An infinitesimal proof of the Riemann hypothesis on nontrivial zeros of the zeta function

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Abstract

The Riemann hypothesis on nontrivial zeros of the Riemann zeta function is proved.

A complex number $s_0 = \sigma_0 + it_0$ is a nontrivial zero iff (σ_0, t_0) is a solution to a system of two equations of two real variables σ and t.

Considering one of that two equations, we found that one side of it is increasing and the other one is nonincreasing as functions on the set of so called *critical values* $\sigma \in (0; 1)$ at the "height" $t = t_0$, so (σ_0, t_0) is a unique solution at $t = t_0$. As nontrivial zeros are symmetric about the line Re $s = \dfrac{1}{2}$, it follows that $\sigma_0 = \dfrac{1}{2}$.

Keywords: the Riemann hypothesis, zeta function, nontrivial zeros.

Setting the problem

Let $s = \sigma + it$ be a complex variable, where $\sigma = \operatorname{Re} s, t = \operatorname{Im} s$, and
$x \in \mathbb{R}$ be a real variable.

For $\operatorname{Re} s > 0, s \neq 1$, it is known [1] that the Riemann zeta function $\zeta(s)$ can be expressed by the formula

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}dx. \tag{1}$$

Here, $\{x\}$ denotes the fractional part of a number x.

Let us rewrite equality 1 in the form

$$\zeta(s) = s\left(\frac{1}{s-1} - \int_1^\infty \frac{\{x\}}{x^{s+1}}dx\right).$$

Thus, to obtain nontrivial zeros of the function $\zeta(s)$, we must solve the following equation:

$$\int_1^\infty \frac{\{x\}}{x^{s+1}} = \frac{1}{s-1}. \tag{2}$$

This implies two equations:

$$\frac{1}{x^{s+1}} = \frac{1}{x^{\sigma+1}}\left(\cos(t\ln x) - i\sin(t\ln x)\right),$$
$$\frac{1}{s-1} = \frac{\sigma-1}{(\sigma-1)^2+t^2} - i\frac{t}{(\sigma-1)^2+t^2}.$$

Therefore, equation 2 is equivalent to the following system:

$$\begin{cases} \int_1^\infty \frac{\{x\}}{x^{\sigma+1}}\cos(t\ln x)dx = \frac{\sigma-1}{(\sigma-1)^2+t^2}, \\ \int_1^\infty \frac{\{x\}}{x^{\sigma+1}}\sin(t\ln x)dx = \frac{t}{(\sigma-1)^2+t^2}. \end{cases} \tag{3}$$

It is known that nontrivial zeros are symmetric about the real axis, therefore we consider only the case $t > 0$.

We always assume that $0 < \sigma < 1,\ t > 0$.

Let $s_0 = \sigma_0 + it_0$ be a nontrivial zero.

The Riemann hypothesis states that $\sigma_0 = \frac{1}{2}$.

Left and right sides of the equations of system 3

Let us introduce four useful functions as follows:

$$u_1(\sigma,t) = \int_1^\infty \frac{\{x\}}{x^{\sigma+1}}\cos(t\ln x)dx,$$
$$v_1(\sigma,t) = \int_1^\infty \frac{\{x\}}{x^{\sigma+1}}\sin(t\ln x)dx,$$
$$u_2(\sigma,t) = \frac{\sigma-1}{(\sigma-1)^2+t^2},$$
$$v_2(\sigma,t) = \frac{t}{(\sigma-1)^2+t^2}.$$

Equation 2 can be expressed as follows:

$$u_1(\sigma,t) - iv_1(\sigma,t) = u_2(\sigma,t) - iv_2(\sigma,t).$$

We represent system 3 in the form

$$\begin{cases} u_1(\sigma, t) = u_2(\sigma, t), \\ v_1(\sigma, t) = v_2(\sigma, t). \end{cases} \tag{4}$$

$s = \sigma + it$ is a nontrivial zero if and only if (σ, t) is a solution to system 4.

