Some questions of the theory of computational complexity from the point of view of elementary theory of models

*The presented study considers one of the most famous problems of computational complexity theory: what is the ratio of complexity classes $\mathcal{NP}$ and $\mathbf{co}-\mathcal{NP}$? To answer this question, the well-known fundamental concept of model completeness of the theory under study, a section of mathematics "**Model Theory**", was rethought and reformulated accordingly. The purpose of reformulating this fundamental concept was to describe the ratio of complexity classes $\mathcal{NP}$ and $\mathbf{co}-\mathcal{NP}$, from a model-theoretical point of view. It is a well-known fact: the hierarchy of properties in any model of a model-complete theory breaks off at the first level. This key idea has been the basis for a fruitful study of the relationship between the complexity classes $\mathcal{NP}$ and $\mathbf{co}-\mathcal{NP}$. It is a well-known fact that there exists an oracle $\mathbb{A}$ such that the complexity class $\mathcal{NP}(\mathbb{A})$ differs from the complexity class $\mathbf{co}-\mathcal{NP}(\mathbb{A})$. By developing oracle computations in an appropriate manner and formalizing them in the class of primitive recursive algorithms, and then using the theoretical-model relationship between the specified classes, it was possible to relate the relationship between the complexity classes of computations $\mathcal{NP}$ and $\mathbf{co}-\mathcal{NP}$ with the relationship between the complexity classes $\mathcal{NP}(\mathbb{A})$ and $\mathbf{co}-\mathcal{NP}(\mathbb{A})$, which then made it possible to establish that the complexity class $\mathcal{NP}$ is not a Boolean algebra. In formalizing oracle computations in the class of primitive recursive algorithms, a number of interesting theorems were proved, one of which is an analogue of the fixed point theorem, which was used in the key theorem that allowed establishing that the complexity class $\mathcal{NP}$ is not a Boolean algebra. After reading the presented research, one can understand why the relativization effect prevents one from obtaining high lower bounds or separating one complexity class from another complexity class of computations using the methods of "Discrete Mathematics"*[1]

The presented study is original, and many important concepts that are used in this study have not been encountered in any studies known to me.

Part I

Equality calculus for calculating closed terms

Let the alphabets $\mathscr{L}_1, \ldots, \mathscr{L}_6$ be such that: $\mathscr{L}_1 = \{S, I, \mathbf{Z}, \boldsymbol{\delta}, \mathbf{Length}, \dot{-}, \mathbf{Concat}, \mathbf{D}, \}$, $\mathscr{L}_2 = \{\Lambda, x\}$, $\mathscr{L}_3 = \{R, J\}$, $\mathscr{L}_4 = \{=, |, ,\}$, $\mathscr{L}_5 = \{[,], (,)\}$, $\mathscr{L}_6 = \{\mathbf{U}\}$, $\mathscr{L} = \bigcup\limits_{i=1}^{i=5} \mathscr{L}_i$, $\mathscr{L}(\mathbf{U}) = \mathscr{L} \cup \mathscr{L}_6$. The alphabet $\mathscr{L}(\mathbf{U})$ will formalize the calculation of closed terms for the class of oracle primitive recursive word functions [1, p. 204],[2].

For the sake of completeness, we present a few fairly traditional definitions.

Definition of a functor and its localities:

1) Words of the form: $\mathbf{Z}, \boldsymbol{\delta}, \mathbf{Length}$, $S|S, S||S, \ldots, S\underbrace{||, \ldots, |}_{m \text{ times}}S$ - one-place functors. functor having the form $S\underbrace{|, \ldots, |}_{k \text{ times}}S$ will be denoted as $\mathbf{S}_k$.

2) Words of the form: $I\underbrace{|, \ldots, |}_{m \text{ times}},\underbrace{|, \ldots, |}_{n \text{ times}}$ - n - place functor, which will be denoted traditionally $\mathbf{I^n_m}$, at $1 \leq m \leq n$.

3) Word $\mathbf{U}$ - one-place functor. In what follows, the functor $\mathbf{U}$ will also be called undefined function symbol or oracular symbol.

4) Words of the form: $\dot{-}, \mathbf{Concat}, \mathbf{D}$ - two-place functors.

[1] By methods of Discrete Mathematics, I mean those proofs that can be expressed in the standard model of arithmetic, for example, the proof of Consis cannot be expressed in the standard model of arithmetic, although this sentence is true in this model, Not all statements that are subject to relativization can be proved using the methods of Discrete Mathematics. And this is demonstrated in this research.

5) If Φ - k - place functor, $\Psi_1, \ldots, \Psi_k$ - n - place functors, then the word $[J\Phi\Psi_1, \ldots, \Psi_k]$ - n - place functor. This functor will be called the superposition functor.

Let us introduce the following important concept:

a) Λ - the argument word, which is called the empty word;

б) if α - is an argument word, then the word $\mathbf{S}_k(\alpha)$ is an argument word, which will be denoted as αa_k.

An argument word α is called k - alphabetic if this word does not contain functors $\mathbf{S}_l$, for $l > k$.

A set B of argument words is called k - alphabetic if each word $\alpha \in B$ is k - alphabetic. Let $k > 1$. The number of all k - alphabetic words with length l is equal to k^l. The number of all k - alphabetic words with length at most $l > 0$ is equal to $\dfrac{k^{l+1} - 1}{k - 1}$.

Argument words that do not contain the functor $\mathbf{S}_k$ for $k > 1$ are called natural numbers.

6) If α is an argument word, $\Phi_1, \ldots, \Phi_m$ are 2-place functors, then the word $[R\alpha\Phi_1, \cdots, \Phi_m]$ - 1 - place functor.

7) If Φ - k is a place functor, $\Psi_1, \ldots, \Psi_m$ - $k+2$ are place functors, then the word $[R\Phi\Psi_1, \ldots, \Psi_m]$ - $k+1$ is a place functor. The functors of items 6 and 7 will be called recursive functors, and the functors $\Phi, \Phi_1, \ldots \Psi_m$ will be called component functors, the number m is called the branching degree of the recursive functor under consideration.

The functors of items 1-4 will be called the original functors.

Words of the form: $x \underbrace{|, \ldots, |}_{l-\text{ times}} x$ are variables. Let us denote these words traditionally as x_l.

The concept of the term.

1) Every argument word and every variable is a term.

2) If $t_1, \ldots, t_k$ are terms, Ψ - k is a place functor, then the word $\Psi(t_1, \ldots, t_k)$ is a term.

For any subset $\mathbb{A}$ of argument words, we introduce the following equalities (defining equalities), as axioms of the formalization to be defined for calculating oracle closed terms:

1) $\mathbf{T} = \mathbf{T}$, where $\mathbf{T}$ is an arbitrary term,

2) $\mathbf{Z}(x_1) = \Lambda$,

3) $\mathbf{I}^n_k(x_1, \ldots, x_n) = x_k$,

4) $\boldsymbol{\delta}(\Lambda) = \Lambda$,

5) $\boldsymbol{\delta}(x_1 a_k) = x_1$,

6) $\mathbf{Length}(\Lambda) = \Lambda$,

7) $\mathbf{Length}(x_1 a_k) = \mathbf{Length}(x_1) a_1$,

8) $x_1 \dot{-} \Lambda = x_1$,

9) $x_1 \dot{-} x_2 a_k = \boldsymbol{\delta}(x_1 \dot{-} x_2)$,

10) $\mathbf{Concat}(x_1, \Lambda) = x_1$,

11) $\mathbf{Concat}(x_1, x_2 a_k) = \mathbf{Concat}(x_1, x_2) a_k$,

12) $\mathbf{D}(x_1, \Lambda) = \Lambda$,

13) $\mathbf{D}(x_1, x_2 a_k) = \mathbf{Concat}(x_1, \mathbf{D}(x_1, x_2))$,

14) $[J\Phi\Psi_1, \ldots, \Psi_k](\overline{x}) = \Phi(\Psi_1(\overline{x}), \ldots, \Psi_k(\overline{x}))$,

15) $[R\alpha\Phi_1, \ldots, \Phi_m](\Lambda) = \alpha$,

16)$[R\alpha\Phi_1, \ldots, \Phi_m](xa_k) = \Phi_k(x, [R\alpha\Phi_1, \ldots, \Phi_m](x))$, at $k \leqslant m$,

17)$[R\alpha\Phi_1, \ldots, \Phi_m](xa_k) = [R\alpha\Phi_1, \ldots, \Phi_m](x)$, at $k > m$,

18) $[R\Phi\Psi_1, \ldots, \Psi_m](\overline{x}, \Lambda) = \Phi(\overline{x})$,

19) $[R\Phi\Psi_1, \ldots, \Psi_m](\overline{x}, ya_k) = \Psi_k(\overline{x}, y, [R\Phi\Psi_1, \ldots, \Psi_m](\overline{x}, y))$, at $k \leqslant m$,

20) $[R\Phi\Psi_1, \ldots, \Psi_m](\overline{x}, ya_k) = [R\Phi\Psi_1, \ldots, \Psi_m](\overline{x}, y))$, at $k > m$.

Let $\mathbb{A}$ - be some set of argument words. Axioms of interpretation of the undefined function symbol $\mathbf{U}$

a) $\mathbf{U}(\alpha) = \Lambda$, if $\alpha \in \mathbb{A}$,

б) $\mathbf{U}(\alpha) = a_1$, if $\alpha \notin \mathbb{A}$.

Equalities a) and b) are called axioms of interpretation, which correspond to the set $\mathbb{A}$, the set of argument words $\mathbb{A}$ is called the interpretation set.

Rules of inference Calculus of equalities of closed terms

$$\mathbf{Sb}: \frac{\mathbf{T_1} = \mathbf{Q_1}, \mathbf{T_2} = \mathbf{Q_2}}{[\mathbf{T_2}]^x_{\mathbf{T_1}} = [\mathbf{Q_2}]^x_{\mathbf{Q_1}}}, \ \mathbf{Cut_1}: \frac{\mathbf{T_1} = \mathbf{T_2}, \mathbf{T_2} = \mathbf{T_3}}{\mathbf{T_1} = \mathbf{T_3}}, \ \mathbf{Cut_2}: \frac{\mathbf{T_1} = \mathbf{T_2}, \mathbf{T_3} = \mathbf{T_2}}{\mathbf{T_1} = \mathbf{T_3}}.$$

In the **Sb** rule, the variable x is the rule's own variable.

Remark. To calculate the values of closed terms, it is sufficient to use only the $\mathbf{Cut_1}$ rule.

To prove the equalities of closed terms, the $\mathbf{Cut_2}$ rule is added. You can do without the $\mathbf{Cut}_2$ rule by replacing the **Sb** rule with the rule $\dfrac{\mathbf{T_1} = \mathbf{Q_1}, \mathbf{T_2} = \mathbf{Q_2}}{[\mathbf{T_2}]^x_{\mathbf{Q_1}} = [\mathbf{Q_2}]^x_{\mathbf{T_1}}}$.

Definition of proof. Let $\mathbb{A}$ - some set of argument words. Sequence of equalities $\mathbf{T}_1 = \mathbf{Q}_1, \ldots, \mathbf{T}_n = \mathbf{Q}_n$ is a derivation (proof of $\mathbf{T}_n = \mathbf{Q}_n$), if for each $i = 1, 2, \ldots, n$, $\mathbf{T}_i = \mathbf{Q}_i$ is either an axiom or an axiom of the interpretation of $\mathbf{U}(\alpha) = \Lambda, \mathbf{U}(\alpha) = a_1$, which correspond to the set $\mathbb{A}$, or obtained from the previous equalities according to one of the inference rules.

If the proof $\mathbf{P}$ is such that it contains the interpretation axioms $\mathbf{U}(\alpha) = \Lambda$ or $\mathbf{U}(\beta) = a_1$, then we will say that the words α, β were used in this output, and the word α was used positively, the word β is used negatively. With the proof of $\mathbf{P}$, given the interpretation set $\mathbb{A}$, we associate a pair of sets: $(\mathbb{A}^+)_\mathbf{P}$ - the set of all positively of interviewed words in the output $\mathbf{P}$, $(\mathbb{A}^-)_\mathbf{P}$ is the set of all negatively interviewed words in the output $\mathbf{P}$. If the output $\mathbf{P}$ does not contain interpretive axioms, then it will be output in the alphabet $\mathscr{L}$. The definition of the derived equality is assumed to be traditional.

The sequence of equalities $t_1 = q_1, \ldots, t_n = q_n$ is a quasi-inference, if each equality in this sequence is either a derivable equality or is obtained from the previous equalities according to one of the inference rules.

Note. The idea of the above calculus of equalities for computing closed oracle terms was borrowed in [3-5].

The length of the proof $\mathbf{P}$ is the number of equalities in the proof $\mathbf{P}$. This number is denoted as $l_\mathbf{P}$.

The total length of the proof $\mathbf{P}$ is the length of the word obtained by joining all equalities in the proof $\mathbf{P}$, separated by a comma. This number is denoted by $Fl(\mathbf{P})$.

Denote the resulting calculus of closed terms as **CalcEq**, $\mathbf{CalcEq_U}$ in the alphabet $\mathscr{L}$, $\mathscr{L}(\mathbf{U})$ respectively.

The notation $\vdash t = r$ - the equality $t = r$ is derivable in the calculus $\mathbf{CalcEq_U}$, for any interpretation of the functor $\mathbf{U}$.

The notation $\mathbb{A} \vdash t = r$ - the equality $t = r$ is derivable in the calculus $\mathbf{CalcEq_U}$, with interpretation axioms corresponding to the set $\mathbb{A}$.

A functor Φ, a term t are said to be $n > 0$ alphabetic if the presented words do not contain functors S_l, for $l > n$. All original functors are n alphabetic for every n.

For each n and any argument word α one can construct n - place functor, denoted as $\mathbf{Const}^n_\alpha$, which yields the equality $\mathbf{Const}^n_\alpha(\beta_1, \ldots, \beta_n) = \alpha$, for any sequence $\overline{\beta}$ of argument words.

Theorem 1.1. One can compose an algorithm such that for each term t one can construct a functor Φ_t such that $\vdash \Phi_t(\overline{y}) = t$, where $\overline{y}$ - list of variables containing the variables of term t. See [3, p. 62] for the proof and full formulation of this theorem.

Theorem 1.2. For any closed n alphabetical term t, for a given interpretation $\mathbb{A}$ of the functor $\mathbf{U}$, there exists a unique such argument word α of the same alphabet that $\mathbb{A} \vdash t = \alpha$. The proof is carried out by induction on the construction of the functor, then by induction on the construction of the term t.

The proof of the $\mathbf{P}$ equality of the form $t = \alpha$, where t is a closed term and α is some argument word, will be called the calculation of the term t.

Note. For any term $t(\overline{x})$, for any sequence of argument words $\overline{\alpha}$, there exists a positive integer k such that for any interpretation set of argument words $\mathbb{A}$, it is possible to construct such a calculation of the term $t(\overline{x})$

on $\overline{\alpha}$, in which no more than k argument words will be used.

Theorem 1.3. Let $[J\Phi\Psi_1,\ldots,\Psi_k]$ - n - be a place functor, $\alpha_1,\ldots,\alpha_n$ - some sequence of argument words. Let $\mathbf{P}_{\Psi_i,\overline{\alpha}}$ be the calculation of the functor Ψ_i on the sequence of argument words $\alpha_1,\ldots,\alpha_n$, with the result of the calculation β_i, in the interpretation set $\mathbb{A}$, $1\le i\le n$. Let $\mathbf{P}_{\Phi,\overline{\beta}}$ be the calculation of the functor Φ on the sequence of argument words $\beta_1,\ldots,\beta_k$, with the result calculation γ, in the interpretation set $\mathbb{A}$, then it is possible to construct the calculation $\mathbf{P}_{[J\Phi\Psi_1,\ldots,\Psi_k]}$ of the functor $[J\Phi\Psi_1,\ldots,\Psi_k]$ on the sequence of argument words $\alpha_1,\ldots,\alpha_n$, for which:

$(\mathbb{A}^+)_{\mathbf{P}_{[J\Phi\Psi_1,\ldots,\Psi_k],\overline{\alpha}}}=\bigcup\limits_{i=1}^{k}(\mathbb{A}^+)_{\mathbf{P}_{\Psi_i},\overline{\alpha}}\bigcup(\mathbb{A}^+)_{\mathbf{P}_{\Phi},\overline{\beta}}$, $(\mathbb{A}^-)_{\mathbf{P}_{[J\Phi\Psi_1,\ldots,\Psi_k],\overline{\alpha}}}=\bigcup\limits_{i=1}^{k}(\mathbb{A}^-)_{\mathbf{P}_{\Psi_i},\overline{\alpha}}\bigcup(\mathbb{A}^-)_{\mathbf{P}_{\Phi},\overline{\beta}}$.

Proof. We compose the following sequence of equalities: $\mathbf{P}_{\Psi_1},\ldots,\mathbf{P}_{\Psi_k}$,

$[J\Phi\Psi_1,\ldots,\Psi_k](x_1,\ldots,x_n)=\Phi(\Psi_1(\overline{x}),\ldots,\Psi_k(\overline{x})),\ldots,$

$[J\Phi\Psi_1,\ldots,\Psi_k](\alpha_1,\ldots,\alpha_n)=\Phi(\Psi_1(\overline{\alpha}),\ldots,\Psi_k(\overline{\alpha})),\Phi(x_1,\ldots,x_n)=\Phi(x_1,\ldots,x_n),\ldots,$

$\Phi(\Psi_1(\overline{\alpha}),\ldots,\Psi_k(\overline{\alpha}))=\Phi(\beta_1,\ldots,\beta_n),\mathbf{P}_{\Phi},\Phi(\Psi_1(\overline{\alpha}),\ldots,\Psi_k(\overline{\alpha}))=\gamma,[J\Phi\Psi_1,\ldots,\Psi_k](\alpha_1,\ldots,\alpha_n)=\gamma$ - calculation of the functor $[J\Phi\Psi_1,\ldots,\Psi_k]$ on the sequences $\alpha_1,\ldots,\alpha_n$, in the interpretation set $\mathbb{A}$, with the specified set of positively and negatively interrogated words.

Theorem 1.4. Let $n+1$ - a place functor $[R\Phi\Psi_1,\ldots,\Psi_k]$, $\alpha_1,\ldots,\alpha_n,\beta a_i$ - some sequence of argument words.

Let $\mathbf{P}_{\Phi}$ be the computation of the functor Φ on the sequence of argument words $\alpha_1,\ldots,\alpha_n$, in the interpretation set $\mathbb{A}$.

Let $\mathbf{P}_{[R\Phi\Psi_1,\ldots,\Psi_k],\alpha_1,\ldots,\alpha_n,\beta}$ be the computation functor $[R\Phi\Psi_1,\ldots,\Psi_k]$ on the sequence of argument words $\alpha_1,\ldots,\alpha_n,\beta$, with the result of computing γ, in the interpretation set $\mathbb{A}$.

Let $\mathbf{P}_{\Psi_i}$ be the computation of the functor Ψ_i on the sequence of argument words $\alpha_1,\ldots,\alpha_n,\beta,\gamma$, with the result of the calculation η, in the interpretation set $\mathbb{A}$, then you can construct the calculation

$\mathbf{P}_{[R\Phi\Psi_1,\ldots,\ Psi_k],\alpha_1,\ldots,\alpha_n,\beta a_i}$ of the functor $[R\Phi\Psi_1,\ldots,\Psi_k]$ on the sequence of argument words $\alpha_1,\ldots,\alpha_n,\beta a_i$, in the interpretation set $\mathbb{A}$, for which:

1. $(\mathbb{A}^+)_{\mathbf{P}_{[R\Phi\Psi_1,\ldots,\Psi_k],\alpha_1,\ldots,\alpha_n,\Lambda}}=(\mathbb{A}^+)_{\mathbf{P}_{\Phi}}$,

   $(\mathbb{A}^-)_{\mathbf{P}_{[R\Phi\Psi_1,\ldots,\Psi_k],\alpha_1,\ldots,\alpha_n,\Lambda}}=(\mathbb{A}^-)_{\mathbf{P}_{\Phi}}$

2. $(\mathbb{A}^+)_{\mathbf{P}_{[R\Phi\Psi_1,\ldots,\Psi_k],\alpha_1,\ldots,\alpha_n,\beta a_i}}=(\mathbb{A}^+)_{\mathbf{P}_{\Psi_i},,\alpha_1,\ldots,\alpha_n,\beta,\gamma}\bigcup(\mathbb{A}^+)_{\mathbf{P}_{[R\Phi\Psi_1,\ldots,\Psi_k],\alpha_1,\ldots,\alpha_n,\beta}}$,

   $(\mathbb{A}^-)_{\mathbf{P}_{[R\Phi\Psi_1,\ldots,\Psi_k],\alpha_1,\ldots,\alpha_n,\beta a_i}}=(\mathbb{A}^-)_{\mathbf{P}_{\Psi_i},\alpha_1,\ldots,\alpha_n,\beta,\gamma}\bigcup(\mathbb{A}^-)_{\mathbf{P}_{[R\Phi\Psi_1,\ldots,\Psi_k],\alpha_1,\ldots,\alpha_n,\beta}}$

Proof. Point (1) is obvious. Point (2). We compose the following sequence of equalities:

$\mathbf{P}_{[R\Phi\Psi_1,\ldots,\Psi_k],\alpha_1,\ldots,\alpha_n,\beta}$,

$[R\Phi\Psi_1,\ldots,\Psi_k](x_1,\ldots,x_n,za_i)=\Psi(x_1,\ldots,x_n,z,[R\Phi\Psi_1,\ldots,\Psi_k](x_1,\ldots,x_n,z)),\ldots,$

$[R\Phi\Psi_1,\ldots,\Psi_k](\alpha_1,\ldots,\alpha_n,\beta a_i) = \Psi_i(\alpha_1,\ldots,\alpha_n,\beta,[R\Phi\Psi_1,\ldots,\Psi_k](\alpha_1,\ldots,\alpha_n,\beta))$,

$\Psi_i(x_1,\ldots,x_n,z,u) = \Psi_i(x_1,\ldots,x_n,z,u),\ldots,$

$\Psi_i(\alpha_1,\ldots,\alpha_n,\beta,[R\Phi\Psi_1,\ldots,\Psi_k](\alpha_1,\ldots,\alpha_n,\beta)) = \Psi_i(\alpha_1,\ldots,\alpha_n,\beta,\gamma)$,

$\mathbf{P}_{\Psi_i}$,

$\Psi_i(\alpha_1,\ldots,\alpha_n,\beta,[R\Phi\Psi_1,\ldots,\Psi_k](\alpha_1,\ldots,\alpha_n,\beta)) = \eta$,

$[R\Phi\Psi_1,\ldots,\Psi_k](\alpha_1,\ldots,\alpha_n,\beta a_i) = \eta$ - functor calculation $[R\Phi\Psi_1,\ldots,\Psi_k]$ on the sequence of argument words $\alpha_1,\ldots,\alpha_n,\beta a_i$, in interpretation set $\mathbb{A}$, with the specified set of positively and negatively interrogated words.

Theorem 1.5. Let a functor be given $[R\alpha\Psi_1,\ldots,\Psi_k]$, $\alpha_1,\ldots,\alpha_n,\beta a_i$ - - some sequence of argument words.

Let $\mathbf{P}_{[R\alpha\Psi_1,\ldots,\Psi_k],\beta}$ be the evaluation of the functor $[R\alpha\Psi_1,\ldots,\Psi_k]$ on the argument word β, with the result of the evaluation γ, in the interpretation set $\mathbb{A}$.

Let $\mathbf{P}_{\Psi_i}$ be a computation of the functor Ψ_i on a sequence of argument words β,γ, with the result of the computation η, in the interpretation set $\mathbb{A}$, then we can construct a computation

$\mathbf{P}_{[R\alpha\Psi_1,\ldots,\Psi_k],\beta a_i}$ of the functor $[R\alpha\Psi_1,\ldots,\Psi_k]$ on the argument word βa_i, in the interpretation set $\mathbb{A}$, for which the following is true:

1. $(\mathbb{A}^+)_{\mathbf{P}_{[R\alpha\Psi_1,\ldots,\Psi_k],\Lambda}} = \emptyset$,

$(\mathbb{A}^-)_{\mathbf{P}_{[R\alpha\Psi_1,\ldots,\Psi_k],\Lambda}} = \emptyset$

2. $(\mathbb{A}^+)_{\mathbf{P}_{[R\alpha\Psi_1,\ldots,\Psi_k],\beta a_i}} = (\mathbb{A}^+)_{\mathbf{P}_{\Psi_i},\beta,\gamma} \bigcup (\mathbb{A}^+)_{\mathbf{P}_{[R\alpha\Psi_1,\ldots,\Psi_k],\beta}}$,

$(\mathbb{A}^-)_{\mathbf{P}_{[R\alpha\Psi_1,\ldots,\Psi_k],\beta a_i}} = (\mathbb{A}^-)_{\mathbf{P}_{\Psi_i},\beta,\gamma} \bigcup (\mathbb{A}^-)_{\mathbf{P}_{[R\alpha\Psi_1,\ldots,\Psi_k],\beta}}$

Proof. Point (1) is obvious. Point (2). Let us compose the following sequence of equalities:

$\mathbf{P}_{[R\alpha\Psi_1,\ldots,\Psi_k],\beta}$,

$[R\alpha\Psi_1,\ldots,\Psi_k](za_i) = \Psi_i(z,[R\alpha\Psi_1,\ldots,\Psi_k](z))$

$[R\alpha\Psi_1,\ldots,\Psi_k](\beta a_i) = \Psi_i(\beta,[R\alpha\Psi_1,\ldots,\Psi_k](\beta))$,

$\Psi_i(z,u) = \Psi_i(z,u),\ldots,$

$\Psi_i(\beta,[R\alpha\Psi_1,\ldots,\Psi_k](\beta)) = \Psi_i(\beta,\gamma)$,

$\mathbf{P}_{\Psi_i;\beta,\gamma}$,

$\Psi_i(\beta,[R\Phi\Psi_1,\ldots,\Psi_k](\beta)) = \eta$,

$[R\alpha\Psi_1,\ldots,\Psi_k](\beta a_i) = \eta$ - calculation of the functor $[R\alpha\Psi_1,\ldots,\Psi_k]$ on the sequence of argument words βa_i, in the interpretation set $\mathbb{A}$, with the specified set of positively and negatively interrogated words.

