A proof of the Riemann hypothesis on nontrivial zeros of the Riemann zeta function

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April 04, 2025

Abstract

The Riemann hypothesis on nontrivial zeros of the Riemann zeta function is proved.

A complex number $s_0 = \sigma_0 + it_0$ is a nontrivial zero iff (σ_0, t_0) is a solution to a system of two equations of two real variables σ and t.

Considering one of that two equations, we found that one side of it is strictly increasing and the other one is nonincreasing as functions on the set of so called *critical values* $\sigma \in (0;1)$ at the "height" $t = t_0$, so (σ_0, t_0) is a unique solution at $t = t_0$. As nontrivial zeros are symmetric about the line Re $s = \frac{1}{2}$, it follows that $\sigma_0 = \frac{1}{2}$.

Keywords: the Riemann hypothesis, zeta function, nontrivial zeros.

Setting the problem

Let $s = \sigma + it$ be a complex variable, where $\sigma = \operatorname{Re} s, t = \operatorname{Im} s$, and
$x \in \mathbb{R}$ be a real variable.

For $\operatorname{Re} s > 0, s \neq 1$, it is known [1] that the Riemann zeta function $\zeta(s)$ can be expressed by the formula

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^{\infty} \frac{\{x\}}{x^{s+1}}dx. \tag{1}$$

Here, $\{x\}$ denotes the fractional part of a number x.

Let us rewrite equality 1 in the form

$$\zeta(s) = s\left(\frac{1}{s-1} - \int_1^{\infty} \frac{\{x\}}{x^{s+1}}dx\right).$$

Thus, to obtain nontrivial zeros of the function $\zeta(s)$, we must solve the following equation:

$$\int\limits_1^\infty \frac{\{x\}}{x^{s+1}} = \frac{1}{s-1}. \tag{2}$$

This implies two equations:

$$\frac{1}{x^{s+1}} = \frac{1}{x^{\sigma+1}}\left(\cos(t\ln x) - i\sin(t\ln x)\right),$$
$$\frac{1}{s-1} = \frac{\sigma-1}{(\sigma-1)^2+t^2} - i\frac{t}{(\sigma-1)^2+t^2}.$$

Therefore, equation 2 is equivalent to the following system:

$$\begin{cases} \int\limits_1^\infty \frac{\{x\}}{x^{\sigma+1}}\cos(t\ln x)dx = \frac{\sigma-1}{(\sigma-1)^2+t^2}, \\ \int\limits_1^\infty \frac{\{x\}}{x^{\sigma+1}}\sin(t\ln x)dx = \frac{t}{(\sigma-1)^2+t^2}. \end{cases} \tag{3}$$

It is known that nontrivial zeros are symmetric about the real axis, therefore we consider only the case $t > 0$.

We always assume that $0 < \sigma < 1,\ t > 0$.

Let $s_0 = \sigma_0 + it_0$ be a nontrivial zero.

The Riemann hypothesis states that $\sigma_0 = \frac{1}{2}$.

Left and right sides of the equations of system 3

Let us introduce four useful functions as follows:

$$u_1(\sigma,t) = \int\limits_1^\infty \frac{\{x\}}{x^{\sigma+1}}\cos(t\ln x)dx,$$
$$v_1(\sigma,t) = \int\limits_1^\infty \frac{\{x\}}{x^{\sigma+1}}\sin(t\ln x)dx,$$
$$u_2(\sigma,t) = \frac{\sigma-1}{(\sigma-1)^2+t^2},$$
$$v_2(\sigma,t) = \frac{t}{(\sigma-1)^2+t^2}.$$

Equation 2 can be expressed as follows:

$$u_1(\sigma,t) - iv_1(\sigma,t) = u_2(\sigma,t) - iv_2(\sigma,t).$$

We represent system 3 in the form

$$\begin{cases} u_1(\sigma, t) = u_2(\sigma, t), \\ v_1(\sigma, t) = v_2(\sigma, t). \end{cases} \tag{4}$$

$s = \sigma + it$ is a nontrivial zero if and only if (σ, t) is a solution to system 4.

