^{-\eta\overline{\eta}} d\eta\, d\overline{\eta}. \quad (1.12)$$

These operators are more general than (1.9); it can be shown that in Berezin's formula (1.9) the skew-symmetric matrix $\begin{pmatrix} -\overline{\Psi}\Phi^{-1} & (\Phi^t)^{-1} \\ -\Phi^{-1} & -\Phi^{-1}\Psi \end{pmatrix}$ is contained in the pseudo-unitary group $\mathrm{U}(n,n)$.

For Berezin operators in general position, the product of Berezin operators has a kernel of the same form. But sometimes this is not the case. It is possible to improve the definition (see our Section 2), and to obtain a semigroup of Berezin operators.

In this paper we introduce 'Gauss–Berezin integral operators', which unify bosonic 'Gaussian operators' and fermionic 'Berezin operators'. Our main result is a construction of a canonical bijection between the set of all Gauss–Berezin operators and a certain domain in Lagrangian super-Grassmannian; also we propose a geometric interpretation of products of Gauss–Berezin operators.

Formula (1.10) for superspinor representation of the supergroup $\mathrm{OSp}(2p|2q)$ is a byproduct of the geometric construction.

1.5. Aims of this paper. Possible applications. The spinor and oscillator representations are important at least for the following two reasons.

First, they are a basic tool in representation theory of infinite-dimensional groups, see, e.g., [61], [56], [50], in particular, for classical groups, the group of diffeomorphisms of the circle, loop groups[12].

Second, the Howe duality for spinors is an important topic of classical representation theory, see, for instance, [30], [35], [1].

Possible applications of our work in similar directions are discussed in Section 10.

1.6. Structure of this paper. I tried to write a self-contained paper, no preliminary knowledge of the super-mathematics or representation theory is assumed. But this implies a necessity of various preliminaries, which I provide. Also, we try to minimize the vocabulary of this text.

We start with an exposition of spinor and oscillator representations in Sections 2 and 3. These sections contain also a discussion of Gaussian integral operators and Berezin (fermionic Gaussian) operators (a detailed exposition of these topics is contained in the books [50] and [53].

In Sections 6 – 8 we discuss supergroups $\mathrm{OSp}(2p|2q)$, super-Grassmannians and super-linear relations. I am trying to use minimally necessary tools, these

[12]The parabolic induction, which is a main tool for construction of representations of semisimple Lie groups does not work in these situations.

sections do not contain an introduction to super-science and its basic definitions. For generalities of super-mathematics, see [17], [8], [42], [40], [18], [12], [68], [11].

Section 4 contains a discussion of super-analogue of the Gaussian integral, this is a simple imitation of the well-known formula

$$\int_{\mathbb{R}^n} \exp\Bigl\{-\frac{1}{2}xAx^t + bx^t\Bigr\}\,dx = (2\pi)^{n/2}\det(A)^{-1/2}\exp\Bigl\{-\frac{1}{2}bA^{-1}b^t\Bigr\}. \quad (1.13)$$

Apparently, these calculations are written somewhere, but I do not know references.

The Gauss–Berezin integral operators are introduced in Section 5. Our main construction is the canonical one-to-one correspondence between super-linear relations and Gauss–Berezin operators, which is obtained in Section 9. This immediately produces a representation of real supergroups $\mathrm{OSp}(2p|2q,\mathbb{R})$. For g being in an 'open dense' subset in the supergroup, the operators of the representation are of the form (1.10).

In the last section we discuss some open problems.

Acknowledgements. I am grateful to D. V. Alekseevski, A. L. Onishchik, and A. S. Losev for explanations of super-algebra and super-analysis. The preliminary variant [51] of this paper was revised after a discussion with D. Westra, who proposed numerous suggestions to improve the text.

2 A survey of orthogonal spinors. Berezin operators and Lagrangian linear relations

This Section is subdivided into 3 parts.

In Part A we develop the standard formalism of Grassmann algebras Λ_n and define Berezin operators $\Lambda_n \to \Lambda_m$, which are in a some sense morphisms of Grassmann algebras (but not morphisms in the category of algebras!).

In Part B we describe the 'geometrical' category **GD**, which is equivalent to the category of Berezin operators. Morphisms of the category **GD** are certain Lagrangian subspaces.

In Part C we describe explicitly the correspondence between Lagrangian subspaces and Berezin operators.

In some cases we present proofs or explanations, for a coherent treatment see [50], Chapter 2.

A. Grassmann algebras and Berezin operators.

2.1. Grassmann variables and Grassmann algebra. We denote by ξ_1, ..., ξ_n Grassmann variables,

$$\xi_i\xi_j = -\xi_j\xi_i,$$

in particular, $\xi_i^2 = 0$. Denote by Λ_n the algebra of polynomials *with complex coefficients* in these variables, evidently, $\dim \Lambda_n = 2^n$. The monomials

$$\xi_{j_1}\xi_{j_2}\dots\xi_{j_\alpha}, \qquad \text{where } \alpha = 0, 1, \dots, n \text{ and } j_1 < j_2 < \dots < j_\alpha, \quad (2.1)$$

form a basis of Λ_n. Below elements of Grassmann algebra are called *functions.*

2.2. Derivatives. We define *left differentiations* in ξ_j as usual. Exactly, if $f(\xi)$ does not depend on ξ_j, then

$$\frac{\partial}{\partial \xi_j} f(\xi) = 0, \qquad \frac{\partial}{\partial \xi_j} \xi_j f(\xi) = f(\xi).$$

Evidently,

$$\frac{\partial}{\partial \xi_k} \frac{\partial}{\partial \xi_l} = -\frac{\partial}{\partial \xi_l} \frac{\partial}{\partial \xi_k}, \qquad \left(\frac{\partial}{\partial \xi_k}\right)^2 = 0.$$

2.3. Exponentials. Let $f(\xi)$ be an *even* function, i.e., $f(-\xi) = f(\xi)$. We define its exponential as usual,

$$\exp\bigl\{f(\xi)\bigr\} := \sum_{j=0}^{\infty} \frac{1}{j!} f(\xi)^j.$$

Even functions f, g commute, $fg = gf$, therefore

$$\exp\{f(\xi) + g(\xi)\} = \exp\{f(\xi)\} \cdot \exp\{g(\xi)\}.$$

2.4. Berezin integral. Let $\xi_1, \ldots, \xi_n$ be Grassmann variables. The *Berezin integral*

$$\int f(\xi)\, d\xi = \int f(\xi_1, \ldots, \xi_n)\, d\xi_1 \ldots d\xi_n$$

is a linear functional on Λ_n defined by

$$\int \xi_1 \xi_2 \ldots \xi_n\, d\xi_1 \ldots d\xi_n = 1,$$

integrals of all other monomials are zero.

The following formula for integration by parts holds

$$\int f(\xi) \cdot \frac{\partial}{\partial \xi_k} g(\xi)\, d\xi = -\int \frac{\partial f(-\xi)}{\partial \xi_k} \cdot g(\xi)\, d\xi.$$

2.5. Integrals with respect 'to odd Gaussian measure'. Let $\xi_1, \ldots, \xi_q$ be as above. Let $\overline{\xi}_1, \ldots, \overline{\xi}_q$ be another collection of Grassmann variables,

$$\overline{\xi}_k \overline{\xi}_l = -\overline{\xi}_l \overline{\xi}_k, \qquad \xi_k \overline{\xi}_l = -\overline{\xi}_l \xi_k.$$

We need the following Gaussian expression

$$\begin{aligned} \exp\bigl\{-\xi\overline{\xi}^t\bigr\} &:= \exp\{-\xi_1\overline{\xi}_1 - \xi_2\overline{\xi}_2 - \ldots\} = \\ &= \exp\{\overline{\xi}_1\xi_1\} \exp\{\overline{\xi}_2\xi_2\} \cdots = (1 + \overline{\xi}_1\xi_1)(1 + \overline{\xi}_2\xi_2)(1 + \overline{\xi}_3\xi_3) \ldots \end{aligned}$$

Denote

$$d\overline{\xi}\, d\xi = d\overline{\xi}_1\, d\xi_1\, d\overline{\xi}_2\, d\xi_2 \ldots$$

Therefore,

$$\int \Big(\prod_{k=1}^{m} (\overline{\xi}_{\alpha_k}\xi_{\alpha_k})\Big)\, e^{-\xi\overline{\xi}^t}\, d\overline{\xi}\, d\xi = 1, \tag{2.2}$$

and the integral is zero for all other monomials. For instance,

$$\int \xi_1\xi_2\overline{\xi}_3\, e^{-\xi\overline{\xi}^t}\, d\overline{\xi}\, d\xi = 0, \qquad \int e^{-\xi\overline{\xi}^t}\, d\overline{\xi}\, d\xi = +1,$$

$$\int \xi_1\overline{\xi}_1\overline{\xi}_{33}\xi_{33}\, e^{-\xi\overline{\xi}^t}\, d\overline{\xi}\, d\xi = -\int (\overline{\xi}_1\xi_1)\,(\overline{\xi}_{33}\xi_{33})\, e^{-\xi\overline{\xi}^t}\, d\overline{\xi}\, d\xi = -1.$$

Evidently,

$$\int f(\overline{\xi})\,\cdot\, \frac{\partial}{\partial\xi_k} g(\xi)\, e^{-\xi\overline{\xi}^t}\, d\overline{\xi}\, d\xi = \int \overline{\xi}_k f(-\overline{\xi})\,\cdot\, g(\xi)\, e^{-\xi\overline{\xi}^t}\, d\overline{\xi}\, d\xi,$$

$$\int \frac{\partial}{\partial\overline{\xi}_k} f(\overline{\xi})\,\cdot\, g(\xi)\, e^{-\xi\overline{\xi}^t}\, d\overline{\xi}\, d\xi = \int f(-\overline{\xi})\cdot \xi_k g(\xi)\, e^{-\xi\overline{\xi}^t}\, d\overline{\xi}\, d\xi.$$

2.6. Integral operators. Now, consider Grassmann algebras Λ_p and Λ_q consisting of polynomials in Grassmann variables $\xi_1, \dots, \xi_p$ and $\eta_1, \dots, \eta_q$, respectively. For a function $K(\xi,\overline{\eta})$ we define an *integral operator*

$$A_K : \Lambda_q \to \Lambda_p$$

by

$$A_K f(\xi) = \int K(\xi,\overline{\eta}) f(\eta)\, e^{-\eta\overline{\eta}^t}\, d\overline{\eta}\, d\eta.$$

Proposition 2.1 *The map $K \mapsto A_K$ is a one-to-one correspondence of the set of all polynomials $K(\xi,\overline{\eta})$ and the set of all linear maps $\Lambda_q \to \Lambda_p$.*

A proof of Proposition 2.1 is trivial. Indeed, expand

$$K(\xi,\eta) = \sum a_{i_1,\dots,i_l\, j_1,\dots,j_l} \xi_{i_1}\dots\xi_{i_k}\overline{\eta}_{j_1}\dots\overline{\eta}_{j_l}.$$

Then $a_{\dots}$ are the matrix elements[13] of A_K in the standard basis (2.1).

Proposition 2.2 *If $A : \Lambda_q \to \Lambda_p$ is determined by the kernel $K(\xi,\eta)$ and $B : \Lambda_p \to \Lambda_r$ is determined by the kernel $L(\zeta,\overline{\xi})$, then the kernel of BA is*

$$M(\zeta,\overline{\eta}) = \int L(\zeta,\overline{\xi}) K(\xi,\overline{\eta}) e^{-\xi\overline{\xi}^t}\, d\overline{\xi}\, d\xi.$$

2.7. Berezin operators in the narrow sense. A *Berezin operator $\Lambda_q \to \Lambda_p$ in the narrow sense* is an operator of the form

$$\mathfrak{B}\begin{bmatrix} A & B \\ -B^t & C\end{bmatrix} f(\xi) := \\ = \int \exp\Big\{\frac{1}{2}\begin{pmatrix}\xi & \overline{\eta}\end{pmatrix}\begin{pmatrix} A & B \\ -B^t & C\end{pmatrix}\begin{pmatrix}\xi^t \\ \overline{\eta}^t\end{pmatrix}\Big\} f(\eta)\, e^{-\eta\overline{\eta}^t}\, d\overline{\eta}\, d\eta, \tag{2.3}$$

[13] up to signs.

where $A = -A^t$, $C = -C^t$. Let us explain the notation.

1. $\begin{pmatrix}\xi & \eta\end{pmatrix}$ denotes the row-matrix

$$\begin{pmatrix}\xi & \overline{\eta}\end{pmatrix} := \begin{pmatrix}\xi_1 & \dots & \xi_p & \overline{\eta}_1 & \dots & \overline{\eta}_q\end{pmatrix}.$$

Respectively, $\begin{pmatrix}\xi^t \\ \overline{\eta}^t\end{pmatrix}$ denotes the transposed column-matrix.

2. The $(p+q) \times (p+q)$-matrix $\begin{pmatrix} A & B \\ -B^t & C\end{pmatrix}$ is skew-symmetric. The whole expression for the kernel has the form

$$\exp\Bigl\{\frac{1}{2}\sum_{k\leqslant p, l\leqslant p} a_{kl}\xi_k\xi_l + \sum_{k\leqslant p, m\leqslant q} b_{km}\xi_k\overline{\eta}_m + \frac{1}{2}\sum_{m\leqslant q, j\leqslant q} c_{mj}\overline{\eta}_m\overline{\eta}_j\Bigr\}.$$

2.8. Product formula.

Theorem 2.3 *Let*

$$\mathfrak{B}[S_1] = \mathfrak{B}\begin{bmatrix} P & Q \\ -Q^t & R\end{bmatrix} : \Lambda_p \to \Lambda_q, \qquad \mathfrak{B}[S_2] = \mathfrak{B}\begin{bmatrix} K & L \\ -L^t & M\end{bmatrix} : \Lambda_q \to \Lambda_r$$

be Berezin operators in the narrow sense. Assume $\det(1 - MP) \neq 0$. *Then*

$$\mathfrak{B}[S_2]\,\mathfrak{B}[S_1] = \operatorname{Pfaff}\begin{pmatrix} M & 1 \\ -1 & P\end{pmatrix}\mathfrak{B}[S_2 \circ S_1], \tag{2.4}$$

where

$$\begin{pmatrix} K & L \\ -L^t & M\end{pmatrix} \circ \begin{pmatrix} P & Q \\ -Q^t & R\end{pmatrix} = \\ = \begin{pmatrix} K + LP(1-MP)^{-1}L^t & L(1-PM)^{-1}Q \\ -Q^t(1-MP)^{-1}L^t & R - Q^t(1-MQ)^{-1}MQ\end{pmatrix}. \tag{2.5}$$

The symbol Pfaff$(\cdot)$ denotes the Pfaffian, see the next subsection.

A calculation is not difficult, see Subsection 4.9. The formula (2.5) is clarified below, Theorem 2.10.

2.9. Pfaffians and odd Gaussian integrals. Let R be a skew-symmetric $2n \times 2n$ matrix. Its *Pfaffian* $\operatorname{Pfaff}(R)$ is defined by the condition

$$\frac{1}{n!}\Bigl(\frac{1}{2}\sum_{kl} r_{kl}\xi_k\xi_l\Bigr)^n = \operatorname{Pfaff}(R)\,\xi_1\xi_2\dots\xi_{2n-1}\xi_{2n}$$

(where $\xi_1, \dots, \xi_{2n}$ are Grassmann variables). In other words,

$$\operatorname{Pfaff}(R) = \frac{1}{n!}\int\Bigl(\frac{1}{2}\sum_{kl} r_{kl}\xi_k\xi_l\Bigr)^n d\xi = \int \exp\Bigl\{\frac{1}{2}\xi R\xi^t\Bigr\}\,d\xi.$$

Recall that

$$\mathrm{Pfaff}(R)^2 = \det R.$$

More generally, let $\xi_1, \ldots, \xi_{2n}, \theta_1, \ldots, \theta_{2n}$ be pairwise anticommuting variables. Then (see [50], Theorem II.4.4)

$$\int \exp\Big\{\frac{1}{2}\xi R \xi^t + \sum_{j=1}^{2n} \theta_j \xi_j\Big\}\, d\xi = \mathrm{Pfaff}(R) \exp\Big\{\frac{1}{2}\theta R^{-1}\theta^t\Big\}. \tag{2.6}$$

Theorem 2.3 follows from this formula in a straightforward way.

2.10. The definition of Berezin operators. Theorem 2.3 suggests an extension of the definition of Berezin operators. Indeed, this theorem is perfect for operators in general position. If $\det(1 - MP) = 0$, then we get an indeterminate of the form $0 \cdot \infty$ in the product formula (2.4), (2.5).

For this reason, consider the cone[14] $\mathcal{C}$ of all the operators of the form $s \cdot \mathfrak{B}\begin{bmatrix} A & B^t \\ -B^t & C \end{bmatrix}$, where s ranges in $\mathbb{C}$. Certainly, the cone $\mathcal{C}$ is not closed. Indeed,

$$\lim_{\varepsilon \to 0} \varepsilon \exp\Big\{\frac{1}{\varepsilon}\xi_1\xi_2\Big\} = \xi_1\xi_2;$$
$$\lim_{\varepsilon \to 0} \varepsilon^2 \exp\Big\{\frac{1}{\varepsilon}(\xi_1\xi_2 + \xi_2\xi_4)\Big\} = \xi_1\xi_2\xi_3\xi_4.$$

This suggests the following definition.

Berezin operators are operators $\Lambda_q \to \Lambda_p$, whose kernels have the form (cf. [59])

$$s \cdot \prod_{j=1}^{m} (\xi u_j^t + \overline{\eta} v_j^t) \cdot \exp\Big\{\frac{1}{2}\begin{pmatrix} \xi & \overline{\eta} \end{pmatrix} \begin{pmatrix} A & B \\ -B^t & C \end{pmatrix} \begin{pmatrix} \xi^t \\ \overline{\eta}^t \end{pmatrix}\Big\}, \tag{2.7}$$

where

1. $s \in \mathbb{C}$;
2. $u_j \in \mathbb{C}^p$, $v_j \in \mathbb{C}^q$ are row-matrices;
3. m ranges in the set $\{0, 1, \ldots, p+q\}$;
4. $\begin{pmatrix} A & B \\ -B^t & C \end{pmatrix}$ is a skew-symmetric $(p+q) \times (p+q)$-matrix.

2.11. The space of Berezin operators.

Proposition 2.4 a) *The cone of Berezin operators is closed in the space of all linear operators $\Lambda_q \to \Lambda_p$.*

b) *Denote by* $\mathrm{Ber}^{[m]}$ *the set of all Berezin operators with a given number m of linear factors in* (2.7). *Then the closure of* $\mathrm{Ber}^{[m]}$ *is*

$$\mathrm{Ber}^{[m]} \cup \mathrm{Ber}^{[m+2]} \cup \mathrm{Ber}^{[m+4]} \cup \ldots$$

[14] Below, a *cone* in $\mathbb{C}^N$ is a subset invariant with respect to homotheties $z \mapsto sz$, where $s \in \mathbb{C}$.

Therefore, the cone of all Berezin operators $\Lambda_q \to \Lambda_p$ consists of two components, namely

$$\bigcup_{m \text{ is even}} \mathrm{Ber}^{[m]} \qquad \text{and} \qquad \bigcup_{m \text{ is odd}} \mathrm{Ber}^{[m]}.$$

They are closures of $\mathrm{Ber}^{[0]}$ and $\mathrm{Ber}^{[1]}$, respectively. Also, kernels $K(\xi, \overline{\eta})$ of Berezin operators satisfies

$$K(-\xi, -\overline{\eta}) = K(\xi, \overline{\eta}), \qquad \text{or} \qquad K(-\xi, -\overline{\eta}) = -K(\xi, \overline{\eta}),$$

respectively.

REMARK. Kernels $K(\xi, \eta)$ of Berezin operators $\Lambda_q \to \Lambda_p$ are canonically defined polynomials in ξ, η. However, expressions (2.7) for the kernels of Berezin operators are non-canonical if numbers m of linear factors are $\geqslant 1$. For instance,

$$(\xi_1 + \xi_{33})(\xi_1 + 7\xi_{33}) = 6\xi_1\xi_{33}, \qquad \xi_1 \exp\{\xi_1\xi_2 + \xi_3\xi_4\} = \xi_1 \exp\{\xi_3\xi_4\}.$$

A quadratic form in the exponential in (2.7) also is not defined canonically. ⊠

2.12. Examples of Berezin operators.

a) The identity operator is a Berezin operator. Its kernel is $\exp\bigl\{\sum \xi_j\overline{\eta}_j\bigr\}$.

b) More generally, operators with kernels $\exp\bigl\{\sum_{ij} b_{ij}\xi_i\overline{\eta}_j\bigr\}$ are operators $\Lambda_q \to \Lambda_p$ defined by the substitution $\eta_i = \sum_j b_{ji}\xi_j$.

c) The operator with kernel $\xi_1 \exp\{\sum \xi_j\overline{\eta}_j\}$ is the operator $f \mapsto \xi_1 f$.

e) The operator with kernel $\overline{\eta}_1 \exp\{\sum \xi_j\overline{\eta}_j\}$ is the operator $f \mapsto \frac{\partial}{\partial\xi_1} f$.

f) The operator in Λ_n determined by the kernel $(\xi_1 + \overline{\eta}_1)(\xi_2 + \overline{\eta}_2)\dots$ is Hodge $\star$-operator. Namely, let $i_1 < i_2 < \dots < i_k$ be a subset in $\{1, 2, \dots, n\}$. Let $j_1 < j_2 < \dots < j_{n-k}$ be the complementary subset. Then

$$\star\bigl(\xi_1\xi_2\dots\xi_{i_k}\bigr) = \xi_{j_1}\dots\xi_{j_{n-k}}.$$

g) The operator with kernel $K(\xi, \eta) = 1$ is the projection to the vector $f(\xi) = 1 \in \Lambda_n$.

h) The operator $\mathfrak{B}$ with the kernel

$$\exp\Bigl\{\frac{1}{2} a_{ij}\xi_i\xi_j + \sum \xi_j\overline{\eta}_j\Bigr\}$$

is the multiplication operator

$$\mathfrak{B} f(\xi) = \exp\Bigl\{\frac{1}{2} a_{ij}\xi_i\xi_j\Bigr\} f(\xi).$$

i) A product of Berezin operators is a Berezin operator (see below).

j) Operators of the spinor representation of $\mathrm{O}(2n, \mathbb{C})$ are Berezin operators.