Given that $\mathbf{Length}(\Lambda) = \Lambda$, $\mathbf{Length}(\alpha a_i) = \mathbf{S}_1(\alpha)$. $|\Lambda| = \Lambda$, $|\alpha a_i| = |\alpha| + 1 = S(|\alpha|)$, $|x|$ is a function of the length of the word x, then expression of the form $\mathbf{Length}(t)$, will be denoted as $|t|$. Obviously, the argument word α is a natural number if and only if $\vdash |\alpha| = \alpha$, $\mathbf{Length}(\alpha)$ is a natural number.

A term containing only functors of the form: $\mathbf{Concat}, \mathbf{D}$, as well as natural numbers, is called a word polynomial, word polynomials will be denoted as $\mathbf{P}(\overline{x})$.

From properties: $|\mathbf{Concat}(x, y)| = |\mathbf{Concat}(|x|, y)| = |\mathbf{Concat}(x, |y|)| = \mathbf{Concat}(|x|, |y|) = |x| + |y|$, $|\mathbf{D}(x, y)| = \mathbf{D}(|x|, y) = |\mathbf{D}(x, |y|)| = \mathbf{D}(|x|, |y|) = |x| \cdot |y| = \mathbf{D}(|y|, x)$, we get $|\mathbf{P}(x_1, \ldots, x_n)| = \mathbf{P}(|x_1|, \ldots, |x_n|)$, $\forall \alpha\beta\gamma[\mathbf{Concat}(|\alpha|, |\beta|) = \gamma \vee \mathbf{D}(|\alpha|, \beta) = \gamma]$, then γ is a natural number.

For any word polynomial $\mathbf{P}(\overline{x})$, one can construct polynomial with natural coefficients $P^*(\overline{x})$, which is true $P^*(|\overline{x}|) = |\mathbf{P}(\overline{x})|$.

Let us compose a $3 \leq n$ - place functor of the form $[J\mathbf{Concat}\mathbf{I}^n_1[J\mathbf{Concat}\mathbf{I}^n_{n-1}[J\mathbf{Concat}\mathbf{I}^n_{n-1}\mathbf{I}_n]\ldots]$. For this functor in the calculus **CalcEq** the following equality holds

$[J\mathbf{Concat}\mathbf{I}^n_1[J\mathbf{Concat}\mathbf{I}^n_2\ldots[J\mathbf{Concat}\mathbf{I}^n_{n-1}\mathbf{I}^n_n]\ldots](x_1\ldots x_n) = \mathbf{Concat}(x_1, \mathbf{Concat}(x_2, \ldots \mathbf{Concat}(x_{n-1}, x_n)\ldots))$.

Let's $\mathbf{Concat}^n \rightleftharpoons [J\mathbf{Concat}\mathbf{I}^n_1[J\mathbf{Concat}\mathbf{I}^n_2\ldots[J\mathbf{Concat}\mathbf{I}^n_{n-1}\mathbf{I}^n_n]\ldots]$, at $n \geq 3$, then $\vdash \mathbf{Concat}^n(x_1, \ldots x_n) = \mathbf{Concat}(x_1, \mathbf{Concat}(x_2, \ldots \mathbf{Concat}(x_{n-1}, x_n)\ldots))$. At $n = 2$, $\mathbf{Concat}^2 \rightleftharpoons \mathbf{Concat}$, at $n = 1$, $\mathbf{Concat}^1 \rightleftharpoons \mathbf{I}^1_1$ and the following equations hold : $\vdash \mathbf{Concat}^2(x_1, x_2) = \mathbf{Concat}(x_1, x_2)$, $\vdash \mathbf{Concat}^1(x_1) = \mathbf{I}^1_1(x_1) = x_1$.
$\vdash \mathbf{Concat}^{n+1}(x_1, \ldots x_{n+1}) = \mathbf{Concat}(x_1, \mathbf{Concacat}^n(x_2, \ldots x_{n+1}))$.

Definition. Let $\mathscr{D}$ be some set of n - alphabetical functors. For each set $\mathbb{A}$ of argument words, we define the concept of a *standard word model*, which we denote as $\mathbf{WordM}_{n,\mathbb{A},\mathscr{D}}$.

The universe of this model is all argument words, or n - alphabetic argument words. For each $k-$ place functor $\Phi \in \mathscr{D}$ we define an operation, denoted f_Φ and defined as $\forall\overline{\alpha}\forall\beta[f_\Phi(\overline{\alpha}) = \beta \Longleftrightarrow \mathbb{A} \vdash \Phi(\overline{\alpha}) = \beta]$.

If the set of functors $\mathscr{D}$ coincides with the set of all primitive recursive functors, then the standard model will be denoted as $\mathbf{WordM}_{n,\mathbb{A}}$, $\mathbf{WordM}_n$, in the alphabet $\mathscr{L}(\mathbf{U})$, $\mathscr{L}$ respectively, or more simply $\mathbf{WordM}_\mathbb{A}$, **WordM**.

If every functor belonging to the set $\mathscr{D}$ is a functor of the alphabet $\mathscr{L}$, then the standard word model corresponding to the set of functors $\mathscr{D}$ will be denoted as $\mathbf{WordM}_\mathscr{D}$. The model $\mathbf{WordM}_{\mathbb{A},\mathscr{D}}$ will also be referred to as the model in signature $\mathscr{D}$ of the alphabet $\mathscr{L}(\mathbf{U})$.

Remark. Note that the set of all operations of the standard word model $\mathbf{WordM}_n$ coincides with the class of word functions $Pr(\Sigma)$, where Σ is an alphabet consisting of n different symbols[2, p.220, Definition 3].

Theorem 1.6. There is an algorithm, executing which, according to an arbitrary formula $\mathcal{A}(x_1, \ldots, x_n)$ for propositional calculus, in which elementary propositions are propositions of the form $r = q$, where r, q - terms of the alphabet $\mathscr{L}(\mathbf{U})$, one can construct n - a place functor $\Phi_\mathcal{A}$ such that
$\forall\overline{\alpha}[\mathbf{WordM}_\mathbb{A} \models \mathcal{A}(\overline{\alpha}) \Leftrightarrow \mathbb{A} \vdash \Phi_\mathcal{A}(\overline{\alpha}) = \Lambda]$ $(\mathbf{WordM}_\mathbb{A} \models \forall\overline{x}[\mathcal{A}(\overline{x}) \equiv \Phi_\mathcal{A}(\overline{x}) = \Lambda])$[5].

Part II

Simple calculation of functor

With each n - ary functor Φ and a sequence of argument words $\alpha_1, \ldots, \alpha_n$, hereinafter denoted as $\overline{\alpha}$, we associate a simple (canonical) computation of the functor Φ on the sequence of argument words $\alpha_1, \ldots, \alpha_n$, denoted as $\mathbf{P}_{\Phi;\alpha_1,\ldots\alpha_n}$. We construct this simple computation by induction on the construction of the functor Φ, and inside this induction, for a recursive functor by induction on the construction of the argument word. With each simple calculation we indicate the sets $(\mathbb{A}^+)_{\mathbf{P},\Phi}$ and $(\mathbb{A}^-)_{\mathbf{P},\Phi}$ and the length of the calculation $l_{\mathbf{P},\Phi}(\overline{\alpha})$ - the number of equalities in the output $\mathbf{P}_{\Phi;\alpha_1,\ldots\alpha_n}$.

For original functors: $\mathbf{S}_k, \mathbf{Z}, \mathbf{I}^n_m, \boldsymbol{\delta}, \mathbf{Length}, \dot{-}, \mathbf{Concat}, \mathbf{D}$:

For the functor: $\mathbf{S}_k$:

1. $\mathbf{S}_k(x) = \mathbf{S}_k(x)$,

2. $\mathbf{S}_k(\alpha) = \mathbf{S}_k(\alpha)$.

$(\mathbb{A}^+)_{\mathbf{P},\mathbf{S}_k} = \emptyset$,

$(\mathbb{A}^-)_{\mathbf{P},\mathbf{S}_k} = \emptyset$.

Calculation length $l_{\mathbf{P},\mathbf{S}_k}(x) = 2$.

For the functor $\mathbf{Z}$:

1. $\mathbf{Z}(x_1) = \Lambda$,

2. $\mathbf{Z}(\alpha) = \Lambda$.

Calculation length $l_{\mathbf{P},\mathbf{Z}}(x) = 2$.

$(\mathbb{A}^+)_{\mathbf{P},\mathbf{Z}} = \emptyset$,

$(\mathbb{A}^-)_{\mathbf{P},\mathbf{Z}} = \emptyset$.

For the functor $\mathbf{U}$:

$\mathbf{U}(\alpha) = \Lambda$, if $\alpha \in \mathbb{A}$, otherwise,

$\mathbf{U}(\alpha) = a_1$.

Calculation length $l_{\mathbf{P},\mathbf{U}}(x) = 1$.

$(\mathbb{A}^+)_{\mathbf{P},\mathbf{U}} = \{\alpha\}$, $(\mathbb{A}^-)_{\mathbf{P},\mathbf{U}} = \emptyset$, if $\alpha \in \mathbb{A}$,

$(\mathbb{A}^+)_{\mathbf{P},\mathbf{U}} = \emptyset$, $(\mathbb{A}^-)_{\mathbf{P},\mathbf{U}} = \{\alpha\}$, if $\alpha \notin \mathbb{A}$

For the functor $\mathbf{I}^n_m$:

1. $\mathbf{I}^n_m(x_1, \ldots, x_n) = x_m$,

2. $\mathbf{I}^n_m(\alpha_1, x_2, \ldots, x_n) = x_m, \ldots,$,

$n+1.\ \mathbf{I}^n_m(\alpha_1,\dots,\alpha_n)=\alpha_m.$

Calculation length $l_{\mathbf{P},\mathbf{I}^n_m}(x_1,\dots,x_n)=n+1.$

$(\mathbb{A}^+)_{\mathbf{P},\mathbf{I}^n_m}=\emptyset,$

$(\mathbb{A}^-)_{\mathbf{P},\mathbf{I}^n_m}=\emptyset.$

For the functor $\boldsymbol{\delta}$:

1. $\boldsymbol{\delta}(\Lambda)=\Lambda,$

2. $\boldsymbol{\delta}(x_1a_i)=x_1,$

3. $\boldsymbol{\delta}(\alpha a_i)=\alpha.$

The length of the calculation is given by the defining equalities:

$l_{\mathbf{P}_{\boldsymbol{\delta}}}(\Lambda)=1,$

$l_{\mathbf{P}_{\boldsymbol{\delta}}}(x_1)=2$, at $\alpha\neq\Lambda$,

$(\mathbb{A}^+)_{\mathbf{P},\boldsymbol{\delta}}=\emptyset,$

$(\mathbb{A}^-)_{\mathbf{P},\boldsymbol{\delta}}=\emptyset.$

For the functor **Length**:

1. $|\Lambda|=\Lambda,$

1.$|x_1a_k|=|x_1|a_1,$

2.$|\alpha a_k|=|\alpha|a_1,$

[Let $\mathbf{P}_{\mathbf{Length};\alpha}$ - simple calculation of functor **Length** on the argument word α, Next, we write out this simple calculation $\mathbf{P}_{\mathbf{Length},\alpha}$, at the end of this conclusion is the equality $|\alpha|=\gamma$, continue]

$\mathbf{P}_{\mathbf{Length};\alpha},$

3. $\mathbf{S}_1(x_1)=\mathbf{S}_1(x_1),$

4. $\mathbf{S}_1(|\alpha|)=\mathbf{S}_1(\gamma),$

5. $|\alpha a_k|=\gamma a_1.$

The length of the calculation is given by the defining equalities:

$l_{\mathbf{P}}(\Lambda)=1,$

$l_{\mathbf{P},\mathbf{Length}}(x_1a_k)=l_{\mathbf{P}}(x_1)+5.$

$(\mathbb{A}^+)_{\mathbf{P},\mathbf{Length}}=\emptyset,$

$(\mathbb{A}^-)_{\mathbf{P},\mathbf{Length}}=\emptyset.$

For the functor $\dot{-}$:

1. $x_1\dot{-}\Lambda=x_1,$

2. $\alpha\dot{-}\Lambda=\alpha,$

1. $x_1 \dot{-} x_2 a_k = \boldsymbol{\delta}(x_1 \dot{-} x_2)$,

2. $\alpha \dot{-} x_2 a_k = \boldsymbol{\delta}(\alpha \dot{-} x_2)$,

3. $\alpha \dot{-} \beta a_k = \boldsymbol{\delta}(\alpha \dot{-} \beta)$,

[Let $\mathbf{P}_{\dot{-};\alpha,\beta}$ - simple calculation of functor $\dot{-}$ on the argument words α, β, next, we write out this simple calculation $\mathbf{P}_{\dot{-};\alpha,\beta}$, at the end of this calculation is the equality $\alpha \dot{-} \beta = \gamma$, continue]

$\mathbf{P}_{\dot{-};\alpha,\beta}$,

4. $\boldsymbol{\delta}(z) = \boldsymbol{\delta}(z)$,

5. $\boldsymbol{\delta}(\alpha \dot{-} \beta) = \boldsymbol{\delta}(\gamma)$.

[Let $\mathbf{P}_{\boldsymbol{\delta};\gamma}$ - simple calculation of functor $\boldsymbol{\delta}$ on the argument word γ, at the end of this calculation there is an equality of the form $\boldsymbol{\delta}(\gamma) = \eta$, continue]

$\mathbf{P}_{\boldsymbol{\delta},\gamma}$,

6. $\boldsymbol{\delta}(\alpha \dot{-} \beta) = \eta$.

7. $\alpha \dot{-} \beta a_k = \eta$,

The length of the calculation is given by the defining equalities:

$l_{\mathbf{P}_{\dot{-}}}(x_1, \Lambda) = 2$,

$l_{\mathbf{P}_{\dot{-}}}(x_1, x_2 a_k) = l_{\mathbf{P}_{\dot{-}}}(x_1, x_2) + l_{\mathbf{P}_{\boldsymbol{\delta}}}(x_1 \dot{-} x_2) + 7$.

$(\mathbb{A}^+)_{\mathbf{P},\dot{-}} = \emptyset$,

$(\mathbb{A}^-)_{\mathbf{P},\dot{-}} = \emptyset$.

For the functor **Concat**:

1. $\mathbf{Concat}(x_1, \Lambda) = x_1$,

2. $\mathbf{Concat}(\alpha, \Lambda) = \alpha$,

1. $\mathbf{Concat}(x_1, x_2 a_k) = \mathbf{Concat}(x_1, x_2) a_k$,

2. $\mathbf{Concat}(\alpha, x_2 a_k) = \mathbf{Concat}(\alpha, x_2) a_k$,

3. $\mathbf{Concat}(\alpha, \beta a_k) = \mathbf{Concat}(\alpha, \beta) a_k$,

[Let $\mathbf{P}_{\mathbf{Concat};\alpha,\beta}$ - simple calculation of functor **Concat** on the argument word α and word β, at the end of this calculation there is an equality of the form $\mathbf{Concat}(\alpha, \beta) = \gamma$, continue]

$\mathbf{P}_{\mathbf{Concat};\alpha,\beta}$,

4. $\mathbf{S}_k(x_1) = \mathbf{S}_k(x_1)$,

5. $\mathbf{S}_k(\mathbf{Concat}(\alpha, \beta)) = \mathbf{S}_k(\gamma)$,

6. $\mathbf{Concat}(\alpha, \beta a_k) = \mathbf{S}_k(\gamma)$.

The length of the calculation is given by the defining equalities:

$l_{\mathbf{P_{Concat}}}(\alpha, \Lambda) = 2,$

$l_{\mathbf{P_{Concat}}}(\alpha, \beta a_k) = l_{\mathbf{P_{Concat}}}(\alpha, \beta) + 6.$

$(\mathbb{A}^+)_{\mathbf{P},\mathbf{Concat}} = \emptyset,$

$(\mathbb{A}^-)_{\mathbf{P},\mathbf{Concat}} = \emptyset.$

For the functor $\mathbf{D}$:

1. $\mathbf{D}(x_1, \Lambda) = \Lambda,$

2. $\mathbf{D}(\alpha, \Lambda) = \Lambda,$

1. $\mathbf{D}(x_1, x_2 a_k) = \mathbf{Concat}(x_1, \mathbf{D}(x_1, x_2)),$

2. $\mathbf{D}(\alpha, x_2 a_k) = \mathbf{Concat}(\alpha, \mathbf{D}(\alpha, x_2)),$

3. $\mathbf{D}(\alpha, \beta a_k) = \mathbf{Concat}(\alpha, \mathbf{D}(\alpha, \beta)),$

[Let $\mathbf{P}_{\mathbf{D};\alpha,\beta}$ - simple calculation of functor $\mathbf{D}$ on the argument word α and word β, at the end of this calculation there is an equality of the form $\mathbf{D}(\alpha, \beta) = \gamma$, continue]

$\mathbf{P}_{\mathbf{D};\alpha,\beta},$

4. $\mathbf{Concat}(x_1, x_2) = \mathbf{Concat}(x_1, x_2),$

5. $\mathbf{Concat}(\alpha, x_2) = \mathbf{Concat}(\alpha, x_2),$

6. $\mathbf{Concat}(\alpha, \mathbf{D}(\alpha, \beta)) = \mathbf{Concat}(\alpha, \gamma),$

[Let $\mathbf{P}_{\mathbf{Concat};\alpha,\gamma}$ - simple calculation of functor $\mathbf{Concat}$ on the argument word α, γ, at the end of this calculation there is an equality of the form $\mathbf{Concat}(\alpha, \gamma) = \eta$, continue].

$\mathbf{P}_{\mathbf{Concat};\alpha,\gamma},$

7. $\mathbf{Concat}(\alpha, \mathbf{D}(\alpha, \beta)) = \eta,$

8. $\mathbf{D}(\alpha, \beta a_k) = \eta.$

The length of the calculation is given by the defining equalities:

$l_{\mathbf{P_D}}(\alpha, \Lambda) = 2,$

$l_{\mathbf{P_D}}(\alpha, \beta a_k) = l_{\mathbf{P_D}}(\alpha, \beta) + l_{\mathbf{P_{Concat}}}(\alpha, \mathbf{D}(\alpha, \beta)) + 8.$

$(\mathbb{A}^+)_{\mathbf{P},\mathbf{D}} = \emptyset,$

$(\mathbb{A}^-)_{\mathbf{P},\mathbf{D}} = \emptyset.$

For the functor $[J\Phi\Psi_1, \ldots, \Psi_k]$:

Let $\mathbf{P}_{\Psi_1;\overline{\alpha}}$ - simple calculation of functor Ψ_1 on a sequence of argument words $\overline{\alpha}, \ldots,$ $\mathbf{P}_{\Psi_k;\overline{\alpha}}$ - simple calculation of functor Ψ_k on a sequence of argument words $\overline{\alpha}$.

Let's compose a sequence of equalities:

$\mathbf{P}_{\Psi_1;\overline{\alpha}}, \ldots, \mathbf{P}_{\Psi_k;\overline{\alpha}},$

[at the end of this calculation $\mathbf{P}_{\Psi_i}$ at the end of this calculation there is an equality of the form $\Psi_i(\overline{\alpha}) = \gamma_i$, continue]

$s+1$. $\Phi(x_1,\ldots,x_k) = \Phi(x_1,\ldots,x_k)$,

$s+2$. $\Phi(\Psi_1(\overline{\alpha}), x_2,\ldots,x_k) = \Phi(\gamma_1, x_2,\ldots,x_k),\ldots,$

$s+\mathbf{k}+1$. $\Phi(\Psi_1(\overline{\alpha}),\ldots,\Psi_k(\overline{\alpha})) = \Phi(\gamma_1,\ldots,\gamma_k)$,

[Let $\mathbf{P}_{\Phi;\overline{\gamma}}$ - simple calculation of functor Φ on a sequence of argument words $\overline{\gamma}$, at the end of this calculation is the equality $\Phi(\gamma_1,\ldots,\gamma_k) = \eta$, continue]

$\mathbf{P}_{\Phi;\overline{\gamma}}$,

$s+\mathbf{k}+r+2$. $\Phi(\Psi_1(\overline{\alpha}),\ldots,\Psi_1(\overline{\alpha})) = \eta$,

$s+\mathbf{k}+r+3$. $[J\Phi\Psi_1,\ldots,\Psi_k](x_1,\ldots,x_n) = \Phi(\Psi_1(x_1,\ldots,x_n),\ldots,\Psi_k(x_1,\ldots,x_n))$,

$s+\mathbf{k}+r+4$. $[J\Phi\Psi_1,\ldots,\Psi_k](\alpha_1,\ldots,x_n) = \Phi(\Psi_1(\alpha_1,\ldots,x_n),\ldots,\Psi_k(\alpha_1,\ldots,x_n)),\ldots,$,

$s+\mathbf{k}+r+\mathbf{n}+3$. $[J\Phi\Psi_1,\ldots,\Psi_k](\alpha_1,\ldots,\alpha_n) = \Phi(\Psi_1(\alpha_1,\ldots,\alpha_n),\ldots,\Psi_k(\alpha_1,\ldots,\alpha_n))$,

$s+\mathbf{k}+r+\mathbf{n}+4$. $[J\Phi\Psi_1,\ldots,\Psi_k](\alpha_1,\ldots,\alpha_n) = \eta$ - the resulting sequence of equalities - simple calculation of functor $[J\Phi\Psi_1,\ldots,\Psi_k]$ on a sequence of argument words $\overline{\alpha}$.

The length of the calculation is given by the defining equalities:

$l_{\mathbf{P}_{[J\Phi\Psi_1,\ldots,\Psi_k]}}(\overline{\alpha}) = l_{\mathbf{P}_{\Psi_1}}(\overline{\alpha})+,\ldots,+l_{\mathbf{P}_{\Psi_k}}(\overline{\alpha}) + l_{\mathbf{P}_{\Phi}}(\Psi_1(\overline{\alpha}),\ldots,\Psi_k(\overline{\alpha})) + \mathbf{n}+\mathbf{k}+4.$

$(\mathbb{A}^+)_{\mathbf{P}_{[J\Phi\Psi_1,\ldots,\Psi_k]};\overline{\alpha}} = \bigcup\limits_{i=1}^{k}(\mathbb{A}^+)_{\mathbf{P}_{\Psi_i};\overline{\alpha}} \bigcup (\mathbb{A}^+)_{\mathbf{P}_{\Phi};\overline{\gamma}}$,

$(\mathbb{A}^-)_{\mathbf{P}_{[J\Phi\Psi_1,\ldots,\Psi_k]};\overline{\alpha}} = \bigcup\limits_{i=1}^{k}(\mathbb{A}^-)_{\mathbf{P}_{\Psi_i};\overline{\alpha}} \bigcup (\mathbb{A}^-)_{\mathbf{P}_{\Phi};\overline{\gamma}}$(see Theorem 1.3).