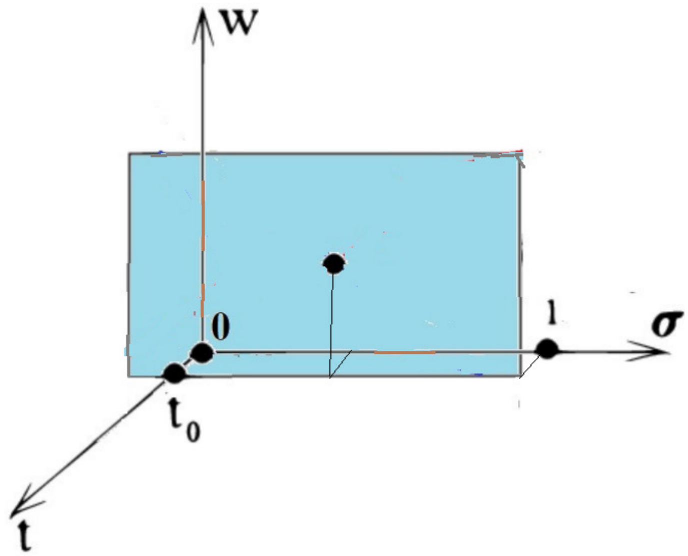

Figure 1: The plane $t = t_0$

Let $s_0 = \sigma_0 + it_0$ be a nontrivial zero.

Lemma 1. *The function $w = v_2(\sigma, t_0)$ increases as a function of one variable $\sigma \in (0; 1)$.*

Proof. It follows from the inequality

$$\frac{dv_2}{d\sigma} = -\frac{2(\sigma - 1)t_0}{(t_0^2 + (\sigma - 1)^2)^2} > 0.$$

□

The range of the function $w = v_2(\sigma, t_0)$ is $U = \left(\frac{t_0}{1 + t_0^2}, \frac{1}{t_0}\right)$.

Obviously, the graph of the function $w = v_2(\sigma, t_0)$ lies in the rectangle $\Pi = \left\{(\sigma, w) \,\middle|\, \sigma \in (0; 1), w \in U\right\}$.

We consider the part of the graph of the function $v_1(\sigma, t_0)$ that lies in this rectangle.

Definition 1. *A rectangle Π is called critical.*

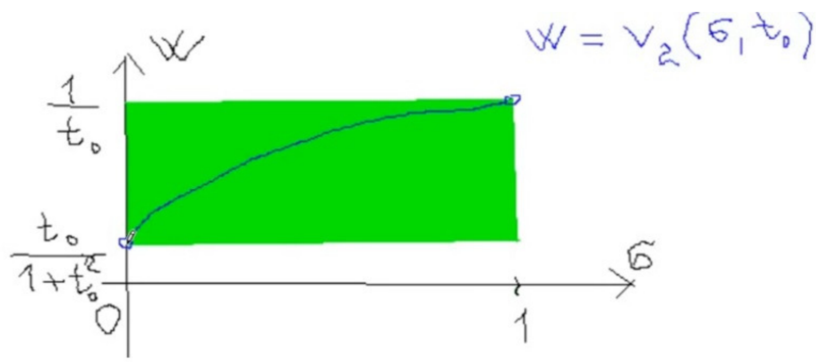

Figure 2: A critical rectangle

Remark 1. *Critical rectangles are very thin, their width equals* $\frac{1}{t_0} - \frac{t_0}{1+t_0^2} = \frac{1}{(1+t_0^2)t_0}$. *Take the nontrivial zero with the least positive imaginary part* $t_0 = 14.134725141...$ *and get the width 0.0003523461812...*

Definition 2. σ *is critical if* $(\sigma, v_1(\sigma, t_0)) \in \Pi$.

Thus the value σ_0 is critical. The graphs of $v_1(\sigma, t_0)$ and $v_2(\sigma, t_0)$ intersect in the point $(\sigma_0, v_1(\sigma_0, t_0)) \in \Pi$.

This implies the inequality

$$v_1(\sigma_0, t_0) = \int_1^{+\infty} \frac{\{x\}}{x^{\sigma_0+1}} \sin(t_0 \ln x) dx = \frac{t_0}{\sigma_0^2 + t_0^2} > 0.$$

Moreover, by definition, we get $v_1(\sigma, t_0) \in \left(\frac{t_0}{1+t_0^2}, \frac{1}{t_0} \right)$ for all critical σ; this implies that $v_1(\sigma, t_0) > 0$.