2.13. Another definition of Berezin operators. Denote by $\mathfrak{D}[\xi_j]$ the following operators in Λ_p

$$\mathfrak{D}[\xi_j]f(\xi) := \Bigl(\xi_j + \frac{\partial}{\partial \xi_j}\Bigr) f(\xi). \tag{2.8}$$

If g, h do not depend on ξ_j, then

$$\mathfrak{D}[\xi_j]\bigl(g(\xi) + \xi_j h(\xi)\bigr) = \xi_j g(\xi) + h(\xi)$$

In the same way, we define the operators $\mathfrak{D}[\eta_j] : \Lambda_q \to \Lambda_q$. Obviously

$$\mathfrak{D}[\xi_i]^2 = 1, \qquad \mathfrak{D}[\xi_i]\,\mathfrak{D}[\xi_j] = -\mathfrak{D}[\xi_j]\,\mathfrak{D}[\xi_i], \quad \text{for } i \neq j. \tag{2.9}$$

A *Berezin operator* $\Lambda_q \to \Lambda_p$ is any operator that can be represented in the form

$$\mathfrak{D}[\xi_{k_1}]\dots\mathfrak{D}[\xi_{k_\alpha}] \cdot \mathfrak{B} \cdot \mathfrak{D}[\eta_{m_1}]\dots\mathfrak{D}[\eta_{m_\beta}], \tag{2.10}$$

where

— $\mathfrak{B}$ is a Berezin operator in the narrow sense,

— α ranges in the set $\{0, 1, \dots, p\}$ and β ranges in $\{0, 1, \dots, q\}$

By (2.9) we can assume $k_1 < \dots < k_\alpha$, $m_1 < \dots < m_\beta$.

Proposition 2.5 *The two definitions of Berezin operators are equivalent.*

REMARK. Usually, a Berezin operator admits many representations in the form (2.10). In fact, the space of all Berezin operators is a smooth cone, and formula (2.10) determines 2^{p+q} coordinate systems on this cone. Any collection of $2^{p+q} - 1$ charts have a non-empty complement. ⊠

2.14. The category of Berezin operators. Groups of automorphisms.

Theorem 2.6 *Let*

$$\mathfrak{B}_1 : \Lambda_q \to \Lambda_p, \qquad \mathfrak{B}_2 : \Lambda_p \to \Lambda_r$$

be Berezin operators. Then $\mathfrak{B}_2\mathfrak{B}_1 : \Lambda_q \to \Lambda_r$ is a Berezin operator. If $\mathfrak{B}$ is an invertible Berezin operator, then $\mathfrak{B}^{-1}$ is a Berezin operator.

By Theorem 2.6 we get a category, whose objects are Grassmann algebras Λ_0, Λ_1, Λ_2, ... and whose morphisms are Berezin operators.

Denote by G_n the group of all invertible Berezin operators $\Lambda_n \to \Lambda_n$. By the definition, it contains the group $\mathbb{C}^\times$ of all scalar non-zero operators.

Theorem 2.7 $G_n/\mathbb{C}^\times \simeq \mathrm{O}(2n, \mathbb{C})/\{\pm 1\}$.

Here $\mathrm{O}(2n,\mathbb{C})$ denotes the usual group of orthogonal transformations in $\mathbb{C}^{2n}$.

The map sending each element of $\mathrm{O}(2n,\mathbb{C})/\{\pm 1\}$. Moreover, this isomorphism sending an orthogonal matrix to the corresponding Berezin operator in Λ_n (determined up to a factor) is nothing but the spinor representation of $\mathrm{O}(2n,\mathbb{C})$.

Our next aim is to describe explicitly the category of Berezin operators. In fact, we intend to clarify the strange matrix multiplication (2.5).

B. Linear relations and the category GD.

2.15. Linear relations. Let V, W be linear spaces over $\mathbb{C}$. A *linear relation* $P : V \rightrightarrows W$ is a linear subspace $P \subset V \oplus W$.

Remark. Let $A : V \to W$ be a linear operator. Its *graph* $\mathrm{graph}(A) \subset V \oplus W$ consists of all vectors $v \oplus Av$. By the definition, $\mathrm{graph}(A)$ is a linear relation,

$$\dim \mathrm{graph}(A) = \dim V.$$

2.16. Product of linear relations. Let $P : V \rightrightarrows W$, $Q : W \rightrightarrows Y$ be linear relations. Informally, a product QP of linear relations is a product of many-valued maps. If P takes a vector v to a vector w and Q takes the vector w to a vector y, then QP takes v to y.

Now, we present a formal definition. The *product* QP is a linear relation $QP : V \rightrightarrows Y$ consisting of all $v \oplus y \in V \oplus Y$ such that there exists $w \in W$ satisfying $v \oplus w \in P$, $w \oplus y \in Q$.

In fact, the multiplication of linear relations extends the usual matrix multiplication.

2.17. Imitation of some standard definitions of matrix theory.

1°. The *kernel* $\ker P$ consists of all $v \in V$ such that $v \oplus 0 \in P$. In other words,

$$\ker P = P \cap (V \oplus 0).$$

2°. The *image* $\mathrm{im}\, P \subset W$ is the projection of P to $0 \oplus W$.

3°. The *domain* $\mathrm{dom}\, P \subset V$ of P is the projection of P to $V \oplus 0$.

4°. The *indefinity* $\mathrm{indef}\, P \subset W$ of P is $P \cap (0 \oplus W)$.

The definitions of a kernel and an image extend the corresponding definitions for linear operators. The definition of a domain extends the usual definition of the domain of an unbounded operator in an infinite-dimensional space. For a linear operator, $\mathrm{indef} = 0$.

2.18. Lagrangian Grassmannian and orthogonal groups. First, let us recall some definitions. Let V be a linear space equipped with a non-degenerate symmetric (or skew-symmetric) bilinear form M. A subspace H is *isotropic* with respect to the bilinear form M, if $M(h,h') = 0$ for all h, $h' \in H$. The dimension of an isotropic subspace satisfies

$$\dim H \leqslant \frac{1}{2} \dim V.$$

A *Lagrangian subspace*[15] is an isotropic subspace whose dimension is precisely $\frac{1}{2}\dim V$. By $\mathrm{Lagr}(V)$ we denote the *Lagrangian Grassmannian*, i.e., the space of all Lagrangian subspaces in V.

Consider a space $\mathcal{V}_{2n} = \mathbb{C}^{2n}$ equipped with the symmetric bilinear form $L = L_n$ determined by the matrix $\begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$. Denote by $\mathrm{O}(2n, \mathbb{C})$ the group of all linear transformations g of $\mathbb{C}^{2n}$ preserving L, i.e., g must satisfy the condition

$$g \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} g^t = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}.$$

Let m, $n = 0, 1, 2, \dots$. Equip the space $\mathcal{V}_{2n} \oplus \mathcal{V}_{2m}$ with the symmetric bilinear form $L^\ominus$ given by

$$L^\ominus(v \oplus w, v' \oplus w') = L_n(v, v') - L_m(w, w') \ , \tag{2.11}$$

where v, $v' \in \mathcal{V}_{2n}$ and w, $w' \in \mathcal{V}_{2m}$.

Observation 2.8 [16] *Let g be an operator in $\mathbb{C}^{2n}$. Then $g \in \mathrm{O}(2n, \mathbb{C})$ if and only if its graph is an $L^\ominus$-Lagrangian subspace in $\mathcal{V}_{2n} \oplus \mathcal{V}_{2n}$.*

This is obvious.

2.19. Imitation of orthogonal groups. Category GD. Now we define the category **GD**. The objects are the spaces $\mathcal{V}_{2n}$, where $n = 0, 1, 2, \dots$. There are two types of morphisms $\mathcal{V}_{2n} \to \mathcal{V}_{2m}$:

a) $L^\ominus$-Lagrangian subspaces $P \subset \mathcal{V}_{2n} \oplus \mathcal{V}_{2m}$; we regard them as linear relations.

b) a distinguished morphism[17] denoted by $\mathsf{null}_{2n,2m}$.

Now we define a product of morphisms.

— Product of null and any morphism is null.

— Let $P : \mathcal{V}_{2n} \rightrightarrows \mathcal{V}_{2m}$, $Q : \mathcal{V}_{2m} \rightrightarrows \mathcal{V}_{2k}$ be Lagrangian linear relations. Assume that

$$\ker Q \cap \mathrm{indef}\, P = 0 \qquad \text{or, equivalently,} \quad \mathrm{im}\, P + \mathrm{dom}\, Q = \mathcal{V}_{2m}. \tag{2.12}$$

Then QP is the product of linear relations.

— If the condition (2.12) is not satisfied, then $QP = \mathsf{null}$.

Theorem 2.9 *The definition is self-consistent, i.e., a product of morphisms is a morphism and the multiplication is associative.*

[15] Mostly, the term 'Lagrangian subspace' is used for spaces equipped with a skew-symmetric bilinear forms; however, my usage is a usual slang.

[16] I use term 'Observation' for statements, which are important for understanding and became trivial or semi-trivial being formulated.

[17] It is not identified with any linear relation.

At first glance the appearance of null seems strange; its necessity will be transparent immediately (null correspond to zero operators in the next theorem). Also, null will be the source of some our difficulties below.

Theorem 2.10 *The category of Berezin operators defined up to scalar factors and the category* **GD** *are equivalent.*

In fact, there is a map that takes each Lagrangian linear relation $P : \mathcal{V}_{2n} \rightrightarrows \mathcal{V}_{2m}$ to a nonzero Berezin operator $\mathrm{spin}(P) : \Lambda_n \to \Lambda_m$ such that for each $R : \mathcal{V}_{2q} \rightrightarrows \mathcal{V}_{2p}$, $Q : \mathcal{V}_{2p} \rightrightarrows \mathcal{V}_{2r}$

$$\mathrm{spin}(Q)\,\mathrm{spin}(R) = \lambda(Q, R)\,\mathrm{spin}(QR),$$

where $\lambda(Q, R)$ is a constant. Moreover,

$$\lambda(Q, R) = 0 \text{ if and only if } QR = \mathsf{null}.$$

We describe the correspondence between the category of Berezin operators and the category **GD** explicitly in Theorems 2.16, 2.17. First, we need some auxiliary facts concerning Lagrangian Grassmannians (for a detailed introduction to Lagrangian Grassmannians, see [2], Sect. 43, [53], Chapter 3).

C. Explicit correspondence.

2.20. Coordinates on the Lagrangian Grassmannian. Recall that $\mathcal{V}_{2n}$ is the space $\mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$ equipped with the bilinear form $L = L_n$ with the matrix $\begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$. We write $\mathcal{V}_{2n} = \mathbb{C}^{2n}$ as

$$\mathcal{V}_{2n} = \mathcal{V}_n^+ \oplus \mathcal{V}_n^- = \mathbb{C}^n \oplus \mathbb{C}^n,$$

in this decomposition the summands are Lagrangian subspaces.

Lemma 2.11 *Let $H \subset \mathcal{V}_{2n}$ be an n-dimensional subspace such that $H \cap \mathcal{V}_n^- = 0$. Under this condition H is the graph of an operator*

$$T_H : \mathcal{V}_n^+ \to \mathcal{V}_n^-.$$

The following conditions are equivalent

— the matrix T_H is skew-symmetric;

— the subspace H is Lagrangian

PROOF. We write the bilinear form L as

$$L(v, w) = L(v^+ \oplus v^-, w^+ \oplus w^-) = v^+(w^-)^t + w^+(v^-)^t.$$

By the definition, $v \in H$ if and only if $v^- = v^+ T_H$. For v, $w \in P$, we evaluate

$$L(v^+ \oplus v^+T, w^+ \oplus w^+T) = v^+(w^+T_H)^t + v^+T_H(w^+)^t = v^+(T_H + T_H^t)w^+.$$

Now the statement becomes obvious. □

Lemma 2.11 defines a coordinate system in $\mathrm{Lagr}(\mathcal{V}_{2n})$. Certainly, this coordinate system does not cover the whole space $\mathrm{Lagr}(\mathcal{V}_{2n})$.

2.21. Atlas on the Lagrangian Grassmannian. Denote by e_1^+, ..., e_n^+, e_1^-, ..., e_n^- the standard basis in $\mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$. Let J be a subset in $\{1, 2, \dots, n\}$. Denote by $\overline{J}$ its complement. We define the subspaces

$$\mathcal{V}_n^+[J] := \big(\oplus_{j\in J}\mathbb{C}e_j^+\big) \oplus \big(\oplus_{j\notin J}\mathbb{C}e_j^-\big),$$
$$\mathcal{V}_n^-[J] := \big(\oplus_{j\notin J}\mathbb{C}e_j^+\big) \oplus \big(\oplus_{j\in J}\mathbb{C}e_j^-\big),$$

then $\mathcal{V}_{2n} = \mathcal{V}_n^+[J] \oplus \mathcal{V}_n^-[J]$. Denote by $\mathcal{M}[J]$ the set of all Lagrangian subspaces H in $\mathcal{V}_{2n}$ such that $H \cap \mathcal{V}_n^-[J] = 0$. Such subspaces are precisely graphs of symmetric operators $\mathcal{V}_n^+[J] \to \mathcal{V}_n^-[J]$.

Proposition 2.12 *The 2^n charts $\mathcal{M}[J]$ cover the whole Lagrangian Grassmannian* $\mathrm{Lagr}(\mathcal{V}_{2n})$.

2.22. Atlas on the Lagrangian Grassmannian. Elementary reflections. We can describe the same maps in a slightly different way. For $i = 1, 2, \dots, n$, define an elementary reflection $\sigma_i : \mathcal{V}_{2n} \to \mathcal{V}_{2n}$ by

$$\sigma_i e_i^+ = e_i^-, \qquad \sigma_i e_i^- = e_i^+, \qquad \sigma_i e_j^\pm = e_j^\pm \qquad \text{for } i \ne j. \tag{2.13}$$

Observation 2.13

$$\mathcal{M}[J] = \Big(\prod_{i\in J}\sigma_i\Big)\mathcal{M}[\varnothing]. \tag{2.14}$$

2.23. Components of Lagrangian Grassmannian.

Observation 2.14 *The Lagrangian Grassmannian in the space $\mathcal{V}_{2n}$ consists of two connected components.*[18]

We propose two proofs to convince the reader.

1. The group $\mathrm{O}(n, \mathbb{C})$ is dense in $\mathrm{Lagr}(\mathcal{V}_{2n})$. This group has two components.
2. It can be readily checked that $\mathrm{Lagr}(\mathcal{V}_{2n})$ is a homogeneous space

$$\mathrm{Lagr}(\mathcal{V}_{2n}) \simeq \mathrm{O}(2n, \mathbb{C})/\mathrm{GL}(n, \mathbb{C}).$$

The group $\mathrm{O}(2n, \mathbb{C})$ consists of two components and the group $\mathrm{GL}(n, \mathbb{C})$ is connected. □

Remark. Two components of the Lagrangian Grassmannian correspond to two components of the space of Berezin operators. ⊠

2.24. Coordinates on the set of morphisms of GD. We can apply the reasoning of Section 2.20 to $\mathrm{Lagr}(\mathcal{V}_{2n} \oplus \mathcal{V}_{2m})$. Due to the minus in formula (2.11) we must take care of the signs in Lemma 2.11.

[18] Recall that $\mathcal{V}_{2n}$ is equipped with a symmetric bilinear form. The usual Lagrangian Grassmannian discussed in the next section is connected. The orthosymplectic Lagrangian Grassmannian (see Section 7) consists of two components.

Lemma 2.15 *Decompose* $\mathcal{V}_{2n} \oplus \mathcal{V}_{2m}$ *as*[19]

$$\mathcal{V}_{2n} \oplus \mathcal{V}_{2m} = (\mathcal{V}_n^- \oplus \mathcal{V}_m^+) \oplus (\mathcal{V}_n^+ \oplus \mathcal{V}_m^-).$$

Let P *be an* $(m+n)$*-dimensional subspace such that* $P \cap (\mathcal{V}_n^+ \oplus \mathcal{V}_m^-) = 0$*, i.e.,* P *is the graph of an operator*

$$\mathcal{V}_n^- \oplus \mathcal{V}_m^+ \to \mathcal{V}_n^+ \oplus \mathcal{V}_m^-.$$

Then the following conditions are equivalent

— $P \in \mathrm{Lagr}(\mathcal{V}_{2n} \oplus \mathcal{V}_{2m})$*;*

— P *is the graph of an operator having the form*

$$\begin{pmatrix} A & B \\ B^t & C \end{pmatrix}, \qquad \text{where } A = -A^t,\ C = -C^t. \tag{2.15}$$

2.25. Creation-annihilation operators. Let $\mathcal{V}_{2n}$ be as above. Decompose

$$\mathcal{V}_{2n} = \mathcal{V}_n^+ \oplus \mathcal{V}_n^-, \qquad \text{where } \mathcal{V}_n^+ := \mathbb{C}^n \oplus 0, \quad \mathcal{V}_n^- := 0 \oplus \mathbb{C}^n.$$

Let us write elements of $\mathcal{V}_{2n}$ as

$$v := \begin{pmatrix} v_1^+ & \dots & v_n^+ & v_1^- & \dots & v_n^- \end{pmatrix}.$$

For each $v \in \mathcal{V}_{2n}$, we define a creation-annihilation operator $\widehat{a}(v)$ in Λ_n by

$$\widehat{a}(v) f(\xi) := \Bigl(\sum_j v_j^+ \xi_j + \sum_j v_j^- \frac{\partial}{\partial \xi_j}\Bigr) f(\xi).$$

Evidently,

$$\widehat{a}(v)\widehat{a}(w) + \widehat{a}(w)\widehat{a}(v) = L(v,w) \cdot 1.$$

2.26. A construction of the correspondence. Let $\mathfrak{B} : \Lambda_q \to \Lambda_p$ be a nonzero Berezin operator. Consider the subspace $P = P[\mathfrak{B}] \subset \mathcal{V}_{2q} \oplus \mathcal{V}_{2p}$ consisting of $v \oplus w$ such that

$$\widehat{a}(w)\, \mathfrak{B} = \mathfrak{B}\, \widehat{a}(v). \tag{2.16}$$

Theorem 2.16 a) $P[\mathfrak{B}]$ *is a morphism of the category* **GD***, i.e., a Lagrangian subspace.*

b) *The map* $\mathfrak{B} \mapsto P$ *is a bijection*

$$\left\{\begin{matrix} \textit{Set of nonzero Berezin operators } \Lambda_q \to \Lambda_p \\ \textit{defined up to a scalar factor} \end{matrix}\right\} \longleftrightarrow$$

$$\longleftrightarrow \left\{\begin{matrix} \textit{The set of non-'}\mathsf{null}\textit{' morphisms } \mathcal{V}_{2q} \to \mathcal{V}_{2p} \\ \textit{of the category } \mathbf{GD} \end{matrix}\right\}.$$

[19] We emphasize that the source space $\mathcal{V}_{2n}$ and the target space $\mathcal{V}_{2m}$ are mixed in the next row.

c) *Let* $\mathfrak{B}_1 : \Lambda_q \to \Lambda_p$, $\mathfrak{B}_2 : \Lambda_p \to \Lambda_r$ *be Berezin operators. Then*

$$\mathfrak{B}_2\mathfrak{B}_1 = 0 \quad \textit{if and only if} \quad P[\mathfrak{B}_2]\, P[\mathfrak{B}_1] = \mathsf{null}\,. \tag{2.17}$$

d) Otherwise,

$$P[\mathfrak{B}_2\mathfrak{B}_1] = P[\mathfrak{B}_2]P[\mathfrak{B}_1].$$

Sketch of proof. 1. First, let us consider a Berezin operator in the narrow sense. We write the equation (2.16)

$$\Bigl(\sum w_j^+\xi_j + \sum w_j^- \frac{\partial}{\partial \xi_j}\Bigr)\int \exp\Bigl\{\frac{1}{2}\begin{pmatrix}\xi & \overline{\eta}\end{pmatrix}\begin{pmatrix} A & B\\ -B^t & C\end{pmatrix}\begin{pmatrix}\xi^t\\ \overline{\eta}^t\end{pmatrix}\Bigr\} f(\eta) e^{-\eta\overline{\eta}^t}\, d\overline{\eta}\, d\eta =$$
$$= \int \exp\Bigl\{\frac{1}{2}\begin{pmatrix}\xi & \overline{\eta}\end{pmatrix}\begin{pmatrix} A & B\\ -B^t & C\end{pmatrix}\begin{pmatrix}\xi^t\\ \overline{\eta}^t\end{pmatrix}\Bigr\}\Bigl(\sum v_k^+\eta_k + \sum v_k^- \frac{\partial}{\partial \eta_k}\Bigr) f(\eta) e^{-\eta\overline{\eta}^t}\, d\overline{\eta}\, d\eta,$$

or, equivalently,

$$\Bigl(\sum w_j^+\xi_j + \sum w_j^- \frac{\partial}{\partial \xi_j} - \sum v_k^-\overline{\eta}_k - \sum v_k^+ \frac{\partial}{\partial \eta_k}\Bigr)\times$$
$$\times \exp\Bigl\{\frac{1}{2}\begin{pmatrix}\xi & \overline{\eta}\end{pmatrix}\begin{pmatrix} A & B\\ -B^t & C\end{pmatrix}\begin{pmatrix}\xi^t\\ \overline{\eta}^t\end{pmatrix}\Bigr\} = 0$$

We differentiate the exponential and get

$$\Bigl[\sum_j w_j^+\xi_j + \sum_j w_j^-\Bigl(\sum_i a_{ji}\xi_i + \sum_m b_{jk}\overline{\eta}_m\Bigr) - \sum_k v_k^-\overline{\eta}_k - \sum_k v_k^+$$
$$+\Bigl(-\sum_l b_{lk}\xi_l + \sum_m c_{km}\overline{\eta}_m\Bigr)\Bigr]\times$$
$$\times \exp\Bigl\{\frac{1}{2}\begin{pmatrix}\xi & \overline{\eta}\end{pmatrix}\begin{pmatrix} A & B\\ -B^t & C\end{pmatrix}\begin{pmatrix}\xi^t\\ \overline{\eta}^t\end{pmatrix}\Bigr\} = 0.$$

Therefore,

$$\begin{pmatrix}v^+ & w^-\end{pmatrix} = \begin{pmatrix}v^- & w^+\end{pmatrix}\begin{pmatrix} A & -B\\ -B^t & -C\end{pmatrix}.$$

By Lemma 2.15, P is a Lagrangian subspace in $\mathcal{V}_{2q}\oplus\mathcal{V}_{2p}$.