For the functor $[R\Phi\Psi_1,\ldots,\Psi_m]$:

Let's compose a sequence of equalities:

1. $[R\Phi\Psi_1,\ldots,\Psi_m](x_1,\ldots,x_n,\Lambda) = \Phi(x_1,\ldots,x_n)$,

2. $[R\Phi\Psi_1,\ldots,\Psi_m](\alpha_1,\ldots,x_n,\Lambda) = \Phi(\alpha_1,\ldots,x_n),\ldots,$,

$\mathbf{n}+1$. $[R\Phi\Psi_1,\ldots,\Psi_m](\alpha_1,\ldots,\alpha_n,\Lambda) = \Phi(\alpha_1,\ldots,\alpha_n)$,

[Let $\mathbf{P}_{\Phi;\overline{\alpha}}$ - simple calculation of functor Φ on a sequence of argument words $\overline{\alpha}$, at the end of this calculation is the equality $\Phi(\alpha_1,\ldots,\alpha_n) = \gamma$, continue]

$\mathbf{P}_{\Phi;\overline{\alpha}}$,

$\mathbf{n}+r+2$. $[R\Phi\Psi_1,\ldots,\Psi_m](\alpha_1,\ldots,\alpha_n,\Lambda) = \eta$,

[Let $\mathbf{P}_{[R\Phi\Psi_1,\ldots,\Psi_m];\alpha_1,\ldots,\alpha_{n+1}}$- simple calculation of functor $[R\Phi\Psi_1,\ldots,\Psi_m]$ on a sequence of argument words $\alpha_1,\ldots,\alpha_{n+1}$, at the end of this calculation is the equality $[R\Phi\Psi_1,\ldots,\Psi_m](\alpha_1,\ldots,\alpha_n,\alpha_{n+1}) = \beta$, continue]

$\mathbf{P}_{[R\Phi\Psi_1,\ldots,\Psi_m];\alpha_1,\ldots,\alpha_{n+1}}$, [Let $\mathbf{P}_{\Psi_k;\alpha_1,\dots,\alpha_{n+1},\beta}$- simple calculation of functor Ψ_k] on a sequence of argument words $\alpha_1,\dots,\alpha_{n+1},\beta$, at the end of this calculation is the equality [$\Psi_k(\alpha_1,\dots,\alpha_n,\alpha_{n+1},\beta)=\theta$, continue]

$\mathbf{P}_{\Psi_k;\alpha_1,\dots,\alpha_{n+1},\beta}$,

$s+t+1$. $[R\Phi\Psi_1,\dots,\Psi_m](x_1,\dots,x_n,x_{n+1}a_k)=\Psi_k(x_1,\dots,x_n,x_{n+1},[R\Phi\Psi_1,\dots,\Psi_m](x_1,\dots,x_n,x_{n+1}))$,

$s+t+2$. $[R\Phi\Psi_1,\dots,\Psi_m](\alpha_1,\dots,x_n,x_{n+1}a_k)=\Psi_k(\alpha_1,\dots,x_n,x_{n+1},[R\Phi\Psi_1,\dots,\Psi_m](\alpha_1,\dots,x_n,x_{n+1})),\dots$,

$\mathbf{n}+s+t+2$. $[R\Phi\Psi_1,\dots,\Psi_m](\alpha_1,\dots,\alpha_n,\alpha_{n+1}a_k)=\Psi_k(\alpha_1,\dots,\alpha_n,\alpha_{n+1},[R\Phi\Psi_1,\dots,\Psi_m](\alpha_1,\dots,\alpha_n,\alpha_{n+1}))$,

$\mathbf{n}+s+t+3$. $\Psi_k(x_1,\dots,x_{n+2})=\Psi_k(x_1,\dots,x_{n+2})$,

$\mathbf{n}+s+t+4$. $\Psi_k(\alpha_1,\dots,x_{n+2})=\Psi_k(\alpha_1,\dots,x_{n+2}),\dots$,

$2\mathbf{n}+s+t+4$. $\Psi_k(\alpha_1,\dots,\alpha_{n+1},x_{n+2})=\Psi_k(\alpha_1,\dots,\alpha_{n+1},x_{n+2})$,

$2\mathbf{n}+s+t+5$. $\Psi_k(\alpha_1,\dots,\alpha_{n+1},[R\Phi\Psi_1,\dots,\Psi_m](\alpha_1,\dots,\alpha_n,\alpha_{n+1}))=\Psi_k(\alpha_1,\dots,\alpha_{n+1},\beta)$,

$2\mathbf{n}+s+t+6$. $\Psi_k(\alpha_1,\dots,\alpha_{n+1},[R\Phi\Psi_1,\dots,\Psi_m](\alpha_1,\dots,\alpha_n,\alpha_{n+1}))=\theta$,

$2\mathbf{n}+s+t+7$. $[R\Phi\Psi_1,\dots,\Psi_m](\alpha_1,\dots,\alpha_n,\alpha_{n+1}a_k)=\theta$ - this sequence of equalities is a simple calculation of the functor $[R\Phi\Psi_1,\dots,\Psi_m]$ on a sequence of argument words $\alpha_1,\dots,\alpha_{n+1}$.

The length of the calculation is given by the defining equalities:

$l_{\mathbf{P}_{[R\Phi\Psi_1,\dots,\Psi_k]}}(\overline{\alpha},\Lambda)=l_{\mathbf{P}_\Phi}(\overline{\alpha})+\mathbf{n}+2$,

$l_{\mathbf{P}_{[R\Phi\Psi_1,\dots,\Psi_k]}}(\overline{\alpha},\alpha_{n+1}a_k)=l_{\mathbf{P}_{[R\Phi\Psi_1,\dots,\Psi_k]}}(\overline{\alpha},\alpha_{n+1})+l_{\mathbf{P}_{\Psi_k}}(\overline{\alpha},\alpha_{n+1},[R\Phi\Psi_1,\dots,\Psi_k](\overline{\alpha},\alpha_{n+1}))+2\mathbf{n}+7$

1. $(\mathbb{A}^+)_{\mathbf{P}_{[R\Phi\Psi_1,\dots,\Psi_k],\alpha_1,\dots,\alpha_n,\Lambda}}=(\mathbb{A}^+)_{\mathbf{P}_\Phi;\alpha_1,\dots,\alpha_n}$,

$(\mathbb{A}^-)_{\mathbf{P}_{[R\Phi\Psi_1,\dots,\Psi_k],\alpha_1,\dots,\alpha_n,\Lambda}}=(\mathbb{A}^-)_{\mathbf{P}_\Phi;\alpha_1,\dots,\alpha_n}$,

2. $(\mathbb{A}^+)_{\mathbf{P}_{[R\Phi\Psi_1,\dots,\Psi_k],\alpha_1,\dots,\alpha_n,\beta a_i}}=(\mathbb{A}^+)_{\mathbf{P}_{\Psi_i}}\bigcup(\mathbb{A}^+)_{\mathbf{P}_{[R\Phi\Psi_1,\dots,\Psi_k],\alpha_1,\dots,\alpha_n,\beta}}$,

$(\mathbb{A}^-)_{\mathbf{P}_{[R\Phi\Psi_1,\dots,\Psi_k],\alpha_1,\dots,\alpha_n,\beta a_i}}=(\mathbb{A}^-)_{\mathbf{P}_{\Psi_i}}\bigcup(\mathbb{A}^-)_{\mathbf{P}_{[R\Phi\Psi_1,\dots,\Psi_k],\alpha_1,\dots,\alpha_n,\beta}}$(see Theorem 1.4).

The case when the recursion functor has the form $[R\alpha\Phi_1,\dots,\Phi_m]$ is treated similarly.

For $[R\alpha\Phi_1,\dots,\Phi_m]$, the defining equality lengths of the simple calculus are as follows:

$l_{\mathbf{P}_{[R\alpha\Psi_1,\dots,\Psi_k]}}(\Lambda)=1$,

$l_{\mathbf{P}_{[R\alpha\Psi_1,\dots,\Psi_k]}}(\alpha a_k)=l_{\mathbf{P}_{[R\alpha\Psi_1,\dots,\Psi_k]}}(\alpha)+l_{\mathbf{P}_{\Psi_k}}(\alpha,[R\alpha\Psi_1,\dots,\Psi_k](\alpha))+7$(see Theorem 1.5).

Properties of simple the calculation functor:

1. The last equality of a simple calculation of the $n-$ place functor Φ on a sequence of argument words $\overline{\alpha}$ is an equality of the form $\Phi(\overline{\alpha})=\beta$, where β is an argument word, which is called the result of a simple calculation of the $n-$ place functor Φ on the sequence of argument words $\overline{\alpha}$.

2. A simple calculation of the $n-$ place functor Φ on a sequence of argument words $\overline{\alpha}$ consists only of those functors that are subfunctors of the functor Φ.

3. All the words queried in a simple computation of the functor $[J\Phi\Psi_1,\dots,\Psi_m]$ on the sequence of argument words $\alpha_1,\dots,\alpha_k$, consist of interrogated words that enter into a simple calculation of the functor Ψ_1 on the sequence of argument words $\alpha_1,\dots,\alpha_k$, and so on. from the interrogated words that are included in the simple calculation of the functor Ψ_m on the sequence of argument words $\alpha_1,\dots,\alpha_k$, from the interrogated words that are included in the simple calculation of the functor Φ on the sequence of argument words $\gamma_1,\dots,\gamma_m$, where γ_i is the result of calculating the functor Ψ_i on the sequence of argument words $\alpha_1,\dots,\alpha_k$.

4. All interrogated words, when simply calculating the functor $[R\Phi\Psi_1,\dots,\Psi_m]$ on the sequence of argument words $\overline{\alpha},\beta a_k$, consist from the interrogated words that are included in the simple calculation of the functor Φ on the sequence of argument words $\overline{\alpha}$, from the interrogated words of the functor $[R\Phi\Psi_1,\dots,\Psi_m]$ in a simple calculation on the sequence of argument words $\overline{\alpha},\beta$ (previous step), from the interrogated words that are included in the simple calculation of the functor Ψ_k on the sequence of argument words $\overline{\alpha},\beta,\gamma$ where γ is the result of calculating the functor $[R\Phi\Psi_1,\dots,\Psi_m]$ on the word sequence $\overline{\alpha},\beta$.

5. Any $n-$ place functor Φ can be interpreted as some algorithm, executing which, it is possible to calculate the value of this functor on a given sequence of argument words $\alpha_1,\dots,\alpha_n$. A simple calculation of this functor on the specified sequence of argument words is an implementation of this algorithm.

Part III

Bounded recursion functor. PPr functors

An equality of the form $x\dot{-}y=\Lambda$ will be denoted as $x\leqslant y$. Given the property $x\dot{-}y=\Lambda \Longleftrightarrow |x|\dot{-}|y|=\Lambda$, a formula of the form $x\leqslant y$, will also be written as $|x|\leqslant|y|$.

$$x\dot{-}y=\begin{cases}\Lambda, & if\ |x|\le|y|;\\ \\ z, & other\end{cases},$$

where z - is such a word which is the beginning of word x and whose length is $|x|-|y|$.

For any word polynomial $\mathbf{P}(\overline{y})$, given that $|\mathbf{P}(\overline{y})|=\mathbf{P}(|y|)$, we have: $|x|\leqslant|\mathbf{P}(\overline{y})| \Longleftrightarrow |x|\leqslant\mathbf{P}(|\overline{y}|)$.

Denote the two-place functor $J[\dot{-}\mathbf{I}_1^2J[\dot{-}\mathbf{I}_1^2\mathbf{I}_2^2]]$ as **min**. For this functor of the alphabet $\mathscr{L}$, in the calculus **CalcEq** we derive the equality $\mathbf{min}(x_1,x_2)=x_1\dot{-}(x_1\dot{-}x_2)$.

Properties: $\vdash\mathbf{min}(x_1,x_2)=\mathbf{min}(x_1,|x_2|)$, $\vdash|\mathbf{min}(x_1,x_2)|=\mathbf{min}(|x_1|,|x_2|)$

$$\mathbf{min}(x,y)=\begin{cases}x, & if\ |x|\le|y|;\\ \\ z, & other\end{cases},$$

where z is such a word which is the beginning of word x and whose length is $|y|$.

$\forall\alpha\beta\mathbf{WordM}\models|\mathbf{min}(\alpha,\beta)|\le|\beta|$.

Let Φ be an arbitrary n -place functor. Compose the functor $[J\Phi\mathbf{I}_1^{n+1},\dots\mathbf{I}_n^{n+1}]$ - introducing $n+1$ fictitious variable, this functor will be denoted as $[J\Phi_{n+1}]$.

Let Φ be an arbitrary n -place functor. Compose the functor $[J\Phi\mathbf{I}_1^{n+1},\dots\mathbf{I}_n^{n+1}]$ - introducing $n+1$ fictitious variable, this functor will be denoted as $[J\Phi_{n+1}]$.

Let $\mathbf{P}$ - n-ary polynomial functor, Φ - n- ary functor, Ψ - $n+1$ - ary functor. Compose functors: $[J\mathbf{min}\Phi,\mathbf{P}]$, $[J\mathbf{min}\Psi[J\mathbf{P}_{n+1}]]$ - functors restrictions, respectively, without the introduction of a dummy variable and with the introduction of a dummy variable. We denote these functors as $\mathbf{Bound}(\Phi,\mathbf{P})$.

Let Φ - n - ary functor, $\Psi_1,\dots,\Psi_k$ - $n+2$ -ary functors, $\mathbf{P},\mathbf{P}_1$ - respectively n, $n+1$ - ary polynomial functors. Compose a functor $[R\mathbf{Bound}(\Phi,\mathbf{P}),\mathbf{Bound}(\Psi_1,\mathbf{P}_1),\dots,\mathbf{Bound}(\Psi_k,\mathbf{P}_1)]$ is a bounded recursion functor.

For each functor of bounded recursion $\Gamma \rightleftharpoons [R\mathbf{Bound}(\Phi,\mathbf{P}),\mathbf{Bound}(\Psi_1,\mathbf{P}_1),\dots,\mathbf{Bound}(\Psi_k,\mathbf{P}_1)]$ the following equations are derivable:

$\vdash \Gamma(x_1,\dots,x_n,\Lambda)=\mathbf{min}(\Phi(x_1,\dots,x_n),\mathbf{P}(x_1,\dots,x_n))$,

$\vdash \Gamma(x_1,\dots,x_n,\mathbf{S}_k(x_{n+1}))=\mathbf{min}(\Psi_k(x_1,\dots,x_n,x_{n+1},\Gamma(x_1,\dots,x_{n+1})),\mathbf{P}_1(x_1,\dots,x_n,x_{n+1}))$,

for any set of argument words $\mathbb{A}$ the following is true: $\forall\overline{\alpha},\forall\beta\neq\Lambda\ \ \mathbb{A}\vdash|\Gamma(\overline{\alpha},\Lambda)|\leqslant|\mathbf{P}(\overline{\alpha})|$,

$\mathbb{A}\vdash|\Gamma(\overline{\alpha},\beta)|\leqslant|\mathbf{P}_1(\overline{\alpha},\beta)|$.

We inductively define a set of functors, denoted as $\mathbf{PPr}(\mathbf{U})$:

1) Words of the form $\mathbf{U}$ - polynomial program;

2) Words of the form $\mathbf{Z}$, $\boldsymbol{\delta},\mathbf{Length}$, $\mathbf{S}_k,\mathbf{I}_n^m$, $\dot{-},\mathbf{Concat},\mathbf{D}$ - polynomial programs;

3) If Φ - k is a place functor, $\Psi_1,\dots,\Psi_k$ - n are place functors and are polynomial programs, then the functor $[J\Phi\Psi_1,\dots,\Psi_k]$ - polynomial program, i.e. belongs to the set $\mathbf{PPr}(\mathbf{U})$;

4) If Φ - n is a place functor, $\Psi_1,\dots,\Psi_k$ - $n+2$ are place functors and are polynomial programs, $\mathbf{P},\mathbf{P}_1$ - respectively n, $n+1$ are place polynomial functors, then the functor

$[R\mathbf{Bound}(\Phi,\mathbf{P}),\mathbf{Bound}(\Psi_1,\mathbf{P}_1),\dots,\mathbf{Bound}(\Psi_k,\mathbf{P}_1)]$ - polynomial program.

The set of functors defined according to items 2-4 will also be called polynomial programs, but in the alphabet $\mathscr{L}$. This set of functors will also be denoted as $\mathbf{PPr}$. It will be clear from the context in which alphabet $\mathscr{L}$ or $\mathscr{L}(\mathbf{U})$, the set $\mathbf{PPr}$ is considered.

The set of all operations of the standart word model $\mathbf{WordM}_{n,\mathbf{PPr}}$ coincides with the class of function $E_2(\Sigma)$[1, p.220. Definition 7], where Σ is an alphabet consisting of n different symbols $a_1,\dots,a_n$.

Theorem 3.1. Let Φ is an n-place polynomial program, i.e. $\Phi\in\mathbf{PPr}$, then there exists (can be constructed) such a word polynomial $\mathbf{P}(\overline{x})$ of the same place as for any set of argument words $\overline{\alpha}$ is true:

a) $\forall\mathbb{A}\mathbf{WordM}_{\mathbb{A}}\models|\Phi(\overline{\alpha})|<|\mathbf{P}(\overline{\alpha})|$.

b) $\forall \mathbb{A} \mathbf{WordM}_{\mathbb{A}} \models l_{\mathbf{P}_\Phi}(\overline{\alpha}) < |\mathbf{P}(\overline{\alpha})|$;

c) $\forall \mathbb{A} \mathbf{WordM}_{\mathbb{A}} \models Fl_{\mathbf{P}_\Phi}(\overline{\alpha}) < |\mathbf{P}(\overline{\alpha})|$.

Proof. See the definition of a simple evaluation of the functor Φ - pp. 8-13.

Note. Let $\Phi \in \mathbf{PPr}$ is an n-place functor, $\mathbb{A}$ - set of argument words, then there exists a word polynomial $\mathbf{P}(\overline{x})$ such that for any sequences argument words $\alpha_1, \ldots, \alpha_n$, the length of all used words in a simple calculation of the functor Φ on $\alpha_1, \ldots, \alpha_n$ and the number of interrogated words is limited to $|\mathbf{P}(\alpha_1, \ldots, \alpha_n)|$.

Note. Let $\mathbf{MT}$ is an oracle Turing machine with input alphabet $A = \{a_1, \ldots, a_k\}$, and oracle set B, whose running time is bounded by some polynomial $P(x_1, \ldots, x_n)$ with natural coefficients. Let $f_{\mathbf{MT}}(x_1, \ldots, x_n)$ is a vocabulary function, which is generated by the oracle $\mathbf{MT}$. Then we can construct such a functor $\Phi \in \mathbf{PPr}(\mathbf{U})$ of the same place, whose set of input words is the set of argument words $\{\mathbf{S}_1(\Lambda), \ldots, \mathbf{S}_k(\Lambda)\}$, which is true $\forall \alpha_1, \ldots, \alpha_k, \beta \ \mathbb{B} \vdash \Phi(\alpha_1 \ldots, \alpha_k) = \beta \Longleftrightarrow f_{\mathbf{MT}}(\alpha_1, \ldots, \alpha_k) = \beta$[2 p. 224. Theorem 6][2][1, Theorem 1 p. 228][3].

Note. Let $\Phi \in \mathbf{PPr}(\mathbf{U})$ -n - ary functor whose set of input words, is the set of argument words $\{\mathbf{S}_1(\Lambda), \ldots, \mathbf{S}_k(\Lambda)\}$. Let $\mathbb{B}$ be the interpretation of the oracle symbol $\mathbf{U}$. Then we can construct an oracle Turing machine $\mathbf{MT}$ - with input alphabet $A = \{a_1, \ldots, a_k\}$ and oracle set B, whose running time is bounded by some polynomial $P(x_1, \ldots, x_n)$ with natural coefficients, that for the dictionary function $f_{\mathbf{MT}}(x_1, \ldots, x_n)$ which is generated by the oracle $\mathbf{MT}$ under consideration it is true that

$\forall \alpha_1, \ldots, \alpha_k, \beta \ \mathbb{B} \vdash \Phi(\alpha_1 \ldots, \alpha_k) = \beta \Longleftrightarrow f_{\mathbf{MT}}(\alpha_1, \ldots, \alpha_k) = \beta$[[2 p. 224. Theorem 7][1, Theorem 1 p. 230].

Let $\mathbb{A}$ is the interpretation of the oracle symbol $\mathbf{U}$. Let us inductively define the set of functors, denoted as $\mathbf{PPr}(\mathbb{A})$:

1) Words of the form $\mathbf{U}$ - $\mathbf{PPr}(\mathbb{A})$ - program;

2) Words of the form $\mathbf{Z}$, $\boldsymbol{\delta}, \mathbf{Length}$, $\mathbf{S}_k, \mathbf{I}_n^m$, $\dot{-}, \mathbf{Concat}, \mathbf{D}$ - $\mathbf{PPr}(\mathbb{A})$ programs;

3) If Φ - k - ary functor, $\Psi_1, \ldots, \Psi_k$ - n -ary functors and are $\mathbf{PPr}(\mathbb{A})$ programs, then the functor $[J\Phi\Psi_1, \ldots, \Psi_k]$ - $\mathbf{PPr}(\mathbb{A})$ - program;

4) If Φ - n - ary functor, $\Psi_1, \ldots, \Psi_k$ - $n+2$ - ary funktors and are $\mathbf{PPr}(\mathbb{A})$ programs, $\mathbf{P}$ - $n+1$ - ary word polynomial, then if it is true $\mathbf{WordM}_{\mathbb{A}} \models \forall \overline{x}, y\{|[R\Phi\Psi_1, \ldots, \Psi_k](\overline{x}, y)| \leq |\mathbf{P}(\overline{x}, y)|\}$, then functor $[R\Phi\Psi_1, \ldots, \Psi_k]$ is a $\mathbf{PPr}(\mathbb{A})$ - program.

Note. If $\Phi \in \mathbf{PPr}(\mathbb{A})$, then we can construct such a word polynomial $\mathbf{P}(\overline{x})$, which is true

$\mathbf{WordM}_{\mathbb{A}} \models \forall \overline{x}\{|\Phi(\overline{x})| \leq |\mathbf{P}(\overline{x})|\}$.

The proof is by induction on the construction of the functor Φ, and within this induction, by induction on

[2]This theorem is easily transferred to the case when the Turing machine under consideration is an oracle machine

[3]All the word functions mentioned on pages 212-215 are **PPr** functions of the alphabet $\mathscr{L}$, so the theorem under consideration is easily transferred to an oracle Turing machine

the construction of the argument word.

Note. For any functor $\Phi \in \mathbf{PPr}(\mathbf{U})$, for any oracle $\mathbb{A}$, $\Phi \in \mathbf{PPr}(\mathbb{A})$ is true.

Note. Let $\mathbb{A}$ be the interpretation of the oracle symbol $\mathbf{U}$. For any functor $\Phi \in \mathbf{PPr}(\mathbb{A})$, we can construct such a functor $\Psi \in \mathbf{PPr}(\mathbf{U})$, which is true $\mathbf{WordM}_{\mathbb{A}} \models \forall \overline{x}[\Phi(\overline{x}) = \Psi(\overline{x})]$.

The proof is by induction on the construction of the functor Φ, and within this induction, by induction on the construction of the argument word.

For each natural number $k > 1$ we write the following defining equalities:

$exp_{\mathbf{k}}(\Lambda) = a_1$,

$exp_{\mathbf{k}}(\alpha a_i) = \mathbf{D}(\underbrace{a_1, \ldots, a_1}_{k-\text{раз}}, exp_{\mathbf{k}}(\alpha))$.

There is a primitive recursive word functor that satisfies these defining equalities. Let's denote it as $\mathbf{exp_k}$. For the functor $\mathbf{exp_k}$ true $\mathbf{WordM} \models \forall x[|\mathbf{exp_k}(x)| = k^{|x|}]$, $\forall \alpha\beta[\mathbf{exp_k}(\alpha) = \beta]$, then β is a natural number.

For $k > 1$ $\mathbf{exp_k}(\alpha)$ is the number of k - alphabetic words whose length is equal to the length of the word α, $\dfrac{\mathbf{exp_k}(\alpha a_1) - 1}{\mathbf{k} - 1}$ - number k - alphabetic words preceding in the lexicographic ordering of the word $|\alpha a_1|$.

Note. Note that for the relation $\mathbf{exp_k}(x) = y$ one can compose a functor $\mathbf{EXP_k}$ belonging to $\mathbf{PPr}$ such that $\mathbf{WordM} \models \forall xy[\mathbf{exp_k}(x) = y \Leftrightarrow \vdash \mathbf{EXP_k}(x, y) = \Lambda]$.

Part IV

Function words and their properties

Let's compose the following word term $\mathbf{Concat}(|\alpha|, \mathbf{Concat}(a_2, \mathbf{Concat}(\alpha, \mathbf{Concat}(\beta, \mathbf{Concat}(a_2, a_2)))))$. Let 1 be the designation of the argument word $S_1(\Lambda)$, 2 be the designation of the argument word $S_2(\Lambda)$, then the word term $\mathbf{Concat}(|\alpha|, \mathbf{Concat}(a_2, \mathbf{Concat}(\alpha, \mathbf{Concat}(\beta, \mathbf{Concat}(a_2, a_2)))))$ For clarity, we will denote in the form $\underbrace{1, \ldots, 1}_{|\alpha|\text{- раз}} 2\alpha\beta 22$.

Let $\mathbf{c}$ - such a functor for which in the calculus $\mathbf{CalcEq}$ we derive the equality

$\mathbf{c}(x, y) = \underbrace{1, \ldots, 1}_{|x|\text{- times}} 2xy22 = |x|2xy22.$

Let an arbitrary sequence of pairs of argument words be given $(\alpha_1, \gamma_1) \ldots, (\alpha_n, \gamma_n)$. This sequence will be called functional if the following conditions are met:

1. $\forall i[\gamma_i = \Lambda \vee \gamma_i = a_1]$,

2. $\forall i, j[\alpha_i = \alpha_j \to \gamma_i = \gamma_j]$.

Let us introduce a concept that will be of great importance in what follows.

Definition. 1. Λ is a function word.

2. If a sequence of pairs $(\alpha_1, \gamma_1) \dots, (\alpha_n, \gamma_n)$ is a functional, then a word of the form $\mathbf{Concat}(\mathbf{c}(\alpha_1, \gamma_1), \mathbf{Concat}(\mathbf{c}(\alpha_2, \gamma_2), \dots, \mathbf{Concat}(\mathbf{c}(\alpha_k, \gamma_k), \Lambda)), \dots,)$ - function word, where $1 \le k \le n$.

Visually, a functional word can be written in the form $|\alpha_1|2\alpha_1\gamma_1 22, \dots, |\alpha_k|2\alpha_k\gamma_k 22$.

The words of the sequence $\alpha_1, \dots, \alpha_k$ will be called the domain of definition of the functional word under consideration, and the words of the sequence $\gamma_1, \dots, \gamma_k$ will be called the corresponding values.

Note. Any functional word θ will be interpreted as a word according to its definition and as a function with the same name. Domain of definition and set of values of the function θ - domain of definition and set of values of the functional word θ, moreover, $\theta(\alpha) = \Lambda$, if and only if the word $\mathbf{c}(\alpha, \Lambda)$ is a subword of θ and $\theta(\alpha) = a_1$ if and only if the word $\mathbf{c}(\alpha, a_1)$ - subword of the word θ.

The domain of definition of the functional word θ will be denoted as $dom(\theta)$.

Let $\theta \subseteq \theta_1 (\theta_1 \supseteq \theta) \rightleftharpoons \forall x \in dom(\theta)[\theta(x) = \theta_1(x)]$.