Let us introduce the function

$$\Psi(\sigma, x) = \frac{\{x\}}{x^{\sigma+1}} \sin(t_0 \ln x).$$

Then we have the equality

$$v_1(\sigma, t_0) = \int_1^{\infty} \Psi(\sigma, x) dx.$$

Lemma 2. *The function* $v_1(\sigma, t_0)$ *does not increase on the set of all critical* σ.

Proof. Let σ' be a positive number such that $\sigma + \sigma'$ is critical.

We must prove that $v_1(\sigma, t_0) \geq v_1(\sigma + \sigma', t_0)$.

It is obvious that

$$\Psi(\sigma + \sigma', x) = \frac{1}{x^{\sigma'}} \Psi(\sigma, x).$$

Then we get

$$v_1(\sigma+\sigma', t_0) = \int_1^{\infty} \frac{1}{x^{\sigma'}} \Psi(\sigma, x) dx.$$

Since σ and $\sigma+\sigma'$ are critical, we obtain $v_1(\sigma, t_0) > 0$ and $v_1(\sigma+\sigma', t_0) > 0$. This implies that there exists a X_0 such that for all $X > X_0$ we get the inequalities

$$\int_1^X \Psi(\sigma, x) dx > 0 \text{ and } \int_1^X \frac{1}{x^{\sigma'}} \Psi(\sigma, x) dx > 0.$$

We must prove the inequality

$$\int_1^X \frac{1}{x^{\sigma'}} \Psi(\sigma, x) dx \leq \int_1^X \Psi(\sigma, x) dx. \tag{5}$$

The proof consists of two parts.

Part 1

Let $\Re[a, b]$ be the set of Riemann-integrable functions on an interval $[a, b]$.

We use the following[2]

Theorem (the second mean-value theorem for the integral[1]). *If $f, g \in \Re[a, b]$ and g is a monotonic function on [a,b], then there exists a point $\xi \in [a, b]$ such that*

$$\int_a^b f(x)g(x)dx = g(a)\int_a^{\xi} f(x)dx + g(b)\int_{\xi}^b f(x)dx.$$

If $g(x) = \dfrac{1}{x^{\sigma'}}$ and $f(x) = \Psi(\sigma, x)$, then there exists a point $\xi = \xi(X) \in [1, X]$ such that

$$\int_1^X \frac{1}{x^{\sigma'}} \Psi(\sigma, x) dx = A + \gamma B,$$

$$\text{where } \gamma = \frac{1}{X^{\sigma'}}, A = A(\xi) = \int_1^{\xi} \Psi(\sigma, x) dx, \text{ and } B = B(\xi) = \int_{\xi}^X \Psi(\sigma, x) dx.$$

We have $0 < \gamma < 1, A + B > 0, A + \gamma B > 0$.

Let us prove inequality 5; this implies Lemma 2.

If $\xi = 1,$ then $A = 0$. It follows from $A + \gamma B > 0$ that $\gamma B > 0$. As $B > 0$, we have $\gamma B < B$, and inequality 5 is true.

[1]It states the equality which is often colled Bonnet's formula

If $\xi = X$, then $B = 0$, we get $A + \gamma B = A$, and inequality 5 is true as well.

Assume that $1 < \xi < X$.

If $A \leq 0$, then $B > 0$, otherwise it would be $A + B \leq 0$. Inequality 5 is true as well.

If $A > 0$ and $B \geq 0$, then inequality 5 is true.

Case remainded is $A > 0, B < 0$. In the sequel it turnes out impossimle.

Part 2

Let us introduce the function $\Phi_1(x) = \int\limits_x^{\xi} \Psi(\sigma, x)dx + \gamma B$, defned on $[1, \xi]$.

As $\Phi_1(1) > 0, \Phi_1(\xi) = B < 0$, there exists $\xi' \in (1, \xi)$ such that $\Phi_1(\xi') = 0$.

Then

$$\int\limits_1^X \frac{1}{x^{\sigma'}} \Psi(\sigma, x)dx = \underbrace{\int\limits_1^{\xi'} \Psi(\sigma, x)dx + \int\limits_{\xi'}^{\xi} \Psi(\sigma, x)dx}_{A} + \gamma B = \int\limits_1^{\xi'} \Psi(\sigma, x)dx + \underbrace{\int\limits_{\xi'}^{\xi} \Psi(\sigma, x)dx + \gamma B}_{\Phi_1(\xi')=0}.$$

We get

$$\int\limits_1^X \frac{1}{x^{\sigma'}} \Psi(\sigma, x)dx = \int\limits_1^{\xi'} \Psi(\sigma, x)dx), \tag{6}$$

herewith

$$\int\limits_{\xi'}^{\xi} \Psi(\sigma, x)dx + \gamma B = 0. \tag{7}$$

Now let us introduce the function $\Phi_2(x) = \int\limits_x^X \Psi(\sigma, x)dx$.

