Fermat Primes.

César Aguilera.

April 1, 2025

Abstract

Properties related to Fermat Primes, their connectedness with Mersenne Numbers and the reason why there are not infinite many Mersenne Primes and Perfect Numbers.

Introduction.

The Fermat Primes, discovered by the french Mathematician Pierre de Fermat, are prime numbers of the form $\mathbf{2^{2^n}+1}$ for $\mathbf{n \geq 0}$, there are only five known numbers of this type (3,5,17,257,65537).

Theorem. (1)

Given the matrix $A = \begin{vmatrix} n & 1 \\ n+1 & 2n+1 \end{vmatrix}$ with Determinant $|A|$, then:

$$A = \begin{vmatrix} n & 1 \\ n+1 & 2n+1 \end{vmatrix} = \begin{vmatrix} n & 1 \\ n+1 & F_k \end{vmatrix} \Rightarrow \begin{vmatrix} |A|+1 & 1 \\ |A|+2 & F_{(k+1)} \end{vmatrix} \qquad (1)$$

This is:

$$\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1 \Rightarrow \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = 7 \Rightarrow \begin{vmatrix} 8 & 1 \\ 9 & 17 \end{vmatrix} = 127 \Rightarrow \begin{vmatrix} 128 & 1 \\ 129 & 257 \end{vmatrix} = 32767 \Rightarrow$$
$$\begin{vmatrix} 32768 & 1 \\ 32769 & 65537 \end{vmatrix}$$

The matrix **A** is a Fibonacci Box,[1] and they have the next properties:

$$A = \underbrace{\begin{bmatrix} x & w \\ y & z \end{bmatrix}}_{[a,b,c]} \Rightarrow \begin{array}{c} y = x + w \\ z = y + x \\ a = (z)(w) \\ b = (2)(x)(y) \\ c = (z)(x) + (y)(w) \end{array}$$

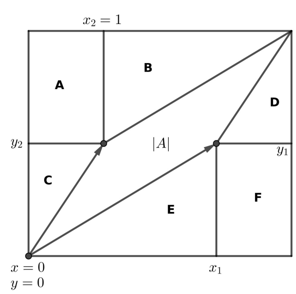
so we take $z = 2^{2^{(n-1)}} + 1 = F_k$ and $w = 1$ and then we can find the other terms x and y.

From matrix A we have $\begin{vmatrix} 2^{(2^{(n-1)}-1)} & 1 \\ 2^{(2^{(n-1)}-1)}+1 & 2^{2^{(n-1)}}+1 \end{vmatrix}$ where $|A| = 2^{(2^n-1)} - 1$
thus, we can find the Primitive Pythagorean Triple [a,b,c]:
$a = (z)(w) = (2^{2^{(n-1)}} + 1)(1) = 2^{2^{(n-1)}} + 1 = F_k$
$b = (2)(x)(y) = (2)(2^{(2^{(n-1)}-1)})(2^{(2^{(n-1)}-1)} + 1) = 2^{2^{n-1}} + 2^{2^n-1}$
$c = (z)(x) + (y)(w)$
$= (2^{2^{(n-1)}} + 1)(2^{(2^{(n-1)}-1)}) + (2^{(2^{(n-1)}-1)} + 1)(1) = 2^{2^{n-1}} + 2^{2^n-1} + 1$

So, if:
Fermat primes $F_k = 2^{2^{(n-1)}} + 1 = \{3, 5, 17, 257, 65537\}$
and
$|A| = 2^{(2^n-1)} - 1 = \{0, 1, 7, 127, 32767\}$ then, from (1): $2|A| + 3 = F_{(k+1)}$

Also if
$|A|_k = n_k \to |A|_{(k+1)} = ((F_{(k+1)})(|A|_k + 1)) - (|A|_k + 2)(1)$
$= 2(|A|_k)^2 + 4|A|_k + 1$ or $n_{(k+1)} = 2(n_k)^2 + 4n_k + 1.$ (2)
In other words, if the polynomial $2n^2 + 4n + 1$ takes the values of $n = |A_k|$, we obtain $|A_{k+1}|$.

Primitive Pythagorean Triples related to Areas,
Graphic representation of $\mathbf{A} = \begin{vmatrix} n & 1 \\ n+1 & 2n+1 \end{vmatrix}$ **as a determinant.**



Areas of segments.

$$A = F = x_2 \cdot y_1 = n + 1$$

$$B = E = \frac{x_1 \cdot y_1}{2} = \frac{n \cdot (n+1)}{2} = \frac{n^2+n}{2}$$

$$C = D = \frac{x_2 \cdot y_2}{2} = \frac{y_2}{2} = \frac{2n+1}{2}$$

Figure 1: Determinant.

[2]

As the matrix is a Fibonacci box, we calculate the Primitive Pythagorean Triple.

$a = \{2C \vee 2D\} = 2 \cdot \frac{2n+1}{2} = 2n+1 = F_k = C + D.$
$b = \{4B \vee 4E\} = 4 \cdot \frac{n^2+n}{2} = 2n^2 + 2n$
$c = \{4B \vee 4E\} + 1 = 4 \cdot \frac{n^2+n}{2} + 1 = 2n^2 + 2n + 1$

So, the Pythagorean Triple related to Areas is: $[\underbrace{2n+1}_{a}, \underbrace{2n^2+2n}_{b}, \underbrace{2n^2+2n+1}_{c}]$

now, we can perform Pythagorean Triples whose shortest leg is the n-th Fermat number, we can see them in the next table with only Fermat Primes.

Table 1

$a = F_k = 2n+1$	$b = 2n^2 + 2n$	$2n^2 + 2n + 1$
3	4	5
5	12	13
17	144	145
257	33024	33025
65537	2147549184	2147549185

note that:
$\sqrt{(2n^2+2n)+(2n^2+2n+1)} = \sqrt{(2n^2+2n+1)^2-(2n^2+2n)^2} = 2n+1$
this is the short leg F_k

To obtain these Pythagorean Triples, we need to transform from:
$2^{2^{(n-1)}} + 1$ to the short leg $2n+1 = a$
$2^{2^n} + 2^{2^{n+1}-1}$ to the long leg $2n^2 + 2n = b$.
$2^{2^n} + 2^{2^{n+1}-1} + 1$ to the hypotenuse $2n^2 + 2n + 1 = c$.

As these Pythagorean Triples are related to areas, we can represent them this way

$$a = 2n+1 = F_k, b = sqrt\left[\int_1^{F_k} (x^3 - x)\,dx\right], c = sqrt\left[\int_1^{F_k} (x^3 - x)\,dx + 1\right]$$

[3]

Properties related to the Dot product $|\tilde{\mathbf{a}} \cdot \tilde{\mathbf{b}}|$.
Given two vectors **a** and **b**, such that:

$\mathbf{a} = (2^{(2^{(n-1)}-1)}, 2^{(2^{(n-1)}-1)}+1) = (n, n+1)$ and $\mathbf{b} = (1, 2^{2^{(n-1)}}+1) = (1, 2n+1)$ then $\frac{|\vec{a} \cdot \vec{b}| - |A|}{2} = F_k$,

this is, the dot product of $\underbrace{\begin{bmatrix} n \\ n+1 \end{bmatrix}}_{\vec{a}} \cdot \underbrace{\begin{bmatrix} 1 \\ 2n+1 \end{bmatrix}}_{\vec{b}}$ minus $|A|$, and all that divided by 2

is a Fermat Number, so we have:
$|\vec{a} \cdot \vec{b}| = 2n^2 + 4n + 1, |A| = 2n^2 - 1$