2. Next, let $\mathfrak{C}$ be a Berezin operator having the form (2.10),

$$\mathfrak{C} = \mathfrak{D}[\xi_{k_1}]\dots\mathfrak{D}[\xi_{k_\alpha}]\cdot\mathfrak{B}\cdot\mathfrak{D}[\eta_{m_1}]\dots\mathfrak{D}[\eta_{m_\beta}]. \tag{2.18}$$

We note that

$$\widetilde{a}(v)\,\mathfrak{D}[\xi_i] = \mathfrak{D}[\xi_i]\,\widetilde{a}(\sigma_i v),$$

where σ_i is an elementary reflection (2.13). Let the operator $\mathfrak{B}$ satisfy

$$\widehat{a}(w)\,\mathfrak{B} = \mathfrak{B}\,\widehat{a}(v),$$

where $v \oplus w$ ranges in $P[\mathfrak{B}]$. Then the operator $\mathfrak{C}$ satisfies

$$\widehat{a}(\sigma_{k_1} \dots \sigma_{k_\alpha} w)\, \mathfrak{C} = \mathfrak{C}\, \widehat{a}(\sigma_{m_1} \dots \sigma_{m_\beta} v).$$

Therefore, the corresponding linear relation is

$$\sigma_{k_1} \dots \sigma_{k_\alpha} P \sigma_{m_1} \dots \sigma_{m_\beta} \tag{2.19}$$

and is Lagrangian. This proves a).

3. By Proposition 2.12, the sets (2.19) sweep the whole Lagrangian Grassmannian. This proves b).

4. Let $v \oplus w \in P[\mathfrak{B}_1]$, $w \oplus y \in P[\mathfrak{B}_2]$, i.e.,

$$\widehat{a}(w)\, \mathfrak{B}_1 = \mathfrak{B}_1\, \widehat{a}(v), \qquad \widehat{a}(y)\, \mathfrak{B}_2 = \mathfrak{B}_2\, \widehat{a}(w).$$

Then

$$\mathfrak{B}_2 \mathfrak{B}_1 \widehat{a}(v) = \mathfrak{B}_2 \widehat{a}(w) \mathfrak{B}_1 = \widehat{a}(y) \mathfrak{B}_2 \mathfrak{B}_1$$

and this proves c).

We omit a proof of d), which is more difficult (see [50], Subsect. II.6.5). □

2.27. Explicit correspondence. Another description. The proof above implies also the following theorem (see [50], Theorem II.6.11).

Theorem 2.17 *Let P satisfy Lemma* 2.15. *Then the corresponding Berezin operator $\mathfrak{B}[P]$ has the kernel*

$$\exp\left\{ \frac{1}{2} \begin{pmatrix} \xi & \overline{\eta} \end{pmatrix} \begin{pmatrix} A & -B \\ B^t & -C \end{pmatrix} \begin{pmatrix} \xi^t \\ \overline{\eta}^t \end{pmatrix} \right\}.$$

3 A survey of the oscillator representation. The category of Gaussian integral operators

In Subsections 3.1–3.2, we define the bosonic Fock space. In Subsections 3.3–3.4, we introduce Gaussian integral operators. The algebraic structure of the category of Gaussian integral operators is described in Subsections 3.6–3.8. For a detailed exposition, see [50], Chapter 4, or [53], Sect. 5.1-5.2

3.1. Fock space. Denote by $d\lambda(z)$ the Lebesgue measure on $\mathbb{C}^n$ normalized as

$$d\lambda(z) := \pi^{-n} dx_1 \dots dx_n\, dy_1 \dots dy_n, \qquad \text{where } z_j = x_j + i y_j.$$

The bosonic Fock space $\mathbf{F}_n$ is the space of entire functions on $\mathbb{C}^n$ satisfying the condition

$$\int_{\mathbb{C}^n} |f(z)|^2 e^{-|z|^2}\, d\lambda(z) < \infty.$$

We define the inner product in $\mathbf{F}_n$ by the formula

$$\langle f, g \rangle := \int_{\mathbb{C}^n} f(z) \overline{g(z)} e^{-|z|^2}\, d\lambda(z).$$

Theorem 3.1 *The space* $\mathbf{F}_n$ *is complete, i.e., it is a Hilbert space.*

Proposition 3.2 *The monomials* $z_1^{k_1}\dots z_n^{k_n}$ *are pairwise orthogonal and*

$$\|z_1^{k_1}\dots z_n^{k_n}\|^2 = \prod k_j!$$

3.2. Operators.

Theorem 3.3 *For each bounded operator* $A:\mathbf{F}_n\to\mathbf{F}_m$*, there is a function* $K(z,\overline{u})$ *on* $\mathbb{C}^m\oplus\mathbb{C}^n$ *holomorphic in* $z\in\mathbb{C}^m$ *and antiholomorphic in* $u\in\mathbb{C}^n$ *such that*

$$Af(z)=\int_{\mathbb{C}^n} K(z,\overline{u})\, f(u)\, e^{-|u|^2}\, d\lambda(u)$$

(*the integral absolutely converges for all* f).

3.3. Gaussian operators.

Fix m, $n=0,1,2,\dots$ Let $S=\begin{pmatrix}K & L\\ L^t & M\end{pmatrix}$ be a symmetric $(m+n)\times(m+n)$-matrix, i.e., $S=S^t$. A Gaussian operator

$$\mathfrak{B}[S]=\mathfrak{B}\begin{bmatrix}K & L\\ L^t & M\end{bmatrix}:\mathbf{F}_n\to\mathbf{F}_m$$

is defined by

$$\mathfrak{B}\begin{bmatrix}K & L\\ L^t & M\end{bmatrix} f(z)=\int_{\mathbb{C}^n}\exp\Bigl\{\frac12\begin{pmatrix}z & \overline{u}\end{pmatrix}\begin{pmatrix}K & L\\ L^t & M\end{pmatrix}\begin{pmatrix}z^t\\ \overline{u}^t\end{pmatrix}\Bigr\} f(u)\, e^{-|u|^2}\, d\lambda(u),$$

where

$$z:=\begin{pmatrix}z_1 & \dots & z_n\end{pmatrix},\qquad \overline{u}:=\begin{pmatrix}\overline{u}_1 & \dots & \overline{u}_n\end{pmatrix}$$

are row-matrices.

For an operator A from the standard Euclidean space $\mathbb{C}^l$ to the standard Euclidean space $\mathbb{C}^k$ denote by $\|A\|$ the operator norm of A,

$$\|A\|=\max_{v\in\mathbb{C}^l,\,\|v\|=1}\|Av\| = \Bigl\{\text{the maximal eigenvalue of } A^*A\Bigr\}^{1/2}.$$

Theorem 3.4 (*G. I. Olshanski*) *An operator* $\mathfrak{B}[S]$ *is bounded if and only if*[20]

1. $\|S\|\leqslant 1$,
2. $\|K\|<1$, $\|M\|<1$.

3.4. Product formula.

[20]The condition 1 implies $\|K\|\leqslant 1$, $\|M\|\leqslant 1$, the condition 2 strengthen these inqualities.

Theorem 3.5 *Let*

$$\mathfrak{B}[S_1] = \mathfrak{B}\begin{bmatrix} P & Q \\ Q^t & R \end{bmatrix} : \mathbf{F}_n \to \mathbf{F}_m, \qquad \mathfrak{B}[S_2] = \mathfrak{B}\begin{bmatrix} K & L \\ L^t & M \end{bmatrix} : \mathbf{F}_m \to \mathbf{F}_k$$

be bounded Gaussian operators. Then their product is

$$\det(1 - MP)^{-1/2}\, \mathfrak{B}[S_2 * S_1],$$

where $S_2 * S_1$ is given by

$$\begin{pmatrix} K & L \\ L^t & M \end{pmatrix} * \begin{pmatrix} P & Q \\ Q^t & R \end{pmatrix} = \\ = \begin{pmatrix} K + LP(1-MP)^{-1}L^t & L(1-PM)^{-1}Q \\ Q^t(1-MP)^{-1}L^t & R + Q^t(1-MQ)^{-1}MQ \end{pmatrix}. \qquad (3.1)$$

Theorem 3.6 *Denote by G_n the set of unitary $(n+n)\times(n+n)$ symmetric matrices $\begin{pmatrix} K & L \\ L^t & M \end{pmatrix}$ satisfying $\|K\| < 1$, $\|M\| < 1$. Then G_n is closed with respect to the $*$-multiplication and is isomorphic to the group* $\mathrm{Sp}(2n, \mathbb{R})$.

Observe that formula (3.1) almost coincides with formula (2.5). Again, formula (3.1) hides a product of linear relations.

First, we define analogues of the spaces $\mathcal{V}_{2n}$ from the previous section.

3.5. Complexification of a linear space with bilinear form. Denote by $\mathcal{W}_{2n}$ the space $\mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$. Let us denote its elements by $v = \begin{pmatrix} v^+ & v^- \end{pmatrix}$. We equip $\mathcal{W}_{2n}$ with two forms

— the skew-symmetric bilinear form

$$\Lambda(v,w) = \Lambda_n(v,w) := \begin{pmatrix} v^+ & v^- \end{pmatrix} \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \begin{pmatrix} (w^+)^t \\ (w^-)^t \end{pmatrix} = v^+(w^-)^t - v^-(w^+)^t; \qquad (3.2)$$

— the indefinite Hermitian form

$$M(v,w) = M_n(v,w) := \begin{pmatrix} v^+ & v^- \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix} \begin{pmatrix} (\overline{v}^+)^t \\ (\overline{w}^-)^t \end{pmatrix} = v^+(\overline{v}^+)^t - w^-(\overline{w}^-)^t. \qquad (3.3)$$

Remark. Let us explain the origin of the definition. Consider a real space $\mathbb{R}^{2n}$ equipped with a nondegenerate skew-symmetric bilinear form $\{\cdot,\cdot\}$. Consider the space $\mathbb{C}^{2n} \supset \mathbb{R}^{2n}$. We can extend $\{\cdot,\cdot\}$ to $\mathbb{C}^{2n}$ in the following two ways. First, we can extend it as a bilinear form,

$$\widetilde{\Lambda}(x+iy, x'+iy') := \{x,x'\} - \{y,y'\} + i\big(\{x,y'\} + \{x',y\}\big).$$

Next, we extend $\{\cdot,\cdot\}$ as a sesquilinear form

$$\widetilde{M}(x+iy, x'+iy') := \{x,x'\} + \{y,y'\} + i\big(\{y,x'\} - \{x,y'\}\big).$$

This form is anti-Hermitian, $\widetilde{M}(u,z) = -\overline{\widetilde{M}(z,u)}$, it is more convenient to pass to a bilinear form $\Lambda := i\widetilde{\Lambda}$, and a Hermitian form $M := i\widetilde{M}$. Thus, we get a space endowed with two forms, skew symmetric and Hermitian. Denoting by e_l the standard basis in $\mathbb{C}^{2n}$ and passing to a new basis $(e_j + e_{n+j})/\sqrt{2}$, $(e_j - e_{n+j})/\sqrt{2}$, where $j \leqslant n$, we arrive at expressions (3.2), (3.3). ⊠

3.6. The category Sp. Objects of the category **Sp** are the spaces $\mathcal{W}_{2n}$, where $n = 0, 1, 2, \dots$.

For $m, n = 0, 1, 2, \dots$, we equip the direct sum $\mathcal{W}_{2n} \oplus \mathcal{W}_{2m}$ with two forms,

$$\Lambda^{\ominus}(v \oplus w, v' \oplus w') := \Lambda_n(v,v') - \Lambda_m(w,w'),$$
$$M^{\ominus}(v \oplus w, v' \oplus w') := M_n(v,v') - M_m(w,w'). \tag{3.4}$$

A morphism $\mathcal{W}_{2n} \to \mathcal{W}_{2m}$ is a linear relation $P : \mathcal{W}_{2n} \rightrightarrows \mathcal{W}_{2m}$ satisfying the following conditions.

1. P is $\Lambda^{\ominus}$-Lagrangian.

2. The form $M^{\ominus}$ is non-positive on P.

3. The form $M_{\mathcal{W}_{2n}}$ is strictly negative on $\ker P$ and the form $M_{\mathcal{W}_{2m}}$ is strictly positive on $\operatorname{indef} P$.

A product of morphisms is the usual product of linear relations[21].

Observation 3.7 *The group of automorphisms of $\mathcal{W}_{2n}$ is the real symplectic group* $\mathrm{Sp}(2n,\mathbb{R})$.

This follows from the remark given in the previous subsection. The group of operators preserving both the forms $\widetilde{\Lambda}$, $\widetilde{M}$ on $\mathbb{C}^{2n}$ preserves also the real subspace $\mathbb{R}^{2n}$.

3.7. Construction of Gaussian operators from linear relations. Recall that $\mathcal{W}_{2n}$ is $\mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$. Denote this decomposition by

$$\mathcal{W}_{2n} = \mathcal{W}_n^+ \oplus \mathcal{W}_n^- .$$

Represent a linear relation P as the graph of an operator

$$S = S(P) : \mathcal{W}_m^- \oplus \mathcal{W}_n^+ \to \mathcal{W}_m^+ \oplus \mathcal{W}_n^- .$$

This is possible, because $M^{\ominus}$ is negative semidefinite on the subspace P and is strictly positive on the subspace $\mathcal{W}_m^+ \oplus \mathcal{W}_n^-$; therefore $P \cap (\mathcal{W}_m^+ \oplus \mathcal{W}_n^-) = 0$.

Proposition 3.8 *A matrix S has the form $S(P)$ if and only if it is symmetric and satisfies the Olshanski conditions from Theorem* 3.4.

[21] Under the condition 3, for morphisms $P : \mathcal{W}_{2n} \rightrightarrows \mathcal{W}_{2m}$, $Q : \mathcal{W}_{2m} \rightrightarrows \mathcal{W}_{2k}$, we have $\operatorname{im} P \cap \ker Q = 0$. For this reason null does not appear in the category **Sp**.

Theorem 3.9 *For each morphisms*

$$P : \mathcal{W}_{2n} \rightrightarrows \mathcal{W}_{2m}, \qquad Q : \mathcal{W}_{2m} \rightrightarrows \mathcal{W}_{2k},$$

the corresponding Gaussian operators

$$\mathfrak{B}\bigl[S(P)\bigr] : \mathbf{F}_n \to \mathbf{F}_m, \qquad \mathfrak{B}\bigl[S(Q)\bigr] : \mathbf{F}_m \to \mathbf{F}_k$$

satisfy

$$\mathfrak{B}\bigl[S(Q)\bigr]\,\mathfrak{B}\bigl[S(P)\bigr] = \lambda(Q,P)\,\mathfrak{B}\bigl[S(QP)\bigr],$$

where QP is the product of linear relations and $\lambda(Q,P)$ is a nonzero scalar.

As formulated, the theorem can be proved by direct force.

3.8. Construction of linear relations from Gaussian operators. For

$$\begin{pmatrix} v_1^+ & \dots & v_n^+ & v_1^- & \dots & v_n^- \end{pmatrix} \in \mathcal{V}_{2n},$$

we define the differential operator (a *creation-annihilation operator*)

$$\widehat{a}(v) f(z) = \Bigl(\sum_j v_j^+ z_j + \sum_j v_j^- \frac{\partial}{\partial z_j}\Bigr) f(z).$$

For a given bounded Gaussian operator $\mathfrak{B}[S] : \mathbf{F}_n \to \mathbf{F}_m$ we consider the set P of all $v \oplus w \in \mathcal{W}_{2n} \oplus \mathcal{W}_{2m}$ such that

$$\widehat{a}(w)\,\mathfrak{B}[S] = \mathfrak{B}[S]\,\widehat{a}(v).$$

Theorem 3.10 *The linear relation P is a morphism of the category* **Sp**.

3.9. Details. An analogue of the Schwartz space. We define the Schwartz–Fock space $\boldsymbol{\mathcal{S}}\mathbf{F}_n$ as the subspace in $\mathbf{F}_n$ consisting of all

$$f(z) = \sum c_{j_1,\dots,j_n} z^{j_1} \dots z^{j_n}$$

such that for each N

$$\sup_j |c_{j_1,\dots,j_n}| \prod_k j_k!\, {j_k}^N < \infty.$$

Theorem 3.11 *The subspace $\boldsymbol{\mathcal{S}}\mathbf{F}_n$ is a common invariant domain for all Gaussian bounded operators and for all creation-annihilation operators.*

See [53], Subsect. 4.2.4 and Theorem 5.1.5.

3.10. Details. The Olshanski semigroup $\Gamma\mathrm{Sp}(2n,\mathbb{R})$. The *Olshanski semigroup* $\Gamma\mathrm{Sp}(2n,\mathbb{R})$ is defined as the subsemigroup in $\mathrm{Sp}(2n,\mathbb{C})$ consisting of complex matrices g satisfying the condition

$$g\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} g^* - \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \leqslant 0,$$

where $g^* = \overline{g}^t$ denotes the adjoint matrix (see [55]).

Equivalently, $g \in \Gamma\mathrm{Sp}(2n, \mathbb{R})$ if and only if

$$M(ug, ug) \leqslant M(u, u) \quad \text{for all } u \in \mathbb{C}^{2n}.$$

The Olshanski semigroup is a subsemigroup in the semigroup of endomorphisms of the object $\mathcal{W}_{2n}$. For details, see [53], Sect. 2.7, 3.5.

4 Gauss–Berezin integrals

Here we discuss super-analogues of Gaussian integrals. Actually, the final formulas are not used, but their structure is important for us.

Apparently, these integrals are evaluated somewhere, but I do not know a reference. Calculations in the fermionic case are contained in [59].

4.1. Phantom algebra. *Phantom generators* $\mathfrak{a}_1$, $\mathfrak{a}_2$, ... are anticommuting variables,

$$\mathfrak{a}_k \mathfrak{a}_l = -\mathfrak{a}_l \mathfrak{a}_k, \qquad \mathfrak{a}_j^2 = 0.$$

We define a *phantom algebra* $\mathcal{A}$ as the algebra of polynomials in the variables $\mathfrak{a}_j$. *For the sake of simplicity, we assume that the number of variables is infinite.* We also call elements of $\mathcal{A}$ *phantom constants.*

The phantom algebra has a natural $\mathbb{Z}$-grading by degree of monomials,

$$\mathcal{A} = \oplus_{j=0}^{\infty} \mathcal{A}_j.$$

Therefore, $\mathcal{A}$ admits a $\mathbb{Z}_2$-grading, namely

$$\mathcal{A}_{\mathrm{even}} := \oplus \mathcal{A}_{2j}, \qquad \mathcal{A}_{\mathrm{odd}} := \oplus \mathcal{A}_{2j+1}.$$

We define the automorphism $\mu \mapsto \mu^\sigma$ of $\mathcal{A}$ by the rule

$$\mu^\sigma = \begin{cases} \mu & \text{if } \mu \text{ is even,} \\ -\mu & \text{if } \mu \text{ is odd} \end{cases} \tag{4.1}$$

(equivalently, $\mathfrak{a}_j^\sigma = -\mathfrak{a}_j$).

The algebra $\mathcal{A}$ is *supercommutative* in the following sense:

$$\mu \in \mathcal{A},\ \nu \in \mathcal{A}_{\mathrm{even}} \quad \Longrightarrow \quad \mu\nu = \nu\mu,$$
$$\mu \in \mathcal{A}_{\mathrm{odd}},\ \nu \in \mathcal{A}_{\mathrm{odd}} \quad \Longrightarrow \quad \mu\nu = -\nu\mu.$$

Also,

$$\mu \in \mathcal{A},\ \nu \in \mathcal{A}_{\mathrm{odd}} \Longrightarrow \quad \nu\mu = \mu^\sigma \nu. \tag{4.2}$$

Next, represent $\mu \in \mathcal{A}$ as $\mu = \sum_{j \geqslant 0} \mu_j$, where $\mu_j \in \mathcal{A}_j$. We define the map

$$\pi_\downarrow : \mathcal{A} \to \mathbb{C}$$

by

$$\pi_\downarrow(\mu) = \pi_\downarrow\Bigl(\sum\nolimits_{j\geqslant 0} \mu_j\Bigr) := \mu_0 \in \mathbb{C}.$$

Evidently,

$$\pi_\downarrow(\mu_1\mu_2) = \pi_\downarrow(\mu_1)\,\pi_\downarrow(\mu_2), \qquad \pi_\downarrow(\mu_1+\mu_2) = \pi_\downarrow(\mu_1) + \pi_\downarrow(\mu_2).$$

Take $\varphi \in \mathcal{A}$ such that $\pi_\downarrow(\varphi) = 0$. Then $\varphi^N = 0$ for sufficiently large N. Therefore,

$$(1+\varphi)^{-1} := \sum_{n\geqslant 0} (-\varphi)^n, \tag{4.3}$$

actually, the sum is finite. In particular, if $\pi_\downarrow(\mu) \neq 0$, then μ is invertible.