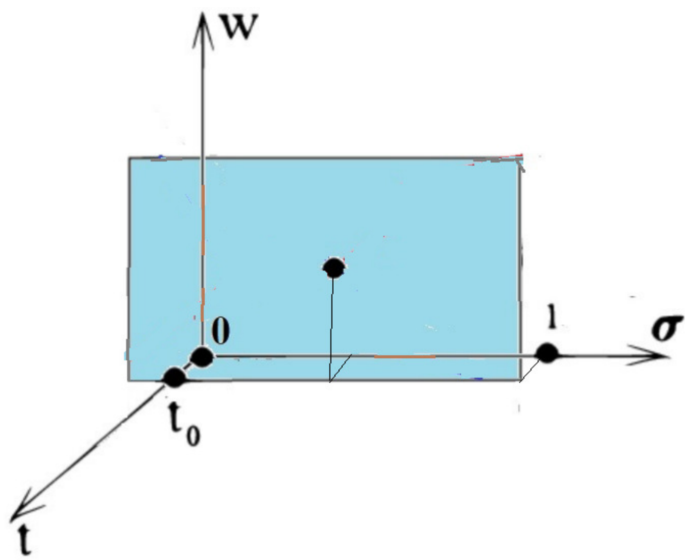

Figure 1: The plane $t = t_0$

Let $s_0 = \sigma_0 + it_0$ be a nontrivial zero.

Lemma 1. *The function $w = v_2(\sigma, t_0)$ increases as a function of one variable $\sigma \in (0; 1)$.*

Proof. It follows from the inequality

$$\frac{\mathrm{d}v_2}{\mathrm{d}\sigma} = -\frac{2(\sigma - 1)t_0}{(t_0^2 + (\sigma - 1)^2)^2} > 0.$$

□

The range of the function $w = v_2(\sigma, t_0)$ is $U = \left(\frac{t_0}{1 + t_0^2}, \frac{1}{t_0} \right)$.

Obviously, the graph of the function $w = v_2(\sigma, t_0)$ lies in the rectangle $\Pi = \left\{ (\sigma, w) \;\middle|\; \sigma \in (0; 1), w \in U \right\}$.

We consider the part of the graph of the function $v_1(\sigma, t_0)$ that lies in this rectangle.

Definition 1. *A rectangle Π is called critical.*

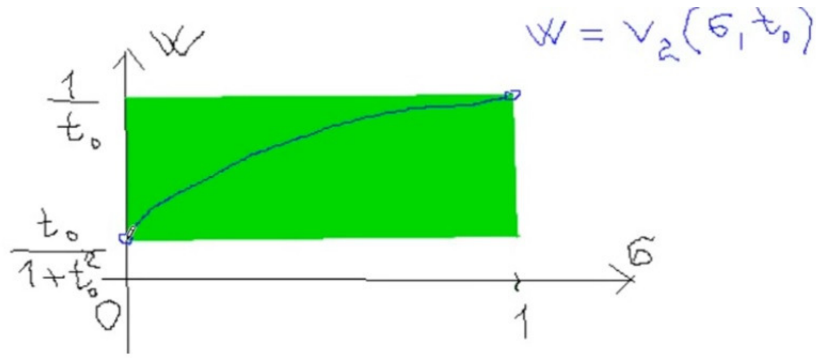

Figure 2: A critical rectangle

Remark. *Critical rectangles are very thin, their width equals* $\frac{1}{t_0} - \frac{t_0}{1+t_0^2} = \frac{1}{(1+t_0^2)t_0}$. *Take the nontrivial zero with the least positive imaginary part* $t_0 = 14.134725141...$ *and get the width 0.0003523461812...*

Definition 2. σ *is critical if* $(\sigma, v_1(\sigma, t_0)) \in \Pi$.

Thus the value σ_0 is critical. The graphs of $v_1(\sigma, t_0)$ and $v_2(\sigma, t_0)$ intersect in the point $(\sigma_0, v_1(\sigma_0, t_0)) \in \Pi$.

