Line and surface integrals in the hypercomplex numbers concept of Clifford algebra

Alimzhan Kholmuratovich Babaev[1, 2, *]

PhD of Physical and Math Sciences. E-mail: prepadamira@gmail.com (Alimzhan Kh. Babaev)

[1]Department Physics, National University of Uzbekistan, Tashkent, Republic of Uzbekistan

[2]Department of Applied Mathematics and Computer Science, Novosibirsk State Technical University, Novosibirsk, Russian Federation

ORCID

0000-0002-4071-5524 Alimzhan Babaev

Abstract

The article presents a generalization theory of functions of a complex variable for 3-dimensional Euclidean space and for Minkowski's space: Cauchy's integral theorem, Cauchy's integral formula, its integral representation for derivatives, and Stoker's and Ostrogradsky-Gauss theorems. Also, line and surface integrals were combined and generalized within the framework of the concept of hypercomplex numbers. A bijection (one-to-one correspondence) was established between multidimensional vectors and hypercomplex numbers, i.e., the Pauli matrices (σ_i) for 3-dimensional Euclidean space and the Dirac matrices (γ^i) for Minkowski space were used as hypercomplex numbers and basis vectors. The results of the calculations were used to study the laws of physics; in particular, dual integration was applied – replacing surface integrals over "time" planes with integration over "purely spatial" planes. The law of conservation of 4-electromagnetic currents has been proven within the framework of the Clifford algebra concept.

Keywords

Line integral; surface integral; hypercomplex number; Cauchy's integral formula; Generalized Stokes theorem; divergence theorem;

1. Introduction

A bijection (one-to-one correspondence) is obvious [1] between vectors and complex numbers on the plane (E^2). This correspondence does not seem obvious in the case of a multidimensional space (E^n, $n>2$).

In this article, we will consider the relationship between line and surface integrals in 3- and 4-dimensional Euclidean (pseudo-Euclidean) spaces within the framework of Clifford algebra. In this case, we will establish a correspondence between hypercomplex numbers and Dirac matrices (γ^i), which are used as unit vectors in Minkowski's space $E^{4,1}$. We use the Pauli matrices ($\sigma_\alpha = \gamma^\alpha\gamma^0$) as a basis in the space E^3, which is a special case of $E^{4,1}$.

The choice of Dirac matrices in 4-dimensional space $E^{4,1}$ is related to the fact that $E^{4,1}$ has a signature + - - -. In the zeroth approximation, all laws of physics (special theory of relativity, quantum mechanics, etc.) occur precisely in Minkowski space. Therefore, Dirac matrices will be used both as an orthonormal basis for vector space $E^{4,1}$ and as hypercomplex numbers.

The 3-dimensional Euclidean space has the signature - - -, but for convenience we will use +++. Therefore, as basis vectors, we will use the Pauli matrices σ_i (i=0, 1, 2, 3), where σ_0 is the identity 2x2 matrix.

2. Results

2.1. Line integrals in 3-dimensional Euclidean space.

Let a positively oriented surface $\boldsymbol{D}$, bounded by a contour $\boldsymbol{l}$, be given in a 3-dimensional Euclidean space (Figure 1). The vector normal n to the surface D forms angles α, β, γ with the coordinate axes x, y, z.

The projections of the surface D on the planes xy, yz, zx are $\boldsymbol{D}_{xy}$, $\boldsymbol{D}_{yz}$,, $\boldsymbol{D}_{zx}$ with contours $\boldsymbol{l}_{xy}$, $\boldsymbol{l}_{yz}$, $\boldsymbol{l}_{zx}$.

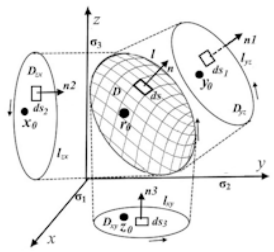

Figure 1

Let the vector function $\boldsymbol{R}(\boldsymbol{r})$ be given in a region V (volume) bounded by a surface $\boldsymbol{D}$:

$\boldsymbol{R}(\boldsymbol{r})=\sigma_1 X(\boldsymbol{r})+ \sigma_2 Y(\boldsymbol{r})+ \sigma_3 Z(\boldsymbol{r})$ **(1)**

Let us consider the line $_l\oint \boldsymbol{R}(\boldsymbol{r})d\boldsymbol{l}$ and surface $_D\oiint \boldsymbol{R}(\boldsymbol{r})d\boldsymbol{S}$ integrals in this region.

Here

$\boldsymbol{r} = (x, y, z)$ is the radius vector;

$d\boldsymbol{l} = d\boldsymbol{x} + d\boldsymbol{y} + d\boldsymbol{z} = \sigma_1 dx + \sigma_2 dy + \sigma_3 dz$ is an arc element;

$d\boldsymbol{S} = d\boldsymbol{x}\wedge d\boldsymbol{y}\, cos\alpha + d\boldsymbol{y}\wedge d\boldsymbol{z}\, cos\beta + d\boldsymbol{z}\wedge d\boldsymbol{x}\, cos\gamma = \boldsymbol{n}dS$ is a surface element;

$\boldsymbol{n} = \sigma_1\sigma_2\, cos\gamma + \sigma_2\sigma_3\, cos\alpha + \sigma_3\sigma_1\, cos\beta$ is the positively oriented normal to surface $\boldsymbol{D}$;

$$d\boldsymbol{x}\wedge d\boldsymbol{y} = \sigma_1\sigma_2\, dxdy = i\sigma_3 dxdy$$
$$d\boldsymbol{y}\wedge d\boldsymbol{z} = \sigma_2\sigma_3\, dydz = i\sigma_1\, dydz$$
$$d\boldsymbol{z}\wedge d\boldsymbol{x} = \sigma_3\sigma_1\, dzdx = i\sigma_2\, dzdx$$

$\boldsymbol{\nabla} = \sigma_1\partial/\partial x + \sigma_1\partial/\partial y + \sigma_1\partial/\partial z$ is the nabla operator.

σ_0, σ_1, σ_2, σ_3 are Pauli matrices.

Remark 1.

Depending on the expediency and simplicity, we will use surface integrals of either the first type or the second type, which, we hope, will not create inconvenience for the reader.

Theorem 1.

The following formulas are valid:

$$_l\oint \boldsymbol{R}(\boldsymbol{r})d\boldsymbol{l} = 0, \text{ if } \boldsymbol{R}(\boldsymbol{r}) \text{ is analytic} \quad \textbf{(2.1)}$$

$$_l\oint \boldsymbol{R}(\boldsymbol{r})d\boldsymbol{l} = 2\pi\, i\, \boldsymbol{f}(\boldsymbol{r}_0)\, \boldsymbol{n}, \text{ если } \boldsymbol{R}(\boldsymbol{r}) = \boldsymbol{f}(\boldsymbol{r})/(\boldsymbol{r} - \boldsymbol{r}_0) \quad \textbf{(2.2)}$$

and

$$ {}_{l}\oint \boldsymbol{R}\bullet d\boldsymbol{l} = {}_{D}\oiint(\nabla\wedge\boldsymbol{R})\bullet d\boldsymbol{S} \qquad \textbf{(3.1)} $$

$$ {}_{l}\oint \boldsymbol{R}\wedge d\boldsymbol{l} = {}_{D}\oiint(\nabla\bullet\boldsymbol{R})d\boldsymbol{S} \qquad \textbf{(3.2)} $$

• and ∧ are the symbols in the inner and outer product of vectors [2].

$\boldsymbol{R}(\boldsymbol{r}) = \boldsymbol{f}(\boldsymbol{r})/(\boldsymbol{r} - \boldsymbol{r}_0)$ means that the vector function $\boldsymbol{R}(\boldsymbol{r})$ has a pole at the point $\boldsymbol{r}_0$ [3].

$\boldsymbol{f}(\boldsymbol{r}_0)$ is the value of the function $\boldsymbol{f}(\boldsymbol{r})$ at the point $\boldsymbol{r}_0$.

i is the imaginary unit.

<u>Remark 2</u>.

For simplicity, we will assume that the region $\boldsymbol{D}$ is simply connected. We will not dwell on methods for dividing non-simply connected regions into simply connected ones [4].

<u>Remark 3</u>.

It is obvious that

$$ (\nabla\boldsymbol{R})d\boldsymbol{S} = (\nabla\bullet\boldsymbol{R})d\boldsymbol{S} + (\nabla\wedge\boldsymbol{R})\bullet d\boldsymbol{S} + (\nabla\wedge\boldsymbol{R})\wedge d\boldsymbol{S}. $$

But the dimension of $(\nabla\wedge\boldsymbol{R})\wedge d\boldsymbol{S}$ exceeds the dimension of 3-dimensional space (>3), i.e., ${}_{D}\oiint(\nabla\wedge\boldsymbol{R})\wedge d\boldsymbol{S}$ has no meaning.

<u>Proof.</u>

Surface $\boldsymbol{D}$ is the sum of surfaces $\boldsymbol{D}_{xy}$, $\boldsymbol{D}_{yz}$ and $\boldsymbol{D}_{zx}$. Therefore, it is sufficient to prove Theorem 1 on one of the projections, for example, on $\boldsymbol{D}_{xy}$, and then add the integrals over all projections.