Note. The relation $x \in dom(\theta)$ can be expressed using the **PPr** functor.

The function word θ is consistent with the set $\mathbb{A}$ $(\theta \subset \mathbb{A})$ if $\forall x \in dom(\theta)[\theta(x) = \Lambda \iff x \in \mathbb{A}]$.

Remark. For any function words θ_1, θ_2 that are compatible with the set $\mathbb{A}$, there exists function word θ consistent with $\mathbb{A}$ and $\theta_1 \subseteq \theta$, $\theta_2 \subseteq \theta$, e.g. $\theta_1 \cup \theta_2 (\mathbf{Concat}(\theta_1, \theta_2))$.

For the relation $\theta \subseteq \theta_1$, there exists a **PPr** functor ϕ of the alphabet $\mathscr{L}$ that is true $\mathbf{WordM} \models [\theta \subseteq \theta_1 \iff \phi(\theta, \theta_1) = \Lambda]$.

Let θ be some function word. For this functional word, we construct a set of argument words defined as $\mathbb{A}_\theta = \{\alpha : \alpha \in dom(\theta) \text{ and } \theta(\alpha) = \Lambda\}$.

For any term $t(\overline{x})$, for any sequence of argument words $\overline{\alpha}$, for any set of argument words $\mathbb{A}$, one can construct such a functional word $\theta_{\overline{\alpha},\mathbb{A},t}$, consistent with the set $\mathbb{A}$, that $\mathbb{A} \vdash t(\overline{\alpha}) = \beta \Leftrightarrow \mathbb{A}_{\theta_{\overline{\alpha},\mathbb{A},t}} \vdash t(\overline{\alpha}) = \beta$. To do this, it suffices to construct a calculation of the closed term $t(\overline{\alpha})$ on the set $\mathbb{A}$, collect all the interrogated words in this calculation, and use the obtained interrogated words to compose the corresponding functional word. Of course, the function word constructed in this way depends on the constructed calculation of the term $t(\overline{\alpha})$, but in this case the following property will be fulfilled: for any functional word $\theta \supseteq \theta_{\overline{\alpha},\mathbb{A},t}$, true $\mathbb{A} \vdash t(\overline{\alpha}) = \beta \Leftrightarrow \mathbb{A}_\theta \vdash t(\overline{\alpha}) = \beta$. This property is true for any quantifier-free sentence Φ (a sentence composed using logical connectives from equalities of closed terms): $\mathbf{WordM}_{\mathbb{A}} \models \Phi \Leftrightarrow \mathbf{WordM}_{\mathbb{A}_\theta} \models \Phi$.

Let **Fw** be a functor of the alphabet $\mathscr{L}$ for which:

1. $\forall \alpha[\vdash \mathbf{Fw}(\alpha) = \Lambda \vee \vdash \mathbf{Fw}(\alpha) = a_1]$;

2. $\vdash \mathbf{Fw}(\alpha) = \Lambda \Leftrightarrow \alpha$ - functional word.

3. Functor **Fw** - is a **PPr** functor.

Let us introduce the binary relation $\theta \subseteq \theta_1$:

$\theta \subseteq \theta_1 \Longleftrightarrow \mathbf{WordM} \models \mathbf{Fw}(\theta) \wedge \mathbf{Fw}(\theta_1) \wedge \forall x \in dom(\theta)[\theta(x) = \theta_1(x)]$.

The relation $\theta \subseteq \theta_1$ belongs to **PPr** of the alphabet $\mathscr{L}$.

Let us introduce a binary relation $\approx$:

$\mathbf{WordM} \models \theta \approx \theta_1 \Longleftrightarrow \mathbf{Fw}(\theta) \wedge \mathbf{Fw}(\theta_1) \wedge dom(\theta) = dom(\theta_1) \wedge \forall x \in dom(\theta)\theta(x) = \theta_1(x)$.

The relation $x \approx y$ belongs to **PPr** of the alphabet $\mathscr{L}$.

Remark. If $\theta \approx \theta_1$ then $\mathbf{Concat}(\theta, \theta_1) \approx \mathbf{Concat}(\theta_1, \theta)$, $\mathbf{Concat}(\theta, \theta_1) \approx \theta$.

Let $\mathbf{G}$ be a two-place functor of the alphabet $\mathscr{L}$ that satisfies the following conditions:

1. If θ is a function word, $\alpha \in dom(\theta)$ and $\theta(\alpha) = \gamma$, then $\vdash \mathbf{G}(\theta, \alpha) = \gamma$.

2. If θ is a function word, $\alpha \notin dom(\theta)$, then $\vdash \mathbf{G}(\theta, \alpha) = a_1$.

3. If θ is not a function word, then $\forall \alpha \vdash \mathbf{G}(\theta, \alpha) = a_1$.

4. Functor **G** - is a **PPr** functor.

The functor **G** has the following properties:

1. For any function words θ, θ_1 such that $\theta \subseteq \theta_1$, $\forall \alpha \in dom(\theta) \vdash [\mathbf{G}(\theta, \alpha) = \mathbf{G}(\theta_1, \alpha)]$.

2. Relation $x \subset \mathbf{U} \rightleftharpoons \mathbf{Fw}(x) = \Lambda \wedge \forall z \in dom(x))[(\mathbf{G}(x, z) = \Lambda \rightarrow \mathbf{U}(z) = \Lambda) \wedge (\mathbf{G}(x, z) = a_1 \rightarrow \mathbf{U}(z) = a_1)]$ belong **PPr**, i.e. there is a one-place **PPr** functor $\boldsymbol{\varphi}$ alphabet $\mathscr{L}(\mathbf{U})$ such that

$\mathbf{WordM}_{\mathbb{A}} \models \forall x[x \subset \mathbf{U} \Leftrightarrow \boldsymbol{\varphi}(x) = \Lambda]$.

Definition. Given a functor Φ, a sequence of argument words $\overline{\alpha}$, and an interpretation set $\mathbb{A}$. Let **P** be a simple computation of the functor Φ, on the sequence $\overline{\alpha}$, in the interpretation of $\mathbb{A}$. Then, using a simple calculation of **P**, we compose a function word:

1. Let's write out all the words from the set $(\mathbb{A}^+)_{\mathbf{P}_{\Phi,\overline{\alpha}}}$. Let these be the words $\beta_1, \ldots, \beta_k$, arrange them, for example, in lexicographic order.

2. Let's write out all the words from the set $(\mathbb{A}^-)_{\mathbf{P}_{\Phi,\overline{\alpha}}}$. Let these be the words $\gamma_1, \ldots, \gamma_s$, arrange them also in lexicographic order.

3. Let's make a functional word

$\mathbf{Concat}(\mathbf{c}(\beta_1, \Lambda), \ldots, \mathbf{Concat}(\mathbf{c}(\beta_k, \Lambda), \mathbf{Concat}(\mathbf{c}(\gamma_1, a_1), \ldots, \mathbf{Concat}(\mathbf{c}(\gamma_{s-1}, a_1), \mathbf{c}(\gamma_s, a_1))), \ldots,)$. A functional word composed in this way is called a functional word composed according to a simple calculation **P** functor Φ on the sequence $\overline{\alpha}$, in the interpretation of $\mathbb{A}$. Denote such a function word as $\theta_{\mathbf{SimpleFw},\Phi,\overline{\alpha},\mathbb{A}}$.

Definition. Terms of the form $|x|2x\mathbf{U}(x)22|z|2z\mathbf{U}(z)22, \ldots, |\mathrm{v}|2\mathrm{v}\mathbf{U}(\mathrm{v})22$ will be called functional terms of the alphabet $\mathscr{L}(\mathbf{U})$. The set of functional terms constructed in this way will be denoted as $\mathbf{F}\mathrm{term}_{\mathbf{U}}$, and the specific functional term of this set as $f_{\mathbf{term}}(x, z, \ldots, \mathrm{v})$.

Definition. Terms of the form $|x|2x\mathbf{G}(\mathbf{y},x)22|z|2z\mathbf{G}(\mathbf{y},z)22,\dots,|\mathrm{v}\ |2\mathrm{v}\mathbf{G}(\mathbf{y},\mathrm{v})22$ will be called functional terms of the alphabet $\mathscr{L}$. The set of functional terms constructed in this way will be denoted as **F**term, and the specific functional term of this set as $f^*_{\mathbf{term}}(\mathbf{y},x,z,\dots,\mathrm{v})$.

Properties. 1. $f^*_{\mathbf{term}}(f_{\mathbf{term}}(x,z,\dots,\mathrm{v}),x,z,\dots,\mathrm{v}) = f_{\mathbf{term}}(x,z,\dots,\mathrm{v})$ - like words.

2. $f^*_{\mathbf{term}}(f^*_{\mathbf{term}}(x,z,\dots,\mathrm{v}),x,z,\dots,\mathrm{v}) = f^*_{\mathbf{term}}(x,z,\dots,\mathrm{v})$ - like words.

3. For any function word θ true $\forall x,z,\dots,\mathrm{v} \in dom(\theta) f^*_{\mathbf{term}}(\theta,x,z,\dots,\mathrm{v}) \subseteq \theta$.

Definition of a functor for constructing a function word

For each n - place functor Φ, we define the functional word construction functor associated with this functor.

We will carry out the definition by induction on the construction of the functor Φ..

1.For original functors: $\mathbf{S}_k, \mathbf{Z}, \boldsymbol{\delta}, \mathbf{Length}, \dot{-}, \mathbf{Concat}, \mathbf{D}, \mathbf{I}^n_k$, $\mathbf{U}$:

$\Theta_{\mathbf{S}_k} = \mathbf{Z}$, $\Theta_{\mathbf{Z}} = \mathbf{Z}$, $\Theta_{\boldsymbol{\delta}} = \mathbf{Z}$, $\Theta_{\mathbf{Length}} = [J\mathbf{ZI}^2_2]$, $\Theta_{\dot{-}} = [J\mathbf{ZI}^2_2]$, $\Theta_{\mathbf{Concat}} = [J\mathbf{ZI}^2_2]$, $\Theta_{\mathbf{D}} = [J\mathbf{ZI}^2_2]$, $\Theta_{\mathbf{I}^n_k} = [J\mathbf{ZI}^n_k]$, $\Theta_{\mathbf{U}} = [J\mathbf{cI}^1_1\mathbf{U}]$.

For these functors in the calculus **CalcEq** the equalities are derivable: $\Theta_{\mathbf{S}_k}(x_1) = \Lambda$, $\Theta_{\mathbf{Z}}(x_1) = \Lambda$, $\Theta_{\boldsymbol{\delta}} = \Lambda$, $\Theta_{\mathbf{Length}}(x_1) = \Lambda$, $\Theta_{\dot{-}}(x_1,x_2) = \Lambda$, $\Theta_{\mathbf{Concat}}(x_1,x_2) = \Lambda$, $\Theta_{\mathbf{D}}(x_1,x_2) = \Lambda$, $\Theta_{\mathbf{I}^n_k}(x_1,\dots,x_n) = \Lambda$, $\vdash \Theta_{\mathbf{U}}(x_1) = \mathbf{c}(x_1,\mathbf{U}(x_1))$ - in calculus $\mathbf{CalcEq_U}$ and the expression $\forall\alpha\ \forall\mathbb{A}\ \mathbb{A} \vdash \mathbf{G}(\Theta_{\mathbf{U}}(\alpha),\alpha) = \mathbf{U}(\alpha)$ is true.

2. For the functor $[J\Phi\Psi_1,\dots,\Psi_k]$. Let for the functor Φ functor built Θ_Φ, for the functor Ψ_1 functor built Θ_{Ψ_1}, etc. for the functor Ψ_k functor built Θ_{Ψ_k}, then $\Theta_{[J\Phi\Psi_1,\dots,\Psi_k]} \rightleftharpoons [J\mathbf{Concat}^{k+1}[J\Theta_\Phi\Psi_1\dots\Psi_k],\Theta_{\Psi_1}\dots\Theta_{\Psi_k}]$ at $k \geq 2$. The obtained functor is n - place and the following provable equations are true for it

$\vdash [J\mathbf{Concat}^{k+1}[J\Theta_\Phi\Psi_1\dots\Psi_k],\Theta_{\Psi_1}\dots\Theta_{\Psi_k}](x_1,\dots,x_n) =$

$\mathbf{Concacat}^{k+1}([J\Theta_\Phi\Psi_1\dots\Psi_k](x_1,\dots,x_n),\Theta_{\Psi_1}(x_1,\dots,x_n)\dots\Theta_{\Psi_k}(x_1,\dots,x_n))$

$\vdash \mathbf{Concat}^{k+1}([J\Theta_\Phi\Psi_1\dots\Psi_k](x_1,\dots,x_n),\Theta_{\Psi_1}(x_1,\dots,x_n)\dots\Theta_{\Psi_k}(x_1,\dots,x_n)) =$

$\mathbf{Concat}^{k+1}(\Theta_\Phi(\Psi_1(x_1,\dots,x_n)\dots\Psi_k(x_1,\dots,x_n)),\Theta_{\Psi_1}(x_1,\dots,x_n)\dots\Theta_{\Psi_k}(x_1,\dots,x_n))$.

So, we have $\vdash \Theta_{[J\Phi\Psi_1\dots\Psi_k]}(x_1,\dots,x_n) = [J\mathbf{Concat}^{k+1}[J\Theta_\Phi\Psi_1\dots\Psi_k],\Theta_{\Psi_1}\dots\Theta_{\Psi_k}](x_1,\dots,x_n) =$

$\mathbf{Concat}^{k+1}(\Theta_\Phi(\Psi_1(x_1,\dots,x_n)\dots\Psi_k(x_1,\dots,x_n)),\Theta_{\Psi_1}(x_1,\dots,x_n)\dots\Theta_{\Psi_k}(x_1,\dots,x_n))$,

$\vdash \Theta_{[J\Phi\Psi_1\dots\Psi_k]}(x_1,\dots,x_n) = \mathbf{Concat}^{k+1}(\Theta_\Phi(\Psi_1(x_1\dots,x_n)\dots\Psi_k(x_1,\dots,x_n)),\Theta_{\Psi_1}(x_1,\dots,x_n)\dots\Theta_{\Psi_k}(x_1,\dots,x_n))$,

$\vdash \Theta_{[J\Phi\Psi_1\dots\Psi_k]}(\overline{x}) = \mathbf{Concat}([J\Theta_\Phi\Psi_1,\dots,\Psi_k](\overline{x}),(\mathbf{Concat}(\Theta_{\Psi_1}(\overline{x}),\dots,\mathbf{Concat}(\Theta_{\Psi_{k-1}}(\overline{x}),\Theta_{\Psi_k}(\overline{x}))),\dots,)$.

Let Φ - k - place functor($k=1$), then $\Theta_{[J\Phi\Psi_1]} \rightleftharpoons [J\mathbf{Concat}[J\Theta_\Phi\Psi_1]\Theta_{\Psi_1}]$, then

$\Theta_{[J\Phi\Psi_1]}(x_1,\dots,x_n) \rightleftharpoons [J\mathbf{Concat}[J\Theta_\Phi\Psi_1]\Theta_{\Psi_1}](x_1,\dots,x_n) = \mathbf{Concat}(\Theta_\Phi(\Psi_1(x_1,\dots,x_n)),\Theta_\Psi(x_1,\dots,x_n))$

3. For the functor $[R\alpha\Psi_1\dots\Psi_k]$ and funtor $[R\Phi\Psi_1\dots\Psi_k]$. Let for the functor Φ functor built Θ_Φ, for functor Ψ_1 functor built Θ_{Ψ_1}, etc. for the functor Ψ_k functor built Θ_{Ψ_k}, then for the functor $\Theta_{[R\Phi\Psi_1,\dots,\Psi_k]}$in calculus

$\mathbf{CalcEq_U}$ holds (defining equality)(see theorem 1)[4]:

$\vdash \Theta_{[R\alpha\Psi_1\ldots\Psi_k]}(\Lambda) = \Lambda.$

$\vdash \Theta_{[R\alpha\Psi_1\ldots\Psi_k]}(x_1 a_i) = \mathbf{Concat}(\Theta_{\Psi_i}(x_1, [R\Phi\Psi_1, \ldots, \Psi_k](x_1)), \Theta_{[R\alpha\Psi_1,\ldots,\Psi_k]}(x_1))$, at $i \leq k$.

$\vdash \Theta_{[R\alpha\Psi_1\ldots\Psi_k]}(x_1 a_i) = \Theta_{[R\alpha\Psi_1,\ldots,\Psi_k]}(x_1)$, at $i > k$

$\vdash \Theta_{[R\Phi\Psi_1\ldots\Psi_k]}(\overline{x}, \Lambda) = \Theta_\Phi(\overline{x}).$

$\vdash \Theta_{[R\Phi\Psi_1\ldots\Psi_k]}(\overline{x}, x_{n+1} a_i) = \mathbf{Concat}(\Theta_{\Psi_i}(\overline{x}, x_{n+1}, [R\Phi\Psi_1, \ldots, \Psi_k](\overline{x}, x_{n+1})), \Theta_{[R\Phi\Psi_1\ldots\Psi_k]}(\overline{x}, x_{n+1}))$,

at $i \leq k$.

$\vdash \Theta_{[R\Phi\Psi_1\ldots\Psi_k]}(\overline{x}, x_{n+1} a_i) = \Theta_{[R\Phi\Psi_1\ldots\Psi_k]}(\overline{x}, x_{n+1}))$, at $i > k$.

Note. Let Φ be an arbitrary functor that belongs to **PPr**, then the functor Θ_Φ belongs to **PPr**. See the corollary 2.1-2.3 of the definition of a simple calculation pp.7-13 and Theorem 3.1 p.16.

Note. Let Φ be a functor of the alphabet $\mathscr{L}$, then in the calculus **CalEq** it is true $\vdash \Theta_\Phi(\overline{x}) = \Lambda$.

Theorem 4.1. Let Φ be an arbitrary n - place functor. For any interpretation set $\mathbb{A}$, any sequence of argument words $\overline{\alpha}, \beta$, the following is true:

If $\mathbb{A} \vdash \Theta_\Phi(\overline{\alpha}) = \beta$ then β is a function word and $\theta_{\mathbf{SimpleFw},\Phi,\overline{\alpha},\mathbb{A}} \subseteq \beta \subset \mathbb{A}$;

Proof. The proof is carried out by induction on the construction of the functor.

Induction basis. For original functors $\mathbf{S}_k, \mathbf{Z}, \boldsymbol{\delta}, \mathbf{Length}, \dot{-}, \mathbf{Concat}, \mathbf{D}, \mathbf{I}_k^n$ the proof is immediate. For the functor $\mathbf{U}$, we get: $\mathbb{A} \vdash \Theta_\mathbf{U}(\alpha) = \beta$, if and only if

$(\beta = |\alpha| 2\alpha 22 \& \alpha \in \mathbb{A}\ ) \vee (\beta = |\alpha| 2\alpha a_1 22 \& \alpha \notin \mathbb{A})$, then $\beta = \theta_{\mathbf{SimpleFw},\mathbf{U},\alpha,\mathbb{A}}$ and $\beta \subset \mathbb{A}$.

Inductive assumption. 1. Let the theorem be true for the functor Φ, functors $\Psi_1, \ldots, \Psi_k$. Let us prove that the theorem is true for the functor $[J\Phi\Psi_1, \ldots, \Psi_k]$.

By the inductive hypothesis, we have: if $\mathbb{A} \vdash \Theta_{\Psi_1}(\overline{\alpha}) = \gamma_1, \ldots, A \vdash \Theta_{\Psi_k}(\overline{\alpha}) = \gamma_k$, then γ_i - function words и $\theta_{\mathbf{SimpleFw},\Psi_1,\overline{\alpha},\mathbb{A}} \subseteq \gamma_1 \subset \mathbb{A}, \ldots, \theta_{\mathbf{SimpleFw},\Psi_k,\overline{\alpha},\mathbb{A}} \subseteq \gamma_k \subset \mathbb{A}$,

Function word $\theta_{\mathbf{SimpleFw},\Psi_i,\overline{\alpha},\mathbb{A}}$, composed according to sets $(\mathbb{A}^+)_{\mathbf{P}_{\Psi_i},\overline{\alpha}}$, $(\mathbb{A}^-)_{\mathbf{P}_{\Psi_i},\overline{\alpha}}$. then

$\bigcup_{i=1}^{k} \theta_{\mathbf{SimpleFw},\Psi_i,\overline{\alpha},\mathbb{A}} \subseteq \mathbf{Concat}(\Theta_{\Psi_1}(\overline{\alpha}), \ldots, \mathbf{Concat}(\Theta_{\Psi_{k-1}}(\overline{\alpha}), \Theta_{\Psi_k}(\overline{\alpha})), \ldots,).$

For $\mathbb{A} \vdash \Psi_i(\overline{\alpha}) = \beta_i$, if $\mathbb{A} \vdash \Theta_\Phi(\beta_1, \ldots, \beta_k) = \eta$, then η - function word and $\theta_{\mathbf{SimpleFw},\Phi,\beta_1,\ldots,\beta_k,\mathbb{A}} \subseteq \eta \subset \mathbb{A}$, Function word $\theta_{\mathbf{SimpleFw},\Phi,\overline{\beta},\mathbb{A}}$ composed according to the set $(\mathbb{A}^+)_{\mathbf{P}_\Phi,\overline{\beta}}$, $(\mathbb{A}^-)_{\mathbf{P}_\Phi,\overline{\beta}}$, then

$\theta_{\mathbf{SimpleFw},\Phi,\beta_1,\ldots,\beta_k,\mathbb{A}} \subseteq \mathbf{Concat}(\Theta_\Phi(\beta_1, \ldots, \beta_k), \mathbf{Concat}(\Theta_{\Psi_1}(\overline{\alpha}), \ldots, \mathbf{Concat}(\Theta_{\Psi_{k-1}}(\overline{\alpha}), \Theta_{\Psi_k}(\overline{\alpha})), \ldots,)$, then

$\theta_{\mathbf{SimpleFw},\Phi,\beta_1,\ldots,\beta_k,\mathbb{A}} \subseteq \mathbf{Concat}(\Theta_\Phi(\Psi_1(\overline{\alpha}), \ldots, \Psi_л(\overline{\alpha})), \mathbf{Concat}(\Theta_{\Psi_1}(\overline{\alpha}), \ldots, \mathbf{Concat}(\Theta_{\Psi_{k-1}}(\overline{\alpha}), \Theta_{\Psi_k}(\overline{\alpha})), \ldots,).$

Function word $\theta_{\mathbf{SimpleFw},[J\Phi\Psi_1,\ldots,\Psi_k],\overline{\alpha},\mathbb{A}}$ composed according to set

$(\mathbb{A}^+)_{\mathbf{P}_{[J\Phi\Psi_1,\ldots,\Psi_k]},\overline{\alpha}} = \bigcup_{i=1}^{k} (\mathbb{A}^+)_{\mathbf{P}_{\Psi_i},\overline{\alpha}} \bigcup (\mathbb{A}^+)_{\mathbf{P}_\Phi,\overline{\beta}}$, $(\mathbb{A}^-)_{\mathbf{P}_{[J\Phi\Psi_1,\ldots,\Psi_k]},\overline{\alpha}} = \bigcup_{i=1}^{k} (\mathbb{A}^-)_{\mathbf{P}_{\Psi_i},\overline{\alpha}} \bigcup (\mathbb{A}^-)_{\mathbf{P}_\Phi,\overline{\beta}}$, then, according

[4] see Application

to the defining equality for $\Theta_{[J\Phi\Psi_1,\dots,\Psi_k]}$, we get $\theta_{\mathbf{SimpleFw},[J\Phi\Psi_1,\dots,\Psi_k],\overline{\alpha},\mathbb{A}} \subseteq \Theta_{[J\Phi\Psi_1,\dots,\Psi_k]}(\overline{\alpha}) \subset \mathbb{A}$.

2. Let the theorem be true for the functor Φ, the functors $\Psi_1,\dots,\Psi_k$. Let us prove that the theorem is true for the functor $[R\Phi\Psi_1,\dots,\Psi_k]$.

According to the defining equalities for the functor $\Theta_{[R\Phi\Psi_1,\dots,\Psi_k]}$, we have:

$\vdash \Theta_{[R\Phi\Psi_1,\dots,\Psi_k]}(\overline{x},\Lambda) = \Theta_\Phi(\overline{x})$.

$\vdash \Theta_{[R\Phi\Psi_1,\dots,\Psi_k]}(\overline{x},za_i) = \mathbf{Concat}(\Theta_{\Psi_i}(\overline{x},\beta,[R\Phi\Psi_1,\dots,\Psi_k](\overline{x},z)),\Theta_{[R\Phi\Psi_1,\dots,\Psi_k]}(\overline{x},z))$.

Induction basis.

By the inductive hypothesis, we have:

$\mathbb{A} \vdash \Theta_\Phi(\overline{\alpha}) = \beta$ - function word and $\theta_{\mathbf{SimpleFw},\Phi,\overline{\alpha},\mathbb{A}} \subseteq \beta \subset \mathbb{A}$. Function word $\theta_{\mathbf{SimpleFw},\Phi,\overline{\alpha},\mathbb{A}}$ built on sets: $(\mathbb{A}^+)_{\mathbf{P}_{\Phi,\overline{\alpha}}}$, $(\mathbb{A}^-)_{\mathbf{P}_{\Phi,\overline{\alpha}}}$. Function word $\theta_{\mathbf{SimpleFw},[R\Phi\Psi_1,\dots,\Psi_k],\overline{\alpha},\Lambda,\mathbb{A}}$ also built on sets: $(\mathbb{A}^+)_{\mathbf{P}_{\Phi,\overline{\alpha}}}$, $(\mathbb{A}^-)_{\mathbf{P}_{\Phi,\overline{\alpha}}}$, taking into account the defining equality, $\Theta_{[R\Phi\Psi_1,\dots,\Psi_k]}(\overline{\alpha},\Lambda) = \Theta_\Phi(\overline{\alpha})$, we obtain $\theta_{\mathbf{SimpleFw},[R\Phi\Psi_1,\dots,\Psi_k],\overline{\alpha},\Lambda,\mathbb{A}} \subseteq \beta$ и $\beta \subset \mathbb{A}$.