This implies the inequality

$$v_1(\sigma_0, t_0) = \int_1^{+\infty} \frac{\{x\}}{x^{\sigma_0+1}} \sin(t_0 \ln x) dx = \frac{t_0}{\sigma_0^2 + t_0^2} > 0.$$

Moreover, by definition, we get $v_1(\sigma, t_0) \in \left(\frac{t_0}{1+t_0^2}, \frac{1}{t_0}\right)$ for all critical σ; this implies that $v_1(\sigma, t_0) > 0$.

Let us introduce the function

$$\Psi(\sigma, x) = \frac{\{x\}}{x^{\sigma+1}} \sin(t_0 \ln x).$$

Then we have the equality

$$v_1(\sigma, t_0) = \int_1^{\infty} \Psi(\sigma, x) dx.$$

Lemma 2. *The function* $v_1(\sigma, t_0)$ *does not increase on the set of all critical* σ.

Proof. Let σ' be a positive number such that $\sigma + \sigma'$ is critical.

We must prove that $v_1(\sigma, t_0) \geq v_1(\sigma + \sigma', t_0)$.

It is obvious that

$$\Psi(\sigma + \sigma', x) = \frac{1}{x^{\sigma'}} \Psi(\sigma, x).$$

Then we get

$$v_1(\sigma+\sigma',t_0)=\int\limits_1^{\infty}\frac{1}{x^{\sigma'}}\Psi(\sigma,x)dx.$$

Since σ and $\sigma+\sigma'$ are critical, we obtain $v_1(\sigma,t_0)>0$ and $v_1(\sigma+\sigma',t_0)>0$. This implies that there exists a X_0 such that for all $X>X_0$ we get the inequalities

$$\int\limits_1^{X}\Psi(\sigma,x)dx>0 \text{ and } \int\limits_1^{X}\frac{1}{x^{\sigma'}}\Psi(\sigma,x)dx>0.$$

We must prove the inequality

$$\int\limits_1^{X}\frac{1}{x^{\sigma'}}\Psi(\sigma,x)dx\le\int\limits_1^{X}\Psi(\sigma,x)dx. \tag{5}$$

Let $\Re[a,b]$ be the set of Riemann-integrable functions on an interval $[a,b]$.

We use the following[2]

Theorem (the second mean-value theorem for the integral[1]). *If $f,g\in\Re[a,b]$ and g is a monotonic function on [a,b], then there exists a point $\xi\in[a,b]$ such that*

$$\int\limits_a^{b}f(x)g(x)dx=g(a)\int\limits_a^{\xi}f(x)dx+g(b)\int\limits_{\xi}^{b}f(x)dx.$$

If $g(x)=\dfrac{1}{x^{\sigma'}}$ and $f(x)=\Psi(\sigma,x)$, then there exists a point $\xi=\xi(X)\in[1,X]$ such that

$$\int\limits_1^{X}\frac{1}{x^{\sigma'}}\Psi(\sigma,x)dx=A+\gamma B,$$

$$\text{where } \gamma=\frac{1}{X^{\sigma'}}, A=A(\xi)=\int\limits_1^{\xi}\Psi(\sigma,x)dx,\ \text{and } B=B(\xi)=\int\limits_{\xi}^{X}\Psi(\sigma,x)dx.$$

We have $0<\gamma<1, A+B>0, A+\gamma B>0$.

Let us prove inequality 5; this implies Lemma 2.

According to the nonstandard transfer principle, the second mean-value theorem for the integral takes place on nonstandard numbers as well.

Let now σ' be a positive infinitely small number, and let a nonstadard ξ correspond to it.

If $\xi=1,$ then $A=0$. It follows from $A+\gamma B>0$ that $\gamma B>0$. As $B>0$, we have $\gamma B<B$, and inequality 5 is true.

[1]It states the equality which is often colled Bonnet's formula

If $\xi = X$, then $B = 0$, we get $A + \gamma B = A$, and inequality 5 is true as well.

Assume that $1 < \xi < X$.