$|\vec{a}\cdot\vec{b}| - |A| = 4n+2 \Rightarrow \frac{4n+2}{2} = 2n+1.$
The dot product $|\vec{a}\cdot\vec{b}|$ is also (2) and have the same value of a Determinant so,
$\frac{|\vec{a}\cdot\vec{b}|-|A|}{2} = F_k$ turns to $\frac{|A|_k - |A|_{k-1}}{2} = F_k$ this is:
$\frac{2n^2+4n+1-(2(n-1)^2+4(n-1)+1)}{2} = \frac{4n+2}{2} = 2n+1 = F_k$
Note that $2n+1$ which represents the Fermat Numbers becomes the *short leg* of the pythagorean triangle (a,b,c).
another property is: $a+b = \underbrace{2n+1}_{a} + \underbrace{2n^2+2n}_{b} = 2n^2+4n+1$ thus $a+b = |\vec{a}\cdot\vec{b}|$

Also note that:
$(2n^2+2n+1)(4n^2+4n+2) - (2n^2+4n+1)^2 = (2n^2-1)^2$
this is:
$(|\vec{a}||\vec{b}|)^2 - |\vec{a}\cdot\vec{b}|^2 = |A|^2$
from **(1)** we have:

$$A_{k+1} = \begin{vmatrix} \sqrt{(|\vec{a}||\vec{b}|)^2 - |\vec{a}\cdot\vec{b}|^2} + 1 & 1 \\ \sqrt{(|\vec{a}||\vec{b}|)^2 - |\vec{a}\cdot\vec{b}|^2} + 2 & 2\sqrt{(|\vec{a}||\vec{b}|)^2 - |\vec{a}\cdot\vec{b}|^2} + 3 \end{vmatrix}$$

Since $\left\{\prod_1^k F_k\right\} + 2 = F_{k_n} \Rightarrow 3\left\{\prod_1^k 2\sqrt{(|\vec{a_k}||\vec{b_k}|)^2 - |\vec{a_k}\cdot\vec{b_k}|^2} + 3\right\} + 2 = F_{k_n}$

example:
$3((2\sqrt{(\sqrt{5}\sqrt{10})^2 - 7^2} + 3) \times (2\sqrt{(\sqrt{13}\sqrt{26})^2 - 17^2} + 3)$
$\times (2\sqrt{(\sqrt{145}\sqrt{290})^2 - 161^2} + 3) \times (2\sqrt{(\sqrt{33025}\sqrt{66050})^2 - 33281^2} + 3)) + 2$
$= 4294967297$

Euler-Legendré-Gauss **Theorems.**
if every Mersenne number $M_n \equiv 2^n - 1$ we can write:

$$|\vec{a}_k| = \sqrt{\left(2^{\frac{M_{n_k}-1}{2}}\right)^2 + \left(2^{\frac{M_{n_k}-1}{2}} + 1\right)^2}$$

$$|\vec{b}_k| = \sqrt{2}\sqrt{\left(2^{\frac{M_{n_k}-1}{2}}\right)^2 + \left(2^{\frac{M_{n_k}-1}{2}} + 1\right)^2}$$

$$(d_k)^2 = \left(2^{\frac{M_{n_k}-1}{2}}\right)^2 + \left(2^{\frac{M_{n_k}-1}{2}} - 1\right)^2$$

so, from the *Euler-Legendré-Gauss* Quadratic Reciprocity Theorem:
if $(d_k)^2$ is prime then $(d_k)^2 \equiv 1 \pmod 4$
since $(d_k)^2 = |A_{k-1}|^2 + |A_{k-1}+1|^2$
and $\left(|\vec{a}_k\cdot\vec{b}_k| - |\vec{a}_k|^2\right)^2 = |\vec{a}_{k+1}\cdot\vec{b}_{k+1}| - |\vec{a}_{k+1}|^2$
from *Legendrés Sum of Three Squares Theorem*:
$|\vec{a}_k\cdot\vec{b}_k| - (d_k)^2 = \left(|\vec{a}_{k-1}\cdot\vec{b}_{k-1}| - |\vec{a}_{k-1}|^2\right)^2 + \left(|\vec{a}_{k-1}\cdot\vec{b}_{k-1}| - |\vec{a}_{k-1}|^2\right)^2 + \left(|\vec{a}_{k-1}\cdot\vec{b}_{k-1}| - |\vec{a}_{k-1}|^2\right)^2$
or
$|\vec{a}_k\cdot\vec{b}_k| - (d_k)^2 = (F_{k-1}-1)^2 + (F_{k-1}-1)^2 + (F_{k-1}-1)^2$

Given $\begin{vmatrix} n & 1 \\ n+1 & F_k \end{vmatrix} = |A|_k \Rightarrow (d_k)^2 = |A|_k - F_k + 3$ and $\frac{F_k - 3}{2} = |A|_{k-1}$

$\Rightarrow 2|A|_{k-1} + 3 = F_k$, since $(d_k)^2 \equiv 1 \pmod 4$

the recurrent division of $\frac{(d_k)^2 - 1}{4} = |A|_{k-1}$

Example:

$$A = \begin{vmatrix} 32768 & 1 \\ 32769 & 65537_{F_k} \end{vmatrix} \Rightarrow |A| = \underbrace{2147483647}_{\textit{Mersenne prime and n-1}}$$

$$\frac{(d^2)-1}{4} = \frac{2147418113-1}{4} = 536854528 \cdots \frac{131068}{4} = 32767$$

$$A = \begin{vmatrix} 128 & 1 \\ 129 & 257_{F_k} \end{vmatrix} \Rightarrow |A| = 32767$$

$$\frac{(d^2)-1}{4} = \frac{32513-1}{4} = \underbrace{8128}_{\textit{Perfect number}} \cdots \frac{508}{4} = \underbrace{127}_{\textit{Mersenne prime and n-1}}$$

$$A = \begin{vmatrix} 8 & 1 \\ 9 & 17_{F_k} \end{vmatrix} \Rightarrow |A| = 127$$

$$\frac{(d^2)-1}{4} = \frac{113-1}{4} = \underbrace{28}_{\textit{Perfect number}} \cdots \frac{28}{4} = \underbrace{7}_{\textit{Mersenne prime and n-1}}$$

$$A = \begin{vmatrix} 2 & 1 \\ 3 & 5_{F_k} \end{vmatrix} \Rightarrow |A| = 7$$

$$\frac{(d^2)-1}{4} = \frac{5-1}{4} = \frac{4}{4} = 1 \Rightarrow A = \begin{vmatrix} 1 & 1 \\ 2 & 3_{F_k} \end{vmatrix} \Rightarrow |A| = 1$$

while we divide, notice the presence of **Perfect Numbers** and almost **Perfect Numbers** as well **Mersenne Primes** and almost **Mersenne Primes**.
In the matrix of the first non prime Fermat Number we have:

$$A = \begin{vmatrix} 2147483648 & 1 \\ 2147483649 & 4294967297_{F_k} \end{vmatrix} \Rightarrow |A| = 9223372036854775807$$

$$\frac{(d^2)-1}{4} = \frac{9223372032559808512}{4} = \underbrace{2305843008139952128}_{\textit{Perfect Number}} \text{ then}$$

$$\underbrace{2305843009213693951}_{\textit{Mersenne prime}} - \underbrace{2305843008139952128}_{\textit{Perfect Number}} = \underbrace{1073741823}_{\textit{n-1}} \text{ and then}$$

$$A = \begin{vmatrix} 1073741824 & 1 \\ 1073741825 & 2147483649 \end{vmatrix} \Rightarrow |A| = \underbrace{2305843009213693951}_{\textit{Mersenne prime}}$$

The $\underbrace{2147483647}_{\textit{Mersenne prime}}$ corresponds to $\underbrace{2305843008139952128}_{\textit{Perfect Number}}$

Also:

$$A = \begin{vmatrix} 128 & 1 \\ 129 & 257_{F_k} \end{vmatrix} \Rightarrow |A| = 32767$$

$$\frac{(d^2)-1}{4} = \frac{32513-1}{4} = \underbrace{8128}_{\textit{Perfect number}} \Rightarrow \underbrace{8191}_{\textit{Mersenne prime}} - \underbrace{8128}_{\textit{Perfect number}} = \underbrace{63}_{\textit{n-1}}$$

$$A = \begin{vmatrix} 64 & 1 \\ 65 & 129 \end{vmatrix} \Rightarrow |A| = \underbrace{8191}_{\textit{Mersenne prime}}$$

There exists a recurrent triad of functions correlated between them such that their differences generate the Fermat Numbers, this functions are:
$y = 2^x - 1$ (Mersenne Numbers and (Fermat Numbers minus 2), see page 16 (Fermat Numbers with two prime factors)).
$y = 4((2^x - 1)(2^{x-1})) + 1 = 2^{x+1}(2^x - 1) + 1 = d^2$ (squared distance between vectors,).
$y = (2^{x-1})(2^x - 1)$ (contains the Perfect Numbers).