The integral ${}_{xy}\oint\boldsymbol{R}d\boldsymbol{l}$ has a coordinate form:

$$ {}_{xy}\oint\boldsymbol{R}d\boldsymbol{l} = {}_{xy}\oint\boldsymbol{R}\bullet d\boldsymbol{l} + {}_{xy}\oint\boldsymbol{R}\wedge d\boldsymbol{l}, $$

Here

$$ {}_{xy}\oint\boldsymbol{R}\bullet d\boldsymbol{l} = \sigma_0\, {}_{xy}\oint(Xdx + Ydy) \qquad \textbf{(4.1)} $$

$$ {}_{xy}\oint\boldsymbol{R}\wedge d\boldsymbol{l} = i\sigma_3\, {}_{xy}\oint(Xdy - Ydx) \qquad \textbf{(4.2)} $$

The surface integral has a vector form:

$$ {}_{xy}\iint(\nabla\boldsymbol{R})d\boldsymbol{x}\wedge d\boldsymbol{y} = {}_{xy}\iint(\nabla\bullet\boldsymbol{R})d\boldsymbol{x}\wedge d\boldsymbol{y} + {}_{xy}\iint(\nabla\wedge\boldsymbol{R})\bullet d\boldsymbol{x}\wedge d\boldsymbol{y}, $$

Here

$$ {}_{xy}\iint(\nabla\bullet\boldsymbol{R})d\boldsymbol{x}\wedge d\boldsymbol{y} = i\sigma_3\, {}_{xy}\iint(\partial_x X + \partial_y Y)dxdy \qquad \textbf{(5.1)} $$

$$ {}_{xy}\iint(\nabla\wedge\boldsymbol{R})\bullet d\boldsymbol{x}\wedge d\boldsymbol{y} = \sigma_0\, {}_{xy}\iint(\partial_x Y - \partial_y X)dxdy \qquad \textbf{(5.2)} $$

1) We transform the line integral ${}_{xy}\oint\boldsymbol{R}\bullet d\boldsymbol{l}$ (4.1) into the surface one ${}_{xy}\iint(\nabla\wedge\boldsymbol{R})\bullet d\boldsymbol{x}\wedge d\boldsymbol{y}$ (5.2):

$$ {}_{xy}\oint\boldsymbol{R}\bullet d\boldsymbol{l} = {}_{xy}\oint(\sigma_1 X\sigma_1 dx + \sigma_2 Y\sigma_2 dy) = $$
$$ = {}_{xy}\iint(\sigma_2\partial_y(\sigma_1 X) + \sigma_1\partial_x(\sigma_2 Y))\sigma_1\sigma_2 dxdy = $$
$$ = \sigma_0(\, {}_{xy}\iint(\partial_y X - \partial_\chi Y)dxdy $$

or

$$ \sigma_0\, {}_{xy}\oint(Xdx + Ydy) = \sigma_0\, {}_{xy}\iint(\partial_y X - \partial_x Y)dxdy \qquad \textbf{(6)} $$

(6) is Green's theorem [5].

Formulas of type (6) are valid also for integrals on projections $\boldsymbol{D}_{yz}$ and $\boldsymbol{D}_{zx}$.

Since the surface $\boldsymbol{D}$ with contour $\boldsymbol{l}$ is the sum of the integrals over the surfaces $\boldsymbol{D}_{xy}$, $\boldsymbol{D}_{yz}$, and $\boldsymbol{D}_{zx}$ with contours $\boldsymbol{l}_{xy}$, $\boldsymbol{l}_{yz}$, and $\boldsymbol{l}_{zx}$, then, adding the integrals over the projections, we obtain the equation (3.1).

2) We transform the line integral ${}_{xy}\oint\boldsymbol{R}\wedge d\boldsymbol{l}$ (4.2) into the surface one ${}_{xy}\iint(\nabla\bullet\boldsymbol{R})d\boldsymbol{x}\wedge d\boldsymbol{y}$ (5.1):

$$ \sigma_1\sigma_2\, {}_{xy}\oint(Xdy - Ydx) = {}_{xy}\oint(\sigma_1 X\sigma_2 dy + \sigma_2 Y\sigma_1 dx) = $$
$$ = {}_{xy}\iint(\sigma_1\partial_x(\sigma_1 X) + \sigma_2(\partial_y\sigma_2 Y))\sigma_1\sigma_2\, dxdy = $$
$$ = \sigma_1\sigma_2\, {}_{xy}\iint(\partial_x X + \partial_y Y)dxdy $$

or

$$ i\sigma_3\, {}_{xy}\oint(Xdy - Ydx) = i\sigma_3\, {}_{xy}\iint(\partial_x X + \partial_y Y)\, dxdy \qquad \text{(7)} $$

Formulas of type (7) are valid also for integrals in projections $\boldsymbol{D}_{yx}$ and $\boldsymbol{D}_{zx}$. Adding the integrals over all projections $\boldsymbol{D}_{xy}$, $\boldsymbol{D}_{yz}$, and $\boldsymbol{D}_{zx}$, we obtain the equation (3.2).

Formulas (3) of Theorem 1 are proven. Formula (3.1) is Stokes' theorem [6].

To understand the meaning of the formula (3.2), we write it in the form

$$ {}_{l}\oint\boldsymbol{R}\wedge d\boldsymbol{l} = {}_{D}\iint div\boldsymbol{R}d\boldsymbol{S}. $$

In terms of physics, formula (3.2) means that the vector field induction (${}_{l}\oint\boldsymbol{R}\wedge d\boldsymbol{l}$) through a contour $\boldsymbol{l}$ is equal to the field divergence (${}_{D}\iint div\boldsymbol{R}d\boldsymbol{S}$) through a surface $\boldsymbol{D}$ bounded by the same contour.

Looking ahead, let us assume that a closed surface $\boldsymbol{D}$ (for example, a sphere) has a source of divergence inside it (a "sink" or "source").

If $div\boldsymbol{R} = \partial_x X + \partial_y Y > 0$, then the induction vector is directed outward, i.e., the induction vector with $\boldsymbol{n}$ forms an acute angle. This is the source (the lines of force are directed away from the sphere). If $div\boldsymbol{R} < 0$, then the induction vector is directed inward, i.e., this induction vector forms an obtuse angle with $\boldsymbol{n}$. This is a sink (the lines of force are directed inward into the sphere).

Obviously, if the function $\boldsymbol{R}$ is analytic in the domain $\boldsymbol{D}$, then the integrals (3) are equal to zero. Simply put, there is no source of divergence. (3.2) is a special case of the Gauss-Ostrogradsky theorem (for a surface).

If the function $\boldsymbol{R}(x,y)$ is analytic (by Cauchy [7]) in $\boldsymbol{D}$, i.e., it does not have special isolated points (poles), then

$$ {}_{xy}\oint\boldsymbol{R}d\boldsymbol{l} = 0 \qquad \textbf{(8)} $$

Indeed, since $\boldsymbol{R}(x,y)$ is defined and bounded at all points of $\boldsymbol{D}$, there exists an antiderivative $\boldsymbol{F}'_{\boldsymbol{r}} = \boldsymbol{R}(\boldsymbol{r})$. Then both it and the integral (8) are equal to zero over the closed contour:

$$ {}_{xy}\oint\boldsymbol{R}d\boldsymbol{l} = \boldsymbol{F}\,|^{a}_{a} = 0 $$

By adding up all integrals of type (8) over all projections $\boldsymbol{D}_{xy}$, $\boldsymbol{D}_{yz}$, and $\boldsymbol{D}_{zx}$, we obtain the formula (2.1).

Now let us consider the case when the function $\boldsymbol{R}(\boldsymbol{r})$ in the domain $\boldsymbol{D}$ has an isolated singular point r, i.e., a pole of the type

$$ \boldsymbol{R} = \boldsymbol{f}(\boldsymbol{r})/(\boldsymbol{r} - \boldsymbol{r}_0) \qquad \textbf{(9)} $$

We transform the line integral ${}_{xy}\oint\boldsymbol{R}d\boldsymbol{l}$:

$$ \oint_{xy}\boldsymbol{R}\,d\boldsymbol{l} = \oint_{xy}\frac{f(\boldsymbol{r})}{\boldsymbol{r}-\boldsymbol{r}_0}d\boldsymbol{l} = \oint_{xy}\frac{f(\boldsymbol{r})-f(\boldsymbol{r}_0)+f(\boldsymbol{r}_0)}{\boldsymbol{r}-\boldsymbol{r}_0}d\boldsymbol{l} = $$
$$ = \oint_{xy}\frac{f(\boldsymbol{r})-f(\boldsymbol{r}_0)}{\boldsymbol{r}-\boldsymbol{r}_0}d\boldsymbol{l} + \oint_{xy}\frac{f(\boldsymbol{r}_0)}{\boldsymbol{r}-\boldsymbol{r}_0}d\boldsymbol{l} $$

Since $(\boldsymbol{f}(\boldsymbol{r}) - \boldsymbol{f}(\boldsymbol{r}_0))/(\boldsymbol{r} - \boldsymbol{r}_0)$ is the derivative at $\boldsymbol{r} \to \boldsymbol{r}_0$, then the first integral is equal to zero:

$$ {}_{xy}\oint\boldsymbol{f}'d\boldsymbol{l} = \boldsymbol{f}(\boldsymbol{r})\,|^{a}_{\boldsymbol{a}} = 0 $$

Now we transform the second integral, more precisely, ${}_{xy}\oint(\boldsymbol{r} - \boldsymbol{r}_0)^{-1}d\boldsymbol{l}$. By analogy with the theory of complex analysis, we obtain

$$ \oint_{xy}\frac{d\boldsymbol{l}}{\boldsymbol{r}-\boldsymbol{r}_0} = \oint_{xy}\frac{d(\boldsymbol{l}-\boldsymbol{l}_0)}{\boldsymbol{r}-\boldsymbol{r}_0} = \oint_{xy}\frac{\sigma_1 d(\sigma_0 dx+\sigma_1\sigma_2 dy)}{\sigma_1(\sigma_0 x+\sigma_1\sigma_2 y)} $$

or in the exponential form

$$ \oint_{xy}\frac{d(\sigma_0 dx+i\sigma_3 dy)}{\sigma_0 x+i\sigma_3 y} = i\sigma_3\int_0^{2\pi}\frac{|\mathrm{r}|\cdot\exp(i\sigma_3\varphi)}{|\mathrm{r}|\cdot\exp(i\sigma_3\varphi)}d\varphi = 2\pi i\sigma_3 \qquad \textbf{(10)} $$

Equalities of the type (10) also hold for the *zx* and *yz* planes. Adding up all integrals of type (10) over all projections $\boldsymbol{D}_{xy}$, $\boldsymbol{D}_{yz}$, and $\boldsymbol{D}_{zx}$, we obtain the formula (2.2).