Induction hypothesis. Take a sequence of argument words $\alpha_1,\dots,\alpha_n.\beta a_i$.

By the inductive hypothesis, we have:

If $\mathbb{A} \vdash \Theta_{[R\Phi\Psi_1,\dots,\Psi_k]}(\overline{\alpha},\beta) = \gamma$, then γ - function word and $\theta_{\mathbf{SimpleFw},[R\Phi\Psi_1,\dots,\Psi_k],\overline{\alpha},\beta,\mathbb{A}} \subseteq \gamma \subset \mathbb{A}$.

Let $\mathbb{A} \vdash [R\Phi\Psi_1,\dots,\Psi_k](\overline{\alpha},\beta) = \eta$.

If $\mathbb{A} \vdash \Theta_{\Psi_i}(\overline{\alpha},\beta,\eta) = \xi_i$, then ξ_i - function word and $\theta_{\mathbf{SimpleFw},\Psi_i,\overline{\alpha},\beta,\eta,\mathbb{A}} \subseteq \xi_i \subset \mathbb{A}$.

According to the definition of the function word $\theta_{\mathbf{SimpleFw},[R\Phi\Psi_1,\dots,\Psi_k],\overline{\alpha},\beta,\mathbb{A}}$ it is built on sets:
$(\mathbb{A}^+)_{\mathbf{P}_{[R\Phi\Psi_1,\dots,\Psi_k],\overline{\alpha},\beta}}$, $(\mathbb{A}^-)_{\mathbf{P}_{[R\Phi\Psi_1,\dots,\Psi_k],\overline{\alpha},\beta}}$, then
$\theta_{\mathbf{SimpleFw},[R\Phi\Psi_1,\dots,\Psi_k],\overline{\alpha},\beta,\mathbb{A}} \subseteq \mathbf{Concat}(\Theta_{\Psi_i}(\overline{\alpha},\beta,[R\Phi\Psi_1,\dots,\Psi_k](\overline{\alpha},\beta)),\Theta_{[R\Phi\Psi_1,\dots,\Psi_k]}(\overline{\alpha},\beta))$.

Function word $\theta_{\mathbf{SimpleFw},[R\Phi\Psi_1,\dots,\Psi_k],\overline{\alpha},\beta a_i,\mathbb{A}}$ built on sets:
$(\mathbb{A}^+)_{\mathbf{P}_{[R\Phi\Psi_1,\dots,\Psi_k],\overline{\alpha},\beta a_i}} = (\mathbb{A}^+)_{\mathbf{P}_{\Psi_i},\overline{\alpha},\beta,\eta} \bigcup (\mathbb{A}^+)_{\mathbf{P}_{[R\Phi\Psi_1,\dots,\Psi_k],\overline{\alpha},\beta}}$,
$(\mathbb{A}^-)_{\mathbf{P}_{[R\Phi\Psi_1,\dots,\Psi_k],\alpha_1,\dots,\alpha_n,\beta a_i}} = (\mathbb{A}^-)_{\mathbf{P}_{\Psi_i},\overline{\alpha},\beta,\eta} \bigcup (\mathbb{A}^-)_{\mathbf{P}_{[R\Phi\Psi_1,\dots,\Psi_k],\overline{\alpha},\beta}}$, then
$\theta_{\mathbf{SimpleFw},[R\Phi\Psi_1,\dots,\Psi_k],\overline{\alpha},\beta a_i,\mathbb{A}} = \theta_{\mathbf{SimpleFw},\Psi_i,\overline{\alpha},\beta,\eta,\mathbb{A}} \bigcup \theta_{\mathbf{SimpleFw},[R\Phi\Psi_1,\dots,\Psi_k],\overline{\alpha},\beta,\mathbb{A}}$, then
$\theta_{\mathbf{SimpleFw},[R\Phi\Psi_1,\dots,\Psi_k],\overline{\alpha},\beta a_i,\mathbb{A}} \subseteq \mathbf{Concat}(\Theta_{\Psi_i}(\overline{\alpha},\beta,[R\Phi\Psi_1,\dots,\Psi_k](\overline{\alpha},\beta)),\Theta_{[R\Phi\Psi_1,\dots,\Psi_k]}(\overline{\alpha},\beta))$. Considering defining equality $\Theta_{[R\Phi\Psi_1,\dots,\Psi_k]}(\overline{\alpha},\beta a_i) = \mathbf{Concat}(\Theta_{\Psi_i}(\overline{\alpha},\beta,[R\Phi\Psi_1,\dots,\Psi_k](\overline{\alpha},\beta)),\Theta_{[R\Phi\Psi_1,\dots,\Psi_k]}(\overline{\alpha},\beta))$, we get $\theta_{\mathbf{SimpleFw},[R\Phi\Psi_1,\dots,\Psi_k],\overline{\alpha},\beta a_i,\mathbb{A}} \subseteq \Theta_{[R\Phi\Psi_1,\dots,\Psi_k]}(\overline{\alpha},\beta a_i) \subset \mathbb{A}$.

The remaining recursion axioms (15,16,17,20) are treated similarly.

Theorem 4.2 Let Φ be an arbitrary $n-$ place functor of the alphabet $\mathscr{L}(\mathbf{U})$. For any interpretation set $\mathbb{A}$, any sequence of argument words $\overline{\alpha}$, it is true $\mathbf{WordM}_\mathbb{A} \models \Theta_\Phi(\overline{\alpha}) \approx \Theta_{\Theta_\Phi}(\overline{\alpha})$.

The proof is carried out by induction on the construction of the functor Φ[5].

Note. Let Φ be an arbitrary $n-$ place functor of the alphabet $\mathscr{L}(\mathbf{U})$. For any interpretation set $\mathbb{A}$, it is true $\mathbf{WordM}_{\mathbb{A}} \models \forall \overline{x}[\theta_{\mathbf{SimpleFw},\Phi,\overline{x},\mathbb{A}} \approx \Theta_{\Phi}(\overline{x})]$.

Part V

Converting alphabetical expressions $\mathscr{L}(\mathrm{U})$, to alphabetical expressions $\mathscr{L}$

By induction on the construction of a functor, we construct a transformation, denoted as $*$, of functor of the alphabet $\mathscr{L}(\mathbf{U})$ into functor of the alphabet $\mathscr{L}$.

For initial functors:

1. $(\mathbf{S}_k)^* = [J\mathbf{S}_k\mathbf{I}_2^2]$,

2. $(\mathbf{Z})^* = [J\mathbf{Z}\mathbf{I}_2^2]$,

3. $(\boldsymbol{\delta})^* = [J\boldsymbol{\delta}\mathbf{I}_2^2]$,

4. $(\mathbf{U})^* = \mathbf{G}$,

5. $(\mathbf{Length})^* = [J\mathbf{Length}\mathbf{I}_2^3\mathbf{I}_3^3]$,

6. $(\dot{-})^* = [J\dot{-}\mathbf{I}_2^3\mathbf{I}_3^3]$,

7. $(\mathbf{Concat})^* = [J\mathbf{Concat}\mathbf{I}_2^3\mathbf{I}_3^3]$,

8. $(\mathbf{D})^* = [J\mathbf{D}\mathbf{I}_2^3\mathbf{I}_3^3]$,

9. $(\mathbf{I}_k^n)^* = [J\mathbf{I}_k^n\mathbf{I}_2^{n+1}, \ldots, \mathbf{I}_{n+1}^{n+1}]$,

1. $([J\Phi\Psi_1, \ldots, \Psi_k])^* = [J(\Phi)^*\mathbf{I_1^{n+1}}(\Psi_1)^*, \ldots, (\Psi_k)^*]$,

2. $([R\alpha\Phi_1, \ldots, \Phi_m])^* = [R\mathbf{Const}_\alpha^2(\Phi_1)^*, \ldots, (\Phi_m)^*]$,

3. $([R\Phi\Psi_1, \ldots, \Psi_m])^* = [R(\Phi)^*(\Psi_1)^*, \ldots, (\Psi_m)^*]$.

Note. If the functor Φ is a functor of the alphabet $\mathscr{L}$, then the first argument of the functor $(\Phi)^*$ is a dummy variable $\vdash \Phi(\overline{x}) = (\Phi)^*(\mathbf{y}, \overline{x})$.

Theorem 5.1. For any n - place functor Φ, $\forall \mathbb{A}$, $\forall \overline{\alpha}$ $\forall \theta \supseteq \theta_{\mathbf{SimpleFw},\Phi,\overline{\alpha},\mathbb{A}}$ true
$\mathbb{A} \vdash [\Phi(\overline{\alpha}) = (\Phi)^*(\theta, \overline{\alpha})]$ ($\mathbf{WordM}_{\mathbb{A}} \models [\Phi(\overline{\alpha}) = (\Phi)^*(\theta, \overline{\alpha})]$).

Proof. The proof is by induction on the construction of the functor, inside this induction for a recursive functor, the proof is by induction on the construction of the argument word.

Basis of induction. Initial functors

For initial functors: $\mathbf{S}_k, \mathbf{Z}, \boldsymbol{\delta}, \mathbf{Length}, \dot{-}, \mathbf{Concat}, \mathbf{D}, \mathbf{I}_k^n$ can be verified directly by writing out the indicated functor ϕ and functor $(\phi)^*$.

[5]see Application

Let us prove the theorem for the functor $\mathbf{U}$. According to the definition $(\mathbf{U})^* = \mathbf{G}$, need to show $\forall \mathbb{A}\ \forall \alpha$ $\forall \theta \supseteq \theta_{\mathbf{SimpleFw},\mathbf{U},\alpha,\mathbb{A}}\ \ \mathbb{A} \vdash [\mathbf{U}(\alpha) = \mathbf{G}(\theta, \alpha)]$.

Let $\alpha \in \mathbb{A}$, then $\mathbf{U}(\alpha) = \Lambda$ - axiom and is a simple calculation of the functor $\mathbf{U}$ on the word α. As a functional word, we take the word $\theta_{\mathbf{SimpleFw},\mathbf{U},\alpha,\mathbb{A}} = \mathbf{c}(\alpha, \Lambda) = |\alpha|2\alpha 22$, then according to the definition of the functor $\mathbf{G}$, we get $\forall \theta \supseteq \theta_{\mathbf{SimpleFw},\mathbf{U},\alpha,\mathbb{A}} \vdash \mathbf{G}(\theta, \alpha) = \Lambda$. Let $\mathbf{P}_{\theta,\alpha}$ - for example, a simple calculation of the functor $\mathbf{G}$ on a sequence of words θ, α, then sequence of equalities $\mathbf{U}(\alpha) = \Lambda, \mathbf{P}_{\theta,\alpha}, \mathbf{U}(\alpha) = \mathbf{G}(\theta, \alpha)$ - derivation of equality $\mathbf{U}(\alpha) = \mathbf{G}(\theta, \alpha)$, when interpreting the function symbol $\mathbf{U}$ by the set $\mathbb{A}$.

Likewise: let $\alpha \notin \mathbb{A}$, then $\mathbf{U}(\alpha) = a_1$ - is a simple calculation of the functor $\mathbf{U}$ on the word α. As a functional word, we take the word $\theta_{\mathbf{SimpleFw},\mathbf{U},\alpha,\mathbb{A}} = \mathbf{c}(\alpha, a_1) = |\alpha|2\alpha a_1 22(|\alpha|2\alpha 122)$, then according to the definition of the functor $\mathbf{G}$, we get $\forall \theta \supseteq \theta_{\mathbf{SimpleFw},\mathbf{U},\alpha,\mathbb{A}}, \vdash \mathbf{G}(\theta, \alpha) = a_1 (\vdash \mathbf{G}(\theta, \alpha) = 1)$. Let $\mathbf{P}_{\theta,\alpha}$ - a simple calculation of the functor $\mathbf{G}$ on a sequence of words θ, α , then sequence of equalities $\mathbf{U}(\alpha) = a_1, \mathbf{P}_{\theta,\alpha}, \mathbf{U}(\alpha) = \mathbf{G}(\theta, \alpha)$ -derivation of equality $\mathbf{U}(\alpha) = \mathbf{G}(\theta, \alpha)$, when interpreting the function symbol $\mathbf{U}$ by the set $\mathbb{A}$.

(a) Induction hypothesis. Let the theorem be true for functors: $\Phi, \Psi_1, \ldots, \Psi_k$, prove the theorem for the functor $[J\Phi\Psi_1, \ldots, \Psi_k]$. Denote $\mathfrak{f} \leftrightharpoons [J\Phi\Psi_1, \ldots, \Psi_k]$.

We have: i) for the set of argument words $\mathbb{A}$, the sequence of argument words $\alpha_1, \ldots, \alpha_n$, for the functor Ψ_1 true $\forall \theta \supseteq \theta_{\mathbf{SimpleFw},\Psi_1,\overline{\alpha},\mathbb{A}}$, $\mathbb{A} \vdash \Psi_1(\overline{\alpha}) = (\Psi_1)^*(\theta, \overline{\alpha})$,..., for the functor Ψ_k, true $\forall \theta \supseteq \theta_{\mathbf{SimpleFw},\Psi_k,\overline{\alpha},\mathbb{A}}$, $\mathbb{A} \vdash \Psi_k(\overline{\alpha}) = (\Psi_k)^*(\theta, \overline{\alpha})$.

ii) for the set of argument words $\mathbb{A}$, the sequence of argument words $\beta_1, \ldots, \beta_k$, for the functor Φ true $\forall \theta \supseteq \theta_{\mathbf{SimpleFw},\Phi,\overline{\beta},\mathbb{A}}$, $\mathbb{A} \vdash \Phi(\overline{\beta}) = (\Phi)^*(\theta, \overline{\beta})$.

Function word $\theta_{\mathbf{SimpleFw},\mathfrak{f},\overline{\alpha},\mathbb{A}}$, according to his definition, is composed of sets:
$(\mathbb{A}^+)_{\mathbf{P}_{[J\Phi\Psi_1,\ldots,\Psi_k],\overline{\alpha}}} = \bigcup_{i=1}^{k} (\mathbb{A}^+)_{\mathbf{P}_{\Psi_i,\overline{\alpha}}} \bigcup (\mathbb{A}^+)_{\mathbf{P}_{\Phi,\overline{\gamma}}}$, $(\mathbb{A}^-)_{\mathbf{P}_{[J\Phi\Psi_1,\ldots,\Psi_k],\overline{\alpha}}} = \bigcup_{i=1}^{k} (\mathbb{A}^-)_{\mathbf{P}_{\Psi_i,\overline{\alpha}}} \bigcup (\mathbb{A}^-)_{\mathbf{P}_{\Phi,\overline{\gamma}}}]$, then
$\theta_{\mathbf{SimpleFw},\mathfrak{f},\overline{\alpha},\mathbb{A}} \supseteq \theta_{\mathbf{SimpleFw},\Psi_i,\overline{\alpha},\mathbb{A}}$ and $\theta_{\mathbf{SimpleFw},\mathfrak{f},\overline{\alpha},\mathbb{A}} \supseteq \theta_{\mathbf{SimpleFw},\Phi,\overline{\beta},\mathbb{A}}$. Let $\theta_{\mathfrak{f}} \supseteq \theta_{\mathbf{SimpleFw},\mathfrak{f},\overline{\alpha},\mathbb{A}}$.

The following sequence of equalities:

1. $(\Psi_1)^*(\theta_{\mathfrak{f}}, \overline{\alpha}) = \gamma_1, \ldots, (\Psi_k)^*(\theta_{\mathfrak{f}}, \overline{\alpha}) = \gamma_k$ - Calculate

2.$(\Phi)^*(\theta_{\mathfrak{f}}, \gamma_1, \ldots, \gamma_k) = \eta$ - Calculate,

3. $[J(\Phi)^* \mathbf{I}_1^{n+1} (\Psi_1)^*, \ldots, (\Psi)_k^*](\theta_{\mathfrak{f}}, \overline{\alpha}) = (\Phi)^*(\theta_{\mathfrak{f}}, (\Psi_1)^*(\theta_{\mathfrak{f}}, \overline{\alpha}), \ldots, (\Psi)_k^*(\theta_{\mathfrak{f}}, \overline{\alpha}))$ - almost an axiom,

4. $(\Phi)^*(y, x_1, \ldots, x_k) = (\Phi)^*(y, x_1, \ldots, x_k)$ - axiom,

5. $(\Phi)^*(\theta_{\mathfrak{f}}, (\Psi_1)^*(\theta_{\mathfrak{f}}, \overline{\alpha}), \ldots, (\Psi)_k^*(\theta_{\mathfrak{f}}, \overline{\alpha})) = (\Phi)^*(\theta_{\mathfrak{f}}, \gamma_1, \ldots, \gamma_k)$ - from 1,4,

6. $(\Phi)^*(\theta_{\mathfrak{f}}, (\Psi_1)^*(\theta_{\mathfrak{f}}, \overline{\alpha}), \ldots, (\Psi)_k^*(\theta_{\mathfrak{f}}, \overline{\alpha})) = \eta$ - from 2,5,

7. $[J(\Phi)^* \mathbf{I}_1^{n+1} (\Psi_1)^*, \ldots, (\Psi)_k^*](\theta_{\mathfrak{f}}, \overline{\alpha}) = \eta$ - from 3,6,

[Equalities 1-7 are proved in the calculus **CalcEq**]

8. $[J\Phi\Psi_1, \ldots, \Psi_k](\overline{\alpha}) = \Phi(\Psi_1(\overline{\alpha}), \ldots, \Psi_k(\overline{\alpha}))$ - almost an axiom,

9. $\Psi_1(\overline{\alpha}) = (\Psi_1)^*(\theta_{\mathfrak{f}}, \overline{\alpha}), \ldots, \Psi_k(\overline{\alpha}) = (\Psi_k)^*(\theta_{\mathfrak{f}}, \overline{\alpha})$ - induction hypothesis,

10. $\Phi(x_1, \ldots, x_k) = \Phi(x_1, \ldots, x_k)$ - axiom,

11.$\Phi(\Psi_1(\overline{\alpha}), \ldots, \Psi_1(\overline{\alpha})) = \Phi((\Psi_1)^*(\theta_{\mathfrak{f}}, \overline{\alpha}), \ldots, (\Psi_k)^*(\theta_{\mathfrak{f}}, \overline{\alpha}))$ - from 9,10,

12. $\Phi((\Psi_1)^*(\theta_{\mathfrak{f}}, \overline{\alpha}), \ldots, (\Psi_k)^*(\theta_{\mathfrak{f}}, \overline{\alpha})) = \Phi(\gamma_1, \ldots, \gamma_k)$ - from 1,10,

13. $\Phi(\gamma_1, \ldots, \gamma_k) = (\Phi)^*(\theta_{\mathfrak{f}}, \gamma_1, \ldots, \gamma_k)$ - induction hypothesis,

14. $\Phi(\gamma_1, \ldots, \gamma_k) = \eta$ - from 2,13,

15. $\Phi((\Psi_1)^*(\theta_{\mathfrak{f}}, \overline{\alpha}), \ldots, \Psi_k)^*(\theta_{\mathfrak{f}}, \overline{\alpha})) = \eta$ - from 12,14,

16. $\Phi(\Psi_1(\overline{\alpha}), \ldots, \Psi_1(\overline{\alpha})) = \eta$ - from 11,15,

17. $[J\Phi\Psi_1, \ldots, \Psi_k](\overline{\alpha}) = \eta$ - from 8,16,

18. $[J\Phi\Psi_1, \ldots, \Psi_k](\overline{\alpha}) = [J(\Phi)^*\mathbf{I}_1^{n+1}(\Psi_1)^*, \ldots, (\Psi)_k^*](\theta_{\mathfrak{f}}, \overline{\alpha})$- from 7,17

19. $[J\Phi\Psi_1, \ldots, \Psi_k](\overline{\alpha}) = ([J\Phi\Psi_1, \ldots, \Psi_k])^*(\theta_{\mathfrak{f}}, \overline{\alpha})$ - from 18 - quasi-derivation with interpretation set $\mathbb{A}$, in the calculus of $\mathbf{CalcEq}_{\mathbb{A}}$.

(b) Induction hypothesis. Let the theorem be true for functors: $\Phi, \Psi_1, \ldots, \Psi_k$, prove the theorem for the functor $\mathfrak{f} \leftrightharpoons [R\Phi\Psi_1, \ldots, \Psi_k]$.

By the inductive hypothesis, we have:

i) for the set of argument words $\mathbb{A}$, the sequence of argument words $\alpha_1, \ldots, \alpha_n$, for the functor Φ, true $\forall\theta \supseteq \theta_{\mathbf{SimpleFw},\Phi,\overline{\alpha},\mathbb{A}}$, and $\mathbb{A} \vdash \Phi(\overline{\alpha}) = (\Phi)^*(\theta, \overline{\alpha})$;

ii) for the set of argument words $\mathbb{A}$, the sequence of argument words $\alpha_1, \ldots, \alpha_n, \beta, \gamma$, for the functor Ψ_1, true $\forall\theta \supseteq \theta_{\mathbf{SimpleFw},\Psi_1,\overline{\alpha},\beta,\gamma,\mathbb{A}}$, $\mathbb{A} \vdash \Psi_1(\overline{\alpha}) = (\Psi_1)^*(\theta, \overline{\alpha}, \beta, \gamma)$, ..., for the functor Ψ_k, true $\forall\theta \supseteq \theta_{\mathbf{SimpleFw},\Psi_k,\overline{\alpha},\beta,\gamma,\mathbb{A}}$, $\mathbb{A} \vdash \Psi_k(\overline{\alpha}) = (\Psi_k)^*(\theta, \overline{\alpha}, \beta, \gamma)$.

Further, the proof will be carried out by induction on the construction of the argument word.

Induction base. Let us prove that for the set of argument words $\mathbb{A}$, the sequences of argument words $\alpha_1, \ldots, \alpha_n$, true $\forall\theta_{\mathfrak{f}} \supseteq \theta_{\mathbf{SimpleFw},\mathfrak{f},\overline{\alpha},\Lambda}$ $\mathbb{A} \vdash [R\Phi\Psi_1, \ldots, \Psi_k](\overline{\alpha}, \Lambda) = ([R\Phi\Psi_1, \ldots, \Psi_k])^*(\theta_{\mathfrak{f}}, \Lambda)$.

According to the definition of $*$, we have: $([R\Phi\Psi_1, \ldots, \Psi_k])^* = [R(\Phi)^*(\Psi_1)^*, \ldots, (\Psi_k)^*]$, therefore, it is necessary to prove $\forall\theta_{\mathfrak{f}} \supseteq \theta_{\mathbf{SimpleFw},\mathfrak{f},\overline{\alpha},\Lambda}$ $\mathbb{A} \vdash [R\Phi\Psi_1, \ldots, \Psi_k](\overline{\alpha}, \Lambda) = [R(\Phi)^*(\Psi_1)^*, \ldots, (\Psi_k)^*](\theta_{\mathfrak{f}}, \Lambda)$.

The following sequence of equalities:

1. $[R(\Phi)^*(\Psi_1)^*, \ldots, (\Psi_k)^*](y, \overline{x}, \Lambda) = (\Phi)^*(y, \overline{x})$ - axiom,

2. $[R(\Phi)^*(\Psi_1)^*, \ldots, (\Psi_k)^*](\theta_{\mathfrak{f}}, \overline{\alpha}, \Lambda) = (\Phi)^*(\theta_{\mathfrak{f}}, \overline{\alpha})$ - from 1;

[Equalities 1,2 are proved in the calculus **CalcEq**]

[Considering that $\theta_{\mathfrak{f}} \supseteq \theta_{\mathbf{SimpleFw},\mathfrak{f},\overline{\alpha},\Lambda} \supseteq \theta_{\mathbf{SimpleFw},\Phi,\overline{\alpha},\mathbb{A}}$, we get]

3. $[R\Phi\Psi_1,\ldots,\Psi_k](\overline{\alpha},\Lambda)=\Phi(\overline{\alpha})$ - almost an axiom

4. $\Phi(\overline{\alpha})=(\Phi)^*(\theta_{\mathfrak{f}},\overline{\alpha})$ - induction hypothesis,

5. $[R\Phi\Psi_1,\ldots,\Psi_k](\overline{\alpha},\Lambda)=(\Phi)^*(\theta_{\mathfrak{f}},\overline{\alpha})$ - from 1,4,

6. $[R\Phi\Psi_1,\ldots,\Psi_k](\overline{\alpha},\Lambda)=[R(\Phi)^*(\Psi_1)^*,\ldots,(\Psi_k)^*](\theta_{\mathfrak{f},},\overline{\alpha},\Lambda)$ -from 2,5

7. $[R\Phi\Psi_1,\ldots,\Psi_k](\overline{\alpha},\Lambda)=([R\Phi\Psi_1,\ldots,\Psi_k])^*(\theta_{\mathfrak{f},},\overline{\alpha},\Lambda)$ - from 6 - quasi-inference under interpretation set $\mathbb{A}$.