$y = 2^x - 1$	$y = 2^{x+1}(2^x - 1) + 1 = d^2$	$y = (2^{x-1})(2^x - 1)$
1	5	1
$3_{F_k - 2}$	25	6 *Perfect number*
7	113	28 *Perfect number*
$15_{F_k - 2}$	481	120
31 *Mersenne prime*	1985	496 *Perfect number*
...	...	...
9223372036854775807	9223372032559808513	9223372034707292160
$(x = 63)$	$(x = 31)$	$(x = 32)$

In the case of the example above (Mersenne Prime and Perfect Number that looks the same), we have:

$$\underbrace{2305843009213693951}_{Mersenne\ prime}$$

$$\Big\rangle \ 2147483646$$

$$\Big\rangle \ \underbrace{1073741823}_{n-1}$$

$$\underbrace{2305843007066210305}_{d^2}$$

$$\Big\rangle \ \underbrace{1073741823}_{n-1}$$

$$\underbrace{2305843008139952128}_{Perfect\ number}$$

For the matrix of the first non prime Fermat number, we have:

$$A = \begin{vmatrix} 2147483648 & 1 \\ 2147483649 & 4294967297 \end{vmatrix} \Rightarrow |A| = 9223372036854775807$$

$|A| = 9223372036854775807$
$d^2 = 9223372032559808513$
$(2^{x-1})(2^x - 1) = 9223372034707292160$

9223372036854775807

$\rangle\ 4294967294_{F_k-3}$

9223372032559808513

$\rangle\ 2147483647_{M_p}$

$\rangle\ 65534_{F_k-3}$

9223372034707292160 $\quad$ 2147418113

$\rangle\ 32767$

$\rangle\ 254_{F_k-3}$

2147450880 $\quad$ $\underbrace{32513}_{(8128\cdot 4)+1}$

$\rangle\ 127_{M_p}$

$\rangle\ 14_{F_k-3}$

32640 $\quad$ $\underbrace{113}_{(28\cdot 4)+1}$

$\rangle\ 7_{M_p}$

120 $\quad$ 5

6

$7-5=2_{F_k-3}$. Remember that $2|A|_k+3=F_{k+1} \Rightarrow |A|_k=\frac{F_{k+1}-3}{2}$.

Primitive Pythagorean Triples related to Lenghts.
Given the matrix

$$\begin{vmatrix} n & 1 \\ n+1 & 2n+1 \end{vmatrix} \text{ the distance } d = \sqrt{(1-n)^2 + ((2n+1)-(n+1))^2} = \sqrt{2n^2-2n+1}$$

$$\underbrace{(2n-1)^2}_{a} + \underbrace{(2n^2-2n)^2}_{b} = \underbrace{(2n^2-2n+1)^2}_{c}$$

$|A| - d^2 + 3 = F_k = (2n^2-1) - (2n^2+2n+1) + 3 = 2n+1$
$|A| + d^2 + F_k = F_{k+1} = (2n^2-1) + (2n^2-2n+1) + (2n+1) = 4n^2+1$
For areas: $b + d^2 = F_{k+1} = (2n^2+2n) + (2n^2-2n+1) = 4n^2+1$
This is:
$|A| + d^2 + F_k = F_{k+1} \Rightarrow |A| + d^2 + |A| - d^2 + 3 = F_{k+1}$
Comparing polynomials of areas and lenghts we see that n_k related to areas is equal to n_{k+1} related to lenghts.

$$n \begin{array}{l} \nearrow \quad 2n^2-2n+1 = d^2 \\ \searrow \quad 2n^2+2n+1 = \|a\|^2 \end{array}$$

from: $|A|_k = d^2 + F_k - 3$ and knowing that $\mathbf{n_k}$ related to areas is equal to $\mathbf{n_{k+1}}$ related to lenghts, we change values of the lenghts, so we have
$(2n^2+2n+1)(4n^2+4n+2) = (|\vec{a}|_k|\vec{b}|_k)^2$
$(2n^2-2n+1)(4n^2+4n+2) = |A|_{k+1} + 3$
$(2n^2-2n+1)(4n^2-4n+2) = 2(d_k)^4$

Line equations between the two points $(n, n+1)$ **and** $(1, 2n+1)$**.**

$A = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \rightarrow |A| = 1$

$A = \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} \rightarrow |A| = 7, y = \frac{7}{1} - \frac{2x}{1}$

$A = \begin{vmatrix} 8 & 1 \\ 9 & 17 \end{vmatrix} \rightarrow |A| = 127, y = \frac{127}{7} - \frac{8x}{7}$

$A = \begin{vmatrix} 128 & 1 \\ 129 & 257 \end{vmatrix} \rightarrow |A| = 32767, y = \frac{32767}{127} - \frac{128x}{127}$

$A = \begin{vmatrix} 32768 & 1 \\ 32769 & 65537 \end{vmatrix} \rightarrow |A| = 2147483647, y = \frac{2147483647}{32767} - \frac{32768x}{32767},$

so, the line equation is:

$$y_k = \frac{|A|_k}{|A|_{k-1}} - \frac{n_k x}{|A|_{k-1}} = \frac{|A|_k}{|A|_{k-1}} - \frac{(|A|_{k-1}+1)x}{|A|_{k-1}}$$

since all matrices are connected, we can generalize some properties related to (n) for example:
$1^2 + 1^2 = 2$

$2^2 + 2^2 = 8$
$8^2 + 8^2 = 128$
$128^2 + 128^2 = 32768$
$(n_k)^2 + (n_k)^2 = n_{k+1} \qquad (3)$

notice that (n) is the number of *primitive roots* of F_k.
also note that:
$1 = 2^0$
$2 = 2^1$
$8 = 2^3$
$128 = 2^7$
$32768 = 2^{15}$
$0, 1, 3, 7, 15$ are the Mersenne Numbers and $|A|$ is a Mersenne Number, this is, the values of $|A|$ are Mersenne Numbers of the form $2n^2 - 1$.