Since the value of the integral (10) is imaginary, equality (2.2) is satisfied for the imaginary part of the integral ${}_{l}\oint\boldsymbol{R}d\boldsymbol{l}$.

Formula (2.2) is a generalization of the Cauchy integral formula [8] to the case of 3-dimensional Euclidean space. Theorem 1 is proven.

2.2. Surface integrals in 3-dimensional Euclidean space

Let the vector function $\boldsymbol{R}(x,y,z)$ be given in a volume V bounded by a surface $\boldsymbol{D}$ (Figure 1). We will consider the surface integral ${}_D\iint \boldsymbol{R}d\boldsymbol{S} = {}_D\iint \boldsymbol{R}\boldsymbol{n}ds$ in this region.

Theorem 2.

The following formulas are valid:

$${}_D\oint\!\!\!\oint \boldsymbol{R}d\boldsymbol{S} = 0, \text{ if } \boldsymbol{R} \text{ is analytic} \qquad \textbf{(11.1)}$$

$${}_D\oint\!\!\!\oint \boldsymbol{R}d\boldsymbol{S} = 2\pi\, i\, \boldsymbol{f}(\boldsymbol{r}_0)\, \boldsymbol{n}, \text{ if } \boldsymbol{R} = \boldsymbol{f}(\boldsymbol{r})/(\boldsymbol{r} - \boldsymbol{r}_0) \qquad \textbf{(11.2)}$$

and

$${}_D\oint\!\!\!\oint \boldsymbol{R}\wedge d\boldsymbol{S} = {}_V\iiint(\nabla\bullet\boldsymbol{R})dv \qquad \textbf{(12.1)}$$

$${}_D\oint\!\!\!\oint \boldsymbol{R}\bullet d\boldsymbol{S} = {}_V\iiint(\nabla\wedge\boldsymbol{R})\bullet dv \qquad \textbf{(12.2)}$$

<u>Proof</u>.

We have already done the transformation of a line integral into a surface integral and vice versa in Theorem 1 (formula (3)). We cut the closed surface $\boldsymbol{D}$ with a plane parallel to xy and divide it into surface integrals over the "upper" and "lower" surfaces:

$${}_D\oint\!\!\!\oint \boldsymbol{R}d\boldsymbol{S} = {}_{Dup}\oint\!\!\!\oint \boldsymbol{R}d\boldsymbol{S} + {}_{Ddown}\oint\!\!\!\oint \boldsymbol{R}d\boldsymbol{S}$$

Next, we apply formula (3) of Theorem 1 to both the "upper" and "lower" integrals. Formula (3) is valid for both the "lower" and "upper" integrals. By adding the "lower" and "upper" integrals over all projections $\boldsymbol{D}_{xy}$, $\boldsymbol{D}_{yz}$, and $\boldsymbol{D}_{zx}$, we obtain the formula (11).

Since $\boldsymbol{R}$ is analytic everywhere in $\boldsymbol{D}$, then there is an antiderivative $\boldsymbol{F}$ ($\boldsymbol{F}'_{\boldsymbol{r}} = \boldsymbol{R}$). It is also analytic in the domain $\boldsymbol{D}$.

According to Theorem 1,

$${}_{Dup}\iint \boldsymbol{R}d\boldsymbol{S} = {}_l\oint \boldsymbol{F}\wedge d\boldsymbol{l} = 0$$

Formula (11.1) is proven.

Now we consider the case when the function has a pole of type

$$\boldsymbol{R}(\boldsymbol{r}) = \boldsymbol{f}(\boldsymbol{r})/(\boldsymbol{r} - \boldsymbol{r}_0)$$

We transform the surface integrals (11.2):

$$\iint_{xy_{up}} \boldsymbol{R}d\boldsymbol{S} = \iint_{xy_{up}} \frac{f(\boldsymbol{r})}{\boldsymbol{r}-\boldsymbol{r}_0} d\boldsymbol{S} = \iint_{xy_{up}} \frac{f(\boldsymbol{r})-f(\boldsymbol{r}_0)+f(\boldsymbol{r}_0)}{\boldsymbol{r}-\boldsymbol{r}_0} d\boldsymbol{x}\wedge d\boldsymbol{y} =$$

$$= \iint_{xy_{up}} \frac{f(\boldsymbol{r})-f(\boldsymbol{r}_0)}{\boldsymbol{r}-\boldsymbol{r}_0} d\boldsymbol{x}\wedge d\boldsymbol{y} + \boldsymbol{f}(\boldsymbol{r}_0) \iint_{xy_{up}} \frac{d\boldsymbol{x}\wedge d\boldsymbol{y}}{\boldsymbol{r}-\boldsymbol{r}_0}$$

or

$$\iint_{xy_{up}} \frac{f(\boldsymbol{r})}{\boldsymbol{r}-\boldsymbol{r}_0} d\boldsymbol{S} = \iint_{xy_{up}} \nabla \boldsymbol{f}(\boldsymbol{r}) d\boldsymbol{x}\wedge d\boldsymbol{y} + \boldsymbol{f}(\boldsymbol{r}_0) \iint_{xy_{up}} \frac{d\boldsymbol{x}\wedge d\boldsymbol{y}}{\boldsymbol{r}-\boldsymbol{r}_0}, \qquad \textbf{(13)}$$

here

$$\iint_{xy_{up}} \frac{f(\boldsymbol{r})-f(\boldsymbol{r}_0)}{\boldsymbol{r}-\boldsymbol{r}_0} d\boldsymbol{x}\wedge d\boldsymbol{y} = \iint_{xy_{up}} (\nabla \boldsymbol{f}(\boldsymbol{r})) d\boldsymbol{x}\wedge d\boldsymbol{y},$$

since $\lim_{\boldsymbol{r}\to\boldsymbol{r}_0} \frac{f(\boldsymbol{r})-f(\boldsymbol{r}_0)}{\boldsymbol{r}-\boldsymbol{r}_0} = \frac{df(\boldsymbol{r}_0)}{d\boldsymbol{r}} = \nabla \boldsymbol{f}(\boldsymbol{r})$ is the derivative with respect to $\boldsymbol{r}$, i.e., the gradient of a vector function.

According to formulas (3), the first integral on the right-hand side of (13) is transformed into the line integral and is equal to zero:

$${}_{up}\iint(\nabla\boldsymbol{f}(\boldsymbol{r}))dx\wedge d\boldsymbol{y} = {}_{up}\iint(\nabla\boldsymbol{f}(\boldsymbol{r}))d\boldsymbol{S}_{xy} = {}_{up}\iint(\nabla\boldsymbol{f}(\boldsymbol{r}))d\boldsymbol{S}_{xy} =$$

$$= {}_{up}\iint(\nabla\wedge\boldsymbol{f}(\boldsymbol{r}))\bullet d\boldsymbol{S}_{xy} + {}_{up}\iint(\nabla\bullet\boldsymbol{f}(\boldsymbol{r}))d\boldsymbol{S}_{xy} = {}_{xy}\oint \boldsymbol{f}\bullet d\boldsymbol{l} + {}_{xy}\oint \boldsymbol{f}\wedge d\boldsymbol{l} = 0,$$

since the function $\boldsymbol{f}(\boldsymbol{r})$ is analytic at all points of $\boldsymbol{D}$ (both in $\boldsymbol{D}_{up}$ and in $\boldsymbol{D}_{down}$).

According to formula (2.1) of Theorem 1, ${}_{xy}\oint \boldsymbol{f}\bullet d\boldsymbol{l} = 0$.