Since the function $g(x) = \frac{1}{x^{\sigma'}}$ decreases on $[1; +\infty)$, there exists a unique $\eta \in (0; 1]$ such that $\frac{1}{\eta^{\sigma'}} = \xi$.

So we get the equation

$$\int_1^X \frac{1}{x^{\sigma'}} \Psi(\sigma, x) dx = \int_1^{\frac{1}{\eta^{\sigma'}}} \Psi(\sigma, x) dx + \frac{1}{X^{\sigma'}} \int_{\frac{1}{\eta^{\sigma'}}}^X \Psi(\sigma, x) dx. \tag{6}$$

As $\eta > 0$, we have $\eta^{\sigma'} \approx 1$, and $\int_1^{\frac{1}{\eta^{\sigma'}}} \Psi(\sigma, x) dx \approx 0$.

From the inequality $\int_1^X \frac{1}{x^{\sigma'}} \Psi(\sigma, x) dx > 0$ and equality 6, it follows that $\frac{1}{X^{\sigma'}} \int_{\frac{1}{\eta^{\sigma'}}}^X \Psi(\sigma, x) dx > 0$.

Thus we get $\int_{\frac{1}{\eta^{\sigma'}}}^X \Psi(\sigma, x) dx > 0$.

From equality 6 we have as well, that

$$\int_1^X \frac{1}{x^{\sigma'}} \Psi(\sigma, x) dx = \int_1^{\frac{1}{\eta^{\sigma'}}} \Psi(\sigma, x) dx + \frac{1}{X^{\sigma'}} \int_{\frac{1}{\eta^{\sigma'}}}^X \Psi(\sigma, x) dx < \int_1^{\frac{1}{\eta^{\sigma'}}} \Psi(\sigma, x) dx + \int_{\frac{1}{\eta^{\sigma'}}}^X \Psi(\sigma, x) dx = \int_1^X \Psi(\sigma, x) dx.$$

So if $1 < \xi < X$, then

$$\int_1^X \frac{1}{x^{\sigma'}} \Psi(\sigma, x) dx < \int_1^X \Psi(\sigma, x) dx,$$

that is equality 5 is true as well.

Thus we have proved inequality 5 for any $X > X_0$, then it follows that

$$\int_1^\infty \frac{1}{x^{\sigma'}} \Psi(\sigma, x) dx \leqslant \int_1^\infty \Psi(\sigma, x) dx,$$

and this concludes the proof of Lemma 2. □

The proof of the Riemann hypothesis

Theorem. *Let $s_0 = \sigma_0 + it_0$ be a nontrivial zero of the Riemann zeta function, then $\sigma_0 = \frac{1}{2}$.*

Proof. A nontrivial zero of the zeta function is a solution to equation 2, hence the pair (σ_0, t_0) satisfies system 4, and, in particular, its second equality.

From Lemma 2 it follows that this pair is unique. Suppose $\sigma_0 \neq \frac{1}{2}$.

It is known that nontrivial zeros are symmetric about the line $\operatorname{Re} s = \frac{1}{2}$, hence there exists another zero $1 - \sigma_0 + it_0$ at the same "height" $t = t_0$, therefore the pair $(1 - \sigma_0, t_0)$ satisfies the second equality as well.

This contradiction establishes the theorem. □

References

[1] Galochkin A.I., Nesterenko Yu.V., Shidlovskiĭ A.B. : An Introduction to the Theory of Numbers (In Russian). 2nd ed, Lomonosov Moscow State University, 1984.
Галочкин А.И., Нестеренко Ю.В., Шидловский А.Б. Введение в теорию чисел, Изд-во Московского университета,1984.

[2] Zorich V. A.: Mathematical Analysis. Springer-Verlag Berlin Heidelberg, 2004.
Зорич В.А. Математический анализ. Часть I. Изд. 2-е, испр. и доп. М.: ФАЗИС, 1997.

[3] Успенский В.А. Что такое нестандартный анализ? - М.: Наука. Гл. ред. физ.-мат. лит., 1987.

[4] Robinson, Abraham. Non-standard analysis. Princeton University Press, 1996.