Inductive step. By the induction hypothesis, we have: for a set of argument words $\mathbb{A}$, for any sequence of argument words $\alpha_1,\ldots,\alpha_n,\beta$, true

$\forall\theta_{\mathfrak{f}}\supseteq\theta_{\mathbf{SimpleFw},\mathfrak{f},\overline{\alpha},\beta}\ \mathbb{A}\vdash[R\Phi\Psi_1,\ldots,\Psi_k](\overline{\alpha},\beta)=[R(\Phi)^*(\Psi_1)^*,\ldots,(\Psi_k)^*](\theta_{\mathfrak{f},},\overline{\alpha},\beta)$;

ii) For the set of argument words $\mathbb{A}$, sequences of argument words $\alpha_1,\ldots,\alpha_n,\beta,\gamma$, true:

$\forall\theta\supseteq\theta_{\mathbf{SimpleFw},\Psi_i,\overline{\alpha},\beta,\gamma,\mathbb{A}},\ \mathbb{A}\vdash\Psi_i(\overline{\alpha},\beta,\gamma)=(\Psi_i)^*(\theta,\overline{\alpha},\beta,\gamma)$.

It is required to prove that for the functor $\mathfrak{f}$, for the set of argument words $\mathbb{A}$, the sequence of argument words $\alpha_1,\ldots,\alpha_n,\beta a_i$, true

$\forall\theta_{\mathfrak{f}}\supseteq\theta_{\mathbf{SimpleFw},\mathfrak{f},\overline{\alpha},\beta a_i}\ \mathbb{A}\vdash[R\Phi\Psi_1,\ldots,\Psi_k](\overline{\alpha},\beta a_i)=[R(\Phi)^*(\Psi_1)^*,\ldots,(\Psi_k)^*](\theta_{\mathfrak{f},},\overline{\alpha},\beta a_i)$.

Let's compose a functional word according to the sets:

$(\mathbb{A}^+)_{\mathbf{P}_{[R\Phi\Psi_1,\ldots,\Psi_k],\alpha_1,\ldots,\alpha_n,\beta a_i}}=(\mathbb{A}^+)_{\mathbf{P}_{\Psi_i},\alpha_1,\ldots,\alpha_n,\beta,\gamma}\bigcup(\mathbb{A}^+)_{\mathbf{P}_{[R\Phi\Psi_1,\ldots,\Psi_k],\alpha_1,\ldots,\alpha_n,\beta}}$,

$(\mathbb{A}^-)_{\mathbf{P}_{[R\Phi\Psi_1,\ldots,\Psi_k],\alpha_1,\ldots,\alpha_n,\beta a_i}}=(\mathbb{A}^-)_{\mathbf{P}_{\Psi_i},\alpha_1,\ldots,\alpha_n,\beta,\gamma}\bigcup(\mathbb{A}^-)_{\mathbf{P}_{[R\Phi\Psi_1,\ldots,\Psi_k],\alpha_1,\ldots,\alpha_n,\beta}}$, where $\mathbf{P}_{[R\Phi\Psi_1,\ldots,\Psi_k],\alpha_1,\ldots,\alpha_n,\beta a_i}$ - simple calculation of the functor $[R\Phi\Psi_1,\ldots,\Psi_k]$ on a sequence of argument words $\alpha_1,\ldots,\alpha_n,\beta a_i$, $\mathbf{P}_{\Psi_i}$ - simple calculation of the functor Ψ_i on a sequence of argument words $\alpha_1,\ldots,\alpha_n,\beta,\gamma$, we get $\theta_{\mathbf{SimpleFw},\mathfrak{f},\overline{\alpha},\beta a_i}$, then $\theta_{\mathbf{SimpleFw},\mathfrak{f},\overline{\alpha},\beta a_i}\supseteq\theta_{\mathbf{SimpleFw},\Psi_i,\overline{\alpha},\beta,\gamma,\mathbb{A}}$, $\theta_{\mathbf{SimpleFw},\mathfrak{f},\overline{\alpha},\beta a_i}\supseteq\theta_{\mathbf{SimpleFw},\mathfrak{f},\overline{\alpha},\beta}$. Let's take $\theta_{\mathfrak{f}}\supseteq\theta_{\mathbf{SimpleFw},\mathfrak{f},\overline{\alpha},\beta a_i}$

The following sequence of equalities:

1. $[R(\Phi)^*(\Psi_1)^*,\ldots,(\Psi_k)^*](\theta_{\mathfrak{f}},\overline{\alpha},\beta)=\gamma$ - Calculate,

2. $(\Psi_i)^*(\theta_{\mathfrak{f}},\overline{\alpha},\beta,\gamma)=\eta$ - Calculate,

3. $[R(\Phi)^*(\Psi_1)^*,\ldots,(\Psi_k)^*](\theta_{\mathfrak{f}},\overline{\alpha},\beta a_i)=(\Psi_i)^*(\theta_{\mathfrak{f}},\overline{\alpha},\beta,[R(\Phi)^*(\Psi_1)^*,\ldots,(\Psi_k)^*](\theta_{\mathfrak{f}},\overline{\alpha},\beta))$ - almost an axiom

4. $(\Psi_i)^*(y,\overline{x},z,u)=(\Psi_i)^*(y,\overline{x},z,u)$ - axiom,

5. $(\Psi_i)^*(\theta_{\mathfrak{f}},\overline{\alpha},\beta,[R(\Phi)^*(\Psi_1)^*,\ldots,(\Psi_k)^*](\theta_{\mathfrak{f}},\overline{\alpha},\beta))=(\Psi_i)^*(\theta_{\mathfrak{f}},\overline{\alpha},\beta,\gamma))$ - from 1,4,

6. $(\Psi_i)^*(\theta_{\mathfrak{f}},\overline{\alpha},\beta,[R(\Phi)^*(\Psi_1)^*,\ldots,(\Psi_k)^*](\theta_{\mathfrak{f}},\overline{\alpha},\beta))=\eta$ - from 2,5,

7. $[R(\Phi)^*(\Psi_1)^*,\ldots,(\Psi_k)^*](\theta_{\mathfrak{f}},\overline{\alpha},\beta a_i)=\eta$ - from 3,6,

[Equalities 1,7 are proved in the calculus **CalcEq**]

8. $[R\Phi\Psi_1,\ldots,\Psi_k](\overline{\alpha},\beta a_i)=\Psi_i(\overline{\alpha},\beta,[R\Phi\Psi_1,\ldots,\Psi_k](\overline{\alpha},\beta))$ - almost an axiom,

9. $[R\Phi\Psi_1,\ldots,\Psi_k](\overline{\alpha},\beta)=[R(\Phi)^*(\Psi_1)^*,\ldots,(\Psi_k)^*](\theta_{\mathfrak{f},},\overline{\alpha},\beta)$ - induction hypothesis,

10. $\Psi_i(\overline{x},u,v)=\Psi_i(\overline{x},u,v)$ - axiom,

11. $\Psi_i(\overline{\alpha}, \beta, [R\Phi\Psi_1, \ldots, \Psi_k](\overline{\alpha}, \beta)) = \Psi_i(\overline{\alpha}, \beta, [R(\Phi)^*(\Psi_1)^*, \ldots, (\Psi_k)^*](\theta_{\mathfrak{f}}, \overline{\alpha}, \beta))$ - from 9,10,

12. $\Psi_i(\overline{\alpha}, \beta, [R(\Phi)^*(\Psi_1)^*, \ldots, (\Psi_k)^*](\theta_{\mathfrak{f}}, \overline{\alpha}, \beta)) = \Psi_i(\overline{\alpha}, \beta, \gamma)$ - from 1,10,

13. $\Psi_i(\overline{\alpha}, \beta, \gamma) = (\Psi_i)^*(\theta_{\mathfrak{f}}, \overline{\alpha}, \beta, \gamma)$ - induction hypothesis,

14. $\Psi_i(\overline{\alpha}, \beta, [R\Phi\Psi_1, \ldots, \Psi_k](\overline{\alpha}, \beta)) = \Psi_i(\overline{\alpha}, \beta, \gamma)$ - from 11,12,

15. $\Psi_i(\overline{\alpha}, \beta, [R\Phi\Psi_1, \ldots, \Psi_k](\overline{\alpha}, \beta)) = (\Psi_i)^*(\theta_{\mathfrak{f}}, \overline{\alpha}, \beta, \gamma)$ - from 13,14,

16. $\Psi_i(\overline{\alpha}, \beta, [R\Phi\Psi_1, \ldots, \Psi_k](\overline{\alpha}, \beta)) = \eta$ - from 2,15,

17. $[R\Phi\Psi_1, \ldots, \Psi_k](\overline{\alpha}, \beta a_i) = \eta$ - from 8,16,

18. $[R\Phi\Psi_1, \ldots, \Psi_k](\overline{\alpha}, \beta a_i) = [R(\Phi)^*(\Psi_1)^*, \ldots, (\Psi_k)^*](\theta_{\mathfrak{f},}, \overline{\alpha}, \beta a_i)$ - from 7,17,

19. $[R\Phi\Psi_1, \ldots, \Psi_k](\overline{\alpha}, \beta a_i) = ([R\Phi\Psi_1, \ldots, \Psi_k)])^*(\theta_{\mathfrak{f},}, \overline{\alpha}, \beta a_i)$ - from 18 - quasi-inference under interpretation set $\mathbb{A}$, i.e in the calculus $\mathbf{CalcEq}_{\mathbb{A}}$.

The remaining recursion axioms (15,16,17,20) are treated similarly.

Corollary 5.2. For any n - place functor Φ, $\forall\mathbb{A}$, $\forall\overline{\alpha}$ true $\mathbb{A} \vdash [\Phi(\overline{\alpha}) = \Phi^*(\Theta_\Phi(\overline{\alpha}), \overline{\alpha})]$ ($\mathbf{WordM}_{\mathbb{A}} \models [\Phi(\overline{\alpha}) = \Phi^*(\Theta_\Phi(\overline{\alpha}), \overline{\alpha})]$).

2. For any n - place functor Φ, $\forall\mathbb{A}$, $\forall\overline{\alpha}$ $\forall\theta \supseteq \Theta_\Phi(\overline{\alpha})$, true $\mathbb{A} \vdash [\Phi(\overline{\alpha}) = \Phi^*(\theta, \overline{\alpha})]$ ($\mathbf{WordM}_{\mathbb{A}} \models [\Phi(\overline{\alpha}) = \Phi^*(\theta, \overline{\alpha})]$),

3. For any n - place functor Φ, $\forall\mathbb{A}$, $\forall\overline{\alpha}, \beta$ $\forall\theta \supseteq \Theta_\Phi(\overline{\alpha})$, true $\mathbb{A} \vdash \Phi(\overline{\alpha}) = \beta \Leftrightarrow \vdash \Phi^*(\theta, \overline{\alpha}) = \beta$ ($\mathbf{WordM}_{\mathbb{A}} \models \Phi(\overline{\alpha}) = \beta \Leftrightarrow \mathbf{WordM} \models \Phi^*(\theta, \overline{\alpha}) = \beta$).

4. For any n - place functor Φ, $\forall\overline{\alpha}$ true $\mathbf{WordM}_{\mathbb{A}} \models (\Theta_\Phi)^*(\Theta_\Phi(\overline{\alpha}), \overline{\alpha}) = \Theta_\Phi(\overline{\alpha})$ -as equality of words (smallest fixed point: $\forall\theta \supseteq \Theta_\Phi(\overline{\alpha})[(\Theta_\Phi)^*(\theta, \overline{\alpha}) \subseteq \theta]$ $\{\forall\theta \supseteq \Theta_\Phi(\overline{\alpha})[\Theta_\Phi(\overline{\alpha}) = (\Theta_\Phi)^*(\theta, \overline{\alpha})]\}$, moreover, if $\Theta_\Phi(\overline{\alpha}) \neq \Lambda$, then $dom(\Theta_\Phi(\overline{\alpha})) = dom((\Theta_\Phi)^*(\theta, \overline{\alpha}))$ for any any functional word θ.

Theorem 5.3. For arbitrary n - place functor Φ, for arbitrary argument words $\overline{\alpha}$, β, γ, if $\mathbf{WordM}_{\mathbb{A}} \models (\Theta_\Phi)^*(\beta, \overline{\alpha}) = \gamma$, then γ is a function word.

Proof. The proof is by induction on the construction of the functor Φ.

Theorem 5.4. For an arbitrary n - place functor Φ, for an arbitrary set of argument words $\mathbb{A}$, true $\mathbf{WordM}_{\mathbb{A}} \models \forall\overline{\alpha}\forall\beta[\Theta_\Phi(\overline{\alpha}) = \beta \Longrightarrow (\Theta_\Phi)^*(\beta, \overline{\alpha}) = \beta \wedge \beta \subset \mathbb{A}]$.

Proof. For arbitrary argument words $\overline{\alpha}, \beta$ we have $\mathbf{WordM}_{\mathbb{A}} \models \Theta_\Phi(\overline{\alpha}) = \beta \Longleftrightarrow (\Theta_\Phi)^*(\Theta_\Phi(\overline{\alpha}), \overline{\alpha}) = \beta$, then $\mathbf{WordM}_{\mathbb{A}} \models (\Theta_\Phi)^*(\beta, \overline{\alpha}) = \beta \wedge \beta \subset \mathbb{A}$.

Theorem 5.5. Suppose that for n - place functor Φ, for the argumentative words $\alpha_1, \ldots, \alpha_n$, for a set of argument words $\mathbb{A}$, for the function word β is true $\mathbf{WordM}_{\mathbb{A}} \models (\Theta_\Phi)^*(\beta, \alpha_1, \ldots, \alpha_n) \subseteq \beta \wedge \beta \subset \mathbb{A}$, then $\mathbf{WordM}_{\mathbb{A}} \models \Theta_\Phi(\alpha_1, \ldots, \alpha_n) \subseteq \beta$.

Proof. The proof is by induction on the construction of the functor Φ.

Basis of induction. Φ - initial functor, for example $\mathbf{U}$, then $\Theta_{\mathbf{U}} = [J\mathbf{c}\mathbf{I}_1^1\mathbf{U}]$, then $(\Theta_{\mathbf{U}})^* = [J\mathbf{c}^*\mathbf{I}_1^2[J\mathbf{I}_1^1\mathbf{I}_2^2]\mathbf{G}]$, then $\mathbf{WordM}_{\mathbb{A}} \models (\Theta_{\mathbf{U}})^*(\beta,\alpha) = \mathbf{c}^*(\beta,\alpha,\mathbf{G}(\beta,\alpha)) = |\alpha|2\alpha\mathbf{G}(\beta,\alpha)22(\vdash \mathbf{c}^*(x,y,z) = \mathbf{c}(y,z)$, see **Note** стр.23).

Assume $\mathbf{WordM}_{\mathbb{A}} \models (\Theta_{\mathbf{U}})^*(\beta,\alpha) \subseteq \beta$ and $\beta \subset \mathbb{A}$, then $|\alpha|2\alpha\mathbf{G}(\beta,\alpha)22 \subseteq \beta$.

Let's break down the cases:

a) $\mathbf{G}(\beta,\alpha) = \Lambda$, then $\beta(\alpha) = \Lambda$, then $\alpha \in \mathbb{A}$, then $\mathbf{U}(\alpha) = \Lambda$, then $\Theta_{\mathbf{U}}(\alpha) = |\alpha|2\alpha22$, given that $(\Theta_{\mathbf{U}})^*(\beta,\alpha) = |\alpha|2\alpha\mathbf{G}(\beta,\alpha)22 = |\alpha|2\alpha22 \subseteq \beta$, then $\Theta_{\mathbf{U}}(\alpha) \subseteq \beta$;

b) $\mathbf{G}(\beta,\alpha) = 1$, then $\beta(\alpha) = 1$, then $\alpha \notin \mathbb{A}$, then $\mathbf{U}(\alpha) = 1$, then $\Theta_{\mathbf{U}}(\alpha) = |\alpha|2\alpha122$, given that $(\Theta_{\mathbf{U}})^*(\beta,\alpha) = |\alpha|2\alpha\mathbf{G}(\beta,\alpha)22 = |\alpha|2\alpha122 \subseteq \beta$, then $\Theta_{\mathbf{U}}(\alpha) \subseteq \beta$.

For the rest of the initial functors the proof is quite clear.

Inductive step. 1). Let the theorem be true for k - place functor Φ, for n - place functors $\Psi_1,\ldots,\Psi_k$. Let us prove the theorem for the functor $[J\Phi\Psi_1,\ldots,\Psi_k]$. We have

$\vdash \Theta_{[J\Phi\Psi_1,\ldots,\Psi_k]}(\overline{x}) = \mathbf{Concat}([J\Theta_\Phi\Psi_1,\ldots,\Psi_k](\overline{x}),(\mathbf{Concat}(\Theta_{\Psi_1}(\overline{x}),\ldots,\mathbf{Concat}(\Theta_{\Psi_{k-1}}(\overline{x}),\Theta_{\Psi_k}(\overline{x}))),\ldots,)$, then

$\vdash (\Theta_{[J\Phi\Psi_1,\ldots,\Psi_k]})^*(\boldsymbol{y},\overline{x}) = \mathbf{Concat}([J(\Theta_\Phi)^*\mathbf{I}_1^{n+1}(\Psi_1)^*,\ldots,(\Psi_k)^*](\boldsymbol{y},\overline{x}),(\mathbf{Concat}((\Theta_{\Psi_1})^*(\boldsymbol{y},\overline{x}),$ $\ldots,\mathbf{Concat}((\Theta_{\Psi_{k-1}})^*(\boldsymbol{y},\overline{x}),(\Theta_{\Psi_k})^*(\boldsymbol{y},\overline{x}))),\ldots,)$[6], then

$\mathbf{WordM}_{\mathbb{A}} \models (\Theta_{[J\Phi\Psi_1,\ldots,\Psi_k]})^*(\beta,\overline{\alpha}) = \mathbf{Concat}([J(\Theta_\Phi)^*\mathbf{I}_1^{n+1}(\Psi_1)^*,\ldots,(\Psi_k)^*](\beta,\overline{\alpha}),(\mathbf{Concat}((\Theta_{\Psi_1})^*(\beta,\overline{\alpha}),$ $\ldots,\mathbf{Concat}((\Theta_{\Psi_{k-1}})^*(\beta,\overline{\alpha}),(\Theta_{\Psi_k})^*(\beta,\overline{\alpha}))),\ldots,)$.

Let's $(\Theta_{[J\Phi\Psi_1,\ldots,\Psi_k]})^*(\beta,\overline{\alpha}) \subseteq \beta \subset \mathbb{A}$, then $[J(\Theta_\Phi)^*\mathbf{I}_1^{n+1}(\Psi_1)^*,\ldots,(\Psi_k)^*](\beta,\overline{\alpha}) = (\Theta_\Phi)^*(\beta,(\Psi_1)^*(\beta,\overline{\alpha}),\ldots,$ $(\Psi_k)^*(\beta,\overline{\alpha})) \subseteq \beta$ and $(\Theta_{\Psi_1})^*(\beta,\overline{\alpha}) \subseteq \beta,\ldots,(\Theta_{\Psi_k})^*(\beta,\overline{\alpha}) \subseteq \beta$.

By induction assumption we obtain $\Theta_\Phi((\Psi_1)^*(\beta,\overline{\alpha}),\ldots,(\Psi_k)^*(\beta,\overline{\alpha})) \subseteq \beta$, as well as $\Theta_{\Psi_1}(\overline{\alpha}) \subseteq \beta,\ldots,\Theta_{\Psi_k}(\overline{\alpha}) \subseteq \beta$, then $\Psi_1(\alpha) = (\Psi_1)^*(\beta,\overline{\alpha}),\ldots,\Psi_k(\alpha) = (\Psi_k)^*(\beta,\overline{\alpha})$, then $\Theta_\Phi((\Psi_1)^*(\beta,\overline{\alpha}),\ldots,(\Psi_k)^*(\beta,\overline{\alpha})) = \Theta_\Phi(\Psi_1(\alpha),\ldots\Psi_k(\alpha))$, then $\Theta_\Phi(\Psi_1(\alpha),\ldots\Psi_k(\alpha)) \subseteq \beta$, then $\mathbf{Concat}([J\Theta_\Phi\Psi_1,\ldots,\Psi_k](\overline{x}),(\mathbf{Concat}(\Theta_{\Psi_1}(\overline{x}),\ldots,\mathbf{Concat}(\Theta_{\Psi_{k-1}}(\overline{x}),\Theta_{\Psi_k}(\overline{x}))),\ldots,) \subseteq \beta$, then $\Theta_{[J\Phi\Psi_1,\ldots,\Psi_k]}(\overline{\alpha}) \subseteq \beta$.

2). Let the theorem be true for n - place functor Φ, for $n+2$ - place functors $\Psi_1,\ldots,\Psi_k$. Let us prove the theorem for $n+1$ - place functor $[R\Phi\Psi_1,\ldots,\Psi_k]$.

We have: $\vdash \Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{x},\Lambda) = \Theta_\Phi(\overline{x})$.

$\vdash \Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{x},x_{n+1}a_i) = \mathbf{Concat}(\Theta_{\Psi_i}(\overline{x},x_{n+1},[R\Phi\Psi_1,\ldots,\Psi_k](\overline{x},x_{n+1})),\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{x},x_{n+1}))$, then

$\vdash (\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]})^*(\boldsymbol{y},\overline{x},\Lambda) = (\Theta_\Phi)^*(\boldsymbol{y},\overline{x})$,

$\vdash (\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]})^*(\boldsymbol{y},\overline{x},x_{n+1}a_i) =$

[6] see Application

$= \mathbf{Concat}((\Theta_{\Psi_i})^*(\boldsymbol{y}, \overline{x}, x_{n+1}, ([R\Phi\Psi_1, \ldots, \Psi_k])^*(\boldsymbol{y}, \overline{x}, x_{n+1})), (\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]})^*(\boldsymbol{y}, \overline{x}, x_{n+1}))$, then

c) $\mathbf{WordM}_{\mathbb{A}} \models (\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]})^*(\beta, \overline{\alpha}, \Lambda) = (\Theta_\Phi)^*(\beta, \overline{\alpha})$,

d) $\mathbf{WordM}_{\mathbb{A}} \models (\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]})^*(\beta, \overline{\alpha}, \gamma a_i) =$

$= \mathbf{Concat}((\Theta_{\Psi_i})^*(\beta, \overline{\alpha}, \gamma, ([R\Phi\Psi_1, \ldots, \Psi_k])^*(\beta, \overline{\alpha}, \gamma)), (\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]})^*(\beta, \overline{\alpha}, \gamma))$[7].

Case study (***c***). Let's $(\Theta_\Phi)^*(\beta, \overline{\alpha}) \subseteq \beta \subset \mathbb{A}$. By the induction assumption we obtain $\mathbf{WordM}_{\mathbb{A}} \models \Theta_\Phi(\overline{\alpha}) \subseteq \beta$, then $\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{x}, \Lambda) \subseteq \beta$.

Case study (***d***). Let's $(\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]})^*(\beta, \overline{\alpha}, \gamma a_i) \subseteq \beta \subset \mathbb{A}$, then

$\mathbf{Concat}((\Theta_{\Psi_i})^*(\beta, \overline{\alpha}, \gamma, ([R\Phi\Psi_1, \ldots, \Psi_k])^*(\beta, \overline{\alpha}, \gamma)), (\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]})^*(\beta, \overline{\alpha}, \gamma)) \subseteq \beta$, then

$(\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]})^*(\beta, \overline{\alpha}, \gamma) \subseteq \beta$, then By the induction assumption we obtain $\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{\alpha}, \gamma) \subseteq \beta$, then $\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{\alpha}, \gamma) = \delta \Longleftrightarrow (\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]})^*(\beta, \overline{\alpha}, \gamma)) = \delta$ и $\delta \subseteq \beta$. Further considering

$[R\Phi\Psi_1, \ldots, \Psi_k])(\overline{\alpha}, \gamma)) = \eta \Longleftrightarrow ([R\Phi\Psi_1, \ldots, \Psi_k])^*(\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{\alpha}, \gamma), \overline{\alpha}, \gamma)) = \eta$ and $\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{\alpha}, \gamma) \subseteq \beta$, we get $[R\Phi\Psi_1, \ldots, \Psi_k])(\overline{\alpha}, \gamma)) = \eta \Longleftrightarrow ([R\Phi\Psi_1, \ldots, \Psi_k])^*(\beta, \overline{\alpha}, \gamma), \overline{\alpha}, \gamma)) = \eta$, then

$\mathbf{Concat}((\Theta_{\Psi_i})^*(\beta, \overline{\alpha}, \gamma, ([R\Phi\Psi_1, \ldots, \Psi_k])^*(\beta, \overline{\alpha}, \gamma)), (\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]})^*(\beta, \overline{\alpha}, \gamma)) =$

$= \mathbf{Concat}((\Theta_{\Psi_i})^*(\beta, \overline{\alpha}, \gamma, [R\Phi\Psi_1, \ldots, \Psi_k](\overline{\alpha}, \gamma)), \Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{\alpha}, \gamma))$.

We have $(\Theta_{\Psi_i})^*(\beta, \overline{\alpha}, \gamma, ([R\Phi\Psi_1, \ldots, \Psi_k])^*(\beta, \overline{\alpha}, \gamma)) \subseteq \beta$, then $(\Theta_{\Psi_i})^*(\beta, \overline{\alpha}, \gamma, [R\Phi\Psi_1, \ldots, \Psi_k](\overline{\alpha}, \gamma)) \subseteq \beta$, then by induction assumption, we obtain $\Theta_{\Psi_i}(\overline{\alpha}, \gamma, [R\Phi\Psi_1, \ldots, \Psi_k](\overline{\alpha}, \gamma)) \subseteq \beta$, then

$\mathbf{Concat}(\Theta_{\Psi_i}(\overline{\alpha}, \gamma, [R\Phi\Psi_1, \ldots, \Psi_k](\overline{\alpha}, \gamma)), \Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{\alpha}, \gamma)) \subseteq \beta$, then $\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{\alpha}, \gamma a_i) \subseteq \beta$ and $\beta \subset \mathbb{A}$.