We calculate the second integral on the right-hand side of (13):

$$\iint_{xy_{up}} \frac{d\boldsymbol{x}\wedge d\boldsymbol{y}}{\boldsymbol{r}-\boldsymbol{r}_0} = \iint_{xy_{up}} (\nabla Ln(\boldsymbol{r}-\boldsymbol{r}_0)) d\boldsymbol{s} = \oint_{xy} Ln(\boldsymbol{r}-\boldsymbol{r}_0)\, d\boldsymbol{l} \qquad \textbf{(14)}$$

$\nabla Ln(\boldsymbol{r} - \boldsymbol{r}_0) = (\boldsymbol{r} - \boldsymbol{r}_0)^{-1}$, so we expand $(\boldsymbol{r} - \boldsymbol{r}_0)^{-1}$ into a Laurent series [9]:

$$\frac{1}{\boldsymbol{r}-\boldsymbol{r}_0} = \frac{1}{\sigma_1(x-x_0)+\sigma_2(y-y_0)} = \frac{1}{\sigma_1}\cdot\frac{1}{x+i\sigma_2 y - x_0 - i\sigma_2 y_0} = \frac{1}{\sigma_1}\cdot\frac{1}{z-z_0} =$$

$$= \frac{1}{\sigma_1 z}\cdot\frac{1}{1-\frac{z_0}{z}} = \frac{1}{\sigma_1 z}\cdot\sum_{n=1}^{\infty}\left(\frac{z_0}{z}\right)^n = \frac{1}{\sigma_1}\cdot\sum_{n=1}^{\infty}\frac{z_0^n}{z^{n+1}}$$

Now we integrate this series:

$$\int_\tau^\infty \frac{1}{\sigma_1}\sum_{n=1}^{\infty}\frac{z_0^n}{z^{n+1}}dz = \frac{1}{\sigma_1}\cdot\sum_{n=1}^{\infty}\int_\tau^\infty \frac{d\left(\frac{z}{z_0}\right)}{\left(\frac{z}{z_0}\right)^{n+1}} =$$

$$= \frac{1}{\sigma_1}\cdot\sum_{n=1}^{\infty}\frac{1}{n\tau^n}\Big|_\tau^\infty = \frac{1}{\sigma_1}\cdot\sum_{n=1}^{\infty}\frac{1}{n\tau^n}$$

Here we only need the member with number n=1: $c_1 = \frac{1}{\sigma_1 z}$.

Substituting the value of c_1 into (14), we get

$$\iint_{xy_{up}} \frac{dx\wedge dy}{\boldsymbol{r}-\boldsymbol{r}_0} = \frac{1}{\sigma_1}\oint_{xy}\frac{\sigma_1 dz}{z} = i\sigma_3\int_0^{2\pi}d\varphi = 2\pi i\sigma_3$$

Integrating over both the "lower" and "upper" surfaces, we obtain the same result. Since the pole is the same for both surfaces, then, adding the integrals, we get

$${}_{xy}\oint\!\!\!\oint dxdy/(\boldsymbol{r} - \boldsymbol{r})^{-1} = 2\pi\, i\sigma_3 \qquad \textbf{(15)}$$

Equalities of the type (15) are also valid for projections $\boldsymbol{D}_{yz}$ and $\boldsymbol{D}_{zx}$. Adding integrals of the type (15) over all projections $\boldsymbol{D}_{xy}$, $\boldsymbol{D}_{yz}$, and $\boldsymbol{D}_{zx}$, we get the formula (11.2).

Formula (11.2) of Theorem 2 is proven.

Now we will prove the formula (12.1).

It is obvious that

$${}_V\iiint(\nabla\bullet\boldsymbol{R})d\boldsymbol{v} = {}_V\iiint(\nabla\bullet\boldsymbol{R})d\boldsymbol{x}\wedge d\boldsymbol{y}\wedge d\boldsymbol{z} =$$

$$= {}_V\iiint(\partial_x X + \partial_y Y + \partial_z Z)\sigma_1\sigma_2\sigma_3 dxdydz =$$

$$= i\sigma_0\, {}_V\iiint(\partial_x X + \partial_y Y + \partial_z Z)dxdydz$$

or

$${}_V\iiint(\nabla\bullet\boldsymbol{R})dv = i\sigma_0\, {}_V\iiint(\partial_x X + \partial_y Y + \partial_z Z)dv \qquad \textbf{(16)}$$

Applying the Ostrogradsky-Gauss formula [10] to (16), we obtain (12.1).

Now we will prove the formula (12.2).

The triple integral in (12.2) in coordinate form looks like this:

$${}_V\iiint(\nabla\wedge\boldsymbol{R})\bullet d\boldsymbol{v} =$$

$$= {}_V\iiint(\sigma_1(\partial_x Y - \partial_y Z) + \sigma_2(\partial_x Z - \partial_z X) + \sigma_3(\partial_y X - \partial_x Y))dxdydz \qquad \textbf{(17)}$$

We transform the surface integral in (12.2) into a triple one:

$${}_D\oint\!\!\!\oint \boldsymbol{R}\bullet d\boldsymbol{S} =$$

$$= {}_D\oint\!\!\!\oint(\sigma_1 X + \sigma_2 Y + \sigma_3 Z)\bullet(\sigma_1\sigma_2\, dxdy + \sigma_2\sigma_3 dydz + \sigma_3\sigma_1 dzdx) =$$

$$= {}_D\oint\!\!\!\oint(\sigma_1(Zdzdx - Ydxdy) + \sigma_2(Xdxdy - Zdydz) +$$

$$+ \sigma_3(Ydydz - Xdzdx)) =$$

$$= {}_V\iiint(\sigma_1(\partial_y Z - \partial_z Y) + \sigma_2(\partial_z X - \partial_x Z) +$$

$$+ \sigma_3(\partial_x Y - \partial_y X))dxdydz$$

or

$${}_D\oint\!\!\!\oint \boldsymbol{R}\bullet d\boldsymbol{S} = {}_V\iiint(\sigma_1(\partial_y Z - \partial_z Y) + \sigma_2(\partial_z X - \partial_x Z) +$$

$$+ \sigma_3(\partial_x Y - \partial_y X))d\boldsymbol{v} \qquad \textbf{(18)}$$

Comparing (17) and (18), we see that formula (12.2) is correct.

Theorem 2 is proven.

2.3. Line integrals in Minkowski space

We will consider the line ${}_l\int A(r)dl$ and surface ${}_D\iint A(r)dS$ integrals in 4-dimensional pseudo-Euclidean space, where

the vector - function: $\quad A(r) = \gamma^i A_i(r) \qquad$ **(19)**

$r = \{x^i\} = \{t, x, y, z\}$ is the interval (in spacetime),

$$dl = \gamma_i\, dx^i \qquad (\mathbf{20})$$

Dirac matrices γ^i in the following representation:

$$\gamma^0 = \begin{vmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{vmatrix},\ \gamma^\alpha = i\begin{vmatrix} 0 & \sigma_\alpha \\ \sigma_\alpha & 0 \end{vmatrix},\ \alpha = 1, 2, 3. \quad (\mathbf{21})$$

$$\gamma^0\gamma^\alpha = i\begin{vmatrix} 0 & \sigma_\alpha \\ -\sigma_\alpha & 0 \end{vmatrix},\quad \gamma^1\gamma^2 = -i\begin{vmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{vmatrix},$$

$$\gamma^2\gamma^3 = -i\begin{vmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{vmatrix},\quad \gamma^3\gamma^1 = -i\begin{vmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{vmatrix}.$$

$$E = \begin{vmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{vmatrix},\quad \gamma = \gamma^0\gamma^1\gamma^2\gamma^3 = \begin{vmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{vmatrix}.$$

$dS = dt\wedge dx + dt\wedge dy + dt\wedge dz + dx\wedge dy + dy\wedge dz + dz\wedge dx = N ds$ is the hypersurface element;

$N = \gamma^0\gamma^1 \cos\alpha_{01} + \gamma^0\gamma^2 \cos\alpha_{02} + \gamma^0\gamma^3 \cos\alpha_{03} + \gamma^1\gamma^2 \cos\alpha_{12} + \gamma^2\gamma^3 \cos\alpha_{23} + \gamma^3\gamma^1 \cos\alpha_{31}$ is a normal vector;

$\pi/2 - \cos\alpha_{01}$, $\pi/2 - \cos\alpha_{02}$, … are the angles between the normal N and the hyperplanes $tx(\gamma^0\gamma^1)$, $ty(\gamma^0\gamma^2)$, etc.

In other words, $\cos\alpha_{01}$, $\cos\alpha_{02}$, … are direction cosines.

$$dt\wedge dx = \gamma^0\gamma^1 dtdx = \gamma^0\gamma^1 \cos\alpha_{01}\, dS,$$
$$dt\wedge dy = \gamma^0\gamma^2 dtdy = \gamma^0\gamma^2 \cos\alpha_{02}\, dS,$$
$$dt\wedge dz = \gamma^0\gamma^3 dtdz = \gamma^0\gamma^3 \cos\alpha_{03}\, dS,$$
$$dx\wedge dy = \gamma^1\gamma^2 dtdx = \gamma^1\gamma^2 \cos\alpha_{12}\, dS,$$
$$dy\wedge dz = \gamma^2\gamma^3 dydz = \gamma^2\gamma^3 \cos\alpha_{23}\, dS,$$
$$dz\wedge dx = \gamma^3\gamma^1 dzdx = \gamma^3\gamma^1 \cos\alpha_{31}\, dS.$$

The order of inversion (i, j) $\gamma^i\gamma^j$ goes according to the formula

$$\gamma^0\gamma^\alpha = \varepsilon^{0\alpha\lambda\mu}\, \gamma^\lambda\gamma^\nu$$

where $\varepsilon^{0\alpha\lambda\mu}$ is the absolutely antisymmetric unit four-rank tensor (or Levi-Civita symbol) [11].

The hypersurface D and the contour l, with their projections on the hyperplanes tx, ty, tz, xy, yz, and zx, are defined in Minkowski space.

Remark 4.