4.2. A technical comment. The aim of this paper is a specific construction and we use a minimal vocabulary necessary for our aims. We regard supergroups $\mathrm{GL}(p|q)$ and $\mathrm{OSp}(2p|2q)$ as groups of matrices over the algebra $\mathcal{A}$, representations of supergroups are defined in modules over $\mathcal{A}$.

The more common point of view[22] is to consider supergroups as functors from the category of supercommutative algebras to the category of groups. Our constructions require a restriction of a class of supercommutative algebras[23]. Precisely, the algebra $\mathcal{A}$ can be replaced[24] by an arbitrary finitely or countably generated supercommutative algebra $\mathcal{A}'$ over $\mathbb{C}$ satisfying the following properties:

— there is a non-zero homomorphism $\pi_\downarrow : \mathcal{A}' \to \mathbb{C}$, denote $I := \ker \pi_\downarrow$;

— any finitely generated subalgebra in I is nilpotent.

A pass to algebras equipped with the Krull topology make situation more flexible. We can consider supercommutative algebras $\mathcal{A}''$ satisfying the following properties:

— there is a non-zero homomorphism $\pi_\downarrow : \mathcal{A}'' \to \mathbb{C}$, denote $I := \ker \pi_\downarrow$;

— I is finitely or countably generated, and $\cap_n I^n = 0$;

— $\mathcal{A}''$ is complete with respect to I-adic topology (Krull topology, see [19], Subsect. 7.7), i.e., for any sequence $a_n \in \mathcal{A}''/I^n$ such that natural maps $\mathcal{A}''/I^n \to \mathcal{A}''/I^{n-1}$ send a_n to a_{n-1}, there is $a \in \mathcal{A}''$ whose image in each $\mathcal{A}''/I^n$ is a_n.

All our constructions below are functorial with respect to algebras of such types.

Below we use terms '*supergroups*' and *super-Grassmannians* for objects defined over the algebra $\mathcal{A}$ keeping in mind that this can be translated into the functorial language.

[22]On comparison and criticism of different definitions of super-objects, see, e.g., [44], [60]. Our approach follows DeWitt's book [18], we use only linear algebra and integral oprators (without analysis on manifolds). Wider generality discussed in this subsection is similar to Berezin–Kats [9].

[23]We need exponentials of even elements (4.6), $\mathbb{C}$-valued integrals (4.5), and inverses (4.3), (4.8). We also must justify the calculation in Theorem 4.3 and the proof of Lemma 7.1.

[24]with some modifications in proofs.

4.3. Berezinian. Let $\begin{pmatrix} P & Q \\ R & T \end{pmatrix}$ be a block $(p+q)\times(p+q)$-matrix, let P, T be composed of even phantom constants, and Q, R be composed of odd phantom constants. Then the Berezinian (or Berezin determinant) is[25]

$$\operatorname{ber}\begin{pmatrix} P & Q \\ R & T \end{pmatrix} := (\det P)^{-1}\cdot\det(T-QP^{-1}R).$$

We note that P and $T-QP^{-1}R$ are composed of elements of the commutative algebra $\mathcal{A}_{\mathrm{even}}$, therefore their determinants are well-defined. The Berezinian satisfies the multiplicative property of the usual determinant

$$\operatorname{ber}(A)\operatorname{ber}(B)=\operatorname{ber}(AB).$$

4.4. Functions. We consider 3 types of variables:

— real or complex (bosonic) variables, we denote them by x_i, y_j (if they are real) and z_i, u_j (if they are complex);

— Grassmann (fermionic) variables, we denote them by ξ_i, η_j or $\overline{\eta}_j$;

— phantom generators $\mathfrak{a}_j$ as above.

Bosonic variables x_l commute with the fermionic variables ξ_j and phantom constants $\mu\in\mathcal{A}$. We also assume that the fermionic variables ξ_j and the phantom generators $\mathfrak{a}_l$ anticommute,

$$\xi_j\mathfrak{a}_l=-\mathfrak{a}_l\xi_j.$$

Fix a collection of bosonic variables x_1, …, x_p and a collection of fermionic variables ξ_1,…, ξ_q. A *function* is a sum of the form

$$f(x,\xi):=\sum_{m\geqslant 0}\ \sum_{0<i_1<\dots<i_k\leqslant q;\, j_1<\dots<j_m} h_{i_1,\dots,i_k;j_1,\dots,j_m}(x_1,\dots,x_p)\times \\ \times\,\mathfrak{a}_{j_1}\dots\mathfrak{a}_{j_m}\,\xi_{i_1}\dots\xi_{i_k}, \qquad (4.4)$$

where h are smooth functions of $x\in\mathbb{R}^p$. We also write such expressions in the form

$$f(x,\xi)=\sum_{I,J}h_{I,J}(x)\mathfrak{a}^J\xi^I$$

keeping in the mind that I ranges in collections $0<i_1<\dots<i_k\leqslant q$ and J in collections $j_1<\dots<j_m$.

We say, that a function f is *even* (respectively *odd*) if it is an even expression in the total collection ξ_i, $\mathfrak{a}_k$. By $f^\sigma(x,\xi)$ we denote the function obtained from $f(x,\xi)$ by the substitution $\mathfrak{a}_j\mapsto-\mathfrak{a}_j$ for all j.

Remark. Formally, the fermionic variables and the phantom constants have equal rights in our definition. However, below their roles are different: ξ_j serve as variables and elements of $\mathcal{A}$ serve as constants (see (4.7)). ⊠

[25] The usual determinant of a block complex matrix $\begin{pmatrix} P & Q \\ R & T \end{pmatrix}$ is $\det P\cdot\det(T-QP^{-1}R)$.

REMARK. There are three ways to understand expressions (4.4). We consider them as finite sums, but it is possible to consider them as arbitrary formal series in variables $\mathfrak{a}_j$. We also can consider formal series such that each m-th summand depends only on finitely many of $\mathfrak{a}_j$ (considerations below survive in these cases after minor modifications). ⊠

For a given f, we define the function

$$\pi_\downarrow(f) := \sum_I h_{I,\varnothing}(x)\,\xi^I \in C^\infty(\mathbb{R}^p)\otimes\Lambda_q.$$

4.5. Integral. Now, we define the symbols

$$\int f(x,\xi)\,dx \qquad \int f(x,\xi)\,d\xi, \qquad \int f(x,\xi)\,dx\,d\xi.$$

The integration with respect to x is the usual termwise integration in (4.4),

$$\int_{\mathbb{R}^p} f(x,\xi)\,dx := \sum_{I,J}\Bigl(\int h_{I,J}(x)\,dx\Bigr)\mathfrak{a}^J\xi^I. \tag{4.5}$$

The integration with respect to ξ is the usual termwise Berezin integral,

$$\int f(x,\xi)\,d\xi := \sum_J h_{LJ}(x)\mathfrak{a}^J, \qquad \text{where } L=\{1,2,\dots,q\}.$$

4.6. Exponential. Let $f(x,\xi)$ be an *even* expression in ξ, $\mathfrak{a}$, i.e., $f(x,\xi) = f(x,-\xi)^\sigma$. We define its exponential as usual, it satisfies the usual properties. Namely,

$$\exp\{f(x,\xi)\} := \sum_{n=0}^\infty \frac{1}{n!} f(x,\xi)^n. \tag{4.6}$$

Since f_1, f_2 are even, we have $f_1f_2 = f_2f_1$. Therefore, the identity

$$\exp\{f_1+f_2\} = \exp\{f_1\}\exp\{f_2\}$$

holds.

Observation 4.1 *The series* (4.6) *converges.*

Indeed,

$$\exp\bigl\{f(x,\xi)\bigr\} = \exp\bigl\{h_{\varnothing\varnothing}(x)\bigr\}\prod_{(I,J)\ne(\varnothing,\varnothing)}\exp\bigl\{h_{I,J}(x)\mathfrak{a}^I\xi^J\bigr\} =$$
$$= \exp\bigl\{h_{\varnothing\varnothing}(x)\bigr\}\prod_{(I,J)\ne(\varnothing,\varnothing)}(1+h_{I,J}(x)\mathfrak{a}^I\xi^J).$$

Opening brackets, we get a polynomial in $\mathfrak{a}_j$, ξ_k.

4.7. Gauss–Berezin integrals. A special case. Take p real variables x_i and q Grassmann variables ξ_j. Consider the expression

$$I = \iint \exp\Bigl\{\frac{1}{2}\begin{pmatrix} x & \xi \end{pmatrix}\begin{pmatrix} A & B \\ -B^t & C\end{pmatrix}\begin{pmatrix} x^t \\ \xi^t\end{pmatrix}\Bigr\}\,dx\,d\xi =$$
$$= \iint \exp\Bigl\{\frac{1}{2}\sum_{ij} a_{ij}x_i x_j + \sum_{ik} b_{ik}x_i\xi_k + \frac{1}{2}\sum_{kl} c_{kl}\xi_k\xi_l\Bigr\}\,dx\,d\xi. \quad (4.7)$$

The notation ^{t} denotes the transpose as above, also $\begin{pmatrix} x & \xi \end{pmatrix}$ is the row-matrix

$$\begin{pmatrix} x & \xi \end{pmatrix} = \begin{pmatrix} x_1 & \dots & x_p & \xi_1 & \dots & \xi_q \end{pmatrix}.$$

The matrices A and C are composed of even phantom constants, A is symmetric, C is skew-symmetric, and B is a matrix composed of odd phantom constants.[26]

Observation 4.2 *The integral converges if and only if the matrix* $\operatorname{Re}\pi_\downarrow(A)$ *is negative definite.*

Indeed, the integrand $\exp\{\dots\}$ is a finite sum of the form

$$\exp\Bigl\{\frac{1}{2}\sum_{ij}\pi_\downarrow(a_{ij})x_i x_j\Bigr\}\sum_{i_1<\dots<i_k}\ \sum_{j_1<\dots<j_l} P_{i_1,\dots,i_k;\,j_1,\dots,j_l}(x)\xi_{i_1}\dots\xi_{i_k}\mathfrak{a}_{j_1}\dots\mathfrak{a}_{j_l},$$

where $P_{\dots}(x)$ are polynomials. Under the condition $\operatorname{Re}\pi_\downarrow(A) < 0$, a term-wise integration is possible. □

4.8. Evaluation of the Gauss–Berezin integral. Let us evaluate integral (4.7).

Theorem 4.3 *Let* $\operatorname{Re}\pi_\downarrow(A) < 0$. *Then*

$$I = \begin{cases} (2\pi)^{p/2}\det(-A)^{-1/2}\operatorname{Pfaff}(C + B^t A^{-1} B) & \text{if } q \text{ is even,} \\ 0, & \text{otherwise.}\end{cases} \quad (4.8)$$

Recall that q is the number of Grassmann variables.

Remark. The matrix $C + B^t A^{-1} B$ is skew-symmetric and composed of even phantom constants. Therefore, the Pfaffian is well defined. ⊠

Remark. Thus, for q even our expression is a 'hybrid' of a Pfaffian and a Berezinian,

$$I^{-2} = -(2\pi)^p \operatorname{ber}\begin{pmatrix} A & B \\ -B^t & C\end{pmatrix}.$$

Similar (but not precisely same) 'hybrid' appeared in [64] and [39]. ⊠

[26] The argument of the exponential must be even in ξ, $\mathfrak{a}$. This imposes the constraints of parity for A, B, C. The symmetry conditions for A, B, C are the natural conditions for coefficients of a quadratic form in x, ξ.

Proof of Theorem 4.3. First, we integrate with respect to x,

$$\exp\Big\{\frac{1}{2}\xi C\xi^t\Big\}\int_{\mathbb{R}^p}\exp\Big\{\frac{1}{2}xAx^t+xB\xi^t\Big\}\,dx=$$
$$=\exp\Big\{\frac{1}{2}\xi C\xi^t\Big\}\times$$
$$\times\int_{\mathbb{R}^p}\exp\Big\{\frac{1}{2}(x-\xi B^tA^{-1})A(x^t+A^{-1}B\xi^t)\Big\}\exp\Big\{\frac{1}{2}\xi B^tA^{-1}B\xi^t\Big\}\,dx.$$

We substitute

$$y:=x-\xi B^tA^{-1}, \tag{4.9}$$

get

$$\exp\Big\{\frac{1}{2}\xi C\xi^t\Big\}\cdot\exp\Big\{\frac{1}{2}\xi B^tA^{-1}B\xi^t\Big\}\int\exp\Big\{\frac{1}{2}yAy^t\Big\}\,dy,$$

and arrive at the usual Gaussian integral (1.13).

Integrating the result, we get

$$\det(-A)^{-1/2}(2\pi)^{p/2}\int\exp\Big\{\frac{1}{2}\xi(C+B^tA^{-1}B)\xi^t\Big\}\,d\xi,$$

and arrive at the Pfaffian.

We must justify the substitution (4.9). Let Φ be a function on $\mathbb{R}^p$ of Schwartz class, let ν be an even expression in $\mathfrak{a}$, ξ, assume that the constant term of ν is 0. Then

$$\int_{\mathbb{R}^p}\Phi(x+\nu)\,dx=\int_{\mathbb{R}^p}\Phi(x)\,dx.$$

Indeed,

$$\Phi(x+\nu):=\sum_{j=0}^{\infty}\frac{1}{j!}\nu^j\frac{d^j}{dx^j}\Phi(x).$$

Actually, the summation is finite. A termwise integration with respect to x gives zero for all $j\neq 0$. □

4.9. Grassmann Gaussian integral.

Observation 4.4 *Let D be a* complex *skew-symmetric matrix of size N, let ξ_k, ζ_k be Grassmann variables. Then the integral*

$$\int\exp\Big\{\frac{1}{2}\xi D\xi^t+\xi\zeta^t\Big\}\,d\xi$$

can be represented in the form

$$s\cdot\prod_{j=1}^{m}\big(\zeta h_j^t\big)\cdot\exp\Big\{\frac{1}{2}\zeta Q\zeta^t\Big\},$$

where $s\in\mathbb{C}$, Q is a skew-symmetric matrix, and h_j are row-matrices, $m\leqslant N$, $N-m$ is even.

Indeed, one can find a linear substitution $\xi = \eta S$ such that[27]

$$\xi D \xi^t = \sum_{j=1}^{\gamma} \eta_{2j-1}\eta_{2j1}.$$

Then the integral can be reduced to

$$\det S \int \exp\Bigl\{\sum_{j=1}^{\gamma} \eta_{2j-1}\eta_{2j} + \sum_{k=1}^{N} \eta_k \nu_k\Bigr\}\, d\eta,$$

where ν_j are certain linear expressions in ζ_l. So we get

$$\det S \cdot \prod_{j=1}^{\gamma} \int \exp\bigl\{\eta_{2j-1}\eta_{2j} + \eta_{2j-1}\nu_{2j-1} + \eta_{2j}\nu_{2j}\bigr\}\, d\eta_{2j-1}\, d\eta_{2j} \times$$
$$\times \int \exp\Bigl\{\sum_{j=2\gamma+1}^{N} \eta_j \nu_j\Bigr\}\, d\eta_{2\gamma+1} \dots d\eta_N =$$
$$= \pm \det S \cdot \exp\Bigl\{-\sum_{j=1}^{\gamma} \nu_{2j}\nu_{2j+1}\Bigr\} \prod_{k=2\gamma+1}^{N} \nu_k.$$

Recall that ν_j are certain linear expressions[28] in ζ_l. $\square$.

4.10. More general Gauss–Berezin integrals. Consider an expression

$$J = \iint \exp\Bigl\{\frac{1}{2}\begin{pmatrix} x & \xi\end{pmatrix}\begin{pmatrix} A & B \\ -B^t & C\end{pmatrix}\begin{pmatrix} x^t \\ \xi^t\end{pmatrix} + x h^t + \xi g^t\Bigr\}\, dx\, d\xi =$$
$$= \iint \exp\Bigl\{\frac{1}{2}\sum_{ij} a_{ij} x_i x_j + \sum_{ik} b_{ik} x_i \xi_k + \frac{1}{2}\sum_{kl} c_{kl}\xi_k\xi_l +$$
$$+ \sum_j h_j x_j - \sum_k g_k \xi_k\Bigr\}\, dx\, d\xi, \qquad (4.10)$$

here A, B, C are as above and h^t, g^t are column-vectors, $h_j \in \mathcal{A}_{\mathrm{even}}$, $g_k \in \mathcal{A}_{\mathrm{odd}}$.

We propose two ways to evaluate of this integral.

4.11. The first way to evaluate. Substituting

$$\begin{pmatrix} y & \eta\end{pmatrix} = \begin{pmatrix} x & \xi\end{pmatrix} + \begin{pmatrix} h & g\end{pmatrix}\begin{pmatrix} A & B \\ -B^t & C\end{pmatrix}^{-1},$$

[27]In other words, one can reduce a skew-symmetric matrix over $\mathbb{C}$ to a canonical form.

[28]A product of functions ν_m is canonically defined up to a constant factor, equivalently a linear span of functions ν_m is canonically defined (this sentence is a rephrasing of the Plücker embedding of a Grassmannian into an exterior algebra).

we get

$$\exp\Big\{\frac{1}{2}\,(h\quad g)\begin{pmatrix}A & B\\ -B^t & C\end{pmatrix}^{-1}\begin{pmatrix}h^t\\ g^t\end{pmatrix}\Big\}\times$$
$$\times\iint\exp\Big\{\frac{1}{2}\,(y\quad \eta)\begin{pmatrix}A & B\\ -B^t & C\end{pmatrix}\begin{pmatrix}y^t\\ \eta^t\end{pmatrix}\Big\}\,dy\,d\eta$$

and arrive at Gauss–Berezin integral (4.7) evaluated above.

This way is not perfect, because it uses an inversion of a matrix $\begin{pmatrix}A & B\\ -B^t & C\end{pmatrix}$.

Observation 4.5 *A matrix* $\begin{pmatrix}A & B\\ -B^t & C\end{pmatrix}$ *is invertible if and only if* A *and* C *are invertible.*

The necessity is evident; to prove the sufficiency, we note that the matrix

$$T:=\begin{pmatrix}A^{-1} & 0\\ 0 & C^{-1}\end{pmatrix}\begin{pmatrix}A & B\\ -B^t & C\end{pmatrix}-\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

is composed of nilpotent elements of $\mathcal{A}$. We write out $(1+T)^{-1}=1-T+T^2-\dots$, and therefore our initial matrix $\begin{pmatrix}A & 0\\ 0 & C\end{pmatrix}(1+T)$ is invertible. □

The matrix A is invertible, because $\operatorname{Re}A<0$.

But the matrix C is skew-symmetric.

— If q is even, then a $q\times q$ skew-symmetric matrix C in general position is invertible. For noninvertible C, we have a chance to remove uncertainty. This way leads to an expression of the form (4.12) obtained below.

— If q is odd, then C is non-invertible; our approach is not suitable.

4.12. The second way to evaluate of Gauss–Berezin integrals. First, we integrate with respect to x,

$$\exp\Big\{\frac{1}{2}\xi C\xi^t+\xi g^t\Big\}\int_{\mathbb{R}^p}\exp\Big\{\frac{1}{2}xAx^t+xB\xi^t+xh^t\Big\}\,dx=$$
$$=\exp\Big\{\frac{1}{2}\xi C\xi^t+\xi g^t\Big\}\exp\Big\{\frac{1}{2}(h-\xi B^tA^{-1})A(h^t+A^{-1}B\xi^t)\Big\}\times$$
$$\times\int_{\mathbb{R}^p}\exp\Big\{\frac{1}{2}(x+h-\xi B^t)A(x^t+h^t+B\xi^t)\Big\}\,dx.$$

Substituting $y=x+h-\xi B^tA^{-1}$ and integrating with respect to y, we get

$$(2\pi)^{p/2}\det(-A)^{-1/2}\exp\Big\{\frac{1}{2}(h-\xi B^tA^{-1})A(h^t+A^{-1}B\xi^t)+\frac{1}{2}\xi C\xi^t+\xi g^t\Big\}.$$

Next, we must integrate with respect to ξ, our integral has a form

$$\int\exp\Big\{\frac{1}{2}\xi D\xi^t+\xi r^t\Big\}\,d\xi, \tag{4.11}$$

where a matrix D is composed of even phantom constants and a vector r is odd.

If D is invertible, we shift the argument again $\eta^t := \xi^t + D^{-1}r^t$ and get

$$\exp\Bigl\{\frac{1}{2} r D^{-1} r^t\Bigr\} \int \exp\Bigl\{\frac{1}{2}\eta D \eta^t\Bigr\}\, d\eta,$$

the last integral is a Pfaffian. This way is equivalent to the approach discussed in the previous subsection.

Now, consider an arbitrary D. The calculation of Subsection 4.9 does not survive[29].

However, we can write (4.11) explicitly as follows. For any subset

$$I:\, i_1 < \dots < i_{2k}$$

in $\{1,\dots,q\}$ we consider the complementary subset

$$J: j_1 < \dots < j_{q-2k}.$$

Define the constant $\sigma(I) = \pm 1$ as follows

$$\bigl(\xi_{i_1}\xi_{i_2}\dots\xi_{i_{2k}}\bigr)\bigl(\xi_{j_1}\xi_{j_2}\dots\xi_{j_{q-2k}}\bigr) = \sigma(I)\,\xi_1\xi_2\dots\xi_q.$$

Evidently,

$$\int \exp\Bigl\{\frac{1}{2}\xi D\xi^t + \xi r^t\Bigr\}\, d\xi = \\ = \sum_I \sigma(I) \mathrm{Pfaff} \begin{pmatrix} 0 & d_{i_1 i_2} & \dots & d_{i_1 i_{2k}} \\ d_{i_2 i_1} & 0 & \dots & d_{i_2 i_{2k}} \\ \vdots & \vdots & \ddots & \vdots \\ d_{i_{2k} i_1} & d_{i_2 i_{2k}} & \dots & 0 \end{pmatrix} r_{j_1} \dots r_{j_{q-2k}}. \qquad (4.12)$$

Recall that $d_{pq} \in \mathcal{A}_{\mathrm{even}}$ and $r \in \mathcal{A}_{\mathrm{odd}}$.