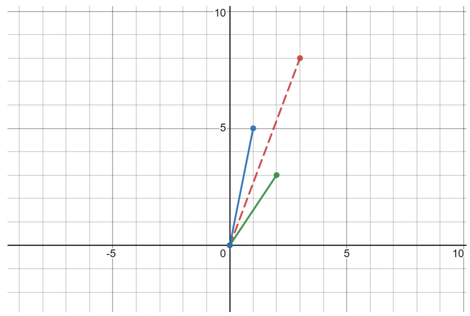
So we can state the next **theorem (2)** for Mersenne Numbers:
If a Mersenne number (Mn_1) have the form $2n^2 - 1$ this is, the equation $(2n^2 - 1 = Mn_1)$ have integer solutions for (n), the next Mersenne Number Mn_2 doesn't have the form $2n^2 - 1$ and $(2 \cdot Mn_1 + 1 = Mn_2)$ then the value of (n) for $(2n^2 - 1 = Mn_2)$ is equal to $n_1\sqrt{2}$.
$M_k = 2n^2 - 1; 2M_k + 1 = M_{k+1}$
$M_k = 2(n_1)^2 - 1 | n_1 \in Z \Rightarrow \{M_{k+1} = 2(n_2)^2 - 1 | n_2 = n_1\sqrt{2}\}$
So, for every Fermat Number F_k corresponds a Mersenne Number Mn_k of the form $2n^2 - 1$,
We can see the next table:

$M_{n1} = 2n^2 - 1$	$M_{n2} \neq 2n^2 - 1$
$1 \to n = 1 = 2^0$	$3 \to n = 1\sqrt{2}$
$7 \to n = 2 = 2^1$	$15 \to n = 2\sqrt{2}$
$31 \to n = 4 = 2^2$	$63 \to n = 4\sqrt{2}$
$M_{n1} = 2(2^k)^2 - 1$	$M_{n2} = 2(2^k\sqrt{2})^2 - 1$

Table 2

Angle between vectors.
Example for $F_k = 5$

$$\underbrace{\begin{bmatrix}2\\3\end{bmatrix}}_{\vec{a}} \ \underbrace{\begin{bmatrix}1\\5\end{bmatrix}}_{\vec{b}} \rightarrow \begin{array}{c} |\vec{a}\cdot\vec{b}| = 17 \\ ||a|| = \sqrt{13} \\ ||b|| = \sqrt{26} \\ \cos\theta = \frac{17}{26}\sqrt{2} \end{array}$$



$$A = \underbrace{\begin{bmatrix}x_k & 1\\ y_k & z_k\end{bmatrix}}_{[a_k, b_k, c_k]} \rightarrow \underbrace{\begin{bmatrix}x_{k+1} & 1\\ y_{k+1} & z_{k+1}\end{bmatrix}}_{[a_{k+1}, b_{k+1}, c_{k+1}]} \begin{array}{c} a = (z)(w) \\ b = (2)(x)(y) \\ c = (z)(x) + (y)(w) \end{array} \rightarrow b_k - (F_k - 1) = x_{k+1}$$

To obtain n_{k+1} given n_k we have: $(2 \cdot x_k \cdot y_k) - z_k = x_{k+1}$ this is
$2 \cdot n \cdot (n+1) - F_k - 1 = n_{k+1} \rightarrow 2(n_k)^2 = n_{k+1}$
To get the matrices and PPT recursively, we have:

$$A_k = \underbrace{\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix}}_{[3,4,5]} \begin{array}{c} n_k = 1 \rightarrow 2(n_k)^2 = 2 = n_{k+1} \\ n_{k+1} + 1 = 3 \\ 2n_{k+1} + 1 = 5 = a_{k+1} \\ b_{k+1} = 2 \cdot n_{k+1} \cdot (n_{k+1} + 1) = 2 \cdot 2 \cdot 3 = 12 \\ c_{k+1} = b_{k+1} + 1 = 12 + 1 + 13 \end{array} \rightarrow A_{k+1} = \underbrace{\begin{bmatrix}2 & 1\\ 3 & 5\end{bmatrix}}_{[5,12,13]}$$

From the theorem (2) above we can factorize the determinant $|A|$ and obtain it recursively:
$|A|_{k+1} = [(2|A|_k + 1) - |A|_k\sqrt{2}][(2|A|_k + 1) + A_k\sqrt{2}]$
as for Mersenne Numbers $2Mn_k = Mn_{k+1}$ we have
$|A|_k = (Mn_k - Mn_{k-1}\sqrt{2})(Mn_k + Mn_{k-1}\sqrt{2})$
This takes us to:

$$\left\{ \begin{array}{l} (F_{k-1} - 1)\sqrt{n+1} \approx \|a\| \\ (F_{k-1} - 1)\sqrt{F_k + 1} \approx \|b\| \end{array} \right\} \quad \text{Table 3}$$

F_K	$\|a\|$	$(F_{k-1} - 1)\sqrt{\frac{F_k+1}{2}}$	$\|b\|$	$(F_{k-1} - 1)\sqrt{F_k + 1}$
5	3.605551275464	3.46410161513775	5.0990195135928	7.3484692283496
17	12.041594578792	12	17.029386365926	21.213203435597
257	181.72781845386	181.72506706560	257.00194551793	273.06043287155
65537	46341.657113229	46341.657113229	65537.000007629	65793.003898591
4294967297	$65536\sqrt{2147483649}$	$65536\sqrt{2147483649}$	$65536\sqrt{4294967298}$	$65536\sqrt{4294967298}$

(From Figure 1) Note that triangle (C) have legs $(2n + 1 = F_k$ and 1) so hypothenuse $\|b\| \approx F_k$.

Cos θ.
Since $\|b\| = \|a\|\sqrt{2}$, $|a \cdot b| = 2n^2 + 4n + 1$, $\|a\|^2 = 2n^2 + 2n + 1$, $|b\|^2 = 4n^2 + 4n + 2$ we have:

$$\frac{|a \cdot b|}{\|a\|\|b\|} = \frac{|a \cdot b|}{\|a\|\|a\|\sqrt{2}} = \frac{|a \cdot b|}{\|a\|^2\sqrt{2}} = \frac{2n^2 + 4n + 1}{(2n^2 + 2n + 1)\sqrt{2}}$$

or

$$\frac{|a \cdot b|}{\|a\|\|b\|} = \frac{|a \cdot b|}{\|b\|^2}\sqrt{2} = \frac{(2n^2 + 4n + 1)\sqrt{2}}{4n^2 + 4n + 2}$$

then
$\cos\theta \approx \lim_{n\to\infty} \frac{(2n^2+4n+1)\sqrt{2}}{4n^2+4n+2} \approx \lim_{n\to\infty} \frac{2n^2+4n+1}{(2n^2+2n+1)\sqrt{2}} \approx \frac{1}{\sqrt{2}} \Rightarrow \theta \approx \frac{\pi}{4} \approx 45°$
note: (the roots of the parabola $(2n^2 - 1)$ are: $-\frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}}$)
$|a \cdot b| = \|a\|\|b\| \cos\theta \Rightarrow \cos\theta\|a\|\|b\| - \frac{1}{\sqrt{2}}\|a\|\|b\| = 2n$.
A PPT conformed by norm values is:

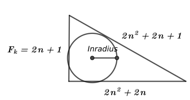
$$a = \sqrt{2\|a\|^2 - 1} = \sqrt{\|b\|^2 - 1} = F_k, b = \|a\|^2 - 1, c = \|a\|^2$$

Primitive Roots, Fermat Primes and their relation with Mersenne Primes and Perfect Numbers.

A Primitive Root modulo (p) is an integer (g) such that $g, g^2, \ldots, g^{p-1}$ constitute a reduced residue system modulo (p), such system is a cyclic group.[4]

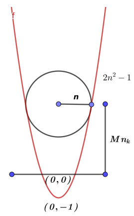
From the PPT polynomials, we can relate the *inradius* of a given pythagorean triangle and it's *primitive roots.*

number of primitive roots=inradius=$\frac{ab}{a+b+c} = \frac{(2n+1)(2n^2+2n)}{(2n+1)+(2n^2+2n)+(2n^2+2n+1)} = n$
circumradius=$\frac{c}{2} = n^2 + n + \frac{1}{2}$
A geometric interpretation is:



$$\Rightarrow \begin{vmatrix} n & 1 \\ n+1 & F_k \end{vmatrix} = Mn_k$$

$$\Downarrow$$



n is the radius of the circle that fits on the parabola $2n^2 - 1$ at a distance $(0, Mn_k)$ from the origin [5].