α_{01}, α_{02}, and α_{03} are the "angles" between the normal N and the "time axis" t. Since these three angles are imaginary, we write them in the form

$$\cos\alpha_{01} = \cos(i\eta_1) = \cosh\eta_1 = (1 - \beta^2_1)^{-0.5}$$
$$\cos\alpha_{02} = \cos(i\eta_2) = \cosh\eta_2 = (1 - \beta^2_2)^{-0.5}$$
$$\cos\alpha_{03} = \cos(i\eta_3) = \cosh\eta_3 = (1 - \beta^2_3)^{-0.5}$$

where η is the rapidity [12], $\Gamma_\lambda = (1 - \beta^2_\lambda)^{-0.5}$ is the Lorentz factor [13], $\beta_\lambda = v_\lambda/c$, v_λ is the projection of the velocity v onto the x_λ axis. c is the speed of light in a vacuum.

α_{12}, α_{23}, α_{31} are the usual spatial angles between the normal N and the spatial axes x, y, z:

$\alpha_{12} = \alpha$, $\alpha_{23} = \beta$, $\alpha_{31} = \gamma$.

Then the normal N has the form:

$$N = \gamma^0\gamma^1\Gamma_1 + \gamma^0\gamma^2\Gamma_2 + \gamma^0\gamma^3\Gamma_3 + \gamma^1\gamma^2\cos\alpha + \gamma^2\gamma^3\cos\beta + \gamma^3\gamma^1\cos\gamma \quad (\mathbf{22})$$

Now we will prove theorems similar Theorems 1 and 2 in the case of Minkowski's space.

Theorem 3.

The following formulas are valid:

$${}_l\oint A dl = 0, \text{ if } A \text{ is analytical} \qquad (\mathbf{23.1})$$
$${}_l\oint A dl = 2\pi\, N f(r_0), \text{ if } A(r) = f(r)/(r - r_0) \qquad (\mathbf{23.2})$$

and

$${}_l\oint A\bullet dl = {}_D\iint (\nabla\wedge A)\bullet dS \qquad (\mathbf{24.1})$$
$${}_l\oint A\wedge dl = {}_D\iint (\nabla\bullet A) dS \qquad (\mathbf{24.2})$$

<u>Proof</u>.

Really, if a function $A(r)$ is defined and bounded everywhere in D, then it has an antiderivative function $F'_r = A(r)$. Then

$${}_l\oint A dl = F|^a_a = 0$$

Suppose that the function has a pole of type

$$A(r) = f(r)/(r - r_0) \qquad (\mathbf{25})$$

We transform the integral:

$$\oint_l \frac{f(r)}{r-r_0} dl = \oint_l \frac{f(r)-f(r_0)}{r-r_0} dl + f(r_0)\oint_l \frac{1}{r-r_0} dl$$

The first integral on the right side of the equation is the derivative $\nabla f(r)$. This integral is equal to zero.

We transform the second integral on the right-hand side of the equation, more precisely, ${}_l\oint (r - r_0)^{-1} dl$:

$$\oint_l \frac{1}{r-r_0} dl = \oint_l \frac{\gamma^0 d(t-t_0)+\gamma^1 d(x-x_0)+\gamma^2 d(y-y_0)+\gamma^3 d(z-z_0)}{\gamma^0(t-t_0)+\gamma^1(x-x_0)+\gamma^2(y-y_0)+\gamma^3(z-z_0)} =$$
$$= \oint_l \frac{\gamma^0 dt+\gamma^1 dx+\gamma^2 dy+\gamma^3 dz}{\gamma^0 t+\gamma^1 x+\gamma^2 y+\gamma^3 z} =$$
$$= \oint_{tx} \frac{\gamma^0 dt+\gamma^1 dx}{\gamma^0 t+\gamma^1 x} + \oint_{ty} \frac{\gamma^0 dt+\gamma^2 dy}{\gamma^0 t+\gamma^2 y} + \oint_{tz} \frac{\gamma^0 dt+\gamma^3 dz}{\gamma^0 t+\gamma^3 z} +$$
$$+ \oint_{xy} \frac{\gamma^1 dx+\gamma^2 dy}{\gamma^1 x+\gamma^2 y} + \oint_{zx} \frac{\gamma^1 dx+\gamma^3 dz}{\gamma^1 x+\gamma^3 z} + \oint_{yz} \frac{\gamma^2 dy+\gamma^3 dz}{\gamma^2 y+\gamma^3 z}$$

By calculating the integrals over all hyperplanes separately, for example, as

$$\oint_{tx} \frac{\gamma^0 dt+\gamma^1 dx}{\gamma^0 t+\gamma^1 x} = \oint_{tx} \frac{E dt+\gamma^0\gamma^1 dx}{E t+\gamma^0\gamma^1 x} =$$
$$= \gamma^0\gamma^1 \int_0^{2\pi} |r_{01}| \frac{\exp(\gamma^0\gamma^1\varphi_{01}) d\varphi_{01}}{|r_{01}| \exp(\gamma^0\gamma^1\varphi_{01})} = 2\pi\gamma^0\gamma^1$$

and, adding up similar ones over all projections, we get

$$\oint_l \frac{1}{r-r_0} dl = 2\pi \cdot N$$

Formulas (23.1) and (23.2) are proven.

Now we will prove formulas (24.1) and (24.2). We will consider line integrals on the planes tx, ty, tz, xy, yz, and tx, ty, tz, xy, yz separately.

According to the Clifford vector product:

$${}_{tx}\oint A dl = {}_{tx}\oint A\bullet dl + {}_{tx}\oint A\wedge dl \qquad (\mathbf{26})$$

We transform the line integral ${}_{tx}\oint A\bullet dl$ into a surface one:

$${}_{tx}\oint A\bullet dl = \gamma^0\gamma^1\, {}_{tx}\iint (\partial_0 A_1 - \partial_1 A_0)\, \gamma^0\gamma^1 dtdx = E\, {}_{tx}\iint (\partial_0 A_1 - \partial_1 A_0) dtdx$$

Then

$${}_{tx}\oint A\bullet dl = E\, {}_{tx}\iint (\partial_0 A_1 - \partial_1 A_0) dtdx \qquad (\mathbf{27})$$

Adding up all integrals of type (27) over all planes, we get the formula (24.1).

Now we transform the line integral ${}_{tx}\oint A\wedge dl$ of (26).

$${}_{tx}\oint A\wedge dl = {}_{tx}\oint (\gamma^0 A_0 + \gamma^1 A_1)\wedge(\gamma^0 dt + \gamma^1 dx) =$$
$$= \gamma^0\gamma^1\, {}_{tx}\oint (A_0\, dx - A_1 dt) = {}_{tx}\iint (\nabla\bullet A) dt\wedge dx =$$
$$= {}_{tx}\iint (\partial_0 A_0 - \partial_1 A_1) dt\wedge dx = \gamma^0\gamma^1\, {}_{tx}\iint (\partial_0 A_0 - \partial_1 A_1) dtdx$$

or

$${}_{tx}\oint A\wedge dl = \gamma^0\gamma^1\, {}_{tx}\iint (\partial_0 A_0 - \partial_1 A_1) dtdx \qquad (\mathbf{28})$$

By adding integrals of the type (28) over all planes, we get the formula (24.2).

Theorem 3 is proven.

2.4. Surface integrals in Minkowski space

Let the function $A(t,x,y,z)$ be given in the domain V bounded by the hypersurface D. We will consider the surface integral ${}_D\oiint A dS$ on this hypersurface.

Theorem 4.

The following formulas are valid:

$$_D\oiint AdS = 0, \text{ if } A \text{ is analytical} \qquad \mathbf{(29.1)}$$

$$_D\oiint AdS = 2\pi f(r_0)\, N, \text{ if } A = f(r)/(r - r_0) \qquad \mathbf{(29.2)}$$

and

$$_D\oiint A\wedge dS = \gamma^0\gamma\ _{xyz}\iiint(\partial_1 A_1 + \partial_2 A_2 + \partial_3 A_3)dxdydz +$$
$$+ \gamma^1\gamma\ _{tyz}\iiint(\partial_0 A_0 - \partial_2 A_2 + \partial_3 A_3)dtdydz +$$
$$+ \gamma^2\gamma\ _{tzx}\iiint(\partial_0 A_0 + \partial_1 A_1 - \partial_3 A_3)dtdzdx +$$
$$+ \gamma^3\gamma\ _{txy}\iiint(\partial_0 A_0 - \partial_1 A_1 + \partial_2 A_2)dtdxdy \qquad \mathbf{(30.1)}$$

$$_D\oiint A\bullet dS =$$
$$= {}_{xyz}\iiint(\gamma^1(\partial_3 A_2 - \partial_2 A_3) + \gamma^2(\partial_1 A_3 - \partial_3 A_1) + \gamma^3(\partial_2 A_1 - \partial_1 A_2))dxdydz +$$
$$+ {}_{tyz}\iiint(\gamma^0(\partial_3 A_2 - \partial_2 A_3) + \gamma^2(\partial_3 A_0 - \partial_0 A_3) + \gamma^3(\partial_0 A_2 - \partial_2 A_0))dtdydz +$$
$$+ {}_{txz}\iiint(\gamma^0(\partial_1 A_3 - \partial_3 A_1) + \gamma^1(\partial_0 A_3 - \partial_3 A_0) + \gamma^3(\partial_1 A_0 - \partial_0 A_1))dtdzdx +$$
$$+ {}_{txy}\iiint(\gamma^0(\partial_2 A_1 - \partial_1 A_2) + \gamma^1(\partial_2 A_0 - \partial_0 A_2) + \gamma^2(\partial_0 A_1 - \partial_1 A_0))dtdxdy \mathbf{(30.2)}$$

Proof.