As $B < 0$, we have $B < \gamma B$. It follows from this that

$$\Phi_2(\xi') = \int\limits_{\xi'}^{\xi} \Psi(\sigma, x)dx + B < \int\limits_{\xi'}^{\xi} \Psi(\sigma, x)dx + \gamma B = 0.$$

Simultaneosly, $\Phi_2(\xi') < 0$ and $\Phi_2(1) > 0$, it follows from this that there exists a point $\xi'' \in (1, \xi')$ such that $\Phi_2(\xi'') = 0$.

So we get

$$\int\limits_{\xi''}^X \Psi(\sigma, x)dx = 0. \tag{8}$$

Denote by $I(a, b)$ the integral $\int\limits_a^b \Psi(\sigma, x)dx$.

Then $0 = I(\xi'', X) = I(\xi'', \xi') + I(\xi', \xi) + I(\xi, X) =$

$= I(\xi'', \xi') + \underbrace{I(\xi', \xi) + \gamma I(\xi, X)}_{=0} + (1-\gamma) I(\xi, X),$

we get $I(\xi'', \xi') + (1-\gamma) I(\xi, X) = 0,$

it follows from this that $I(\xi'', X) = \underbrace{I(\xi'', \xi') + (1-\gamma) I(\xi, X)}_{=0} + I(\xi', \xi) + \gamma I(\xi, X),$

thus $\int\limits_{\xi''}^{X} \Psi(\sigma, x) dx = \int\limits_{\xi'}^{\xi} \Psi(\sigma, x) dx + \gamma \int\limits_{\xi}^{X} \Psi(\sigma, x) dx = 0.$

Consequently, $\int\limits_{1}^{\xi''} \Psi(\sigma, x) dx = \int\limits_{1}^{X} \frac{1}{x^{\sigma'}} \Psi(\sigma, x) dx.$

Taking into account equality 6, we get $\int\limits_{1}^{\xi'} \Psi(\sigma, x) dx = \int\limits_{1}^{\xi''} \Psi(\sigma, x) dx$, thus $\int\limits_{\xi''}^{\xi'} \Psi(\sigma, x) dx) = 0.$

With equality 7 we get $\int\limits_{\xi'}^{\xi} \Psi(\sigma, x) dx + B = \int\limits_{\xi'}^{\xi} \Psi(\sigma, x) dx + \gamma B$, but then $B = \gamma B$, it follows from this that $B = 0$.

As $B < 0$, we got a contradiction, this implies that the case $A > 0, B < 0$ is impossible.

Thus, for arbitrary $X > X_0$ inequality 5 is true, consequently

$$\int\limits_{1}^{X} \frac{1}{x^{\sigma'}} \Psi(\sigma, x) dx \leq \int\limits_{1}^{X} \Psi(\sigma, x) dx.$$

We get the inequality

$$\int\limits_{1}^{\infty} \frac{1}{x^{\sigma'}} \Psi(\sigma, x) dx \leq \int\limits_{1}^{\infty} \Psi(\sigma, x) dx. \tag{9}$$

□

The proof of the Riemann hypothesis

Theorem. *Let $s_0 = \sigma_0 + it_0$ be a nontrivial zero of the Riemann zeta function; then $\sigma_0 = \frac{1}{2}$.*

Proof. A nontrivial zero of the zeta function is a solution to equation 2, hence the pair (σ_0, t_0) satisfies system 4, and, in particular, its second equality.

From Lemma 2 it follows that this pair is unique. Suppose $\sigma_0 \neq \frac{1}{2}$.

It is known that nontrivial zeros are symmetric about the line $\operatorname{Re} s = \frac{1}{2}$, hence there exists another zero $1-\sigma_0+it_0$ at the same "height" $t=t_0$, therefore the pair $(1-\sigma_0, t_0)$ satisfies the second equality as well.

This contradiction establishes the theorem. $\square$

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