5 Gauss–Berezin integral operators

Here we define super hybrids of Gaussian operators and Berezin operators.

5.1. Fock–Berezin spaces. Fix p, $q = 0, 1, 2, \dots$. Let $z_1, \dots, z_p$ be complex variables, $\xi_1, \dots, \xi_q$ be Grassmann variables. We consider expressions

$$f(z,\xi) =: \sum_{I,J} f_{I,J}(z) \mathfrak{a}^J \xi^I,$$

where $r_{I,J}$ are entire functions in z and the summation is finite. We define the map $f \mapsto \pi_\downarrow(f)$ as above.

[29] Let D be a skew-symmetric matrix over $\mathcal{A}_{\mathrm{even}}$. If $\pi_\downarrow(D)$ is degenerate, then we can not reduce D to a normal form.

We define the *Fock–Berezin* space $\boldsymbol{\mathcal{S}}\mathbf{F}_{p,q}(\mathcal{A})$ as the space of all functions $f(z,\xi)$ satisfying the condition: *for each I, J, the function $f_{I,J}(z)$ is in the Schwartz–Fock space $\boldsymbol{\mathcal{S}}\mathbf{F}_p$,* see Subsection 3.9. We say that a sequence $f^{(k)} \in \boldsymbol{\mathcal{S}}\mathbf{F}_{p,q}(\mathcal{A})$ converges[30] to f if

— for all but a finite number of J all $f^{(k)}_{I,J}$ are zero;

— for each I, J, we have a convergence $f^{(k)}_{I,J}(z) \to f_{I,J}(z)$ in $\boldsymbol{\mathcal{S}}\mathbf{F}_p$.

Remark. We can assume that all the $f_{I,J}$ are in the Hilbert–Fock space $\mathbf{F}_p$. But Gauss–Berezin operators defined below can be unbounded in this space; therefore this point of view requires descriptions of domains of operators and examination of products of operators. Our definition admits some variations (we chose an open dense subset in $\mathbf{F}_p$, and our choice is volitional). ⊠

5.2. Another form of the Gauss–Berezin integral. Consider the integral

$$\int \exp\Bigl\{\frac12 \begin{pmatrix} z & \xi\end{pmatrix} \begin{pmatrix} A & B\\ -B^t & C\end{pmatrix} \begin{pmatrix} z^t\\ \xi^t\end{pmatrix} + z\alpha^t + \xi\beta^t\Bigr\}\times$$
$$\times \exp\Bigl\{\frac12 \begin{pmatrix} \overline{z} & \overline{\xi}\end{pmatrix} \begin{pmatrix} K & L\\ -L^t & M\end{pmatrix} \begin{pmatrix} \overline{z}\\ \overline{\xi}\end{pmatrix} + \overline{z}\varkappa + \overline{\xi}\lambda^t\Bigr\} \cdot e^{-z\overline{z}^t - \xi\overline{\xi}^t}\, dz\, d\overline{z}\, d\xi\, d\overline{\xi}, \quad (5.1)$$

where two matrices $\begin{pmatrix} A & B\\ -B^t & C\end{pmatrix}$, $\begin{pmatrix} K & L\\ -L^t & M\end{pmatrix}$ have the same structure as in Subsect. 4.7, row-vectors α, $\varkappa$ are even, the vectors β, λ are odd.

Since $\mathbb{C}^n \simeq \mathbb{R}^{2n}$, this integral is a special case of the Gauss–Berezin integral. We get

$$\mathrm{const}\cdot \exp\left\{ \frac12 \begin{pmatrix} \alpha & \beta & \varkappa & \lambda\end{pmatrix} \begin{pmatrix} -A & -B & 1 & 0\\ B^t & -C & 0 & 1\\ 1 & 0 & -K & -L\\ 0 & -1 & L^t & -M\end{pmatrix}^{-1} \begin{pmatrix} \alpha^t\\ \beta^t\\ \varkappa^t\\ \lambda^t\end{pmatrix}\right\}, \quad (5.2)$$

where the scalar factor is a hybrid of the Pfaffian and the Berezinian mentioned above in Subsect. 4.8.

5.3. Integral operators. We write operators $\boldsymbol{\mathcal{S}}\mathbf{F}_{p,q}(\mathcal{A}) \to \boldsymbol{\mathcal{S}}\mathbf{F}_{r,s}(\mathcal{A})$ as

$$Af(z,\xi) = \int K(z,\xi;\overline{u},\overline{\eta})\, f(u,\eta)\, e^{-z\overline{z}^t - \eta\overline{\eta}^t}\, du\, d\overline{u}\, d\overline{\eta}\, d\eta. \quad (5.3)$$

5.4. Linear and antilinear operators. We say that an operator $A : \boldsymbol{\mathcal{S}}\mathbf{F}_{p,q}(\mathcal{A}) \to \boldsymbol{\mathcal{S}}\mathbf{F}_{r,s}(\mathcal{A})$ is *linear* if

$$A(f_1+f_2) = Af_1 + Af_2, \qquad A(\lambda f) = \lambda A f, \text{ where } \lambda \text{ is a phantom constant,}$$

and *antilinear* if

$$A(f_1+f_2) = Af_1 + Af_2, \qquad A(\lambda f) = \lambda^\sigma A f, \text{ where } \lambda \text{ is a phantom constant,}$$

[30] This convergence corresponds to a topology of inductive limit.

the automorphism $\lambda \mapsto \lambda^\sigma$. Clearly, the operators

$$Af(z,\xi) = \xi_j f(z,\xi), \qquad Bf(z,\xi) = \frac{\partial}{\partial \xi_j} f(z,\xi), \qquad Cf(z,\xi) = \mathfrak{a}_j f(z,\xi)$$

are antilinear.

An integral operator (5.3) is linear if the kernel $K(z,\xi,\overline{u},\overline{\eta})$ is an even function in the total collection of all Grassmann variables ξ, $\overline{\eta}$, $\mathfrak{a}$, i.e..

$$K(z,\xi,\overline{u},\eta) = K(z,-\xi,\overline{u},-\overline{\eta})^\sigma.$$

An operator (5.3) is antilinear if and only if the function K is odd.

Below we meet only linear and antilinear operators.

We define also the (antilinear) operator S of σ-conjugation,

$$\mathsf{S}\, f(z,\xi) = f^\sigma(z,\xi). \tag{5.4}$$

Evidently,

$$\mathsf{S}^2 f = f.$$

5.5. Gauss–Berezin vectors in the narrow sense. A Gauss–Berezin vector (in the narrow sense) is a vector of the form

$$\mathbf{b}\begin{bmatrix} A & B \\ -B^t & C \end{bmatrix} = \lambda \exp\Bigl\{\frac{1}{2}\begin{pmatrix} z & \xi \end{pmatrix} \begin{pmatrix} A & B \\ -B^t & C \end{pmatrix} \begin{pmatrix} z^t \\ \xi^t \end{pmatrix}\Bigr\}, \tag{5.5}$$

where A, B, C are as above, see Subsection 4.7.

Observation 5.1 $\mathbf{b}[\cdot] \in \boldsymbol{\mathcal{S}}\mathbf{F}_{p,q}(\mathcal{A})$ *if and only if* $\|\pi_\downarrow(A)\| < 1$.

5.6. Gauss–Berezin operators in the narrow sense. A *Gauss–Berezin integral operator in the narrow sense* is an integral operator

$$\boldsymbol{\mathcal{S}}\mathbf{F}_{p,q}(\mathcal{A}) \to \boldsymbol{\mathcal{S}}\mathbf{F}_{r,s}(\mathcal{A}),$$

whose kernel as a function in $z_1, \dots, z_r$, $\overline{u}_1, \dots, u_p$, $\xi_1, \dots, \xi_s$, $\overline{\eta}_1$, $\overline{\eta}_q$ is a Gauss–Berezin vector. Precisely, a Gauss–Berezin operator has the form

$$\mathfrak{B} f(z,\xi) = \\ = \lambda \cdot \iint \exp\left\{\frac{1}{2}\begin{pmatrix} z & \xi & \overline{u} & \overline{\eta} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} \begin{pmatrix} z^t \\ \xi^t \\ \overline{u}^t \\ \overline{\eta}^t \end{pmatrix}\right\} f(u,\eta) \times \\ \times e^{-\eta\overline{\eta}^t - u\overline{u}^t}\, du\, d\overline{u}\, d\eta\, d\overline{\eta}, \tag{5.6}$$

where λ is an even phantom constant, A_{ij} is composed of even phantom constants if $(i+j)$ is even, otherwise A_{ij} is composed of odd phantom constants.

They also satisfy the natural symmetry conditions for a matrix of a quadratic form in the variables z, ξ, $\overline{u}$, $\overline{\eta}$.

REMARK. On the other hand, a Gauss–Berezin vector can be regarded as a Gauss–Berezin operator $\boldsymbol{\mathcal{S}}\mathbf{F}_{0,0}(\mathcal{A}) \to \boldsymbol{\mathcal{S}}\mathbf{F}_{p,q}(\mathcal{A})$. ⊠

For Gauss–Berezin operators

$$\mathfrak{B}_1 : \boldsymbol{\mathcal{S}}\mathbf{F}_{p,q}(\mathcal{A}) \to \boldsymbol{\mathcal{S}}\mathbf{F}_{p',q'}(\mathcal{A}), \qquad \mathfrak{B}_2 : \boldsymbol{\mathcal{S}}\mathbf{F}_{p',q'}(\mathcal{A}) \to \boldsymbol{\mathcal{S}}\mathbf{F}_{p'',q''}(\mathcal{A})$$

evaluation of their product is reduced to the Gauss–Berezin integral (5.1). For operators in general position, we can apply formula (5.2). Evidently, in this case the product is a Gauss–Berezin operator again. However, our final Theorem 9.4 avoids this calculation.

Also, considerations of Section 2 suggest an extension of the definition of Gauss–Berezin operators.

5.7. General Gauss–Berezin operators. As above, we define first order differential operators

$$\mathfrak{D}[\xi_j]f := \Bigl(\xi_j + \frac{\partial}{\partial \xi_j}\Bigr) f.$$

If a function f is independent of ξ_j, then

$$\mathfrak{D}[\xi_j]f = \xi_j f, \qquad \mathfrak{D}[\xi_j]\xi_j f = f.$$

Evidently,

$$\mathfrak{D}[\xi_j]^2 = 1, \qquad \mathfrak{D}[\xi_i]\,\mathfrak{D}[\xi_j] = -\mathfrak{D}[\xi_j]\,\mathfrak{D}[\xi_i], \quad , i \neq j.$$

The operators $\mathfrak{D}[\xi_j]$ are antilinear.

A *Gauss–Berezin operator* $\boldsymbol{\mathcal{S}}\mathbf{F}_{p,q}(\mathcal{A}) \to \boldsymbol{\mathcal{S}}\mathbf{F}_{r,s}(\mathcal{A})$ is an operator of the form

$$\mathfrak{C} = \lambda \mathfrak{D}[\xi_{i_1}] \dots \mathfrak{D}[\xi_{i_k}]\,\mathfrak{B}\,\mathfrak{D}[\eta_{m_1}] \dots \mathfrak{D}[\eta_{m_l}] \cdot \mathsf{S}^{k+l}, \tag{5.7}$$

where the operator S is given by (5.4) and

— $\mathfrak{B}$ is a Gauss –Berezin operator in the narrow sense;

— $i_1 < i_2 < \dots < i_k$, $m_1 < m_2 < \dots < m_l$, and k, $l \geqslant 0$;

— λ is an even invertible phantom constant.

Note that a Gauss–Berezin operator is linear.

REMARK. We define the set of Gauss–Berezin operators as a union of 2^{p+q} sets. These sets are not disjoint. Actually, we get a supermanifold consisting of two connected components (according to the parity of $k + l$). Each set (5.7) is open and dense in the corresponding component. This will become obvious below. ⊠

5.8. Operators $\pi_\downarrow(\mathfrak{B})$ and boundedness of Gauss–Berezin operators. Let $K(z, \xi, \overline{u}, \overline{\eta})$ be the kernel of a Gauss–Berezin operator. Then the formula

$$\mathfrak{C}f(z,\xi) = \int \pi_\downarrow\Bigl(K(z,\xi;\overline{u},\overline{\eta})\Bigr)\, f(u,\eta)\, e^{-z\overline{z}^t - \eta\overline{\eta}^t}\, du\, d\overline{u}\, d\overline{\eta}\, d\eta$$

determines an integral operator

$$\pi_\downarrow(\mathfrak{C}):\ \mathbf{F}_p\otimes\Lambda_q\to\mathbf{F}_r\otimes\Lambda_s.$$

Evidently, this operator is a tensor product of a Gaussian operator

$$\pi_\downarrow^+(\mathfrak{C}):\mathbf{F}_p\to\mathbf{F}_r$$

and a Berezin operator

$$\pi_\downarrow^-(\mathfrak{C}):\Lambda_q\to\Lambda_s.$$

For instance, for an operator $\mathfrak{B}$ given by the standard formula (5.6), we get the Gaussian operator

$$\pi_\downarrow^+(\mathfrak{B})f(z)=\int_{\mathbb{C}^n}\exp\Bigl\{\frac12\begin{pmatrix}z&\overline{u}\end{pmatrix}\left[\pi_\downarrow\begin{pmatrix}A_{11}&A_{13}\\A_{31}&A_{33}\end{pmatrix}\right]\begin{pmatrix}z^t\\\overline{u}^t\end{pmatrix}\Bigr\}\,f(z)\,e^{-z\overline{z}^t}\,dz\,d\overline{z}$$

and the Berezin operator

$$\pi_\downarrow^-(\mathfrak{B})g(\xi)=\int\exp\Bigl\{\frac12\begin{pmatrix}\xi&\overline{\eta}\end{pmatrix}\left[\pi_\downarrow\begin{pmatrix}A_{22}&A_{24}\\A_{42}&A_{44}\end{pmatrix}\right]\begin{pmatrix}\xi^t\\\overline{\eta}^t\end{pmatrix}\Bigr\}\,g(\xi)\,e^{-\xi\overline{\xi}^t}\,d\xi\,d\overline{\xi}.$$

Next,

$$\pi_\downarrow^\pm(\mathfrak{C}_1\mathfrak{C}_2)=\pi_\downarrow^\pm(\mathfrak{C}_1)\,\pi_\downarrow^\pm(\mathfrak{C}_2). \tag{5.8}$$

Observation 5.2 *The Gauss–Berezin operator* (5.6) *is bounded in the sense of Fock–Berezin spaces if and only if the operator* $\pi_\downarrow^+(\mathfrak{B})$ *is bounded* (*i.e., satisfies conditions of Theorem* 3.4).

PROOF. Only statement 'if' requires a proof. A kernel of a Gauss–Berezin operator has the form

$$\sum_{I,J,K}\exp\Bigl\{\frac12\begin{pmatrix}z&\overline{u}\end{pmatrix}\left[\pi_\downarrow\begin{pmatrix}A_{11}&A_{13}\\A_{31}&A_{33}\end{pmatrix}\right]\begin{pmatrix}z^t\\\overline{u}^t\end{pmatrix}\Bigr\}P_{I,J,K}(z,\overline{u})\mathfrak{a}^I\xi^J\overline{\eta}^K,$$

where $P_{I,J,K}$ is a polynomial in z, $\overline{u}$. Denote the quadratic form in the exponential by $S(z)$ and decompose $P_{I,J,K}(z,\overline{u})$ as a sum of monomials. It is sufficient to show that the following operators are bounded as operators $\boldsymbol{\mathcal{S}}\mathbf{F}_p\to\boldsymbol{\mathcal{S}}\mathbf{F}_r$:

$$\int\prod z_j^{l_j}\exp\{S(z,\overline{u})\}\prod\overline{u}_i^{k_i}f(u)\,e^{-z\overline{z}}\,d\lambda(u).$$

Integrating by parts (see, e.g., [50], Subsect. V.3.5) we arrive at

$$\prod z_j^{l_j}\cdot\int\exp\{S(z,\overline{u})\}\cdot\prod\Bigl(\frac{\partial}{\partial u_i}\Bigr)^{k_i}f(u)\cdot e^{-z\overline{z}}\,d\lambda(u).$$

It remains to notice that partial differentiations and multiplications by linear functions are bounded operators in the Fock-Schwartz spaces, see Theorem 3.11. □

5.9. Products of Gauss–Berezin operators.

Theorem 5.3 *For each Gauss–Berezin operator*

$$\mathfrak{B}_1 : \boldsymbol{\mathcal{S}}\mathbf{F}_{p,q}(\mathcal{A}) \to \boldsymbol{\mathcal{S}}\mathbf{F}_{p',q'}(\mathcal{A}), \qquad \mathfrak{B}_2 : \boldsymbol{\mathcal{S}}\mathbf{F}_{p',q'}(\mathcal{A}) \to \boldsymbol{\mathcal{S}}\mathbf{F}_{p'',q''}(\mathcal{A}),$$

their product $\mathfrak{B}_2\mathfrak{B}_1$ *is either a Gauss–Berezin operator or*

$$\pi_\downarrow(\mathfrak{B}_2\mathfrak{B}_1 f) = 0 \tag{5.9}$$

for all f.

Clearly, the condition (5.9) also is equivalent to $\pi_\downarrow^-(\mathfrak{B}_2\mathfrak{B}_1 f) = 0$.

For a proof, see Section 9. We also present an interpretation of the product in terms of linear relations.

Remark. In the case (5.9) the kernel of the product has the form (4.12) but it is not a Gauss–Berezin operator in our sense. Possibly, this requires to change our definitions. ⊠

5.10. General Gauss–Berezin vectors. A Gauss–Berezin vector is a vector of the form

$$\mathfrak{D}[\xi_{i_1}] \dots \mathfrak{D}[\xi_{i_k}]\, \mathsf{S}^k \mathfrak{b},$$

where $\mathfrak{b}$ is a Gauss–Berezin vector in the narrow sense.

Remark. We write S^k as in (5.7). However, omitting this factor does not change the definition. ⊠

6 Supergroups $\mathrm{OSp}(2p|2q)$

Here we define a super-analogue of the groups $\mathrm{O}(2n, \mathbb{C})$ and $\mathrm{Sp}(2n, \mathbb{C})$. For a general exposition of supergroups and super-Grassmannians, see books [8], [42], [11], [12].

6.1. Modules $\mathcal{A}^{p|q}$. Let

$$\mathcal{A}^{p|q} := \mathcal{A}^p \oplus \mathcal{A}^q$$

be a direct sum of $(p+q)$ copies of $\mathcal{A}$. We regard elements of $\mathcal{A}^{p|q}$ as row-vectors

$$(v_1, \dots, v_p; w_1, \dots, w_q).$$

We define a structure of $\mathcal{A}$-bimodule on $\mathcal{A}^{p|q}$. The addition in $\mathcal{A}^{p|q}$ is natural. The left multiplication by $\lambda \in \mathcal{A}$ is also natural

$$\lambda \circ (v_1, \dots, v_p; w_1, \dots, w_q) := (\lambda v_1, \dots, \lambda v_p; \lambda w_1, \dots, \lambda w_q).$$

The right multiplications by $\varkappa \in \mathcal{A}$ is

$$(v_1, \dots, v_p; w_1, \dots, w_q) * \varkappa := (v_1\varkappa, \dots, v_p\varkappa; w_1\varkappa^\sigma, \dots, w_q\varkappa^\sigma),$$

where σ is the involution of $\mathcal{A}$ defined above.

We define the *even part* of $\mathcal{A}^{p|q}$ as $(\mathcal{A}_{\text{even}})^p \oplus (\mathcal{A}_{\text{odd}})^q$ and the *odd part* as $(\mathcal{A}_{\text{odd}})^p \oplus (\mathcal{A}_{\text{even}})^q$.

6.2. Matrices. Denote by $\mathrm{Mat}(p|q;\mathcal{A})$ the space of $(p+q)\times(p+q)$ matrices over $\mathcal{A}$, we represent such matrices in the block form

$$Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

We say that a *matrix Q is even* if all matrix elements of A, D are even and all matrix elements of B, C are odd. *A matrix is odd* if elements of A, D are odd and elements of B, C are even.

A matrix Q acts on the space $\mathcal{A}^{p|q}$ as

$$v \to vQ.$$

Such transformations are compatible with the left $\mathcal{A}$-module structure on $\mathcal{A}^{p|q}$, i.e.,

$$(\lambda \circ v)\, Q = \lambda \circ (vQ) \qquad \text{for any } \lambda \in \mathcal{A},\ v \in \mathcal{A}^{p|q}.$$

However, *even matrices also regard the right $\mathcal{A}$-module structure,*

$$(v * \lambda)\, Q = (vQ) * \lambda \qquad \text{for any } \lambda \in \mathcal{A},\ v \in \mathcal{A}^{p|q}.$$

(we use the rule (4.2)).

6.3. Super-transposition. The *supertranspose* of Q is defined by

$$Q^{st} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{st} := \begin{cases} \begin{pmatrix} A^t & C^t \\ -B^t & D^t \end{pmatrix} & \text{if } Q \text{ is even,} \\ \begin{pmatrix} A^t & -C^t \\ B^t & D^t \end{pmatrix} & \text{if } Q \text{ is odd,} \end{cases}$$

and

$$(Q_1 + Q_2)^{st} := Q_1^{st} + Q_2^{st}.$$

The following identity holds

$$(QR)^{st} = \begin{cases} R^{st} Q^{st} & \text{if } Q \text{ or } R \text{ are even,} \\ -R^{st} Q^{st} & \text{if both } R \text{ and } Q \text{ are odd.} \end{cases} \tag{6.1}$$

Below we use only the first row.

6.4. The supergroups $\mathrm{GL}(p|q;\mathcal{A})$. The group $\mathrm{GL}(p|q;\mathcal{A})$ is the group of *even* invertible matrices in $\mathrm{Mat}(p|q;\mathcal{A})$. The following lemma is trivial.