Similar to Theorem 2, we will prove Theorem 4 in 4-dimensional Minkowski space.

If the function $A(t,x,y,z)$ is analytic, then

$$_D\oiint AdS = {}_D\oiint(\nabla F)dS = {}_l\oint Fdl = 0$$

Since A is defined everywhere in D, then there is an antiderivative $F(F'_r = A)$, and it is also analytic in the domain D.

Formula (29.1) is proven.

Now we prove (29.2) in the same way as (11.2) of Theorem 2. Let the function have a pole of type

$$A(r) = f(r)/(r - r_0)$$

We transform the surface integrals (29.2) (over tx, ty, tz, xy, yz, zx):

$$\iint_{tx_{up}} \frac{f(r)}{r-r_0} dt \wedge dx = \iint_{tx_{up}} (\nabla f) dt \wedge dx + f(r_0) \iint_{tx_{up}} \frac{dt \wedge dx}{r-r_0} \quad \mathbf{(31)},$$

where

$$\iint_{tx_{up}} \frac{f(r)-f(r_0)+f(r_0)}{r-r_0} dt \wedge dx = \iint_{tx_{up}} (\nabla f(r)) dt \wedge dx,$$

since $\lim\limits_{r\to r_0} \frac{f(r)-f(r_0)}{r-r_0} = \frac{df(r_0)}{dr} = \nabla f(r)$ is the derivative with respect to r, i.e., the gradient of a vector function.

The first integral on the right side of (31) is transformed into a line integral, and it is equal to zero, since the function $f(r)$ is analytic at all points of D (either by D_{up} and D_{down}).

We will calculate the second integral on the right-hand side of (31). We have already calculated a similar integral in Theorem 2 (formulas (14) – (15)).

$$\iint_{tx_{up}} \frac{dx \wedge dy}{r-r_0} = \iint_{tx_{up}} (\nabla Ln(r-r_0)) ds = \oint_{tx_{up}} Ln(r-r_0)\, dr \quad \mathbf{(32)}$$

Since $\nabla Ln(r - r_0) = (r - r_0)^{-1}$, we will expand $(r - r_0)^{-1}$ in the Laurent series:

$$\frac{1}{r-r_0} = \frac{1}{\gamma^0} \cdot \sum_{n=1}^{\infty} \frac{z_0^n}{z^{n+1}}$$

We integrate this series:

$$\int_{\tau}^{\infty} \frac{1}{\gamma^0} \sum_{n=1}^{\infty} \frac{z_0^n}{z^{n+1}} dz = \frac{1}{\gamma^0} \cdot \sum_{n=1}^{\infty} \frac{1}{n\tau^n} \qquad \mathbf{(33)}$$

In (33) we only need the term with number m=1: $c_1 = \frac{1}{\gamma^0 z}$.

Substituting c_1 into (32), we get

$$\iint_{tx_{up}} \frac{dx \wedge dy}{\boldsymbol{r}-\boldsymbol{r_0}} = \frac{1}{\gamma^0} \oint_{tx_{up}} \frac{\gamma^0 dz}{z} = \gamma^0\gamma^1 \int_0^{2\pi} d\varphi = 2\pi\gamma^0\gamma^1 \qquad \mathbf{(34)}$$

Integrating also over the "lower" surface, we get the same result. Since the pole is the same for both surfaces, adding the integrals, we obtain

$$\oiint_{tx} \frac{dx \wedge dy}{\boldsymbol{r}-\boldsymbol{r_0}} = 2\pi\gamma^0\gamma^1 \qquad \mathbf{(35)}$$

Equalities of the type (35) are also valid for all hyperplanes. By adding integrals of the type (35) over all projections, we get the formula (29.2).

Now we will prove the formula (30.1).

It is obvious that in 4-dimensional space, the elementary volume Ndv consists of the sum of four trivectors:

$$Ndv = dx\wedge dy\wedge dz + dt\wedge dy\wedge dz + dt\wedge dz\wedge dx + dt\wedge dx\wedge dy$$

or

$Ndv = \gamma^1\gamma^2\gamma^3 dxdydz + \gamma^0\gamma^2\gamma^3 dtdydz + \gamma^0\gamma^3\gamma^1 dtdzdx + \gamma^0\gamma^1\gamma^2 dtdxdy$

Trivectors $\gamma^1\gamma^2\gamma^3$, $\gamma^0\gamma^2\gamma^3$, $\gamma^0\gamma^3\gamma^1$ и $\gamma^0\gamma^1\gamma^2$ are dual [14] to pseudovectors $\gamma^0\gamma$, $\gamma^1\gamma$, $\gamma^2\gamma$ and $\gamma^3\gamma$:

$\gamma^1\gamma^2\gamma^3 = \gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^0\gamma$

$\gamma^0\gamma^2\gamma^3 = - \gamma^1\gamma^1\gamma^0\gamma^2\gamma^3 = \gamma^1\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^1\gamma$

$\gamma^0\gamma^3\gamma^1 = -\gamma^2\gamma^2\gamma^0\gamma^3\gamma^1 = -\gamma^2\gamma^2\gamma^0\gamma^3\gamma^1 = \gamma^2\gamma^2\gamma^0\gamma^1\gamma^3 = \gamma^2\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^2\gamma$

$\gamma^0\gamma^1\gamma^2 = -\gamma^3\gamma^3\gamma^0\gamma^1\gamma^2 = \gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^3\gamma$

Then

$Ndv = \gamma^0\gamma dxdydz + \gamma^1\gamma dtdydz + \gamma^2\gamma dtdzdx + \gamma^3\gamma dtdxdy$

Remark 5

Here we took the modulus ("length") of trivectors, bivectors, and pseudovectors as one, or more precisely, as the identity matrix:

$$|\gamma^i| = |\gamma^i\gamma^j| = |\gamma^i\gamma^j\gamma^k| = |E|$$

Using the duality of trivectors and pseudovectors, we write the outer product $A\wedge dS$ in coordinate form:

$$A\wedge dS = \gamma^1 A_1\, \gamma^2\gamma^3 dydz + \gamma^2 A_2\, \gamma^3\gamma^1 dzdx + \gamma^3 A_3\, \gamma^1\gamma^2 dxdy +$$
$$+ \gamma^0 A_0\, \gamma^2\gamma^3 dydz + \gamma^2 A_2\, \gamma^0\gamma^3 dtdz + \gamma^3 A_3\, \gamma^0\gamma^2 dtdy +$$
$$+ \gamma^0 A_0\, \gamma^3\gamma^1 dzdx + \gamma^3 A_3\, \gamma^0\gamma^1 dtdx + \gamma^1 A_1\, \gamma^0\gamma^3 dtdz +$$
$$+ \gamma^0 A_0\, \gamma^1\gamma^2 dxdy + \gamma^1 A_1\, \gamma^0\gamma^2 dtdy + \gamma^2 A_2\, \gamma^0\gamma^1 dtdx \qquad \mathbf{(36)}$$

We transform the $_D\oiint A\wedge dS$ integral into a triple one, for example, for the pseudovector $\gamma^0\gamma = \gamma^1\gamma^2\gamma^3$:

$$\gamma^0\gamma\ _{xyz}\oiint(A_1 dydz + A_2 dxdz + A_3 dxdy) =$$
$$= \gamma^0\gamma\ _{xyz}\iiint(\partial_1 A_1 + \partial_2 A_2 + \partial_3 A_3)dxdydz.$$

In a similar way, we transform the three remaining surface integrals:

$$\gamma^1\gamma\ _{tyz}\oiint(A_0 dydz - A_2 dtdz + A_3 dtdy)$$
$$\gamma^2\gamma\ _{tzx}\oiint(A_0 dzdx + A_1 dtdz - A_3 dtdx)$$
$$\gamma^3\gamma\ _{txy}\oiint(A_0 dxdy - A_1 dtdy + A_2 dtdx)$$

By adding up the triple integrals for all pseudovectors ($\gamma^0\gamma$, $\gamma^1\gamma$, $\gamma^2\gamma$, $\gamma^3\gamma$), we get the formula (30.1).

Now we will prove the formula (30.2).