The remaining axioms of recursion (15,16,17,20) are considered similarly, then

$\forall \overline{x} \forall y \{[(\Theta_\Phi)^*(y, \overline{x}) = y \wedge y \subset \mathbf{U}] \Rightarrow \Theta_\Phi(\overline{x}) \subseteq y\} \in \mathbf{Th}(\mathbf{U})$(see p.34).

Corollary 5.6. For an arbitrary n - place functor Φ, for an arbitrary sequence of argument words $\alpha_1, \ldots, \alpha_n$, for an arbitrary set of argument words $\mathbb{A}$, There is only one function word β, such that

$\mathbf{WordM}_{\mathbb{A}} \models \Theta_\Phi(\alpha_1, \ldots, \alpha_n) = \beta \Longleftrightarrow \mathbf{WordM}_{\mathbb{A}} \models (\Theta_\Phi)^*(\beta, \alpha_1, \ldots, \alpha_n) = \beta \wedge \beta \subset \mathbf{U}$, then

$\forall \overline{x} \exists ! y [(\Theta_\Phi)^*(y, \overline{x}) = y \wedge y \subset \mathbf{U}] \in \mathbf{Th}(\mathbf{U})$. $\forall \overline{x} \exists ! y \{[\Theta_\Phi(\overline{x}) = y \equiv [(\Theta_\Phi)^*(y, \overline{x}) = y \wedge y \subset \mathbf{U}]\} \in \mathbf{Th}(\mathbf{U})$.

Note. Let Φ be an arbitrary n - a place functor of the alphabet $\mathscr{L}(\mathbf{U})$. A simple calculation of the functor Φ on the sequence of argument words $\overline{\alpha}$ can be decomposed into two calculations: a simple calculation of the functor Θ_Φ of the alphabet $\mathscr{L}(\mathbf{U})$ on the sequence of argument words $\overline{\alpha}$ and then a simple calculation of the functor $(\Phi)^*$ of the alphabet $\mathscr{L}$ on the sequence of argument words $\Theta_\Phi(\overline{\alpha}), \overline{\alpha}$. Moreover, the domain of the functional word $\Theta_\Phi(\overline{\alpha})$ consists of those and only those argument words that were used in a simple calculation of the functor Φ on the sequence of argument words $\overline{\alpha}$ and for any extension of the functional word $\Theta_\Phi(\overline{\alpha}) \subseteq \theta$, not necessarily consistent with the oracle set $\mathbb{A}$, the result of a simple calculation of the functor $(\Phi)^*$ on the

[7] see Application

sequence $\Theta_\Phi(\overline{\alpha}), \overline{\alpha}$ will coincide with the result of a simple calculation of the same functor $(\Phi)^*$ on the sequence $\theta, \overline{\alpha}$(analogous to the "Use Principle"("**Use Principle**") of oracle computing, e.g. on **Turing machines**) and, as noted earlier, $(\Theta_\Phi)^*(\theta, \overline{\alpha}) \subseteq \theta$ - "**Use Principle**"will play an important role in the future when transferring (spreading) this fundamental concept, associated with calculations in the standard model, to non-standard models.

If in a simple calculation of the functor Φ on the sequence of argument words $\overline{\alpha}$, each interpretive axiom of the form $\mathbf{U}(\alpha) = \Lambda$ is replaced by an equality of the form $\mathbf{G}(|\alpha|2\alpha 22, \alpha) = \Lambda$ (replace with an equality of the form $\mathbf{G}(\theta, \alpha) = \Lambda$, where $|\alpha|2\alpha 22 \subset \theta$), an axiom of the form $\mathbf{U}(\alpha) = a_1$, replaced by an equality of the form $\mathbf{G}(|\alpha|2\alpha a_1 22, \alpha) = a_1$ (replace with an equality of the form $\mathbf{G}(\theta, \alpha) = a_!$, where $|\alpha|2\alpha a_1 22 \subset \theta$), then the resulting sequence of equalities will be a quasi-inference that does not contain interpretative axioms, and this quasi-inference can be easily transformed into a conclusion by replacing the indicated equalities with their, for example, simple conclusions.

Bounded formulas. Universal functional word.

Let us define the notion of a bounded formula Ψ and accompanying this notion, sets denoted as Bwp_Ψ, Vwp_Ψ.

1) Any quantifier-free formula $\mathcal{A}$ is a bounded formula, $Bwp_\mathcal{A} = \emptyset$, $Vwp_\mathcal{A} = \emptyset$;

2) Let $\mathcal{A}(z, x_1, \ldots, x_n; y_1, \ldots, y_k)$ - bounded formula, $\mathbf{P}(x_1, \ldots, x_n)$ - word polinomial, $z \notin Vwp_\mathcal{A}$, then formula $\mathcal{B} \rightleftharpoons \exists z[|z| \leqslant |\mathbf{P}(x_1, \ldots, x_n)| \& \mathcal{A}(z, x_1, \ldots, x_n; y_1, \ldots, y_k)]$ or

$\mathcal{B} \rightleftharpoons \forall z[|z| \leqslant |\mathbf{P}(x_1, \ldots, x_n)| \supset \mathcal{A}(z, x_1, \ldots, x_n; y_1, \ldots, y_k)]$ - bounded formula, $\mathbf{P}(x_1, \ldots, x_n) \in Bwp_\mathcal{A}$, $\{x_1, \ldots, x_n\} \subset Vwp_\mathcal{A}$, variables $y_1, \ldots, y_k$ - are called the parameters of the bounded formula in question, this list is separated by a semicolon and may be empty. We will denote this formula as $\exists_z^{|\mathbf{P}(x_1,\ldots,x_n)|} \mathcal{A}(z, x_1, \ldots, x_n; y_1, \ldots, y_k)]$ or $\forall_z^{|\mathbf{P}(x_1,\ldots,x_n)|} \mathcal{A}(z, x_1, \ldots, x_n; y_1, \ldots, y_k)]$.

A word polynomial belonging to the set $Bwp_\mathcal{A}$ is called a bounding word polynomial.

A bounded formula $\mathcal{A}$ is called an $\exists(\forall)$ bounded formula if it has the form $\exists_{z_1}^{|\mathbf{P}_1(\overline{x})|}, \ldots \exists_{z_k}^{|\mathbf{P}_k(\overline{x})|} \mathcal{B}(z_1, \ldots z_k, \overline{x}; y_1, \ldots, y_k)$ $(\forall_{z_1}^{|\mathbf{P}_1(\overline{x})|}, \ldots \forall_{z_k}^{|\mathbf{P}_k(\overline{x})|} \mathcal{B}(z_1, \ldots z_k, \overline{x}; y_1, \ldots, y_k))$, where $\mathcal{B}(z_1, \ldots z_k, \overline{x}; y_1, \ldots, y_k)$ -quantifier-free formula.

Note. $\forall \Phi \ \forall \overline{\alpha} \ \forall \mathbf{P}(\overline{x})$:

1. $\mathbf{Word}_\mathbb{A} \models \exists_u^{|\mathbf{P}(\overline{\alpha})|} \Phi(\overline{\alpha}, u) = \Lambda \Leftrightarrow \exists_u^{|\mathbf{P}(\overline{\alpha})|} (\Phi)^*(\Theta_\Phi(\overline{\alpha}, u), \overline{\alpha}, u) = \Lambda$,

2. $\mathbf{Word}_\mathbb{A} \models \forall_u^{|\mathbf{P}(\overline{\alpha})|} \Phi(\overline{\alpha}, u) = \Lambda \Leftrightarrow \forall_u^{|\mathbf{P}(\overline{\alpha})|} (\Phi)^*(\Theta_\Phi(\overline{\alpha}, u), \overline{\alpha}, u) = \Lambda$,

$\forall \beta$:

3. $\mathbf{Word}_\mathbb{A} \models \forall_u^{|\mathbf{P}(\overline{\alpha})|} \Theta_\Phi(\overline{\alpha}, u) \subseteq \beta \Rightarrow \{\exists_u^{|\mathbf{P}(\overline{\alpha})|} \Phi(\overline{\alpha}, u) = \Lambda \Leftrightarrow \forall \theta[\beta \subseteq \theta \Rightarrow \exists_u^{|\mathbf{P}(\overline{\alpha})|} (\Phi)^*(\theta, \overline{\alpha}, u) = \Lambda]\}$,

4. $\mathbf{Word}_{\mathbb{A}} \models \forall_u^{|\mathbf{P}(\overline{\alpha})|}\Theta_\Phi(\overline{\alpha},u) \subseteq \beta \Rightarrow \{\forall_u^{|\mathbf{P}(\overline{\alpha})|}\Phi(\overline{\alpha},u) = \Lambda \Leftrightarrow \forall\theta[\beta \subseteq \theta \Rightarrow \forall_u^{|\mathbf{P}(\overline{\alpha})|}(\Phi)^*(\theta,\overline{\alpha},u) = \Lambda]\}$.

Consider the following word function: Order(α) is a word β such that the number of words preceding the word β in the lexicographic ordering is equal to $|\alpha|$. In [1 p. 217] that this word function is a primitive recursive word function, then for the word function Order there exists a one-place functor **Order** of the alphabet $\mathscr{L}$, which is true:

$\forall\alpha,\beta[\mathrm{Order}(\alpha) = \beta \Leftrightarrow\ \vdash \mathbf{Order}(\alpha) = \beta]$.

Let us write out the defining equations for the functor **Order**:

1). $\mathbf{Order}(\Lambda) = \Lambda$;

2). $\mathbf{Order}(\mathbf{S}_i(\alpha)) = \mathbf{R}(\mathbf{Order}(\alpha))$, where

a). $\mathbf{R}(\Lambda) = a_1$;

b). $\mathbf{R}(\mathbf{S}_i(\alpha)) = \mathbf{S}_{i+1}(\alpha)$, where $1 \le i < p$;

$\hat{c}$). $\mathbf{R}(\mathbf{S}_p(\alpha)) = \mathbf{S}_1(\mathbf{R}(\alpha))$.

Note. For each set of p - alphabetic words, there will be its own p - alphabetic functor $\mathbf{Order}_p$. From the context it will be clear which p - alphabetic funtor is meant. The functor **Order** has the property:

1. $\mathbf{WordM} \models \forall\alpha\beta[|\alpha| = |\beta| \supset \mathbf{Order}(\alpha) = \mathbf{Order}(\beta)]$;

2. At $k \ge 2 \wedge p > 1$ $\mathbf{WordM} \models |\mathbf{Order}_p(k)| < k$;

3. For each $p \ge 2$ true $\mathbf{Order}_p(\dfrac{p^{n+1}-1}{p-1}) = \underbrace{1,\ldots,1}_{n+1-times}$.

4. For each $p \ge 2$ true $\mathbf{WordM} \models \forall x[|x| \ge 2 \Rightarrow \mathbf{Order}_p(p^{|x|}) = \underbrace{(p-1),\ldots,(p-1)}_{|x|-1-times}p]$;

5.According to the defining equalities given in [1 p. 217], it follows that this functor belongs to **PPr**, i.e. $\mathbf{Order} \in \mathbf{PPr}$.

From (3) we get $\mathbf{WordM} \models \mathbf{Order}_p(\dfrac{p^{|\mathbf{P}(\overline{x})|+1}-1}{p-1}) = \underbrace{1,\ldots,1}_{|\mathbf{P}(\overline{x})|+1-times}$.

If for argument words α,β it is true that $\mathbf{WordM} \models \mathbf{Order}(\alpha) = \beta$, then the argument word $|\alpha|$ is a natural number, we will call it the Godel number of the argument word β and denote it by $\ulcorner\beta\urcorner = |\alpha|$.

Obviously, for any argument word β there exists a natural number α such that

$\ulcorner\beta\urcorner = \alpha(\mathbf{WordM} \models \mathbf{Order}(\alpha) = \beta)$, then $\mathbf{WordM} \models \ulcorner\underbrace{(p-1),\ldots,(p-1)}_{|x|-1-times}p\urcorner = p^{|x|}$, for $|x| \ge 2$.

With each $n \ge 1$ -place functor Φ we associate a functor, denoted as $\Theta_{\Phi,\forall}$, satisfying the following defining equalities:

$\Theta_{\Phi,\forall}(\overline{x},\Lambda) = \Theta_\Phi(\overline{x},\Lambda)$;

$\Theta_{\Phi,\forall}(\overline{x},\mathbf{S}_k(x_n)) = \mathbf{Concat}(\Theta_{\Phi,\forall}(\overline{x},x_n),\Theta_\Phi(\overline{x},\mathbf{Order}(\mathbf{S}_k(x_n))))$.

That is right: 1. If $\Phi \in \mathbf{PPr} \Rightarrow \Theta_{\Phi,\forall} \in \mathbf{PPr}$;

2. $\mathbf{WordM}_{\mathbb{A}} \models \forall\alpha,\beta,\overline{x}[|\alpha| = |\beta| \Rightarrow \Theta_{\Phi,\forall}(\overline{x},\alpha) = \Theta_{\Phi,\forall}(\overline{x},\beta)]$;

3. $\mathbf{WordM}_{\mathbb{A}} \models \forall\overline{x}\forall_y^{|z|}[\Theta_{\Phi}(\overline{x},y) \subseteq \Theta_{\Phi,\forall}(\overline{x},\dfrac{p^{|z|+1}-1}{p-1})]$.

Let us define a universal function word denoted as $\Theta_{\mathbb{U}}$.

Defining equalities:

$\Theta_{\mathbb{U}}(\Lambda) = \Lambda$,

$\Theta_{\mathbb{U}}(\alpha a_i) = \mathbf{Concat}(\Theta_{\mathbb{U}}(\alpha), \mathbf{c}(\mathbf{Order}(\alpha), \mathbf{U}(\mathbf{Order}(\alpha))))$. According to the definition, the functor $\Theta_{\mathbb{U}}$ belongs **PPr** alphabet $\mathscr{L}_{\mathbf{U}}$.

We have:

$(\Theta_{\mathbb{U}})^*(\mathbf{y},\Lambda) = \Lambda$,

$(\Theta_{\mathbb{U}})^*(\mathbf{y},\alpha a_i) = \mathbf{Concat}((\Theta_{\mathbb{U}})^*(\mathbf{y},\alpha), \mathbf{c}(\mathbf{Order}(\alpha, \mathbf{G}(\mathbf{y}, \mathbf{Order}(\alpha)))))$,

$\forall\alpha\beta\forall\theta \supseteq \Theta_{\mathbb{U}}(\alpha)\mathbf{WordM}_{\mathbb{A}} \models [\Theta_{\mathbb{U}}(\alpha) = \beta \iff (\Theta_{\mathbb{U}})^*(\theta,\alpha) = \beta]$, in particular

$\mathbf{WordM}_{\mathbb{A}} \models \forall x\forall y[\Theta_{\mathbb{U}}(x) = y \iff (\Theta_{\mathbb{U}})^*(\Theta_{\mathbb{U}}(x),x) = y]$.

Let β be a function word such that for some word α it is true

$\mathbf{WordM}_{\mathbb{A}} \models (\Theta_{\mathbb{U}})^*(\beta,\alpha) \subseteq \beta \subset \mathbb{A}$, then $\mathbf{WordM}_{\mathbb{A}} \models \Theta_{\mathbb{U}}(\alpha) \subseteq \beta$. We have

$\forall\overline{x}\forall y\{\mathbf{Fw}(y) \Rightarrow [[(\Theta_{\mathbb{U}})^*(y,\overline{x}) \subseteq y \wedge y \subset \mathbf{U}] \Rightarrow \Theta_{\mathbb{U}}(\overline{x}) \subseteq y]\} \in \mathbf{Th}(\mathbf{U})$.

Note. For each p, its own p is defined - an alphabetic universal function word $\Theta_{\mathbb{U}}$.

True:

1. $\mathbf{WordM}_{\mathbb{A}} \models \forall\alpha,\beta[|\alpha| = |\beta| \iff \Theta_{\mathbb{U}}(\alpha) = \Theta_{\mathbb{U}}(\beta)]$;

2. $\mathbf{WordM}_{\mathbb{A}} \models \forall\alpha,\beta[|\alpha| \le |\beta| \iff \Theta_{\mathbb{U}}(\alpha) \subseteq \Theta_{\mathbb{U}}(\beta)]$.

3. $\mathbf{WordM}_{\mathbb{A}} \models \forall x\{|x| \le |y| \Rightarrow [\Theta_{\mathbb{U}}(\dfrac{p^{|y|+1}-1}{p-1})](x) = \Lambda \iff x \in \mathbb{A}\}$, where $\mathbb{A}$ - set p - alphabetic argument of words.

4. $\mathbf{WordM}_{\mathbb{A}} \models \forall\alpha[\Theta_{\mathbb{U}}(\alpha) = \Theta_{\Theta_{\mathbb{U}}}(\alpha)]$, using Goodstein's rule, it can be proven that $\vdash \Theta_{\mathbb{U}}(x) = \Theta_{\Theta_{\mathbb{U}}}(x)$.

Let given $n+m$ - ary$(n \ge 1, m \ge 1)$, p - alphabetic functor $\Phi \in \mathbf{PPr}$, interpretative p - alphabetic set of argument words $\mathbb{A}$ be given. For this functor, there exists a word polynomial $\mathbf{P}(x_1,\dots,x_n,y_1\dots y_m)$ such that for $\forall\overline{\alpha},\overline{\beta}$, in a simple calculation of the functor Φ on $\overline{\alpha},\overline{\beta}$, all used words have length not exceeding $|\mathbf{P}(\overline{\alpha},\overline{\beta})|$, then $\Theta_{\Phi}(\overline{\alpha},\overline{\beta}) \subseteq \Theta_{\mathbb{U}}(\dfrac{p^{|\mathbf{P}(\overline{\alpha},\overline{\beta})|+1}-1}{p-1})$.

Next, let $|\gamma_1| \le |\mathbf{P}_1(\overline{\alpha})|,\dots,|\gamma_m| \le |\mathbf{P}_m(\overline{\alpha})|$, then in a simple calculation of the functor Φ on $\overline{\alpha},\overline{\gamma}$, all used words have length at most $|\mathbf{P}(\overline{\alpha},\mathbf{P}_1(\overline{\alpha}),\dots,\mathbf{P}_m(\overline{\alpha}))|$, then we get $\Theta_{\Phi}(\overline{\alpha},\overline{\gamma}) \subseteq \Theta_{\mathbb{U}}(\dfrac{p^{|\mathbf{P}(\overline{\alpha},\mathbf{P}_1(\overline{\alpha}),\dots,\mathbf{P}_m(\overline{\alpha}))|+1}-1}{p-1})$.

Next, consider a formula of the form $\exists_z^{|\mathbf{P}_1(\overline{\alpha})|}[\Phi(\overline{\alpha},z) = \Lambda]$, then for any word γ such that $|\gamma| \le |\mathbf{P}_1(\overline{\alpha})|$, true $\Theta_{\Phi}(\overline{\alpha},\gamma) \subseteq \Theta_{\mathbb{U}}(\dfrac{p^{|\mathbf{P}(\overline{\alpha},\mathbf{P}_1(\overline{\alpha}))|+1}-1}{p-1})$, then

$\mathbf{WordM}_{\mathbb{A}} \models \{\exists_z^{|\mathbf{P}_1(\overline{\alpha})|}\Phi(\overline{\alpha}, z) = \Lambda \Leftrightarrow \exists_z^{|\mathbf{P}_1(\overline{\alpha})|}(\Phi)^*(\Theta_{\mathbb{U}}(\frac{p^{|\mathbf{P}(\overline{\alpha},\mathbf{P}_1(\overline{\alpha}))|+1} - 1}{p-1}), \overline{\alpha}, z) = \Lambda\}.$

Similarly, reasoning, we get

$\mathbf{WordM}_{\mathbb{A}} \models \{\exists_{z_2}^{|\mathbf{P}_2(\overline{\beta})|}\exists_{z_1}^{|\mathbf{P}_1(\overline{\beta})|}\Phi(\overline{\beta}, z_1, z_2) = \Lambda \Leftrightarrow$
$\Leftrightarrow \exists_{z_2}^{|\mathbf{P}_2(\overline{\beta})|}\exists_{z_1}^{|\mathbf{P}_1(\overline{\beta})|}(\Phi)^*(\Theta_{\mathbb{U}}(\frac{p^{|\mathbf{P}(\overline{\beta},\mathbf{P}_1(\overline{\beta}),\mathbf{P}_2(\overline{\beta}))|+1} - 1}{p-1}), \overline{\beta}, z_1, z_2) = \Lambda\}.$

Proposition 5.7 . Let $n + m(n \geq 1, m \geq 1)$ be a place p - alphabetic functor $\Phi \in \mathbf{PPr}$, an interpretive p - an alphabetic set of $\mathbb{A}$ argument words, and word polynomials $\mathbf{P}_1(\overline{x}), \ldots, \mathbf{P}_m(\overline{x})$, then you can construct a word polynomial $\mathbf{P}(\overline{x})$, which

$\mathbf{WordM}_{\mathbb{A}} \models \{\exists_{z_1}^{|\mathbf{P}_1(\overline{x})|} \ldots \exists_{z_m}^{|\mathbf{P}_m(\overline{x})|}\Phi(\overline{x}, z_1, \ldots z_m) = \Lambda \Leftrightarrow$
$\Leftrightarrow \exists_{z_1}^{|\mathbf{P}_1(\overline{x})|}, \ldots, \exists_{z_m}^{|\mathbf{P}_m(\overline{x})|}(\Phi)^*(\Theta_{\mathbb{U}}(\frac{p^{|\mathbf{P}(\overline{x})|+1} - 1}{p-1}), \overline{x}, z_1, \ldots, z_m) = \Lambda\}.$

Likewise.

Proposition 5.8 Let $n + m(n \geq 1, m \geq 1)$ be a place p - alphabetic functor $\Phi \in \mathbf{PPr}$, an interpretative p - an alphabetic set of $\mathbb{A}$ argument words, and word polynomials $\mathbf{P}_1(\overline{x}), \ldots, \mathbf{P}_m(\overline{x})$, then we can construct a dictionary polynomial such $\mathbf{P}(\overline{x})$, that $\mathbf{WordM}_{\mathbb{A}} \models \{\forall_{z_1}^{|\mathbf{P}_1(\overline{x})|} \ldots \forall_{z_m}^{|\mathbf{P}_m(\overline{x})|}\Phi(\overline{x}, z_1, \ldots z_m) = \Lambda \Leftrightarrow$
$\Leftrightarrow \forall_{z_1}^{|\mathbf{P}_1(\overline{x})|}, \ldots, \forall_{z_m}^{|\mathbf{P}_m(\overline{x})|}(\Phi)^*(\Theta_{\mathbb{U}}(\frac{p^{|\mathbf{P}(\overline{x})|+1} - 1}{p-1}), \overline{x}, z_1, \ldots, z_m) = \Lambda\}.$

Basic complexity classes of computational complexity

The set of $\mathbf{B}$ n - of argument words, given the interpretation of the oracle symbol $\mathbf{U}$ by the set of argument words $\mathbb{A}$, is called polynomial, if there exists (can be constructed) such a quantifier-free formula $\mathcal{B}$, which is built from functors belonging to $\mathbf{PPr}$, which is true $\forall\overline{\alpha}[\mathbf{WordM}_{\mathbb{A}} \models \mathcal{B}(\overline{\alpha}) \Longleftrightarrow \overline{\alpha} \in \mathbf{B}]$.

The class of all polynomial sets with respect to the set $\mathbb{A}$ - argument words will be denoted by $\mathcal{P}_{\mathbb{A}}(\mathbf{U})$. This class of word sets is closed with respect to Boolean operations: intersection, union, additionы, and hanging of the limited existence and universal quantifier $(\exists, \forall)$ over subwords.

A set of $\mathbf{B}$ n - of argument words is called a set of type $\sum$, with respect to some set of argument words $\mathbb{A}$, if for some $\exists$ a restricted formula $\mathcal{B}(\overline{x})$ whose quantifier-free formula is built from functors belongs to $\mathbf{PPr}$, $\forall\overline{\alpha}\{\overline{\alpha} \in \mathbf{B} \Longleftrightarrow \mathbf{WordM}_{\mathbb{A}} \models \mathcal{B}(\overline{\alpha})\}$.

Let $\mathbf{co} - \mathcal{NP}_{\mathbb{A}}(\mathbf{U}) = \{C : \overline{C} \in \mathcal{NP}_{\mathbb{A}}(\mathbf{U}\}$. The set belonging to the class $\mathbf{co} - \mathcal{NP}_{\mathbb{A}}(\mathbf{U})$ will be called a set of type $\prod$[7].