We can see the Primitive Roots of the Fermat Primes in the next table. Table 4

F_k	$n = \phi(F_{k_n})$	$\sum \phi(F_{k_n})$	*Factorization*	*Sum of squares*	$\sum \phi(F_{k_n}) - \phi(F_{k_n})$
3	1	2	$1 \cdot 2$	$1^2 + 1^2$	2^0
5	2	5	$1 \cdot 5$	$1^2 + 2^2$	$2^0 \cdot 3$
17	8	68	$4 \cdot 17$	$2^2 + 8^2$	$2^2 \cdot 3 \cdot 5$
257	128	16448	$64 \cdot 257$	$8^2 + 128^2$	$2^6 \cdot 3 \cdot 5 \cdot 17$
65537	32768	1073758208	$16384 \cdot 65537$	$128^2 + 32768^2$	$2^{14} \cdot 3 \cdot 5 \cdot 17 \cdot 257$
			$(\phi(F_{k-1_n}))^2 \cdot (F_{k_n})$	$(\phi(F_{k-1_n}))^2 + (\phi(F_{k_n}))^2$	$(\phi(F_{k-1_n}))^2 \cdot \prod_{k=1}^{4} F_k$

from (1) page 1, and (3) page 4, we have:

$$\sum \phi(F_{k_n}) = (|A|_k + 1)^2 + (|A|_{k-1} + 1)^2 = \left(\frac{F_k - 1}{2}\right)^2 + \left(\frac{F_{k-1} - 1}{2}\right)^2 = 2^{2^n - 2} \cdot F_k$$

Since the Fermat Primes are connected in relation with their Primitive Roots and the next Fermat Number (4294967297) is not prime and it's number of Primitive Roots is cero, there can't exist more Fermat Primes beyond 65537.

Given the **theorem (2)** "for every Fermat Number F_k corresponds a Mersenne Number Mn_k of the form $2n^2 - 1$", we have the next table:

Matrix	*Mn*	*(Mn) Factorization*
$\begin{bmatrix} n & 1 \\ n+1 & 2n+1 \end{bmatrix}$	*(Determinant)*	$[(2n-1) - (n-1)\sqrt{(2)}][(2n-1) + (n-1)\sqrt{(2)}]$
$\begin{bmatrix} \sqrt{2} & 1 \\ 2+\sqrt{2} & 1+2\sqrt{2} \end{bmatrix}$	3	$((2\sqrt{2} - 1) - \sqrt{2}(\sqrt{2} - 1))((2\sqrt{2} - 1) + \sqrt{2}(\sqrt{2} - 1))$
$\begin{bmatrix} 2 & 1 \\ 3_{Fp} & 5 \end{bmatrix}$	7	$(3 - 1\sqrt{2})(3 + 1\sqrt{2})$
$\begin{bmatrix} 4 & 1 \\ 5_{Fp} & 9 \end{bmatrix}$	31	$(7 - 3\sqrt{2})(7 + 3\sqrt{2})$
$\begin{bmatrix} 16 & 1 \\ 17_{Fp} & 33 \end{bmatrix}$	511	$(31 - 15\sqrt{2})(31 + 15\sqrt{2})$
$\begin{bmatrix} 256 & 1 \\ 257_{Fp} & 513 \end{bmatrix}$	131071	$(511 - 255\sqrt{2})(511 + 255\sqrt{2})$
$\begin{bmatrix} 65536 & 1 \\ 65537_{Fp} & 131073 \end{bmatrix}$	8589934591	$(131071 - 65535\sqrt{2})(131071 + 65535\sqrt{2})$

$Mn_k(n^2)$ is the Perfect Number related to the given Mersenne Prime (Mn_k).
$Mn_k - (n-1) = Mn_{k-1}$ triangular number.
$Mn_k = (Mn_{k-1} - (n-1)\sqrt{2})(Mn_{k-1} + (n-1)\sqrt{2})$
$Mn_{k+1} = n^2(Mn_k + 1) - 1.$

n is two times the primitive roots of Fermat primes, so, from (1) page 1 and PPT $(F_k, 2n^2+2n, 2n^2+2n+1)$, page 3, we have:

$$\left\{\begin{array}{c} n + 2\sum\phi(F_{k_n}) = b \\ \Downarrow \\ b - F_k = n_{k+1} - 1 \end{array}\right\} \begin{array}{c} \text{We relate PPT table (page 3)} \\ \text{with Primitive Roots.} \\ \Downarrow \end{array}$$

$2 \cdot b_k - |A|_k = F_{k+1}$, example: PPT $[17, 144, 145] \Rightarrow (2 \cdot 144) - 31 = 257$

We can represent the table this way:

given the values of the function $y = 2^x - 1$

2^2-1	2^3-1	2^4-1	2^5-1	2^6-1	2^7-1	2^8-1	2^9-1	$2^{10}-1$
3	7	15	31	63	127	255	511	1023
✓	✓		✓				✓	

$2^x - 1 = |A|_k \Rightarrow 2^{2x-1} - 1 = |A|_{2k}$.

Example: $2^5 - 1 = 31 \rightarrow 2^{(2\cdot5)-1} - 1 = 511$

The Mersenne Primes and the Perfect Numbers depend on the existence of the Fermat Primes and their primitive roots, every Fermat Prime generates a different number of Mersenne Primes depending on the number of it's Primitive Roots so, there are not infinitely many Mersenne Primes and Perfect Numbers.

The matrices of the table above generate the Mersenne Primes with the primitive roots (2,8,128,32768) of the Fermat Primes and their factors.

$$\begin{bmatrix} 256 & 1 \\ 257_{Fp} & 513 \end{bmatrix} = \overset{2^{17}-1}{131071}_{Mp}$$

$$\Uparrow n^2$$

$$\begin{bmatrix} 16 & 1 \\ 17_{Fp} & 33 \end{bmatrix} = \overset{2^{9}-1}{511}$$

$$\Uparrow n^2$$

$$\begin{bmatrix} 4 & 1 \\ 5_{Fp} & 9 \end{bmatrix} = \overset{2^{5}-1}{31}_{Mp} \xrightarrow{n^3} \begin{bmatrix} \overset{2^6}{64} & 1 \\ 65 & 129 \end{bmatrix} = \overset{2^{13}-1}{8191}_{Mp}$$

$$\Uparrow n^2 \qquad \Uparrow n^2$$

$$\begin{bmatrix} 2 & 1 \\ 3_{Fp} & 5_{Fp} \end{bmatrix} = \overset{2^{3}-1}{7}_{Mp} \xrightarrow{n^3} \begin{bmatrix} 8 & 1 \\ 9 & 17_{Fp} \end{bmatrix} = \overset{2^{7}-1}{127}_{Mp} \xrightarrow{n^3} \begin{bmatrix} \overset{2^9}{512} & 1 \\ 513 & 1025 \end{bmatrix} = \overset{2^{19}-1}{524287}_{Mp}$$

$$\Uparrow n^2 \qquad \Downarrow 2n^2$$

$$\begin{bmatrix} \sqrt{2} & 1 \\ 1+\sqrt{2} & 1+2\sqrt{2} \end{bmatrix} = 3 \quad \begin{bmatrix} 128 & 1 \\ 129 & 257_{Fp} \end{bmatrix} = \overset{2^{15}-1}{32767}$$

$$\Downarrow 2n^2$$

$$\begin{bmatrix} \overset{2^{15}}{32768} & 1 \\ 32769 & 65537_{Fp} \end{bmatrix} = \overset{2^{31}-1}{2147483647}_{Mp} \xrightarrow{8n^4} \underset{170141183460469231731687303715884105727_{Mp}=2^{127}-1}{\begin{bmatrix} \overset{2^{63}=2\cdot 2147483648^2}{9223372036854775808} & 1 \\ 9223372036854775809 & 18446744073709551617 \end{bmatrix}}$$

$$\Downarrow n^2$$

$$\begin{bmatrix} \overset{2^{30}}{1073741824} & 1 \\ 1073741825 & 2147483649 \end{bmatrix} = \overset{2^{61}-1}{2305843009213693951}_{Mp}$$

$$\Downarrow 128^2 n = 16384n = 2^{14}n$$

$$\begin{bmatrix} \overset{2^{44}}{17592186044416} & 1 \\ 17592186044417 & 35184372088833 \end{bmatrix} = \overset{2^{89}-1}{618970019642690137449562111}_{Mp}$$

$$\Downarrow 8^3 n = 512n = 2^9 n$$

$$\begin{bmatrix} \overset{2^{53}}{9007199254740992} & 1 \\ 9007199254740993 & 18014398509481985 \end{bmatrix} = \overset{2^{107}-1}{162259276829213363391578010288127}_{Mp}$$