$_D\oiint A\bullet dS$ in coordinate form has the form

$_D\oiint A\bullet dS = {}_D\oiint(\gamma^0 A_0 + \gamma^1 A_1 + \gamma^2 A_2 + \gamma^3 A_3)\bullet(\gamma^0\gamma^1 dtdx + \gamma^0\gamma^2 dtdy + \gamma^0\gamma^3 dtdz + \gamma^1\gamma^2 dxdy + \gamma^2\gamma^3 dydz + \gamma^3\gamma^1 dzdx)$

or

$$_{xyz}\oiint(\gamma^1(A_2 dxdy - A_3 dxdz) + \gamma^2(A_3 dydz - A_1 dxdy) +$$
$$+ \gamma^3(A_1 dxdz - A_2 dydz)) +$$
$$+ {}_{tyz}\oiint(\gamma^0(A_2 dtdy - A_3 dtdz) + \gamma^2(A_0 dtdy - A_3 dydz) +$$
$$+ \gamma^3(A_2 dydz - A_0 dtdz)) +$$
$$+ {}_{txz}\oiint(\gamma^0(A_3 dtdz - A_1 dtdx) + \gamma^1(A_3 dzdx - A_0 dtdx) +$$
$$+ \gamma^3(A_0 dtdz - A_1 dzdx)) +$$
$$+ {}_{txy}\oiint(\gamma^0(A_1 dtdx - A_2 dtdy) + \gamma^1(A_0 dtdx - A_2 dxdy) +$$
$$+ \gamma^2(A_1 dxdy - A_0 dtdy)) \qquad \mathbf{(37)}$$

The triple integral $_V\iiint(\nabla\wedge A)\bullet dv$ in (30.2) has the expanded form

$_V\iiint(\nabla\wedge A)\bullet dv =$

$= _{xyz}\iiint(\gamma^1(\partial_3A_2 - \partial_2A_3) + \gamma^2(\partial_1A_3 - \partial_3A_1) + \gamma^3(\partial_2A_1 - \partial_1A_2))\, dxdydz +$
$+ _{tyz}\iiint(\gamma^0(\partial_3A_2 - \partial_2A_3) + \gamma^2(\partial_3A_0 - \partial_0A_3) + \gamma^3(\partial_0A_2 - \partial_2A_0))\, dtdydz +$
$+ _{txz}\iiint(\gamma^0(\partial_1A_3 - \partial_3A_1) + \gamma^1(\partial_0A_3 - \partial_3A_0) + \gamma^3(\partial_1A_0 - \partial_0A_1))\, dtdzdx +$
$+ _{txy}\iiint(\gamma^0(\partial_2A_1 - \partial_1A_2) + \gamma^1(\partial_2A_0 - \partial_0A_2) + \gamma^2(\partial_0A_1 - \partial_1A_0))\, dtdxdy$ **(38)**

We transform the integral (38) into the surface one, for example, as

$_{xyz}\iiint(\gamma^1(\partial_3A_2dxdydz - \partial_2A_3dxdydz) + \gamma^2(\partial_1A_3dxdydz - \partial_3A_1dxdydz) + \gamma^3(\partial_2A_1dxdydz - \partial_1A_2dxdydz)) =$
$= _{xyz}\oiint(\gamma^1(A_2dxdy - A_3dxdz) + \gamma^2(A_3dydz - A_1dxdy) + \gamma^3(A_1dxdz - A_2dydz))$

Comparing this with the first integral in (37), we see that they are identical. By transforming the remaining triple integrals in (38), we get the formula (30.2).

Theorem 4 is proven.

2.5. Generalization of the Cauchy integral formula

Now we will consider the case when the function has a pole of the form

$$A = f(r)/(r - r_0)^{-k-1} \tag{39}$$

Theorem 5.

Assume that the function $A(t,x,y,z)$ has a pole of type (39) at point $r_0(t_0, x_0, y_0, z_0)$ in domain D. Then the following formula is valid

$$_l\oint f(r)dl/(r - r_0)^{-k-1} = 2\pi f^{(k)}(r_0)\, N/\Gamma(k + 1) \tag{40}$$

Here $\Gamma(k + 1) = k!$ is the gamma function of an integer non-negative argument [15].

Proof.

Taking r_0 as a parameter, we differentiate the integral (23.2) with respect to it:

$$\oint_l \frac{f(r)dl}{(r-r_0)^2} = \frac{1}{1}\frac{d}{dr_0}\oint_L \frac{f(r)dl}{r-r_0} = \frac{2\pi N}{1} f_{r_0}^{(1)}(r)|_{r_0} = 2\pi N f_{r_0}^{(1)}(r_0)$$

$$\oint_l \frac{f(r)dl}{(r-r_0)^3} = \frac{1}{1\cdot 2}\frac{d^2}{dr_0^2}\oint_L \frac{f(r)dl}{r-r_0} = \frac{2\pi N}{1\cdot 2} f_{r_0}^{(2)}(r)|_{r_0} = \frac{2\pi N}{1\cdot 2} f_{r_0}^{(2)}(r_0)$$

$$\oint_l \frac{f(r)dl}{(r-r_0)^3} = \frac{1}{3!}\frac{d^3}{dr_0^3}\oint_L \frac{f(r)dl}{r-r_0} = \frac{2\pi N}{3!} f_{r_0}^{(3)}(r)|_{r_0} = \frac{2\pi N}{3!} f_{r_0}^{(3)}(r_0)$$

etc.

$$\oint_l \frac{f(r)dl}{(r-r_0)^{k+1}} = \frac{1}{k!}\frac{d^k}{dr_0^k}\oint_L \frac{f(r)dl}{r-r_0} = \frac{2\pi N}{k!} f_{r_0}^{(k)}(r)|_{r_0} = \frac{2\pi N}{k!} f_{r_0}^{(k)}(r_0)$$

Thus, we get the formula (40). Theorem 5 is proven.

Formula (40) is a generalization of the Cauchy-type integral for a multidimensional complex function, i.e., a hypercomplex function for a 4-dimensional pseudo-Euclidean space.

<u>Consequence.</u>

We define the fractional derivative [16] or fractional gradient (for a function of several variables) of order p through a generalized Cauchy-type integral by generalizing formula (40):

$$D^p f(t) = \nabla^p f(t) = \frac{1}{\Gamma(p)}\cdot\oint_L \frac{f(r)dl}{(r-t)^{p+1}} \tag{41}$$

or

$$D^p f(t) = \nabla^p f(t) = \frac{1}{\Gamma(p)}\cdot\oiint_D \frac{f(r)dS}{(r-t)^{p+1}} \tag{42}$$

where p is a positive real number.

3. Application in physics

Formula (24.1) can be examined from the perspective of physics. We write the right side of (24.1) as

$E_D\iint(\nabla\wedge A)\bullet dS = E_D\iint((\partial_0A_1 - \partial_1A_0)dtdx + (\partial_0A_2 - \partial_2A_0)dtdy + (\partial_0A_3 - \partial_3A_0)dtdz + (\partial_2A_1 - \partial_1A_2)dxdy + (\partial_3A_2 - \partial_2A_3)dydz + (\partial_1A_3 - \partial_3A_1)dzdx)$

or

$E_D\oiint(\nabla\wedge A)\bullet dS = E_D\oiint((\partial_0A_1 - \partial_1A_0)\Gamma_1 + (\partial_0A_2 - \partial_2A_0)\Gamma_2 + (\partial_0A_3 - \partial_3A_0)\Gamma_3 + (\partial_2A_1 - \partial_1A_2)\cos\alpha + (\partial_3A_2 - \partial_2A_3)\cos\beta + (\partial_1A_3 - \partial_3A_1)\cos\gamma)dS$

In accordance with the electromagnetic field tensor definition [17]

$$F_{ij} = \partial_i A_j - \partial_j A_i,$$

we write the final integral in the following form:

$$E_D\oiint(\nabla\wedge A)\bullet dS = E_D\oiint(F_{01}\Gamma_1 + F_{02}\Gamma_2 + F_{03}\Gamma_3 + F_{12}\cos\alpha + F_{23}\cos\beta + F_{31}\cos\gamma)dS \tag{43}$$

According to formulas (23), integral (43) is either equal to zero (if the function is analytic) or equal to the total 4-current [18] (if the function has poles). In general, it is constant. In other words, the surface integral over a closed surface (the total electromagnetic 4-current) is constant, i.e., the 4-dimensional electromagnetic current is conserved. This is one of the fundamental laws of physics.

The electromagnetic field tensor can be written as a sum of vectors:

$$F = \gamma^i\gamma^j F_{ij} = \gamma^i\gamma^j(\partial_i A_j - \partial_j A_i) =$$
$$= \gamma^0\gamma^1(\partial_0 A_1 - \partial_1 A_0) + \gamma^0\gamma^2(\partial_0 A_2 - \partial_2 A_0) + \gamma^0\gamma^3(\partial_0 A_3 - \partial_3 A_0) +$$
$$+ \gamma^1\gamma^2(\partial_1 A_2 - \partial_2 A_1) + \gamma^2\gamma^3(\partial_2 A_3 - \partial_3 A_2) + \gamma^3\gamma^1(\partial_3 A_1 - \partial_1 A_3)$$

Using the duality of bivectors ($\gamma^1\gamma^2 = \gamma^0\gamma^3\gamma$, $\gamma^2\gamma^3 = \gamma^0\gamma^1\gamma$, $\gamma^3\gamma^1 = \gamma^0\gamma^2\gamma$), we write the electromagnetic field tensor as

$$F = \gamma^0\gamma^1(\partial_0 A_1 - \partial_1 A_0) + \gamma^0\gamma^2(\partial_0 A_2 - \partial_2 A_0) +$$
$$+ \gamma^0\gamma^3(\partial_0 A_3 - \partial_3 A_0) + \gamma^0\gamma^3\gamma(\partial_1 A_2 - \partial_2 A_1) +$$
$$+ \gamma^0\gamma^1\gamma(\partial_2 A_3 - \partial_3 A_2) +) + \gamma^0\gamma^2\gamma(\partial_3 A_1 - \partial_1 A_3)$$

or

$$F = \gamma^0\gamma^1((\partial_0 A_1 - \partial_1 A_0) + \gamma(\partial_2 A_3 - \partial_3 A_2)) +$$
$$+ \gamma^0\gamma^2((\partial_0 A_2 - \partial_2 A_0) + \gamma(\partial_3 A_1 - \partial_1 A_3)) +$$
$$+ \gamma^0\gamma^3(\partial_0 A_3 - \partial_3 A_0) + \gamma(\partial_1 A_2 - \partial_2 A_1))$$

or in vector form

$$F = E + \gamma B \tag{44}$$

From (44) it is clear that the bivector F consists of three real (polar)

$$E = \gamma^0\gamma^1(\partial_0 A_1 - \partial_1 A_0) + \gamma^0\gamma^2(\partial_0 A_2 - \partial_2 A_0) +$$
$$+ \gamma^0\gamma^3(\partial_0 A_3 - \partial_3 A_0)$$

(electric field strength [19]) and three dual (axial) bivectors (pseudobivector)

$$B = \gamma^0\gamma^3\gamma(\partial_1 A_2 - \partial_2 A_1) + \gamma^0\gamma^1\gamma(\partial_2 A_3 - \partial_3 A_2) +) +$$
$$+ \gamma^0\gamma^2\gamma(\partial_3 A_1 - \partial_1 A_3)$$

(magnetic field induction [20]).