Lemma 6.1 *An even matrix $Q \in \mathrm{Mat}(p|q;\mathcal{A})$ is invertible*

a) *if and only if the matrices A, D are invertible;*

b) *if and only if the matrices $\pi_\downarrow(A)$, $\pi_\downarrow(D)$ are invertible.*

Here $\pi_\downarrow(A)$ denotes the matrix composed of elements $\pi_\downarrow(a_{kl})$.

Also, the map $Q \mapsto \pi_\downarrow(Q)$ is a well-defined epimorphism

$$\pi_\downarrow : \mathrm{GL}(p|q;\mathcal{A}) \to \mathrm{GL}(p,\mathbb{C}) \times \mathrm{GL}(q,\mathbb{C})$$

(because $\pi_\downarrow(B) = 0$, $\pi_\downarrow(D) = 0$).

6.5. The supergroup $\mathrm{OSp}(2p|2q;\mathcal{A})$**.** We define the standard *orthosymplectic form* on $\mathcal{A}^{2p|2q}$ by

$$\mathfrak{s}(u,v) := uJv^{st},$$

where J is a block $(p+p+q+q)\times(p+p+q+q)$ matrix

$$J := \frac{1}{2}\begin{pmatrix} 0 & 1_p & 0 & 0 \\ -1_p & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_q \\ 0 & 0 & 1_q & 0 \end{pmatrix}. \tag{6.2}$$

The group $\mathrm{OSp}(2p|2q;\mathcal{A})$ is the subgroup in $\mathrm{GL}(2p|2q;\mathcal{A})$ consisting of matrices g satisfying

$$\mathfrak{s}(u,v) = \mathfrak{s}(ug, vg).$$

Equivalently,

$$gJg^{st} = J$$

(for this conclusion, we use (6.1); since $g \in \mathrm{GL}(p|q;\mathcal{A})$ is even, a sign does not appear).

We also write elements of $\mathrm{OSp}(2p|2q;\mathcal{A})$ as block $(2p+2q)$-matrices $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. For such matrix, we have

$$\pi_\downarrow(A)\begin{pmatrix} 0 & 1_p \\ -1_p & 0 \end{pmatrix}\pi_\downarrow(A)^t = \begin{pmatrix} 0 & 1_p \\ -1_p & 0 \end{pmatrix}, \qquad \text{i.e., } \pi_\downarrow(A) \in \mathrm{Sp}(2n,\mathbb{C});$$

$$\pi_\downarrow(D)\begin{pmatrix} 0 & 1_q \\ 1_q & 0 \end{pmatrix}\pi_\downarrow(D)^t = \begin{pmatrix} 0 & 1_q \\ 1_q & 0 \end{pmatrix}, \qquad \text{i.e., } \pi_\downarrow(D) \in \mathrm{O}(2n,\mathbb{C}).$$

6.6. The super-Olshanski semigroup $\Gamma\mathrm{OSp}(2p|2q;\mathcal{A})$**.** We define the semigroup $\Gamma\mathrm{OSp}(2p|2q;\mathcal{A})$ as a subsemigroup in $\mathrm{OSp}(2p|2q;\mathcal{A})$ consisting of matrices $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that $\pi_\downarrow(A)$ is contained in the Olshanski semigroup $\Gamma\mathrm{Sp}(2p,\mathbb{R})$, see Subsection 3.10.

7 Super-Grassmannians

This section is a preparation to the definition of super-linear relations.

7.1. Super-Grassmannians. Let $u_1,\dots,u_r$ be even vectors and $v_1,\dots,v_s$ be odd vectors in $\mathcal{A}^{p|q}$. We suppose that

— $\pi_\downarrow(u_j) \in (\mathbb{C}^p \oplus 0)$ are linearly independent,

— $\pi_\downarrow(v_k) \in (0 \oplus \mathbb{C}^q)$ are linearly independent.

A supersubspace of superdimension $r|s$ is a left $\mathcal{A}$-module generated by such vectors. Subspaces also are right $\mathcal{A}$-submodules.

We define the *super-Grassmannian* $\mathrm{Gr}^{r|s}_{p|q}(\mathcal{A})$ as the space of all supersubspaces in $\mathcal{A}^{p|q}$ of superdimension $r|s$.

By the definition, the map $\pi_\downarrow$ projects $\mathrm{Gr}^{r|s}_{p|q}(\mathcal{A})$ to the product $\mathrm{Gr}^r_p \times \mathrm{Gr}^s_q$ of the usual complex Grassmannians. We denote by $\pi^\pm_\downarrow$ the natural projections

$$\pi^+_\downarrow : \mathrm{Gr}^{r|s}_{p|q}(\mathcal{A}) \to \mathrm{Gr}^r_p, \qquad \pi^-_\downarrow : \mathrm{Gr}^{r|s}_{p|q}(\mathcal{A}) \to \mathrm{Gr}^s_q.$$

7.2. Intersections of subspaces. Let us examine superdimensions of intersections of subspaces.

Lemma 7.1 *Let L be a subspace of superdimension $r|s$ in $\mathcal{A}^{p|q}$, M be a subspace of superdimension $\rho|\sigma$. Let the following transversality conditions hold*

$$\pi^+_\downarrow(L) + \pi^+_\downarrow(M) = \mathbb{C}^p, \quad \pi^-_\downarrow(L) + \pi^-_\downarrow(M) = \mathbb{C}^q. \tag{7.1}$$

Then $L \cap M$ is a subspace and its superdimension is $(r+\rho-p)|(s+\sigma-q)$.

Remark. If the transversality conditions are not satisfied, then incidentally $L \cap M$ is not a subspace. For instance, consider $\mathcal{A}^{1|1}$ with a basis e_1, e_2 and subspaces

$$L := \mathcal{A}(e_1 + \mathfrak{a}_1 e_2), \qquad M := \mathcal{A} \cdot e_1.$$

Then $L \cap M = \mathcal{A}\mathfrak{a}_1 e_1$ is not a subspace. ⊠

Proof. Denote by $I \subset \mathcal{A}$ the ideal spanned by all $\mathfrak{a}_j$, i.e., $\mathcal{A}/I = \mathbb{C}$.

It is easy to see that

$$L + M = \mathcal{A}^{p|q}. \tag{7.2}$$

Indeed, denote by $e_1, \dots, e_p, e_{p+1}, \dots, e_{p+q}$ the standard basis in $\mathbb{C}^p \oplus \mathbb{C}^q$. Then for each k the submodule $L + M$ contains a vector of the form

$$E_k = e_k + \sum_{J \neq \varnothing} \sum_m x_{k,m,J} \mathfrak{a}^J e_m,$$

where $x_{k,m,J} \in \mathbb{C}$. Actually, only a finite number of nonzero constants $\mathfrak{a}_j$ are contained in this expression. Without loss of generality, we can assume that this set is $\mathfrak{a}_1, \dots, \mathfrak{a}_N$. Then $L + M$ contains all vectors $\mathfrak{a}_1 \dots \mathfrak{a}_N E_k = \mathfrak{a}_1 \dots \mathfrak{a}_N e_k$ and therefore $L + M \supset \mathfrak{a}_1 \dots \mathfrak{a}_N \cdot \mathbb{C}^p \oplus \mathbb{C}^q$. Next, for each l

$$\mathfrak{a}_1 \dots \mathfrak{a}_{l-1}\mathfrak{a}_{l+1} \dots \mathfrak{a}_N\, E_k - \mathfrak{a}_1 \dots \mathfrak{a}_{l-1}\mathfrak{a}_{l+1} \dots \mathfrak{a}_N\, e_k \;\in\; \mathfrak{a}_1 \dots \mathfrak{a}_N \cdot \mathbb{C}^p \oplus \mathbb{C}^q.$$

Therefore, $\mathfrak{a}_1 \dots \mathfrak{a}_{l-1}\mathfrak{a}_{l+1} \dots \mathfrak{a}_N \cdot \mathbb{C}^p \oplus \mathbb{C}^q \subset L + M$. Repeating this process, we get $\mathbb{C}^p \oplus \mathbb{C}^q \subset L + M$ and this implies (7.2).

Let $v \in \pi_\downarrow^+(L) \cap \pi_\downarrow^+(M)$. Choose $x \in L$, $y \in M$ such that $\pi_\downarrow(x) = v$, $\pi_\downarrow(y) = v$. Then $x - y \in I \cdot \mathcal{A}^{p|q}$. However, $I \cdot L + I \cdot M = I \cdot \mathcal{A}^{p|q}$, therefore we can represent

$$x - y = a - b, \qquad \text{where } a \in I \cdot L,\, b \in I \cdot M.$$

Then

$$(x - a) \in L,\ (y - b) \in M,\ \pi_\downarrow(x - a) = v = \pi_\downarrow(y - b).$$

Thus, for any vector $v \in \pi_\downarrow^+(L) \cap \pi_\downarrow^+(M)$, there is a vector $v^* \in L \cap M$ such that $\pi_\downarrow(v^*) = v$. The same is valid for vectors $w \in \pi_\downarrow^-(L) \cap \pi_\downarrow^-(M)$.

Therefore, $L \cap M$ contains a supersubspace of desired superdimension generated by vectors v^*, w^*. It remains to show that there are no extra vectors in the intersection.

Now, let us vary a phantom algebra $\mathcal{A}$. If $\mathcal{A}$ is an algebra in a finite number of Grassmann constants $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$, then this completes a proof, since (7.2) gives the same superdimension of the intersection over $\mathbb{C}$.

Otherwise, we choose a basis in L and a basis in M. Expressions for basis vectors contain only a finite number of Grassmann constants $\mathfrak{a}_1, \ldots, \mathfrak{a}_k$. After this, we apply the same reasoning to algebras $\mathcal{A}[l]$ generated by Grassmann constants $\mathfrak{a}_1, \ldots, \mathfrak{a}_l$ for all $l \geqslant k$ and observe that $L \cap M$ does not contain extra vectors. □

7.3. Atlas on the super-Grassmannian. Define an atlas on the super-Grassmannian $\mathrm{Gr}_{p|q}^{r|s}(\mathcal{A})$ as usual. Namely, consider the following complementary subspaces

$$V_+ := (\mathcal{A}^r \oplus 0) \oplus (\mathcal{A}^s \oplus 0) \qquad V_- := (0 \oplus \mathcal{A}^{p-r}) \oplus (0 \oplus \mathcal{A}^{q-s})$$

in $\mathcal{A}^{p|q}$. Let $S : V_+ \to V_-$ be an even operator. Then its graph is an element of the super-Grassmannian.

Permuting coordinates in $\mathcal{A}^p$ and $\mathcal{A}^q$, we get an atlas that covers the whole super-Grassmannian $\mathrm{Gr}_{p|q}^{r|s}(\mathcal{A})$.

7.4. Lagrangian super-Grassmannians. Now, equip the space $\mathcal{A}^{2p|2q}$ with the orthosymplectic form $\mathfrak{s}$ as above. We say that a subspace L is *isotropic* if the form $\mathfrak{s}$ is zero on L. A *Lagrangian subspace* L is an isotropic subspace of the maximal possible superdimension, i.e., $\dim L = p|q$.

Observation 7.2 *Let L be a super-Lagrangian subspace. Then*

— $\pi_\downarrow^+(L)$ *is Lagrangian subspace in* $\mathbb{C}^{2p}$ *with respect to the skew-symmetric bilinear form* $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$;

— $\pi_\downarrow^-(L)$ *is a Lagrangian subspace in* $\mathbb{C}^{2q}$ *with respect to the symmetric bilinear form* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

7.5. Coordinates on Lagrangian super-Grassmannian. Consider the following complementary Lagrangian subspaces

$$V_+ := (\mathcal{A}^p \oplus 0) \oplus (\mathcal{A}^q \oplus 0), \qquad V_- := (0 \oplus \mathcal{A}^p) \oplus (0 \oplus \mathcal{A}^q). \tag{7.3}$$

Proposition 7.3 *Consider an even operator $S : V_+ \to V_-$,*

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

The graph of S is a Lagrangian subspace if and only if

$$A = A^t, \quad D = -D^t, \quad C + B^t = 0. \tag{7.4}$$

REMARK. This statement is a super-imitation of Lemma 2.11. ⊠

PROOF. We write out a vector $h \in \mathcal{A}^{2p|2q}$ as

$$h = (u_+, u_-; v_+, v_-) \in \mathcal{A}^p \oplus \mathcal{A}^p \oplus \mathcal{A}^q \oplus \mathcal{A}^q.$$

Then

$$\mathfrak{s}(h', h) = u'_+ \, (u_-)^{st} - u'_- \, (u_+)^{st} + v'_+ \, (v_-)^{st} + v'_- \, (v_+)^{st}.$$

Let h be in the graph of S. Then

$$\begin{pmatrix} u_- & v_- \end{pmatrix} = \begin{pmatrix} u_+ & v_+ \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} u_+ A + v_+ C & u_+ B + v_+ D \end{pmatrix}$$

and

$$\mathfrak{s}(h', h) = u'_+ (u_+ A + v_+ C)^{st} - (u'_+ A + v'_+ C) u_+^{st} + \\ + v'_+ (u_+ B + v_+ D)^{st} + (u'_+ B + v'_+ D) v_+^{st}.$$

Observe that the matrices A, B, C, D are even[31]; for this reason, we write $(u_+ A)^{st} = A^{st} (u_+)^{st}$ etc., see (6.1). We arrive at

$$u'_+ \big[A^{st} (u_+)^{st} + C^{st} (v_+)^{st} \big] - \big[(u'_+ A + v'_+ C \big] u_+^{st} + \\ + v'_+ \big[B^{st} (u_+)^{st} + D^{st} (v_+)^{st} \big] + \big[u'_+ B + v'_+ D \big] v_+^{st}.$$

Next,

$$A^{st} = A^t, \quad B^{st} = -B^t, \quad C^{st} = C^t, \quad D^{st} = D^t.$$

Therefore, we convert our expression to the form

$$u'_+ (A - A^t)(u_+)^{st} + v'_+ (D + D^t)(v_+)^{st} + (u'_+)(B^t + C) v_+^{st} + (v^+)(B + C^t) u_+^{st}.$$

This expression is zero if and only if the conditions (7.4) are satisfied. □

[31] Recall that this means that A, D are composed of even phantom constants and C, B of odd phantom constants.

7.6. Atlas on the Lagrangian super-Grassmannian. Now we imitate the construction of Subsection 2.21.

Consider the standard basis in $\mathcal{A}^{2p|2q}$ consisting of vectors, whose coordinates are 0 except one unit. Denote elements of this basis by

$$e_1, \dots, e_p;\ e'_1, \dots, e'_p;\ f_1, \dots, f_q;\ f'_1, \dots, f'_q.$$

In this basis, the matrix of the orthosymplectic form is (6.2). Consider subsets $I \subset \{1, 2, \dots, p\}$, $J \subset \{1, 2, \dots, q\}$.

We define

$$V_+[I, J] = \big(\oplus_{i\in I}\mathcal{A}e_i\big) \oplus \big(\oplus_{k\notin I}\mathcal{A}e'_k\big) \oplus \big(\oplus_{j\in J}\mathcal{A}f_j\big) \oplus \big(\oplus_{l\notin J}\mathcal{A}f'_l\big), \qquad (7.5)$$

$$V_-[I, J] = \big(\oplus_{i\notin I}\mathcal{A}e_i\big) \oplus \big(\oplus_{k\in I}\mathcal{A}e'_k\big) \oplus \big(\oplus_{j\notin J}\mathcal{A}f_j\big) \oplus \big(\oplus_{l\in J}\mathcal{A}f'_l\big). \qquad (7.6)$$

We denote by $\mathcal{O}[I, J]$ the set of all the Lagrangian subspaces that are graphs of even operators

$$S : V_+[I, J] \to V_-[I, J].$$

In fact, these operators satisfy the same conditions as in Proposition 7.3 (our initial chart is $\mathcal{O}[\varnothing, \varnothing]$). Thus, we get an atlas on the Lagrangian super-Grassmannian.

7.7. Elementary reflections. Now we repeat considerations of Subsection 2.22. We define elementary reflections $\sigma[e_i]$, $\sigma[f_j]$ in $\mathcal{A}^{2p|2q}$ by

$$\sigma[e_i]\, e_i^+ = -e_i^-, \quad \sigma[e_i]\, e_i^- = e_i^+,$$
$$\sigma[e_i]\, e_k^\pm = e_k^\pm \ \text{ for } k \ne i, \quad \sigma[e_i]\, f_j^\pm = f_j^\pm,$$

and

$$\sigma[f_j]\, f_j^+ = f_j^-, \quad \sigma[f_j]\, f_j^- = f_j^+, \qquad (7.7)$$

$$\sigma[f_i]\, f_k^\pm = e_k^\pm \ \text{ for } k \ne j, \quad \sigma[f_j]\, e_i^\pm = e_i^\pm, \qquad (7.8)$$

in the first row, we have an extra change of a sign because we want to preserve the symplectic form. Then

$$\mathcal{O}[I, J] = \prod_{i\in I} \sigma[e_i] \cdot \prod_{j\in J} \sigma[f_j] \cdot \mathcal{O}[\varnothing, \varnothing].$$

8 Superlinear relations

Gauss–Berezin integral operators are enumerated by contractive Lagrangian super-linear relations. These objects are defined in this section.

8.1. Superlinear relations. We define super-linear relations $P : \mathcal{A}^{p|q} \rightrightarrows \mathcal{A}^{r|s}$ as subspaces in $\mathcal{A}^{p|q} \oplus \mathcal{A}^{r|s}$. Products are defined as above, see Subsection 2.16.

Next, for a superlinear relation we define complex linear relations

$$\pi_{\downarrow}^{+}(P) : \mathbb{C}^p \rightrightarrows \mathbb{C}^r, \quad \pi_{\downarrow}^{-}(P) : \mathbb{C}^q \rightrightarrows \mathbb{C}^s$$

in the natural way, we simply project the super-Grassmannian in $\mathcal{A}^{p|q} \oplus \mathcal{A}^{r|s}$ onto the product of the complex Grassmannians.

8.2. Transversality conditions. Let V, W, Y be *complex* linear spaces. We say that linear relations

$$P : V \rightrightarrows W, \; Q : W \rightrightarrows Y$$

are *transversal* if

$$\operatorname{im} P + \operatorname{dom} Q = W, \tag{8.1}$$

$$\operatorname{indef} P \cap \ker Q = 0. \tag{8.2}$$

We met these conditions in Section 2, in what follows they are even more important.

Theorem 8.1 *If $P : V \rightrightarrows W$, $Q : W \rightrightarrows Y$ are transversal, then*

$$\dim QP = \dim Q + \dim P - \dim W.$$

PROOF. We rephrase the definition of the product QP as follows (see [50], Prop. II.7.1). Consider the space $V \oplus W \oplus W \oplus Y$ and the following subspaces

— $P \oplus Q$,

— the subspace H consisting of vectors $v \oplus w \oplus w \oplus y$,

— the subspace $T \subset H$ consisting of vectors $0 \oplus w \oplus w \oplus 0$.

Let us project $(P \oplus Q) \cap H$ on $V \oplus W$ along T. The result is $QP \subset V \oplus W$.

By the first transversality condition (8.1),

$$(P \oplus Q) + H = V \oplus W \oplus W \oplus Y,$$

therefore we know the superdimension of the intersection $S := (P \oplus Q) \cap H$.

By the second condition (8.2) the projection $H \to V \oplus W$ is injective on S. □

8.3. Transversality for super-linear relations. We say that super-linear relations $P : V \rightrightarrows W$ and $Q : W \rightrightarrows Y$ are *transversal* if $\pi_{\downarrow}^{+}(P)$ is transversal to $\pi_{\downarrow}^{+}(Q)$ and $\pi_{\downarrow}^{-}(P)$ is transversal to $\pi_{\downarrow}^{-}(Q)$.

Theorem 8.2 *If $P : V \rightrightarrows W$, $Q : W \rightrightarrows Y$ are transversal super-linear relations, then their product is a super-linear relation and*

$$\dim QP = \dim Q + \dim P - \dim W.$$

PROOF. We follow the proof of the previous theorem. □

8.4. Lagrangian super-linear relations. Consider the spaces $V = \mathcal{A}^{2p|2q}$, $W = \mathcal{A}^{2r|2s}$ endowed with the orthosymplectic forms $\mathfrak{s}_V$, $\mathfrak{s}_W$, respectively. Define the form $\mathfrak{s}^\ominus$ on $V \oplus W$ as

$$\mathfrak{s}^\ominus(v \oplus w, v' \oplus w') := \mathfrak{s}_V(v, v') - \mathfrak{s}_W(w, w').$$

A Lagrangian super-linear relation $P : V \rightrightarrows W$ is a Lagrangian supersubspace in $V \oplus W$.

Observation 8.3 *Let $g \in \mathrm{OSp}(2p|2q; \mathcal{A})$. Then the graph of g is a Lagrangian super-linear relation $\mathcal{A}^{2p|2q} \rightrightarrows \mathcal{A}^{2p|2q}$.*

Theorem 8.4 *Let $P : V \rightrightarrows W$, $Q : W \rightrightarrows Y$ be transversal Lagrangian super-linear relations. Then $QP : V \rightrightarrows Y$ is a Lagrangian super-linear relation.*

PROOF. Let $v \oplus w$, $v' \oplus w' \in P$ and $w \oplus y$, $w' \oplus y' \in Q$. By definition,

$$\mathfrak{s}_V(v, v') = \mathfrak{s}_W(w, w') = \mathfrak{s}_Y(y, y'),$$

therefore QP is isotropic. By the virtue of Theorem 8.2, we know $\dim QP$. □

8.5. Components of Lagrangian super-Grassmannian. As we observed in Subsection 2.23, the orthogonal Lagrangian Grassmannian in the space $\mathbb{C}^{2n}$ consists of two components. The usual symplectic Lagrangian Grassmannian is connected. Therefore, the Lagrangian super-Grassmannian consists of two components.