Note that when n is a perfect square, the Mersenne Prime finishes in 1, otherwise in 7.
We can find other relations between Mersenne Primes.

$$\left(\frac{\overset{2^{44}}{17592186044416}}{\underset{8192}{M_p+1=2^{13}}}\right) - 1 = \overset{2^{31}-1}{2147483647}_{Mp}$$

$$\Downarrow$$

$$\overset{M_p+1=2^{31}}{(2147483648)}\overset{2^{30}}{(1073741824)} - 1 = \overset{2^{61}-1}{2305843009213693951}_{Mp} = \overset{M_p+1=2^{17}}{(131072)}\overset{2^{44}}{(17592186044416)} - 1$$

$$\left(\frac{\overset{M_p+1=2^{107}}{162259276829213363391578010288128}}{\underset{2305843009213693952}{M_p+1=2^{61}}}\right) = \left(\frac{\overset{M_p+1=2^{61}}{2305843009213693952}}{\underset{32768}{2^{15}}}\right)$$

To connect this table with the *primitive roots* we check the *primitive roots tables* of the Fermat Primes.

$F_k = 3$ Root{2}	$\sum$ mod 3
(2)	
1	1
2	2 *(Prim root.)*
$\sum = 3$	

$F_k = 5$ Roots{2,3}		$\sum$ mod 5
(2)	(3)	
1	1	2
2	3	5 *(Primitive roots.)*
4	4	8
3	2	5 *(Primitive roots.)*
$\sum = 10$	$\sum = 10$	

$F_k = 17$ Primitive Roots {3,5,6,7,10,11,12,14}

(3)	(5)	(6)	(7)	(10)	(11)	(12)	(14)	$\sum$ mod 17
1	1	1	1	1	1	1	1	8
3	5	6	7	10	11	12	14	68 (Primitive roots.)
9	8	2	15	15	2	8	9	68
10	6	12	3	14	5	11	7	68 (Primitive roots.)
13	13	4	4	4	4	13	13	68
5	14	7	11	6	10	3	12	68 (Primitive roots.)
15	2	8	9	9	8	2	15	68
11	10	14	12	5	3	7	6	68 (Primitive roots.)
16	16	16	16	16	16	16	16	128
14	12	11	10	7	6	5	3	68 (Primitive roots.)
8	9	15	2	2	15	9	8	68
7	11	5	14	3	12	6	10	68 (Primitive roots.)
4	4	13	13	13	13	4	4	68
12	3	10	6	11	7	14	5	68 (Primitive roots.)
2	15	9	8	8	9	15	2	68
6	7	3	5	12	14	10	11	68 (Primitive roots.)
$\sum = F_k\left(\frac{F_k-1}{2}\right) = 136$ *for all roots.*								

Note that $\{3^n, 5^n, 6^n, 7^n, 10^n, 11^n, 12^n, 14^n\}$ mod(17) are cyclic, so the entire system is cyclic, the summation of all the residues are $\{3, 10, 136, 32896, 2147516416\}$ and from **(3) pages (4,5)** they are connected this way:

$3^2 + 1^2 = 3^2 + (2-1)^2 = 2^1 \cdot 5 = 10$

$10^2 + 6^2 = 10^2 + (8-2)^2 = 2^3 \cdot 17 = 136$

$136^2 + 120^2 = 136^2 + (128-8)^2 = 2^7 \cdot 257 = 32896$

$32896^2 + 32640^2 = 32896^2 + (32768-128)^2 = 2^{15} \cdot 65537 = 2147516416$

$(2, 8, 128, 32768)$ is the summation of the horizontal line with equal values (excepting the first line) on every *Primitive root* and is $\frac{F_{(k+1)}-1}{2}$.

since $2|A| + 3 = F_{(k+1)}$ we have:

$\frac{10}{3^2} = \frac{7+3}{3^2} \Rightarrow \frac{136}{10^2} = \frac{31+3}{5^2} \Rightarrow \frac{32896}{136^2} = \frac{511+3}{17^2} \Rightarrow \frac{2147516416}{32896^2} = \frac{131071+3}{257^2}$

(7,31,511,131071) are the values of the determinants of the matrices that start the table.

Sigma(n)

Knowing that $n = \phi(Fk_n)$ we have:

$\sigma(1) = 1 \Rightarrow \sigma(1) + 2 = 3$

$\sigma(2) = 3 \Rightarrow \sigma(2) + 2 = 5$
$\sigma(8) = 15 \Rightarrow \sigma(8) + 2 = 17$
$\sigma(128) = 255 \Rightarrow \sigma(128) + 2 = 257$
$\sigma(32768) = 65535 \Rightarrow \sigma(32768) + 2 = 65537$
Since $\left\{\prod_1^k F_k\right\} + 2 = F_{k_n} \Rightarrow \sigma(n) = \left\{\prod_1^k F_k\right\} \Rightarrow \sigma\left(\frac{F_k - 1}{2}\right) = F_k - 2$
thus $\sigma(\phi(F_{k_n})) = F_k - 2$
Given $(n) = \{1, 2, 8, 128, 32768\} \Rightarrow (2n_k)^2 = 2n_{(k+1)} \Rightarrow \sigma(2n_k)^2 = \sigma_{(k+1)}$

Fermat Factors.

By now we will assume that all Fermat Factors have the next form:
$\begin{vmatrix} n & 1 \\ n+1 & F_k \end{vmatrix}$ where $F_k = f_1 \cdot f_2 \cdot f_3 \cdots f_k$ then $\forall f_k \, \{f_k \equiv 1 \pmod{24}$ or $f_k \equiv 17 \pmod{24}\}$,from sigma properties and for $F_k = f_1 \cdot f_2$ we have:
$n = \left(\frac{f_2-1}{2}\right)(f_1) + \left(\frac{f_1-1}{2}\right) = \left(\frac{f_1-1}{2}\right)(f_2) + \left(\frac{f_2-1}{2}\right)$
This takes us to:
$\sigma(F_k) - F_k - f_1 - 1 = f_2$
Example: $F_k = 4294967297 = (641 \cdot 6700417), \sigma(F_k) = 4301668356$
$4301668356 - 4294967297 - 6700417 - 1 = 641$
so, given any two factors f_1 and f_2 of the F_k sequence, we have:
$f_1 = (x_1 \cdot 24) + 17$ and $f_2 = (x_2 \cdot 24) + 1$, then:
$f_1 = (x_1 - f_2 + 24x_2 + 17)24 + 17$ and $f_2 = (x_2 - f_1 + 24x_1 + 1)24 + 1$
knowing that $\sigma(n) = \prod_{i=1}^{k}\left(1 + p_i + p_i^2 + \cdots + p_i^{\alpha i}\right)$ we have for a two factor F_k
$\sigma(n) = (24x_1 + 18)(24x_2 + 2) = 576x_1x_2 + 48x_1 + 432x_2 + 36$
example:$\sigma(F_k) = (576 \cdot 26 \cdot 279184) + (48 \cdot 26) + (432 \cdot 279184) + 36 = 4301668356$
notice that for a Fermat Number with three factors $F_k = f_1 \cdot f_2 \cdot f_3$, two of them will be $f_1 \equiv 17 \pmod{24}$ and the other $f_2 \equiv 1 \pmod{24}$ or viceversa, but them all can't have the same residue. this applies for every F_k.