γ is the matrix analogue of the imaginary unit ($\gamma^2 = -1$).

Now we will consider the surface integral $_D\oiint(\nabla\wedge A)\wedge dS$:

$_D\oiint(\nabla\wedge A)\wedge dS = {_D}\oiint(\gamma^0\gamma^1F_{01} + \gamma^0\gamma^2F_{02} + \gamma^0\gamma^3F_{03} + \gamma^1\gamma^2F_{12} + \gamma^2\gamma^3F_{23} + \gamma^3\gamma^1F_{31})\wedge(\gamma^0\gamma^1dtdx + \gamma^0\gamma^2dtdy + \gamma^0\gamma^3dtdz + \gamma^1\gamma^2dxdy + \gamma^2\gamma^3dydz + \gamma^3\gamma^1dzdx)$

or

$$_D\oiint(\nabla\wedge A)\wedge dS = {_D}\oiint\gamma(F_{01}dydz + F_{02}dzdx + F_{03}\,dxdy +$$

$$+ F_{12}\,dtdz + F_{23}\,dtdx + F_{31}dtdy) \qquad \mathbf{(45)}$$

It is difficult to visualize integration in the *tx, ty, tz* planes. Therefore, we introduce the concept of integration in dual space [21], i.e., we replace integration over the planes *tx, ty, tz* with integration over the planes *xy, yz, zx*.

In 3-dimensional Euclidean space, the bivector (antisymmetric tensor of the second rank) $d\mathbf{x}\wedge d\mathbf{y}$ is dual to the pseudovector (axial vector) $i d\mathbf{s}$:

$$d\mathbf{x}\wedge d\mathbf{y} = \sigma_1\sigma_2 dxdy = i\sigma_3 ds = id\mathbf{z}$$

In 4-dimensional space, by analogy with 3-dimensional space, the second-rank antisymmetric tensor (bivector) $dt\wedge dx$ is dual to the second-rank antisymmetric pseudo-tensor (pseudo-bivector) $\gamma dy\wedge dz$ [22]:

$$dt\wedge dx = \gamma^0\gamma^1 dtdx = -\,\gamma\gamma\gamma^0\gamma^1 dydz = -\,\gamma\,\gamma^2\gamma^3 dydz$$

$$dt\wedge dy = \gamma^0\gamma^2 dtdx = -\,\gamma\gamma\gamma^0\gamma^2 dzdx = -\,\gamma\gamma^3\gamma^1 dzdx$$

$$dt\wedge dz = \gamma^0\gamma^3 dtdz = -\,\gamma\gamma\gamma^0\gamma^3 dxdy = -\,\gamma\,\gamma^1\gamma^2 xdy$$

In the general case, we say that γb^* is dual to b, and the following equality holds [23]:

$$a\wedge b = a\bullet\gamma b^* \qquad \mathbf{(46)}$$

Below we present some useful consequences of the formula (46):

$$(a\wedge b)\wedge(dt\wedge dx) = (a\wedge b)\bullet(\gamma dy\wedge dz)$$

In particular,

$$\gamma^2\gamma^3 F_{23}\wedge\gamma^0\gamma^1 dtdx = -\,\gamma^2\gamma^3 F_{23}\bullet\gamma\gamma^2\gamma^3 dydz = \gamma\, F_{23} dydz$$

$$\gamma^3\gamma^1 F_{31}\wedge\gamma^0\gamma^2 dtdy = -\,\gamma^3\gamma^1 F_{31}\bullet\gamma\gamma^3\gamma^1 dzdx = \gamma F_{31} dzdx$$

$$\gamma^1\gamma^2 F_{12}\wedge\gamma^0\gamma^3 dtdz = -\,\gamma^1\gamma^2 F_{12}\bullet\gamma\gamma^1\gamma^2 dxdy = \gamma F_{12} dxdy$$

Taking into account these consequences (46), in formula (45) we replace the surface integrals over the planes *tx, ty, tz* with dual surface integrals over the planes *yz, zx, xy*:

$$_D\oiint(\nabla\wedge A)\wedge dS = {}_D\oiint((F_{01} + \gamma F_{23})\,dydz + (F_{02} + \gamma F_{31})\,dzdx + (F_{03} + \gamma F_{12})\,dxdy) \qquad \mathbf{(47)}$$

or

$$_D\oiint(\nabla\wedge A)\wedge dS = {}_D\oiint((F_{03} + \gamma F_{12})\cos\alpha + (F_{02} + \gamma F_{31})\cos\beta) + (F_{01} + \gamma F_{23})\cos\gamma\,)dS \qquad \mathbf{(48)}$$

Since in (48) F_{ij}, $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are constants, the integral (48) does not change, i.e. the 4-dimensional electromagnetic current over a closed surface is preserved. Thus, we have obtained one of the fundamental laws of physics (the law of conservation of 4-dimensional electromagnetic current) in integral form.

4. Conclusions

1. Theorems of complex analysis (Cauchy's integral theorem, Cauchy's integral formula, and his integral representation for derivatives) are generalized for 3- and 4-dimensional Euclidean (pseudo-Euclidean) space. The Pauli matrices (σ^i) for 3-dimensional Euclidean space and the Dirac matrices (γ^i) for 4-dimensional pseudo-Euclidean (Minkowski) space were used as basis vectors and hypercomplex numbers. Thus, a bijection is established between basis vectors and hypercomplex numbers by definition.

2. The Stokes and Ostrogradsky - Gauss theorems are combined and generalized for 4-dimensional pseudo-Euclidean space.

3. Hypercomplex analysis (Cauchy's theorem and formula and its consequences for derivatives) were performed for both line and surface integrals.

4. The results of the analysis of hypercomplex numbers are applied to the study of the laws of physics: within the framework of Clifford algebra, the law of conservation of 4-dimensional electromagnetic current is derived. The replacement of integrals over "temporal" surfaces *tx, ty, tz* by integrals over "spatial" surfaces *xy, yz, zx* (integration over dual space) is applied.

Acknowledgments

I would like to thank the experts and editors of certain journals for nudging me to write an expanded and in-depth version of the article on this topic.

I would especially like to acknowledge the help of my wife, Lyuba, in creating this article. She helps me, inspires me, and also "pushes forward" and make me when it is necessary for the matter.

In general, I am sincerely grateful to all my readers for their interest to my modest work.

References

[1] Wikipedia. Bijection. https://en.wikipedia.org/wiki/Bijection.

[2] Chris J. L. Doran. Geometric Algebra and its Application to Mathematical Physics. Sidney Sussex College. A dissertation submitted for the degree of Doctor of Philosophy in the University of Cambridge. February 1994, pages 4-6.

[3] Wikipedia. Zeros and poles. https://en.wikipedia.org/wiki/Zeros_and_poles.

[4] Wikipedia. Simple connected space https://en.wikipedia.org/wiki/Simply_connected_space.

[5] Wikipedia. Green's theorem. https://en.wikipedia.org/wiki/Green%27s_theorem.

[6] Wikipedia. Stokes's theorem. https://en.wikipedia.org/wiki/Generalized_Stokes_theorem.

[7] Wikipedia. Analytic function. https://en.wikipedia.org/wiki/Analytic_function.

[8] Wikipedia. Cauchy's integral formula. https://en.wikipedia.org/wiki/Cauchy%27s_integral_formula.

[9] Wikipedia. the Laurent series. https://en.wikipedia.org/wiki/Laurent_series.

[10] Wikipedia. The divergence theorem. https://en.wikipedia.org/wiki/Divergence_theorem.

[11] Wikipedia. Levi-Civita symbol. https://en.wikipedia.org/wiki/Levi-Civita_symbol.

[12] Wikipedia. Rapidity. https://en.wikipedia.org/wiki/Rapidity.

[13] Wikipedia. Lorentz factor. https://en.wikipedia.org/wiki/Lorentz_factor.

[14] Wikipedia. Multivector. https://en.wikipedia.org/wiki/Multivector.

[15] Wikipedia. Gamma function. https://en.wikipedia.org/wiki/Gamma_function.

[16] Wikipedia. Fractional calculus. https://en.wikipedia.org/wiki/Fractional_calculus#Fractional_derivatives.

[17] Wikipedia. Electromagnetic field tensor. https://en.wikipedia.org/wiki/Electromagnetic_tensor.

[18] Wikipedia. Four-current density. https://en.wikipedia.org/wiki/Four-current.

[19] Wikipedia. An electric field. https://en.wikipedia.org/wiki/Electric_field.

[20] Wikipedia. A magnetic field.

https://en.wikipedia.org/wiki/Magnetic_field.
[21] Wikipedia. Dual vector space.
https://en.wikipedia.org/wiki/Dual_space.
[22] Wikipedia. A pseudotensor.
https://en.wikipedia.org/wiki/Pseudotensor.
[23] Wikipedia. the dual numbers.
https://en.wikipedia.org/wiki/Dual_number.