Below we must distinguish them.

Decompose $V = V_+ \oplus V_-$, $W = W_+ \oplus W_-$ as above (7.3). We say that the component containing the linear relation

$$(V_+ \oplus W_-) : V \rightrightarrows W$$

is even; the other component is odd.

8.6. Contractive Lagrangian linear relations. Now, we again (see Section 3) consider the Hermitian form M on $\mathbb{C}^{2p}$, it is defined by a matrix $\begin{pmatrix} 1_p & 0 \\ 0 & -1_p \end{pmatrix}$. Then $\mathbb{C}^{2p}$ becomes an object of the category **Sp**.

We say that a Lagrangian super-linear relation $P : V \rightrightarrows W$ is contractive if $\pi^+_\downarrow(P)$ is a morphism of the category **Sp**.

8.7. Positive domain in the Lagrangian super-Grassmannian. We say that a Lagrangian subspace P in $\mathbb{C}^{2p|2q}$ is positive if the form M defined in the prevous subsection is positive on $\pi^+_\downarrow(P)$.

9 Correspondence between Lagrangian superlinear relations and Gauss–Berezin operators

Here we prove our main results, namely Theorems 9.3, 9.4.

9.1. Creation–annihilation operators. Let $V := \mathcal{A}^{2p|2q}$ be a superlinear space endowed with the orthosymplectic bilinear form $\mathfrak{s}$ defined by the matrix (6.2). For a vector

$$v \oplus w := v_+ \oplus v_- \oplus w_+ \oplus w_- \in \mathcal{A}^{2p|2q},$$

we define the creation-annihilation operator in the Fock–Berezin space $\boldsymbol{\mathcal{S}}\mathbf{F}_{p,q}(\mathcal{A})$ by

$$\widehat{a}(v \oplus w) f(z,\xi) = \Bigl(\sum_i v_+^{(i)} \frac{\partial}{\partial z_i} + \sum_i v_-^{(i)} z_i + \sum_j w_+^{(j)} \frac{\partial}{\partial \xi_i} + \sum_j w_-^{(j)} \xi_j\Bigr) f(z,\xi).$$

9.2. Supercommutator. We say that a vector $v \oplus w$ is even if v is even and w is odd. It is odd if v is odd and w is even. This corresponds to the definition of even/odd for $(1|0) \times (2p|2q)$ matrices. Let $h = v \oplus w$, $h' = v' \oplus w'$. We define the supercommutator $[\widehat{a}(h), \widehat{a}(h')]_{\mathrm{s}}$ as

$$[\widehat{a}(h), \widehat{a}(h')]_{\mathrm{s}} = \begin{cases} [\widehat{a}(h), \widehat{a}(h')] = \widehat{a}(h)\widehat{a}(h') - \widehat{a}(h')\widehat{a}(h) & \text{if } h \text{ or } h' \text{ is even}, \\ \{\widehat{a}(h), \widehat{a}(h')\} = \widehat{a}(h)\widehat{a}(h') + \widehat{a}(h')\widehat{a}(h) & \text{if } h,\ h' \text{ are odd}. \end{cases}$$

Then

$$[\widehat{a}(h), \widehat{a}(h')]_{\mathrm{s}} = \mathfrak{s}(h, h') \cdot 1,$$

where 1 denotes the unit operator.

Also, note that an operator $\widehat{a}(h)$ is linear (see Subsection 5.4) if h is even and antilinear if h is odd.

9.3. Annihilators of Gaussian vectors.

Theorem 9.1 a) *For a Gauss–Berezin vector $\mathbf{b} \in \boldsymbol{\mathcal{S}}\mathbf{F}_{p,q}(\mathcal{A})$ consider the set L of all vectors $h \in \mathcal{A}^{2p|2q}$ such that*

$$\widehat{a}(h)\,\mathbf{b} = 0.$$

Then L is a positive Lagrangian subspace in $\mathcal{A}^{2p|2q}$.

b) *Moreover, the map $\mathbf{b} \mapsto L$ is a bijection*

$$\left\{ \begin{matrix} \textit{The set of all Gauss–Berezin vectors} \\ \textit{defined up to an invertible scalar} \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \textit{The positive} \\ \textit{Lagrangian Grassmannian} \end{matrix} \right\}.$$

Before we begin a formal proof we propose the following (insufficient, but clarifying) argument. Let h, $h' \in L$. If one of them is even, then we write

$$\Bigl(\widehat{a}(h)\widehat{a}(h') - \widehat{a}(h')\widehat{a}(h)\Bigr)\,\mathbf{b}.$$

By the definition of L, this is 0. On the other hand, this is $\mathfrak{s}(h,h')\mathbf{b}$. Therefore, $\mathfrak{s}(h,h') = 0$.

If both h, h' are odd, then we write

$$0 = \Bigl(\widehat{a}(h)\widehat{a}(h') + \widehat{a}(h')\widehat{a}(h)\Bigr)\,\mathbf{b} = \mathfrak{s}(h,h')\mathfrak{b}$$

and arrive at the same result.

PROOF. First, let $\mathfrak{b}(z,\xi)$ have the standard form (5.5). We write out

$$\widehat{a}(v\oplus w)\mathfrak{b}(z,\xi) = \Bigl(\sum_i v_+^{(i)}\frac{\partial}{\partial z_i} + \sum_i v_-^{(i)} z_i + \sum_j w_+^{(j)}\frac{\partial}{\partial \xi_i} + \sum_j w_-^{(j)}\xi_j\Bigr)\times$$
$$\times \exp\Bigl\{\frac{1}{2}\begin{pmatrix} z & \xi\end{pmatrix}\begin{pmatrix} A & B\\ -B^t & C\end{pmatrix}\begin{pmatrix} z^t\\ \xi^t\end{pmatrix}\Bigr\} =$$
$$= \Bigl(v_+(Ax^t + B\xi^t) + v_- z^t + w_+(-B^t z^t + C\xi^t) + w_-\xi^t\Bigr)\cdot\mathfrak{b}(z,\xi) =$$
$$= \Bigl((v_+A - w_+B^t + v_-)z^t + (v_+B + w_+D + w_-)\xi^t\Bigr)\cdot\mathfrak{b}(z,\xi).$$

This is zero if and only if

$$\begin{cases} v_- = -(v_+A - w_+B^t)\\ w_- = -(v_+B + w_+D)\end{cases}.$$

However, this system of equations determines a Lagrangian subspace. The positivity of a Lagrangian subspace is equivalent to $\|\pi_\downarrow(A)\| < 1$ (see, for instance [53]).

Next, consider an arbitrary Gauss–Berezin vector

$$\mathbf{b}(z,\xi) = \mathfrak{D}[\xi_{i_1}]\dots\mathfrak{D}[\xi_{i_k}]\,\mathsf{S}^k\;\mathfrak{b}[T], \tag{9.1}$$

where $\mathfrak{b}[T]$ is a standard Gauss–Berezin vector. We have

$$\widehat{a}\bigl(h\bigr)\,\mathfrak{D}[\xi_1]\,\mathsf{S} = \mathfrak{D}[\xi_1]\,\mathsf{S}\,\widehat{a}\bigl(\sigma[f_1]h\bigr), \tag{9.2}$$

where $\sigma[f_1]$ is an elementary reflection given by (7.7)–(7.8).

If h ranges in a Lagrangian subspace, then $\sigma[f_1]h$ also ranges in (another) Lagrangian subspace. Also, a map $\sigma[f_1]$ takes positive subspaces to positive subspaces. Therefore, the statements a) for vectors

$$\mathfrak{D}[\xi_{i_1}]\,\mathfrak{D}[\xi_{i_2}]\dots\mathfrak{D}[\xi_{i_k}]\,\mathsf{S}^k\;\mathfrak{b}[T] \qquad\text{and}\qquad \mathfrak{D}[\xi_{i_2}]\dots\mathfrak{D}[\xi_{i_k}]\,\mathsf{S}^{k-1}\;\mathfrak{b}[T].$$

are equivalent.

In fact, for fixed $i_1, \ldots, i_k$, all vectors of the form (9.1) correspond to a fixed chart in the Lagrangian super-Grassmannian, namely to

$$\sigma[f_{i_1}]\cdots\sigma[f_{i_k}]\cdot\mathcal{O}[\varnothing,\varnothing]$$

in notation of Subsection 7.7.

But these charts cover the set of all positive Lagrangian subspaces. □

Theorem 9.2 *For a positive Lagrangian subspace $L \subset \mathcal{A}^{2p|2q}$, consider the system of equations*

$$\widehat{a}(v\oplus w)f(z,\xi)=0 \quad \text{for all } v\oplus w\in L, \tag{9.3}$$

for a function $f(z,\xi)$. All its solutions are of the form $\lambda\mathfrak{b}(z,\xi)$, where $\mathfrak{b}(z,\xi)$ is a Gauss–Berezin vector and λ is a phantom constant.

Proof. It suffices to prove the statement for L in the principal chart. Put

$$\varphi(z,\xi):=f(z,\xi)/\mathfrak{b}(z,\xi),$$

i.e.,

$$f(z,\xi)=\mathfrak{b}(z,\xi)\cdot\varphi(z,\xi).$$

By the Leibniz rule,

$$0=\widehat{a}(v\oplus w)\big(\mathfrak{b}(z,\xi)\varphi(z,\xi)\big)=$$
$$=\Big(\widehat{a}(v\oplus w)\mathfrak{b}(z,\xi)\Big)\cdot\varphi(x,\xi)+\mathfrak{b}(z,\xi)\cdot\Big(\sum_j v_+^{(j)}\frac{\partial}{\partial z_i}+\sum_j w_+^{(j)}\frac{\partial}{\partial\xi_i}\Big).\varphi(z,\xi)$$

The first summand is zero by the definition of $\mathfrak{b}(z,\xi)$. Since v_+, w_+ are arbitrary, we get

$$\frac{\partial}{\partial z_i}\varphi(z,\xi)=0,\quad \frac{\partial}{\partial\xi_i}\varphi(z,\xi)=0.$$

Therefore, $\varphi(z,\xi)$ is a phantom constant. □

9.4. Gauss–Berezin operators and superlinear relations. Let $V=\mathcal{A}^{2p|2q}$, $\widetilde{V}=\mathcal{A}^{2r|2s}$ be (super)spaces endowed with orthosymplectic forms.

Theorem 9.3 a) *For each contractive Lagrangian superlinear relation $P: V\rightrightarrows\widetilde{V}$ there exists a linear operator*

$$\mathfrak{B}(P):\boldsymbol{\mathcal{S}}\mathbf{F}_{p,q}(\mathcal{A})\to\boldsymbol{\mathcal{S}}\mathbf{F}_{r,s}(\mathcal{A})$$

such that

1) *The following condition is satisfied*

$$\widehat{a}(\widetilde{h})\,\mathfrak{B}(P)=\mathfrak{B}(P)\,\widehat{a}(h)\quad\text{for all } h\oplus\widetilde{h}\in P.$$

2) *If P is in the even component of Lagrangian super-Grassmannian, then $\mathfrak{B}(P)$ is an integral operator with an even*[32] *kernel. If P is in the odd component, then $\mathfrak{B}(P)\mathsf{S}$ is an integral operator with an odd kernel.*

Moreover, this operator is unique up to a scalar factor $\in \mathcal{A}_{even}$.

b) *The operators $\mathfrak{B}(P)$ are Gauss–Berezin operators and all Gauss–Berezin operators arise in this way.*

PROOF. Let us write differential equations for the kernel $K(z,\xi,\overline{y},\overline{\eta})$ of the operator $\mathfrak{B}(P)$. Denote

$$h = v_+ \oplus v_- \oplus w_+ \oplus w_-, \quad \widetilde{h} = \widetilde{v}_+ \oplus \widetilde{v}_- \oplus \widetilde{w}_+ \oplus \widetilde{w}_-.$$

Then

$$\widehat{a}(\widetilde{h}) \int K(z,\xi,\overline{y},\overline{\eta})\, f(y,\eta)\, e^{-y\overline{y}^t - \eta\overline{\eta}^t} dy\, d\overline{y}\, d\overline{\eta}\, d\eta =$$
$$= \int K(z,\xi,\overline{y},\overline{\eta})\, \widehat{a}(h) f(y,\eta)\, e^{-y\overline{y}^t - \eta\overline{\eta}^t} dy\, d\overline{y}\, d\overline{\eta} d\eta.$$

Let P be even. Integrating by parts in the right-hand side, we get:

$$\Bigl(\sum_i \widetilde{v}_+^{(i)} \frac{\partial}{\partial z_i} + \sum_i \widetilde{v}_-^{(i)} z_i + \sum_j \widetilde{w}_+^{(j)} \frac{\partial}{\partial \xi_i} + \sum_j \widetilde{w}_-^{(j)} \xi_j\Bigr) K(z,\xi,\overline{y},\overline{\eta}) =$$
$$= \Bigl(\sum_i v_+^{(i)} \overline{y}_i + \sum_i v_-^{(i)} \frac{\partial}{\partial \overline{y}_i} + \sum_j w_+^{(j)} \overline{\eta}_j + \sum_j w_-^{(j)} \frac{\partial}{\partial \overline{\eta}_i}\Bigr) K(z,\xi,\overline{y},\overline{\eta}).$$

This system of equations has the form (9.3) and determines a Gaussian.

Evenness condition was essentially used in this calculation. For instance for an odd kernel K we must write $(v_+^{(i)})^\sigma$ instead of $v_+^{(i)}$ in the right hand side.

Now, let P be odd. Let us try to represent $\mathfrak{B}(P)$ as a product

$$\mathfrak{B}(P) = \mathfrak{C} \cdot \mathfrak{D}[\eta_1] \cdot \mathsf{S}.$$

Let L be the kernel of $\mathfrak{C}$.

$$\widehat{a}(\widetilde{h}) \int L(z,\xi,\overline{y},\overline{\eta})\, \mathfrak{D}[\eta_1] \cdot \mathsf{S} \cdot f(y,\eta)\, e^{-y\overline{y}^t - \eta\overline{\eta}^t} dy\, d\overline{y}\, d\overline{\eta}\, d\eta =$$
$$= \int L(z,\xi,\overline{y},\overline{\eta})\, \mathfrak{D}[\eta_1] \cdot \mathsf{S} \cdot \widehat{a}(h) f(y,\eta)\, e^{-y\overline{y}^t - \eta\overline{\eta}^t} dy\, d\overline{y}\, d\overline{\eta} d\eta.$$

Next, we change the order

$$\mathfrak{D}[\eta_1]\, S\widehat{a}(v) = \widehat{a}(\sigma(f_1)v)\, \mathfrak{D}[\eta_1]\, \mathsf{S},$$

where σ is an elementary reflection of the type (7.7)–(7.8).

We again get for L a system of equations determining a Gaussian. □

9.5. Products of Gauss–Berezin operators.

[32] see Subsection 5.4.

Theorem 9.4 a) *Let* $P : \mathcal{A}^{p|q} \rightrightarrows \mathcal{A}^{p'|q'}$, $Q : \mathcal{A}^{p'|q'} \rightrightarrows \mathcal{A}^{p''|q''}$ *be contractive Lagrangian relations. Assume that* P, Q *are transversal. Then*

$$\mathfrak{B}(Q)\,\mathfrak{B}(P) = \lambda \cdot \mathfrak{B}(QP), \tag{9.4}$$

where $\lambda = \lambda(P,Q)$ *is an even invertible phantom constant.*

b) *If* P, Q *are not transversal, then*

$$\pi_{\downarrow}\bigl(\mathfrak{B}(Q)\,\mathfrak{B}(P)\bigr) = 0.$$

PROOF. Let $v \oplus w \in P$, $w \oplus y \in Q$. Then

$$\mathfrak{B}(Q)\mathfrak{B}(P)\widehat{a}(v) = \mathfrak{B}(Q)\widehat{a}(w)\mathfrak{B}(P) = \widehat{a}(y)\mathfrak{B}(Q)\mathfrak{B}(P).$$

On the other hand,

$$\widehat{a}(y)\mathfrak{B}(QP) = \mathfrak{B}(QP)\widehat{a}(v).$$

By Theorem 9.3, these relations define a unique operator and we get (9.4).

It remains to verify conditions of vanishing of

$$\pi_{\downarrow}\bigl(\mathfrak{B}(Q)\,\mathfrak{B}(P)\bigr) : \mathbf{F}_p \otimes \Lambda_q \to \mathbf{F}_{p''} \otimes \Lambda_{q''}.$$

Here we refer to Subsection 5.8. Our operator is a tensor product of

$$\pi_{\downarrow}^{+}\bigl(\mathfrak{B}(Q)\bigr)\,\pi_{\downarrow}^{+}\bigl(\mathfrak{B}(P)\bigr) : \mathbf{F}_p \to \mathbf{F}_{p''} \tag{9.5}$$

and

$$\pi_{\downarrow}^{-}\bigl(\mathfrak{B}(Q)\bigr)\,\pi_{\downarrow}^{-}\bigl(\mathfrak{B}(P)\bigr) : \Lambda_q \to \Lambda_{q''}. \tag{9.6}$$

The operator (9.5) is a product of Gaussian integral operators. By Theorem 3.5, it is nonzero.

The operator (9.6) is a product of Berezin operators. It is nonzero if and only if $\pi_{\downarrow}^{-}(P)$ and $\pi_{\downarrow}^{-}(Q)$ are transversal, here we refer to Theorem 2.16.c. □

Corollary 9.5 *For an element* g *of the Olshanski supersemigroup* $\Gamma\mathrm{OSp}(2p|2q;\mathcal{A})$ (*see Subsect.* 6.6) *denote its graph by* graph(g). *Then* $\mathfrak{B}(P(g))$ *determines a projective representation of* $\Gamma\mathrm{OSp}(2p|2q;\mathcal{A})$ *over* $\mathcal{A}$,

$$\mathfrak{B}(\mathrm{graph}(g_1))\,\mathfrak{B}(\mathrm{graph}(g_2)) = \lambda(g_1,g_2)\mathfrak{B}(\mathrm{graph}(g_1g_2)),$$

where $\lambda(g_1,g_2)$ *is an invertible element of* $\mathcal{A}$.

Denote by $G(2p|2q;\mathcal{A})$ the group of invertible elements of Olshanski supersemigroup. It is easy to see that $G(2p|2q;\mathcal{A})$ consists of all $g \in \mathrm{OSp}(2p|2q;\mathcal{A})$ such that $\pi_{\downarrow}(g) \in \mathrm{Sp}(2q,\mathbb{R}) \times \mathrm{O}(2p,\mathbb{C})$[33].

Corollary 9.6 *The map* $g \mapsto \mathfrak{B}(\mathrm{graph}(g_1))$ *determines a projective representation of the group* $G(2p|2ql\mathcal{A})$ *over* $\mathcal{A}$.

[33]In $\mathrm{Sp}(2q,\mathbb{R}) \times \mathrm{O}(2p,\mathbb{C})$, we have a product of a real Lie group and a complex Lie group, so it is not a real form (on real forms, see [63], [20]). Moreover, $G(2p|2q;\mathcal{A})$ is not a supergroup in the sense of the usual definitions [9], [18], [12], [42], [11] (because there are no intermediate Lie superalgebras between $\mathfrak{osp}(2p|2q,\mathbb{R})$ and $\mathfrak{osp}(2p|2q,\mathbb{C})$). However, $G(2p|2q;\mathcal{A}')$ depends functorially on algebras $\mathcal{A}'$ described in Subsection 4.2.

10 Final remarks

10.1. Extension of notion of Gaussian operators? Our main result seems incomplete, since Theorem 9.4 describes product of integral operators only if superlinear relations are transversal. However, a product of integral operators can be written explicitly in all cases (see Subsection 4.12). We arrive at the following question:

Question 10.1 a) *Is it possible to extend the definitions of Gaussian operators and Lagrangian superlinear relations to make the formula* (9.4) *valid for all* P, Q?

b) *Is it reasonable to consider the expressions* (4.12) *as Gaussians*?

10.2. The infinite-dimensional orthosymplectic supergroup. Recall that constructions of orthogonal and symplectic spinors described in Sect. 2–3 survive well in the infinite-dimensional limit, [50], Chapters IV, VI. However, there remain gaps between sufficient and necessary conditions for boundedness of Gaussian operators in the bosonic Fock spaces with infinite number of degrees of freedom (see [50], Sect. VI.3-4, [57]) and similar gaps in fermionic case (see [50], Sect. IV.2). On the other hand, even for finite-dimensional supergroups 'unitary representations' are realized by unbounded operators in super-Hilbert spaces (for instance, for orthosymplectic spinors discussed in this work).

Question 10.2 *Find natural topologies in Fock–Berezin space* $\mathbf{F}_{\infty,\infty}(\mathcal{A})$ *with infinite number of variables* z_1, z_2, …, ξ_1, ξ_2, … *and conditions of boundedness of Gauss–Berezin operators.*

Such topologies can depend on further applications. See, for instance, the next subsection, a straightforward repetition of our definition of $\boldsymbol{\mathcal{S}}\mathbf{F}_{p,q}$ does not work in that situation.

10.3. The super-Virasoro algebras. It is well-known that highest weight representations of the Virasoro algebra appear in a natural way as restrictions of infinite-dimensional Weil representation and infinite-dimensional spinor representation to the Lie algebra of vector fields on the circle, see [61], [46], for more details, see [50], Chapter VII. Let us explain that a similar phenomenon takes place for super-Virasoro algebras and ortho-symplectic spinors.

Consider the circle S^1 with the coordinate $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$, denote by $C^\infty_+(S^1)$ (resp., $C^\infty_-(S^1)$) the space of smooth complex functions on S^1 such that $f(\varphi+\pi) = f(\varphi)$ (resp., $f(\varphi+\pi) = -f(\varphi)$). Let us denote by $f(\varphi)(d\varphi)^\lambda$ densities of weight λ on the circle, $d\varphi = (d\varphi)^1$, $\frac{d}{d\varphi} = (d\varphi)^{-1}$. Recall that vector fields act on densities by

$$a(\varphi)\frac{d}{d\varphi}\Big(f(\varphi)(d\varphi)^\lambda\Big) = \Big(a(\varphi)f'(\varphi) + \lambda a'(\varphi)f(\varphi)\Big)(d\varphi)^\lambda.$$

Then, $d\varphi = (d\varphi)^1$, $\partial/\partial\varphi = (d\varphi)^{-1}$.