Fermat Numbers with two Prime Factors.

Given a Fermat Factor $\mathbf{f \equiv 1 \pmod{4}}$ we have $f = x^2 + y^2$, for a Fermat Number with two factors $\mathbf{f_1 \cdot f_2}$ then:
$A = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \Rightarrow |\vec{a} \cdot \vec{b}| = F_{k-1} - 1$
if $f^2 = x^2 + y^2$
$A = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \Rightarrow |\vec{a} \cdot \vec{b}| = F_k - 2$
since $F_k = (F_{k-1} - 1)^2 + 1 \Rightarrow F_k = \left(|\vec{a} \cdot \vec{b}|_{k-1}\right)^2 + 1$

Example:
$F_k = 4294967297 = 641 \cdot 6700417$
$641 = 4^2 + 25^2$ and $641^2 = 200^2 + 609^2$
$6700417 = 409^2 + 2556^2$ and $6700417^2 = 2090808^2 + 6365855^2$
so, we have:

$$A=\begin{bmatrix}4 & 409\\ 25 & 2556\end{bmatrix}\Rightarrow|\vec{a}\cdot\vec{b}|=65536$$

and

$$A=\begin{bmatrix}2090808 & 200\\ 6365855 & 609\end{bmatrix}\Rightarrow|\vec{a}\cdot\vec{b}|=4294967295$$

for $f^2=x^2+y^2$, since $|\vec{a}\cdot\vec{b}|\cos(\theta)=f_1\cdot f_2\cos(\theta)\Rightarrow\cos(\theta)=\frac{F_k-2}{F_k}$
since we have the recurrence:

$$A_k=\begin{bmatrix}n & 1\\ n+1 & F_k\end{bmatrix}\Rightarrow A_{k+1}=\begin{bmatrix}|A_k| & 1\\ |A_k|+1 & F_{k+1}\end{bmatrix}$$

$\cos(\theta)=\frac{F_k-2}{F_k}=\frac{1}{1+\frac{1}{|A_{k-1}|+\frac{1}{2}}}$

Fermat Numbers with more than two Prime Factors.

Given a Fermat Number $F_k=f_1\cdot f_2\cdots f_k$ such that $\underbrace{\begin{bmatrix}x_1\\ y_1\end{bmatrix}}_{f_1},\underbrace{\begin{bmatrix}x_2\\ y_2\end{bmatrix}}_{f_2},\underbrace{\begin{bmatrix}x_k\\ y_k\end{bmatrix}}_{f_k}$ where $(f_1)^2=(x_1)^2+(y_1)^2,(f_2)^2=(x_2)^2+(y_2)^2,(f_k)^2=(x_k)^2+(y_k)^2$

we have

$$A=\begin{bmatrix}x_1 & x_2\\ y_1 & y_2\end{bmatrix}\Rightarrow|\vec{a}\cdot\vec{b}|_1=x_1\cdot x_2+y_1\cdot y_2=(f_{1\cdot 2})^2\Rightarrow(f_{1\cdot 2})^2=(x_{1\cdot 2})^2+(y_{1\cdot 2})^2$$

then

$$A_k=\begin{bmatrix}x_{1\cdot 2} & x_k\\ y_{1\cdot 2} & y_k\end{bmatrix}\Rightarrow|\vec{a}\cdot\vec{b}|_k\Rightarrow\frac{f_1\cdot f_2\cdots f_k}{|\vec{a}\cdot\vec{b}|_k}=\frac{(f_1\cdot f_2\cdots f_k)-|\vec{a}\cdot\vec{b}|_k}{|\vec{a}\cdot\vec{b}|_k}+1$$

Multiplying different values of (n).

Given consecutive Mersenne Primes and their respective matrices:

$$\underbrace{\begin{vmatrix}n_1 & 1\\ n_1+1 & 2n_1+1\end{vmatrix}}_{|A_1|=2^{p_1}-1=7},\underbrace{\begin{vmatrix}n_2 & 1\\ n_2+1 & 2n_2+1\end{vmatrix}}_{|A_2|=2^{p_2}-1=31},\ldots,\underbrace{\begin{vmatrix}n_k & 1\\ n_k+1 & 2n_k+1\end{vmatrix}}_{|A_k|=2^{p_k}-1=Mp_k}$$

Then:

$\left(\frac{n_2}{n_1}\right)\cdot\left(\frac{n_3}{n_2}\right)\cdots\left(\frac{n_k}{n_{k-1}}\right)=2^{\frac{p_k-3}{2}}$

This is $\frac{F_k-3}{2}=Mp_k\Rightarrow\frac{2^{2^{n-1}}+1-3}{2}=2^{2^{n-1}-1}-1$

Example:

Given the next matrices and their determinants.

$$\underbrace{\begin{vmatrix}2 & 1\\ 3 & 5\end{vmatrix}}_{|A|=7};\underbrace{\begin{vmatrix}4 & 1\\ 5 & 9\end{vmatrix}}_{|A|=31};\underbrace{\begin{vmatrix}8 & 1\\ 9 & 17\end{vmatrix}}_{|A|=127};\underbrace{\begin{vmatrix}64 & 1\\ 65 & 129\end{vmatrix}}_{|A|=8191}\Rightarrow\left(\frac{4}{2}\right)\cdot\left(\frac{8}{4}\right)\cdot\left(\frac{64}{8}\right)=2\cdot 2\cdot 8=2^{\frac{13-3}{2}}=2^5=32$$

The number 13 is $'p'$ of $2^p-1=\overset{2^{13}-1}{8191}$

Note that from *Pepin's Test:* $3^{\frac{F_k-1}{2}}\equiv -1\mod(F_k)$

from $\begin{vmatrix} n & 1 \\ n+1 & F_k \end{vmatrix} \Rightarrow \frac{F_k-1}{2} = n$, *Pepin's Test* turns to $3^n \equiv -1 \mod (F_k)$
this turns to $F_k \equiv 3 \mod (n-1)$ where $\frac{F_k-3}{n-1} = 2$
then $F_k \equiv 3 \mod (|A|_{k-1}) \Rightarrow F_{(k+1)} = 2|A| + 3$

32768

Given a matrix $A = \begin{vmatrix} n & 1 \\ n+1 & 2n+1 \end{vmatrix} \Rightarrow \begin{vmatrix} n & 1 \\ n+1 & F_k \end{vmatrix} \Rightarrow \begin{vmatrix} 32768 & 1 \\ 32769 & 65537 \end{vmatrix}$

$|A| = \overset{2^{31}-1}{2147483647}_{Mp}$ being **n=32768** the inradius of the Pythagorean Triangle with sides $\underbrace{65537}_{2n+1}, \underbrace{2147549184}_{2n^2+2n}, \underbrace{2147549185}_{2n^2+2n+1}$.The inradius 32768 is connected with all the Mersenne Primes as we can see next.

$\mathbf{Mp_k < 32768}$

$32768 = [(4681)(7)] + 1$
$32768 = [(1057)(31)] + 1$
$32768 = [(258)(127)] + 2$
$32768 = [(4)(8191)] + 4$

$\mathbf{Mp_k > 32768} \quad [(\mathbf{4^n - 1})(\mathbf{32768})] + (\mathbf{32768 - 1})$

$$M_p = \overset{2^{17}-1}{131071} = [(\overset{2^2-1}{3})(32768)] + (32768-1)$$

$$M_p = \underset{(131071 \cdot 4)+3}{\overset{2^{19}-1}{524287}} = \left[(\underset{3\cdot 5}{\overset{2^4-1}{15}})(32768)\right] + (32768-1)$$

$$M_p = \underset{\frac{(3\cdot 5\cdot 17\cdot 257\cdot 65537)-1}{2}}{\overset{2^{31}-1}{2147483647}} = \left[(\underset{3\cdot 5\cdot 17\cdot 257}{\overset{2^{16}-1}{65535}})(32768)\right] + (\underset{\frac{(3\cdot 5\cdot 17\cdot 257)+1}{2}}{32768} - 1)$$

$$M_p = \overset{2^{61}-1}{2305843009213693951}$$
$$= [(\underset{(2147483647)(32768)+(32768-1)}{\overset{2^{46}-1}{70368744177663}})(32768)] + (32768-1)$$

$$M_p = \underset{(2305843009213693951)(32768)\left(\frac{32768}{4}\right)}{\overset{2^{89}-1}{618970019642690137449562111}}$$
$$= [(\underset{(2305843009213693951)\left(\frac{32768}{4}\right)}{\overset{2^{74}-1}{18889465931478580854783}})(32768)] + (32768-1)$$

$$M_p = \underset{(618970019642690137449562111)(32768)(8)}{\overset{2^{107}-1}{162259276829213363391578010288127}}$$
$$= [(\underset{(618970019642690137449562111)(8)}{\overset{2^{92}-1}{4951760157141521099596496895}})(32768)] + (32768-1)$$

$$M_p = \underset{(162259276829213363391578010288127)(32768)(32)}{\overset{2^{127}-1}{170141183460469231731687303715884105727}}$$

$$= [(\underset{(162259276829213363391578010288127)(32)}{\overset{2^{112}-1}{5192296858534827628530496329220095}})(32768)] + (32768 - 1)$$

In short, we can represent a Mersenne Prime $Mp_k > 32768$
$Mp_k = 2^p - 1 = \left[\left(2^{p-15} - 1\right)(32768)\right] + (32768 - 1)$

This can be done with the rest of inradii (2,8,128) of the Pythagorean Triangles related to Fermat Primes, so we have:
$Mp_k > 2$
$Mp_k = 2^p - 1 = \left[\left(2^{p-1} - 1\right)(2)\right] + (2 - 1)$

$Mp_k > 8$
$Mp_k = 2^p - 1 = \left[\left(2^{p-3} - 1\right)(8)\right] + (8 - 1)$

$Mp_k > 128$
$Mp_k = 2^p - 1 = \left[\left(2^{p-7} - 1\right)(128)\right] + (128 - 1)$

$Mp_k > 32768$
$Mp_k = 2^p - 1 = \left[\left(2^{p-15} - 1\right)(32768)\right] + (32768 - 1)$

Note that (1,3,7,15) are the Mersenne Numbers (page 5, table 2).

Following, we can find some congruences:
$31 \equiv (15 \cdot 2) \mod (1)$
$127 \equiv (15 \cdot 8) \mod (7)$
$524287 \equiv (15 \cdot 32768) \mod (32767)$
$8191 \equiv (1023 \cdot 8) \mod (7)$
$131071 \equiv (1023 \cdot 128) \mod (127)$
$31 \equiv (3 \cdot 8) \mod (7)$
$131071 \equiv (3 \cdot 32768) \mod (32767)$

A remark on the non infinity of Mersenne Primes and Perfect Numbers.

We know that $a + b = |\vec{a} \cdot \vec{b}| = \cos(\theta)||\vec{a}||||\vec{b}||$.

We can see that $\cos\theta\|a\|\|b\| - \frac{1}{\sqrt{2}}\|a\|\|b\| = 2n = F_k - 1$ for every $\begin{vmatrix} n & 1 \\ n+1 & F_k \end{vmatrix} = Mn_k$

Given the matrix and it's respective pythagorean triple

$$\underbrace{\begin{bmatrix} n & 1 \\ n+1 & F_k \end{bmatrix}}_{\text{(a,b,c)}} \to b - a = Mp_k$$

As every Perfect Number have the form $Sp_k \cdot Mp_k = Pf_k$, $\left\{\begin{matrix} a = 2n+1 \\ c = (2n+1)n + (n+1) = 2n^2 + 2n + 1 \\ b = c - 1 = 2n^2 + 2n \end{matrix}\right\}$
$Sp_k \cdot Mp_k = n^2 \cdot (b-a) = n^2 \cdot (2n^2 + 2n - 2n - 1) = n^2 \cdot (2n^2 - 1) = 2n^4 - n^2 = Pf_k$

$(Sp_k = \textit{Superperfect Number}, Mp_k = \textit{Mersenne Prime}, Pf_k = \textit{Perfect Number})$

$$\textit{Superperfect Number} = \frac{\textit{Perfect number}}{\textit{Mersenne prime}} = \frac{2n^4 - n^2}{2n^2 - 1} = n^2 \ \forall n \neq \left\{ -\frac{1}{\sqrt{2}} \ \textit{and} \ \frac{1}{\sqrt{2}} \right\}$$

Since the roots of $2n^2 - 1$ are $\left\{ -\frac{1}{\sqrt{2}} \ \textit{and} \ \frac{1}{\sqrt{2}} \right\}$, when $n^2 = Sp_k$ is confined to these limits.

Now we can represent the whole system of Mersenne Primes and Perfect Numbers this way:
Mersenne Prime $= 2n^2 - 1 =$ *Parallelogram.*
Perfect Number $= 2n^4 - n^2 =$ *Parallelepiped volume.*
SuperPerfect Number $= n^2 =$ *Height.*

So, given $'n'$ we have:

$$n \Rightarrow \underbrace{\begin{bmatrix} n & 1 \\ n+1 & 2n+1 \end{bmatrix}}_{Parallelogram} \Rightarrow \underbrace{\begin{bmatrix} 1 & 2n+1 & n \\ n & n+1 & n^2 \\ n+1 & 1 & n \end{bmatrix}}_{Parallelepiped\ volume}$$

Example:

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \Rightarrow |A| = 7 \Rightarrow A' = \begin{bmatrix} 1 & 5 & 2 \\ 2 & 3 & 4 \\ 3 & 1 & 2 \end{bmatrix} \Rightarrow |A'| = 28$$

$$A = \begin{bmatrix} 4 & 1 \\ 5 & 9 \end{bmatrix} \Rightarrow |A| = 31 \Rightarrow A' = \begin{bmatrix} 1 & 9 & 4 \\ 4 & 5 & 16 \\ 5 & 1 & 4 \end{bmatrix} \Rightarrow |A'| = 496$$

$$A = \begin{bmatrix} 8 & 1 \\ 9 & 17 \end{bmatrix} \Rightarrow |A| = 127 \Rightarrow A' = \begin{bmatrix} 1 & 17 & 8 \\ 8 & 9 & 64 \\ 9 & 1 & 8 \end{bmatrix} \Rightarrow |A'| = 8128$$

and so on.

References

[1] *H. Lee Price.*
The Pythagorean Tree: A New Species.
https://arxiv.org/pdf/0809.4324.pdf.

[2] *University of San Antonio.*
The Geometric Interpretation of the Determinant.
https://mathresearch.utsa.edu/wiki/index.php?title=The_Geometric_Interpretation_of_the_Determinant

[3] *On Primitive Pythagorean Triples*
https://hal.science/hal-03247261/document,page 8.

[4] *George E. Andrews.*
Number Theory.
Chapter 7.Primitive Roots,p 97
Dover Publications, Inc. New York.1994.

[5] *Notes on Perfect Numbers*
Geometric Representation of Perfect Numbers.p 15-16.
https://hal.science/hal-01911641v7