Consider a 'super-Witt' Lie superalgebra $\mathfrak{sw}_-$ (see Kirillov [37]), whose elements are

$$a(\varphi)\frac{d}{d\varphi} \oplus b(\varphi)(d\varphi)^{-1/2}, \qquad a \in C_+^\infty(S^1),\, b \in C_-^\infty(S^1), \tag{10.1}$$

the first summand has parity $\overline{0}$, the second summand parity $\overline{1}$. The supercommutator is defined as follows:

— supercommutator of vector fields is the usual commutator;

— supercommutator of a vector field and a density is the natural action of vector fields on densities of weight $-1/2$;

— the anticommutator of densities $b_1(\varphi)(d\varphi)^{-1/2}$, $b_2(\varphi)(d\varphi)^{-1/2}$ is the product $2b_1(\varphi)b_2(\varphi)\frac{d}{d\varphi}$.

This supercommutator is a 'natural differential geometric operation[34], i.e., it is invariant with respect to the action of the group of diffeomorphisms r of S^1 satisfying $r(\varphi+\pi) = r(\varphi)+\pi$.

Consider the superlinear space[35] $W := W_{\overline{0}} \oplus W_{\overline{1}}$ consisting of

$$f(\varphi) \oplus g(\varphi)(d\varphi)^{1/2}, \qquad f \in C_+^\infty(S^1)/\mathbb{C},\, g \in C_-^\infty(S^1). \tag{10.2}$$

We equip W with the orthosymplectic form by

$$\bigl\{f_1(\varphi) \oplus g_1(\varphi)(d\varphi)^{1/2},\, f_2(\varphi) \oplus g_2(\varphi)(d\varphi)^{1/2}\bigr\} = \\ = \frac{1}{4}\int_{S^1}(f_1 df_2 - f_2 df_1) + \int_{S^1} g_1 g_2\, d\varphi.$$

Notice, that the element $1 \oplus 0$ is contained in the kernel of this form and we really get a form on the quotient $W := (C_+^\infty(S^1)/\mathbb{C}) \oplus C_-^\infty(S^1)$. Next, we define the inner product on W by

$$\Bigl\langle \sum p_n e^{2in\varphi} \oplus \sum q_n e^{(2n+1)\varphi}(d\varphi)^{1/2}, \sum p_n' e^{2in\varphi} \oplus \sum q_n' e^{(2n+1)\varphi}(d\varphi)^{1/2}\Bigr\rangle := \\ = \sum |n|\, p_n \overline{p_n'} + \sum q_n \overline{q_n'}.$$

The superalgebra $\mathfrak{sw}_-$ acts in the space W as follows. Vector fields act in $W_{\overline{0}} \oplus W_{\overline{1}}$ in a natural way.

$$a(\varphi)\frac{d}{d\varphi}\Bigl(f(\varphi) \oplus g(\varphi)d\varphi^{1/2}\Bigr) = a(\varphi)f'(\varphi) \oplus \Bigl(a(\varphi)b'(\varphi) + \tfrac{1}{2}a'(\varphi)b(\varphi)\Bigr)(d\varphi)^{1/2}.$$

An element $0 \oplus b(\varphi)(d\varphi)^{-1/2}$ acts as

$$f(\varphi) \oplus g(\varphi) \mapsto b(\varphi)g(\varphi) \,\oplus\, b(\varphi)f'(\varphi)(d\varphi)^{1/2} = \\ = b(\varphi)(d\varphi)^{-1/2}\cdot g(\varphi)(d\varphi)^{1/2} \,\oplus\, df(\varphi)\cdot b(\varphi)(d\varphi)^{-1/2}.$$

[34] On natural differential operations, see, e. g., [38], [27], [28].

[35] Cf. constructions for Virasoro algebra and the group of diffeomorphisms of circle in [50], Subsect.VII.2.2-2.3, 2.5, VII.3, Sect. VII.3, VIII.6.

Again, this action is a 'natural differential geometric operation'.

In this way, we get an embedding of $\mathfrak{sw}_-$ to an infinite-dimensional Lie superalgebra $\mathfrak{osp}(2\infty|2\infty)$. Next, we apply the infinitesimal version of the orthosymplectic spinors setting $p = \infty$, $q = \infty$ in the notation Subsect. 1.1 and assuming even variables x_j be complex. We arrive at a *projective* representation of the Lie superalgebra $\mathfrak{sw}_-$ in a tensor product of the bosonic and fermionic Fock spaces with infinite number degrees of freedom. Its restriction to the Lie superalgebra $\mathfrak{sw}_-$ also is a projective representation. The corresponding central extension is the *Neveu–Schwarz super-Virasoro algebra*. For formulas for operators, see, e.g., [25], Subsect. 4.2.2.

REMARK. The *Ramond super-Virasoro algebra* arises in a similar way: in (10.1) we assume $a,\ b \in C^\infty(S^1)$ and get another 'super-Witt' algebra $\mathfrak{sw}_+$. Next, in (10.2) we assume $f,\ g \in C^\infty(S^1)$. Repeating the construction above we get an embedding of $\mathfrak{sw}_+$ to an infinite-dimensional Lie superalgebra $\mathfrak{osp}(2\infty|2\infty + 1)$, a construction of the spinor representation for this algebra must be slightly modified (cf. [50], Subsect. III.3.4, VII.3.4). We get a projective representation of $\mathfrak{sw}_+$ or a linear representation of the Ramond Lie superalgebra. ⊠

Question 10.3 a) *Integrate these representations of the Neveu–Schwarz and Ramond algebras to actions of the corresponding supergroups by Gauss–Berezin operators.*

b) *Extend highest weight representations of the Ramond and Neveu–Schwarz supergroups to complex semigroups as in* [49], [50], Sect. VII.4-5.

In this context, it is more natural to consider a more general family of super-Virasoro algebras, see [33], [28].

10.4. 'Unitary representations' of supergroups. The Howe duality for orthosymplectic spinors exists and it was a subject of numerous works, e.g., [54], [15], [14], [41], [16]. In particular, this automatically produces many representations of supergroups, which can be extended to Grassmannians.

As far as I know, a notion of *a unitary representation of a Lie supergroup* is commonly recognized. Consider a finite-dimensional *real* Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$, let $G_{\overline{0}}$ be the Lie group, corresponding to $\mathfrak{g}_{\overline{0}}$. We consider a super-Hilbert space $H = H_{\overline{0}} \oplus H_{\overline{1}}$ and a representation of $\mathfrak{g}$ such that the action of $\mathfrak{g}_{\overline{0}}$ corresponds to a unitary representation of $G_{\overline{0}}$, and for each $X \in \mathfrak{g}_{\overline{1}}$ the operator $\sqrt[4]{-1}X$ is self-adjoint.

Under these conditions, for each $X \in \mathfrak{g}_{\overline{1}} = (\sqrt[4]{-1}X)^2$ the operator $\sqrt{-1}X^2$ is positive (semi)definite. On the other hand, $X^2 = \frac{1}{2}[X, X]_s$ is a generator of $\mathfrak{g}_{\overline{0}}$. In unitary representations, generators of Lie algebras rarely have spectra supported by a semi-axis. This places strong constrains on the Lie algebra $\mathfrak{g}_{\overline{0}}$ and its representation τ in H. In the semi-simple case this implies (see [55]) that τ is an element of Harish-Chandra [29] highest weight holomorphic series. In, particular, few semisimple real supergroups have non-trivial unitary representations in this sense[36]. This was observed at least in works by Furutsu and Nishiyama [23], [24]. Such 'unitary representations' of superalgebras are nice counterparts of the Harish-Chandra holomorphic series and admit explicit classification (Jakobsen [31]), which is a counterpart of the Howe–Enright–Wallach classification of unitary highest weight representations.

Conjecture 10.4 *Any unitary representation of a classical Lie supergroup admits an extension to a certain domain in a certain Grassmanian as in our Theorems* 9.3–9.4*.*

References

[1] Adams J. D. *Discrete spectrum of reductive dual pair* $(\mathrm{O}(p,q),\ \mathrm{Sp}\,(2m))$, Inv. Math., **74** (1983), 449–475.

[2] Arnold V. I. *Mathematical methods of classical mechanics.* Springer-Verlag, New York-Heidelberg, 1978.

[3] Barbier S., Claerebout S., De Bie H. *A Fock model and the Segal-Bargmann transform for the minimal representation of the orthosymplectic Lie superalgebra* $\mathrm{osp}(m,2|2n)$. SIGMA 16 (2020), Paper No. 085, 33 pp.

[4] Barbier S., Frahm J. *A minimal representation of the orthosymplectic Lie supergroup.* Int. Math. Res. Not. IMRN 2021 (2021), no. 21, 16359–16422.

[5] Berezin F. A. *Canonical operator transformation in representation of secondary quantization.* (Russian) Dokl. Akad. Nauk SSSR 137 (1961), 311–314; translated as Soviet Physics Dokl. 6 1961 212–215.

[6] Berezin F. A. *The method of second quantization.* (Russian) Nauka, Moscow 1965; English transl.: Academic Press, New York-London, 1966.

[7] Berezin F. A. *Automorphisms of Grassmann algebra.* Math. Notes, 1:3 (1967), 180–184.

[8] Berezin F. A. *Introduction to algebra and analysis with anticommuting variables.* (Russian), edited by A. A. Kirillov and V. P. Palamodov, Moscow State University, Moscow, 1983; English transl.: *Introduction to superanalysis.* Edited and with a foreword by A. A. Kirillov. With an appendix by V. I. Ogievetsky. Translated from the Russian by J. Niederle

[36] Both highest weight and lowest weight representations of a supergroup, say $\mathrm{OSp}(2p|2q;\mathbb{R})$, can be unitary, but they correspond to different $\sqrt[4]{-1}$ (see [23]). For this reason, a tensor product of a highest weight and a lowest weight unitary representation is not unitary. Counterparts of principal series representations for supergroups exist (since there are flag supervarieties and line bundles over them) but they are not unitary. This restricts possible ways for applications of Lie superalgebras to special functions. This is also an obstacle for an extension of Olshanski theory [56] of representations of infinite-dimensional real classical groups to supergroups.
So a picture is poor comparatively to classical representation theory. It remains a minor hope that a notion of unitary representations of supergroups admits some extension.

and R. Kotecký. Translation edited by D. Leĭtes. D. Reidel Publishing Co., Dordrecht, 1987; Second Russian edition: edited by D. A. Leites with additions by D. A. Leites, V. N. Shander, I. M. Shchepochkina, MCCME publishers, Moscow, 2013.

[9] Berezin F. A., Kats G. I. *Lie groups with commuting and anticommuting parameters.* Math. USSR-Sb. 11:3 (1971), 311–325.

[10] Berezin, F. A., Tolstoy, V. N. *The group with Grassmann structure* UOSP(1.2). Comm. Math. Phys. 78 (1980/81), no. 3, 409–428.

[11] Bernstein I. N., Leites D. A., Shander V. N., *Seminar on supersymmetries.* (Russian) MCCME publishers, 2011.

[12] Carmeli C., Caston, L., Fioresi R. *Mathematical Foundations of Supersymmetry,* European Mathematical Society, 2011.

[13] Cartan É. *Les groupes projectifs qui ne laissent invariante aucune multiplicité plane.* (French) S. M. F. Bull. 41 (1913), 53–96.

[14] Cheng Shun-Jen, Wang Weiqiang. *Howe duality for Lie superalgebras.* Compositio Math. 128 (2001), no. 1, 55–94.

[15] Cheng Shun-Jen, Zhang R. B. *Howe duality and combinatorial character formula for orthosymplectic Lie superalgebras.* Adv. Math. 182 (2004), no. 1, 124–172.

[16] Coulembier K. *The orthosymplectic supergroup in harmonic analysis.* J. Lie Theory 23 (2013) 55–83.

[17] Deligne P., Morgan J. W. *Notes on supersymmetry* (*following Joseph Bernstein*). in Deligne P. (ed.) et al., *Quantum fields and strings: a course for mathematicians. Vol. 1,* Providence, RI, Amer. Math. Soc., 1999.

[18] DeWitt B. *Supermanifolds.* 2nd ed. Cambridge University Press, 1992.

[19] Eisenbud D. *Commutative algebra.* Berlin, Springer, 1995.

[20] Fioresi R., Gavarini F. *Chevalley Supergroups*, Memoirs of the AMS 215 (2012), no. 1014.

[21] Frappat L., Sciarrino A., Sorba P. *Dictionary on Lie algebras and superalgebras.* San Diego, CA: Academic Press, 2000

[22] Friedrichs K. O. *Mathematical aspects of the quantum theory of fields.* Interscience Publishers, Inc., London, 1953.

[23] Furutsu H., Nishiyama K. *Classification of irreducible super-unitary representations of* $\mathfrak{su}(p, q/n)$. Comm. Math. Phys. 141:3 (1991), 475–502.

[24] Furutsu H., Nishiyama K. *Realization of irreducible unitary representations of* $\mathfrak{osp}(M/N;\mathbb{R})$ *on Fock spaces.* In *Representation theory of Lie groups and Lie algebras* (*Fuji-Kawaguchiko,* 1990), eds. T. Kawazoe, T. Oshima and S. Sano, 1–21. World Scientific, River Edge, NJ, 1992

[25] Green M. B., Schwarz J. H., Witten E. *Superstring theory. Vol.* 1. *Introduction.* Cambridge University Press, Cambridge, 1987.

[26] Golfand, Yu. A., Likhtman E. P. *Extension of the algebra of Poincaré group generators and violation of* P*-invariance* JETP Lett., 13 (1971), 422–455.

[27] Grozman P. *Invariant bilinear differential operators.* Commun. Math. 30:3 (2022), 129–188.

[28] Grozman P., Leites D., Shchepochkina I. *Lie superalgebras of string theories.* Acta Math. Vietnam. 26:1 (2001), 27–63.

[29] Harish-Chandra, *Representations of semisimple Lie groups.* IV. Amer. J. Math.77 (1955), 743–777.

[30] Howe R. *Transcending classical invariant theory.* J. Amer. Math. Soc. 2:3 (1989), n 535–552.

[31] Jakobsen H. P. *The full set of unitarizable highest weight modules of basic classical Lie superalgebras.* Mem. Amer. Math. Soc.111(1994), no.532.

[32] Kac V. G. *Lie superalgebras.* Advances in Math. 26:1 (1977) 8–96.

[33] Kac V. G., van de Leur, J. W. *On classification of superconformal algebras.* In collection *Strings* '88 (eds. S. J. Gates, Jr., C. R. Preitschopf, and W. Siegel), 77–106, World Scientific Publishing Co., Inc., Teaneck, NJ, 1989.

[34] Karabegov A. Neretin Yu., Voronov Th. *Felix Alexandrovich Berezin and his work.* In *Geometric methods in physics* (eds. P. Kielanowski, S. T. Ali, A. Odzijewicz, M. Schlichenmaier and Th. Voronov), 3–33, Birkhäuser/Springer, Basel, 2013.

[35] Kashiwara M., Vergne M. *On the Segal-Shale-Weil representations and harmonic polynomials.* Invent. Math. 44:1 (1978) 1–47.

[36] Khudaverdian H., Voronov Th. *Thick morphisms of supermanifolds, quantum mechanics, and spinor representation.* J. Geom. Phys. 148 (2020), 103540, 14 pp.

[37] Kirillov A. A. *The orbits of the group of diffeomorphisms of the circle, and local Lie superalgebras.* Funct. Anal. Appl., 15:2 (1981), 135–137.

[38] Kolář, I., Michor P. W., Slovák J. *Natural operations in differential geometry.* Springer-Verlag, Berlin, 1993.

[39] Lavaud P. *Superpfaffian.* J. Lie Theory 16:2 (2006), 271–296.

[40] Leites D. A. *Introduction to the theory of supermanifolds.* Russian Math. Surveys, 35:1 (1980), 1–64.

[41] Leites D., Shchepochkina I. *The Howe duality and Lie superalgebras.* In *Noncommutative structures in mathematics and physics (Kiev, 2000)*, (eds. S. Duplij and J. Wess) 93–111, Kluwer Acad. Publ., Dordrecht, 2001

[42] Manin Yu. I. *Gauge field theory and complex geometry.* Springer-Verlag, Berlin, 1997.

[43] Milnor J. W., Moore J. C. *On the structure of Hopf algebras.* Ann. of Math. (2) 81 (1965), 211–264.

[44] Molotkov V. *Infinite-dimensional and colored supermanifolds,* J.Nonlinear Math.Phys., Vol. 17, Suppl. 1 (2010) 375–446

[45] Nazarov M., Neretin Yu., Olshanskii G. *Semi-groupes engendrés par la représentation de Weil du groupe symplectique de dimension infinie.* C. R. Acad. Sci. Paris Sér. I Math. 309:7 (1989), 443–446.

[46] Neretin Yu. A. *Unitary representations with highest weight of the group of diffeomorphisms of a circle.* Functional Anal. Appl. 17 (1983), no. 3, 235–237.

[47] Neretin Yu. A. *Spinor representation of an infinite-dimensional orthogonal semigroup and the Virasoro algebra.* Funct. Anal. Appl. 23:3 (1990), 196–207.

[48] Neretin Yu. A. *On a semigroup of operators in the bosonic Fock space.* Funct. Anal. Appl. 24:2 (1990), 135–144.

[49] Neretin Yu. A. *Holomorphic continuations of representations of the group of diffeomorphisms of the circle.* Math. USSR-Sb. 67:1 (1990), 75–97.

[50] Neretin Yu. A. *Categories of symmetries and infinite-dimensional groups.* The Clarendon Press, Oxford University Press, New York, 1996.

[51] Neretin Yu. A. *Gauss–Berezin integral operators and spinors over supergroups* $\mathrm{OSp}(2p|2q)$, Preprint ESI–1930, 2007.

[52] Neretin Yu. A. *"Method of second quantization" of Berezin. View* 40 *years after.* In D. Leites, R. A. Minlos, I. Tyutin (eds.) *"Recollections about Felix Alexandrovich Berezin, the discoverer of supersymmetries"*, Moscow, MCCME publishers, 2008; French translation *La méthode de la seconde quantification de F. A. Berezin regards quarante ans plus tard.* In *Les "supermathématiques" et F. A. Berezin* (eds. Anné C., Roubtsov V.), 15–58, Sér. T, Soc. Math. France, Paris, 2018.

[53] Neretin Yu. A. *Lectures on Gaussian integral operators and classical groups,* European Math. Soc., 2011.

[54] Nishiyama K. *Super dual pairs and highest weight modules of orthosymplectic algebras.* Adv.Math., 104 (1994), 66–89.

[55] Olshanski G. I. *Invariant cones in Lie algebras, Lie semigroups and the holomorphic discrete series.* Functional Anal. Appl. 15:4 (1981), 275–285.

[56] Olshanski G. I. *Unitary representations of infinite-dimensional pairs* (G, K) *and the formalism of R. Howe.* In *Representation of Lie groups and related topics* (eds. A. M. Vershik, D. P. Zhelobenko), 269–463, Gordon and Breach, New York, 1990.

[57] Olshanski G. I. *The Weil representation and the norms of Gaussian operators.* Funct. Anal. Appl. 28:1 (1994), 42–54.

[58] Salam A., Strathdee J.A. *Supersymmetry and Nonabelian Gauges.* Phys. Lett. B. 51 (4) (1974), 353–355.

[59] Sato M., Miwa T., Jimbo M. *Studies on holonomic quantum fields. I.* Proc. Japan Acad. Ser. A Math. Sci. 53:1 (1977), 6–10.

[60] Schmitt T. *Supergeometry and quantum field theory, or: what is a classical configuration*? Rev. Math. Phys. 9:8 (1997), 993–1052.

[61] Segal G. *Unitary representations of some infinite-dimensional groups.* Comm. Math. Phys. 80:3 (1981), 301–342.

[62] Segal I.E., *Foundations of the theory of dynamical systems of infinitely many degrees of freedom* (1), Mat.-Fys. Medd. K. Dan. Vidensk. Selsk. 31 (12) (1959), 1–39.

[63] Serganova V. V. *Classification of real simple Lie superalgebras and symmetric superspaces.* Funct. Anal. and Appl. 17:3 (1983), 200–207

[64] Sergeev A. *The invariant polynomials on simple Lie superalgebras.* Represent. Theory, 3 (1999) 250–280.

[65] Serov A. A. *The Clifford-Weyl algebra and the spinor group.* In *Problems in group theory and homological algebra,* ed. A. L. Onishchik (Russian), 143–145, Yaroslav. Gos. Univ., Yaroslavl, 1985.

[66] Shale D. *Linear symmetries of free boson fields.* Trans. Amer. Math. Soc. 103 (1962), 149–167.

[67] Shale D., Stinespring W. F. *Spinor representations of infinite orthogonal groups.* J. Math. Mech. 14 (1965) 315–322.

[68] Varadarajan V. S. *Supersymmetry for mathematicians: an introduction.* Amer. Math. Soc., Providence, RI, 2004.

[69] Volkov D. V., Akulov V. P. *Is the neutrino a Goldstone partile*? Phys. Lett., B46 (1973), 109–112.

[70] Volkov D.V., Soroka V.A. *Higgs effect for Goldstone particles with spin* 1/2, JETP Lett. 18:8 (1973), 312–314.

University of Graz;
High School of Modern Mathematics MIPT;
Moscow State University, MechMath Department.
Math.Dept., University of Vienna;
Institute for Theoretical and Experimental Physics (until 11.2021);

neretin.ua(frog)mipt.ru
URL:www.mat.univie.ac.at/∼neretin