Part VI

Complexity classes and elementary model theory

The known relations between the introduced classes:

a). There is an oracle $\mathbb{A}$, such that $\mathcal{P}_{\mathbb{A}}(\mathbf{U}) = \mathcal{NP}_{\mathbb{A}}(\mathbf{U})$;

b). There is an oracle $\mathbb{B}$, such that $\mathcal{P}_{\mathbb{B}}(\mathbf{U}) \neq \mathcal{NP}_{\mathbb{B}}(\mathbf{U})$;

c). There is an oracle $\mathbb{C}$, such that $\mathcal{P}_{\mathbb{C}}(\mathbf{U}) \neq \mathcal{NP}_{\mathbb{C}}(\mathbf{U})$ и $\mathcal{NP}_{\mathbb{C}}(\mathbf{U}) = \mathbf{co} - \mathcal{NP}_{\mathbb{C}}(\mathbf{U})$;

d). There is an oracle $\mathbb{D}$, such that $\mathcal{NP}_{\mathbb{D}}(\mathbf{U}) \neq \mathbf{co} - \mathcal{NP}_{\mathbb{D}}(\mathbf{U})$;

e). There is an oracle $\mathbb{E}$, such that $\mathcal{P}_{\mathbb{E}}(\mathbf{U}) = \mathcal{NP}_{\mathbb{E}}(\mathbf{U}) \bigcap \mathbf{co} - \mathcal{NP}_{\mathbb{E}}(\mathbf{U})$;

f). There is an oracle $\mathbb{F}$, such that $\mathcal{P}_{\mathbb{F}}(\mathbf{U}) = \mathcal{NP}_{\mathbb{F}}(\mathbf{U}) \bigcap \mathbf{co} - \mathcal{NP}_{\mathbb{F}}(\mathbf{U})$ and $\mathcal{NP}_{\mathbb{F}}(\mathbf{U}) = \mathbf{co} - \mathcal{NP}_{\mathbb{F}}(\mathbf{U})$;

g). There is an oracle $\mathbb{G}$, such that $\mathcal{P}_{\mathbb{G}}(\mathbf{U}) = \mathcal{NP}_{\mathbb{G}}(\mathbf{U}) \bigcap \mathbf{co} - \mathcal{NP}_{\mathbb{G}}(\mathbf{U})$ and $\mathcal{NP}_{\mathbb{G}}(\mathbf{U}) \neq \mathbf{co} - \mathcal{NP}_{\mathbb{G}}(\mathbf{U})$;

h). There is an oracle $\mathbb{H}$, such that $\mathcal{P}_{\mathbb{H}}(\mathbf{U}) \neq \mathcal{NP}_{\mathbb{F}}(\mathbf{U}) \bigcap \mathbf{co} - \mathcal{NP}_{\mathbb{H}}(\mathbf{U})$ and $\mathcal{NP}_{\mathbb{H}}(\mathbf{U}) = \mathbf{co} - \mathcal{NP}_{\mathbb{H}}(\mathbf{U})$

i). There is an oracle $\mathbb{I}$, such that $\mathcal{P}_{\mathbb{I}}(\mathbf{U}) \neq \mathcal{NP}_{\mathbb{I}}(\mathbf{U}) \bigcap \mathbf{co} - \mathcal{NP}_{\mathbb{I}}(\mathbf{U})$ and $\mathcal{NP}_{\mathbb{I}}(\mathbf{U}) \neq \mathbf{co} - \mathcal{NP}_{\mathbb{I}}(\mathbf{U})$.

Every specified ratio in the non-relativized version is a problem.

The main concepts and considered theorems in this section are borrowed from [8-13] and transformed accordingly.

Let $\mathscr{F}(\mathbf{U})$ be some set of functors containing the original functors.

A first-order language, defined by a given set of functors, and denoted as $\mathcal{L}_{\mathscr{F}(\mathbf{U})}$, consists of the function symbols f_Φ, for each functor $\Phi \in \mathscr{F}(\mathbf{U})$ whose locality is equal to the locality of the Φ functor, constant symbol Λ, basic predicate symbol $\leq$.

Note. As a rule, the function symbol f_Φ will be denoted as Φ and interpreted as a function. Constant symbols will also continue to be denoted as a_k. A language $\mathcal{L}_{\mathscr{F}(\mathbf{U})}$ is called k - alphabetic if the set of functors $\mathscr{F}(\mathbf{U})$ is k is alphabetic.

If the set of functors $\mathscr{F}(\mathbf{U})$ consists of the entire set of word primitive recursive functors, then the language $\mathcal{L}_{\mathscr{F}(\mathbf{U})}$ will be denoted as $\mathcal{L}(\mathbf{U})$ $\mathcal{L}$.

For each fixed set of $\mathbf{p} \geq 2$ -alphabetic argument words, we define the following theories:

$\mathbf{Th} = \{\mathcal{A} : \mathbf{WordM} \models \mathcal{A}$- proposition of language $\mathcal{L}\} + \forall xy(x \leq y \equiv |x| \dot{-} |y| = \Lambda)$.

$\mathbf{Th}$ - complete theory in language $\mathcal{L}$.

Let us define a theory in the language $\mathcal{L}(\mathbf{U})$, denoted as $\mathbf{Th}(\mathbf{U})$:

$\mathbf{Th}(\mathbf{U}) = \{\mathcal{A}$: For any set of argument words $\mathbb{A}$ $\mathbf{WordM}_{\mathbb{A}} \models \mathcal{A}$, $\mathcal{A}$ - proposition of language $\mathcal{L}(\mathbf{U})\} + \forall xy(x \leq y \equiv |x| \dot{-} |y| = \Lambda)$.

$\mathbf{Th}(\mathbb{A}) = \{\mathcal{A} : \mathbf{WordM}_{\mathbb{A}} \models \mathcal{A}$, $\mathcal{A}$ - proposition of language $\mathcal{L}_{\mathbb{A}}\} + \forall xy(x \leq y \equiv |x| \dot{-} |y| = \Lambda)$

$\mathbf{Th}(\mathbb{A})$ - complete theory in language $\mathcal{L}_{\mathbb{A}}$.

$\mathbf{Th} \subset \mathbf{Th}(\mathbf{U}) \subset \mathbf{Th}(\mathbb{A})$ takes place.

Theorem 6.1. Let $\mathfrak{A}' \models \mathbf{Th}$. Let $\mathbf{u} : A' \to \{\Lambda, 1\}$, then the model $\mathfrak{A}'$ of the language $\mathcal{L}$ can be enriched to a model $\mathfrak{A}$ of the language $\mathcal{L}(\mathbf{U})$, such that $\mathfrak{A} \models \mathbf{Th}(\mathbf{U})$, $\forall b \in A'$ $\mathbf{u}(b) = \mathbf{U}_{\mathfrak{A}}(b)$ and for any formula $\mathscr{A}(\overline{x})$ in $\mathcal{L}$, for $\forall \overline{a} \in A'$ if $\mathfrak{A}' \models \mathscr{A}(\overline{a})$, then $\mathfrak{A} \models \mathscr{A}(\overline{a})$,

Proof. Let us interpret the oracle function $\mathbf{U}$: for each element $a \in A$, let $\mathbf{U}(a) = \Lambda$ if $\mathbf{u}(a) = \Lambda$ and $\mathbf{U}(a) = 1$ if $\mathbf{u}(a) = 1(\ 1 \leftrightharpoons a_1)$. Let us formulate the theory $\mathbf{Th}(\mathfrak{A}_A)$ (see [9, p. 130]), then let us formulate the theory $\mathbf{Th}(\mathfrak{A}_A) + \mathbf{Th}(\mathbf{U}) + \{\mathbf{U}(a) = b : a \in A'\}$. This theory is consistent, let

$\mathfrak{A} \models \mathbf{Th}(\mathfrak{A}_A) + \mathbf{Th}(\mathbf{U})) + \{\mathbf{U}(a) = b : a \in A'\}$, then for any formula $\mathscr{A}(\overline{x})$ in $\mathcal{L}$, for $\forall \overline{a} \in A'$ if $\mathfrak{A}' \models \mathscr{A}(\overline{a})$, then $\mathfrak{A} \models \mathscr{A}(\overline{a})$.

Let us introduce the following important concept.

Definition. Let $\mathfrak{A}$ be a model of the language $\mathcal{L}(\mathbf{U})$ ($\mathcal{L}$) and $\overline{\boldsymbol{a}} \in A$. A polynomial cut defined by a set of elements $\overline{\boldsymbol{a}} \in A$ is such a model (algebraic system), denoted as $\mathfrak{A}_{\overline{\boldsymbol{a}}}$ supported by the set

$A_{\overline{\boldsymbol{a}}} = \{b :$ for some word polynomial $\mathbf{P}(\overline{x})$ $\mathfrak{A} \models |b| \leq |\mathbf{P}(\overline{\boldsymbol{a}})|\}$, and the signature consists of all those functions f_Φ for which the functor is $\Phi \in \mathbf{PPr}$.

That's right: 1. $\mathfrak{A}_{\overline{\boldsymbol{a}}} \subseteq \mathfrak{A}$[7 p.36].

2. $\forall \overline{a}$ $[\mathfrak{A}_{\overline{\boldsymbol{a}}} \models \mathcal{A}(\overline{\boldsymbol{a}}) \Longleftrightarrow \mathfrak{A} \models \mathcal{A}(\overline{\boldsymbol{a}})]$, where $\mathcal{A}(\overline{x})$ - bounded formula of the signature $\mathbf{PPr}$.

By $\Delta_{\mathfrak{A}}$ - we will denote the diagram of the model $\mathfrak{A}$, in particular $\Delta_{\mathfrak{A}_{\overline{\boldsymbol{a}}}}$ is the diagram of the polynomial cut $\mathfrak{A}_{\overline{\boldsymbol{a}}}$.

Note. Let's $\mathfrak{A}'$ - reduct of the model $\mathfrak{A}$ in lanquaqe $\mathcal{L}(\mathbf{U})$ to the model in language $\mathcal{L}$, we have $\Delta_{\mathfrak{A}'} \subseteq \Delta_{\mathfrak{A}}$, in in particular $\Delta_{\mathfrak{A}'_{\overline{\boldsymbol{a}}}} \subseteq \Delta_{\mathfrak{A}_{\overline{\boldsymbol{a}}}}$, considering Corollary 5.2, page 26, by the diagram $\Delta_{\mathfrak{A}'}$, we can recover the diagram $\Delta_{\mathfrak{A}}$, in this case, it is necessary to know the graph of the oracle $\mathbf{U}$, in the model $\mathfrak{A}$. The diagram $\Delta_{\mathfrak{A}'}$ contains only traces of oracle computations, the full information about oracle computations is contained in the diagram $\Delta_{\mathfrak{A}}$, for example, if $\mathfrak{A} \models \Theta_\Phi(\overline{b}) = c$, then $\mathfrak{A}' \models (\Theta_\Phi)^*(c.\overline{b}) = c$, if $[\Theta_\Phi(\overline{b}) = c] \in \Delta_{\mathfrak{A}_{\overline{\boldsymbol{a}}}}$, then $[(\Theta_\Phi)^*(c.\overline{b}) = c] \in \Delta_{\mathfrak{A}'_{\overline{\boldsymbol{a}}}}$.

Proposition 1. For an arbitrary $n+1 (n \geq 1)$ - ary functor Φ, for an arbitrary model $\mathfrak{A}$ of $\mathcal{L}_{\mathbf{U}}$ such that $\mathfrak{A} \models \mathbf{Th}(\mathbb{A})$, for an arbitrary set $\overline{\boldsymbol{a}} \in \mathfrak{A}$, for an arbitrary word polynomial $\mathbf{P}(\overline{x})$, for an arbitrary function word $\mathbf{b}$ such that $\mathfrak{A} \models \forall_u^{|\mathbf{P}(\overline{\boldsymbol{a}})|} \Theta_\Phi(\overline{\boldsymbol{a}}, u) \subseteq \mathbf{b}$ is true:

1. $\mathfrak{A} \models \forall_u^{|\mathbf{P}(\overline{\boldsymbol{a}})|} \Phi(\overline{\boldsymbol{a}}, u) = \Lambda \Longleftrightarrow \mathfrak{A} \models \forall_u^{|\mathbf{P}(\overline{\boldsymbol{a}})|} (\Phi)^*(\mathbf{b}, \overline{\boldsymbol{a}}, u) = \Lambda$.

2. $\mathfrak{A}_{\overline{\boldsymbol{a}}} \models \forall_u^{|\mathbf{P}(\overline{\boldsymbol{a}})|} \Phi(\overline{\boldsymbol{a}}, u) = \Lambda \Longleftrightarrow \mathfrak{A} \models \forall_u^{|\mathbf{P}(\overline{\boldsymbol{a}})|} (\Phi)^*(\mathbf{b}, \overline{\boldsymbol{a}}, u) = \Lambda$.

Proposition 2. For an arbitrary $n+1-$ place functor Φ, for an arbitrary model $\mathfrak{A}$ of $\mathcal{L}_{\mathbf{U}}$ such that $\mathfrak{A} \models \mathbf{Th}(\mathbb{A})$, for an arbitrary sequence of elements $\overline{\boldsymbol{a}} \in \mathfrak{A}$, for an arbitrary word polynomial $\mathbf{P}(\overline{x})$, there exists a functional element $\mathbf{b}$ with the smallest length such that $\forall_u^{|\mathbf{P}(\overline{\boldsymbol{a}})|} \Theta_\Phi(\overline{\boldsymbol{a}}, u) \subseteq \mathbf{b}$.

Note. Such a functional element is not the only one, but for any such functional elements $\mathbf{b}, \mathbf{c}$ true $|\mathbf{b}| =$

$|\mathbf{c}| \wedge dom(\mathbf{b}) = dom(\mathbf{c}) \wedge \forall x[x \in dom(\mathbf{b}) \Rightarrow \mathbf{b}(x) = \mathbf{c}(x)]$.

Any extension of the polynomial cut $\mathfrak{A}_{\overline{\boldsymbol{a}}} \subseteq \mathfrak{M} \models \mathbf{Th}(\mathbb{A})$, for any functor, any word polynomial $\mathbf{P}(\overline{x})$, produces the following relations:

1. $\mathfrak{A}_{\overline{\boldsymbol{a}}} \subseteq \mathfrak{M}_{\overline{\boldsymbol{a}}}$;

2. $\forall \overline{\boldsymbol{b}} \in A_{\overline{\boldsymbol{a}}}$ true $\Theta_\Phi(\overline{\boldsymbol{b}})_{\mathfrak{A}} = \Theta_\Phi(\overline{\boldsymbol{b}})_{\mathfrak{M}}$;

3. $\forall \overline{\boldsymbol{b}}, c \in A_{\overline{\boldsymbol{a}}}$, if $\mathfrak{A} \models (\Phi)^*(\Theta_\Phi(\overline{\boldsymbol{b}}), \overline{\boldsymbol{b}}) = c$, then $\forall \boldsymbol{f} \in M$, sush that $\Theta_\Phi(\overline{\boldsymbol{b}}) \subseteq \boldsymbol{f}$,true $\mathfrak{M} \models (\Phi)^*(\boldsymbol{f}, \overline{\boldsymbol{b}}) = c$ - "**Use Principle**" when expanding models;

4. Let $\overline{\boldsymbol{b}} \in A_{\overline{\boldsymbol{a}}}$. Let $\boldsymbol{c}_{\mathfrak{A}}$ - the smallest functional element in length, such that $\mathfrak{A} \models \forall_u^{|\mathbf{P}(\overline{\boldsymbol{b}})|} \Theta_\Phi(\overline{\boldsymbol{b}}, u) \subseteq \boldsymbol{c}_{\mathfrak{A}}$.

Let $\boldsymbol{d}_{\mathfrak{M}}$ - the smallest functional element in length, such that $\mathfrak{M} \models \forall_u^{|\mathbf{P}(\overline{\boldsymbol{b}})|} \Theta_\Phi(\overline{\boldsymbol{b}}, u) \subseteq \boldsymbol{d}_{\mathfrak{M}}$. Then:

a). $dom(\boldsymbol{c}_{\mathfrak{A}}) \subset A_{\overline{\boldsymbol{a}}} \wedge dom(\boldsymbol{d}_{\mathfrak{M}}) \subset M_{\overline{\boldsymbol{a}}} \wedge \boldsymbol{c}_{\mathfrak{A}} \subseteq \boldsymbol{d}_{\mathfrak{M}}$;

b). If $|e| \leq |\mathbf{P}(\overline{\boldsymbol{b}})|$ and $\mathfrak{A} \models \Phi(\overline{\boldsymbol{b}}, e) = h$, then $\mathfrak{A} \models (\Phi)^*(\boldsymbol{c}_{\mathfrak{A}}, \overline{\boldsymbol{b}}, e) = h$ и $\mathfrak{M} \models (\Phi)^*(\boldsymbol{d}_{\mathfrak{M}}, \overline{\boldsymbol{b}}, e) = h$;

c). If $\mathfrak{A} \models \forall_u^{|\mathbf{P}(\overline{\boldsymbol{b}})|} (\Phi)^*(\boldsymbol{c}_{\mathfrak{A}}, \overline{\boldsymbol{b}}, u) = \Lambda$, then for any $|f| \leq |\mathbf{P}(\overline{\boldsymbol{b}})|$ we get $\mathfrak{A} \models (\Phi)^*(\boldsymbol{c}_{\mathfrak{A}}, \overline{\boldsymbol{b}}, f) = \Lambda$, then $\mathfrak{M} \models (\Phi)^*(\boldsymbol{d}_{\mathfrak{M}}, \overline{\boldsymbol{b}}, f) = \Lambda$, then $\mathfrak{M} \models \forall u(|u| \leq |\mathbf{P}(\overline{\boldsymbol{b}})| \wedge u \in A_{\overline{\boldsymbol{a}}} \supset (\Phi)^*(\boldsymbol{d}_{\mathfrak{M}}, \overline{\boldsymbol{b}}, u) = \Lambda$. There is also, if $\mathfrak{M} \models \forall u(|u| \leq |\mathbf{P}(\overline{\boldsymbol{b}})| \wedge u \in A_{\overline{\boldsymbol{a}}} \supset (\Phi)^*(\boldsymbol{d}_{\mathfrak{M}}, \overline{\boldsymbol{b}}, u) = \Lambda)$, then $\mathfrak{A} \models \forall_u^{|\mathbf{P}(\overline{\boldsymbol{b}})|} (\Phi)^*(\boldsymbol{c}_{\mathfrak{A}}, \overline{\boldsymbol{b}}, u) = \Lambda$.

"**Use Principle**" when expanding models.

Theorem 6.2. Let $\mathbb{A}$ be an arbitrary oracle. $\mathcal{NP}(\mathbb{A})) = \mathbf{co} - \mathcal{NP}(\mathbb{A})$ if and only if for any bounded $\exists$ formula $\mathcal{A}(x)$ of signature $\mathbf{PPr}(\mathbf{U})$ there exists a bounded $\forall$ formula $\mathcal{B}(x)$ of the same signature such that $\mathbf{Th}(\mathbb{A}) \vdash \forall x[\mathcal{A}(x) \equiv \mathcal{B}(x)]$.

Theorem 6.3. Let $\mathbb{A}$ be an oracle set, $\boldsymbol{\mathcal{A}}(\overline{x})$ be a bounded $\forall$ formula of the signature $\mathbf{PPr}$ of the language $\mathcal{L}(\mathbf{U})$. The following conditions are equivalent:

1. For any model $\mathfrak{A} \models \mathbf{Th}(\mathbb{A})$, for any $\overline{\boldsymbol{a}} \in A$, for any $\overline{\boldsymbol{b}} \in A_{\overline{\boldsymbol{a}}}$, any model $\mathfrak{M} \models \mathbf{Th}(\mathbb{A})$ such that $\mathfrak{A}_{\overline{\boldsymbol{a}}} \subseteq \mathfrak{M}$, if $\mathfrak{A} \models \boldsymbol{\mathcal{A}}(\overline{\boldsymbol{b}})$, then $\mathfrak{M} \models \boldsymbol{\mathcal{A}}(\overline{\boldsymbol{b}})$.

2. For a formula $\boldsymbol{\mathcal{A}}(\overline{x})$ there exists a bounded $\exists$ formula $\boldsymbol{\mathcal{B}}(\overline{x})$ of the language $\mathcal{L}(\mathbf{U})$, such that $\mathbf{Th}(\mathbb{A}) \models \forall \overline{x}[\boldsymbol{\mathcal{A}}(\overline{x}) \equiv \boldsymbol{\mathcal{B}}(\overline{x})]$.

Proof. We will prove that (1) implies (2). The idea of the proof is borrowed from [8 p. 156], [9 p.133-134].

If the formula $\boldsymbol{\mathcal{A}}(\overline{x})$ is such that $\mathbf{Th}(\mathbb{A}) \models \forall \overline{x} \boldsymbol{\mathcal{A}}(\overline{x})$, then $\mathbf{Th}(\mathbb{A}) \models \forall(\overline{x}[\boldsymbol{\mathcal{A}}(\overline{x}) \equiv \boldsymbol{\mathcal{B}}(\overline{x})]$, where $\boldsymbol{\mathcal{B}}(\overline{x})$ is a bounded $\exists$ formula, such that $\mathbf{Th}(\mathbb{A}) \models \forall \overline{x} \boldsymbol{\mathcal{B}}(\overline{x})$.

Let the formula $\boldsymbol{\mathcal{A}}(\overline{x})$ be such that $\mathbf{Th}(\mathbb{A}) \not\models \forall \overline{x} \boldsymbol{\mathcal{A}}(\overline{x})(1)$.

Let's $\boldsymbol{\Gamma}(\overline{c}) = \{\Theta(\overline{c}) : \ \mathbf{Th}(\mathbb{A}) \models \neg \boldsymbol{\mathcal{A}}(\overline{x}) \supset \Theta(\overline{x})\}$, where $\Theta(\overline{x})$ - bounded $\forall$ formula, $\overline{c}$ - new constant symbols. From (1) we obtain that $\mathbf{Th}(\mathbb{A}) + \boldsymbol{\Gamma}(\overline{c})$ - consistent theory. Let's prove it $\mathbf{Th}(\mathbb{A}) + \boldsymbol{\Gamma}(\overline{c}) \models \neg \boldsymbol{\mathcal{A}}(\overline{c})$. Let's $\mathfrak{A} \models \mathbf{Th}(\mathbb{A}) + \boldsymbol{\Gamma}(\overline{c})$, $\overline{c} \in A$. Let $\Delta_{\mathfrak{A}_{\overline{c}}}$ - be a diagram of the model $\mathfrak{A}_{\overline{c}}$.

Set of sentences $\mathbf{Th}(\mathbb{A})+\Delta_{\mathfrak{A}_{\overline{c}}}+\neg\boldsymbol{A}(\overline{c})$ - consistent or inconsistent. If $\mathbf{Th}(\mathbb{A})+\Delta_{\mathfrak{A}_{\overline{c}}}+\neg\boldsymbol{A}(\overline{c})$ - inconsistent, then $\mathbf{Th}(\mathbb{A})+\Delta_{\mathfrak{A}_{\overline{c}}}\models\boldsymbol{A}(\overline{c})$, then $\mathbf{Th}(\mathbb{A})+\bigwedge_{i\le k}\Lambda_i(\overline{c},\overline{d})\models\boldsymbol{A}(\overline{c})$, where $\Lambda_i(\overline{c},\overline{d})\in\Delta_{\mathfrak{A}_{\overline{c}}}$, then $\mathbf{Th}(\mathbb{A})\models\bigwedge_{i\le k}\Lambda_i(\overline{c},\overline{d})\supset\boldsymbol{A}(\overline{c})$, then $\mathbf{Th}(\mathbb{A})\models\neg\boldsymbol{A}(\overline{c})\supset\neg\bigwedge_{i\le k}\Lambda_i(\overline{c},\overline{d})$, then $\mathbf{Th}(\mathbb{A})\models\forall\overline{x}\forall\overline{y}[\neg\boldsymbol{A}(\overline{x})\supset\neg\bigwedge_{i\le k}\Lambda_i(\overline{x},\overline{y})]$, then $\mathbf{Th}(\mathbb{A})\models\forall\overline{x}[\neg\boldsymbol{A}(\overline{x})\supset\forall\overline{y}\neg\bigwedge_{i\le k}\Lambda_i(\overline{x},\overline{y})](2)$.

For $\overline{d}$, there exist such word polynomials $\tilde{\mathbf{P}}(\overline{c})$, that $|\overline{d}|\le|\tilde{\mathbf{P}}(\overline{c})|$. From(2) we obtain $\mathbf{Th}(\mathbb{A})\models\forall\overline{x}[\neg\boldsymbol{A}(\overline{x})\supset\forall_{\overline{y}}^{|\tilde{\mathbf{P}}(\overline{x})|}\neg\bigwedge_{i\le k}\Lambda_i(\overline{x},\overline{y})]$, then $\forall_{\overline{y}}^{|\tilde{\mathbf{P}}(\overline{c})|}\neg\bigwedge_{i\le k}\Lambda_i(\overline{c},\overline{y})\in\mathbf{\Gamma}(c)$, then $\mathfrak{A}\models\forall_{\overline{y}}^{|\tilde{\mathbf{P}}(\overline{c})|}\neg\bigwedge_{i\le k}\Lambda_i(\overline{c},\overline{y})$, then $\mathfrak{A}\models\neg\bigwedge_{i\le k}\Lambda_i(\overline{c},\overline{d})$ - contradiction, hence set sentences $\mathbf{Th}(\mathbb{A})+\Delta_{\mathfrak{A}_{\overline{c}}}+\neg\boldsymbol{A}(\overline{c})$ - consistent.

Let's $\mathfrak{M}\models\mathbf{Th}(\mathbb{A})+\Delta_{\mathfrak{A}_{\overline{c}}}+\neg\boldsymbol{A}(\overline{c})$, then we get $\mathfrak{A}_{\overline{c}}\subseteq\mathfrak{M}$ и $\mathfrak{M}\models\neg\boldsymbol{A}(\overline{c})$, then $\mathfrak{A}\models\neg\boldsymbol{A}(\overline{c})$, consequently $\mathbf{Th}(\mathbb{A})+\mathbf{\Gamma}(\overline{c})\models\neg\Phi(\overline{c})$, then $\mathbf{Th}(\mathbb{A})\models\bigwedge_{j\le m}\Omega_j(\overline{c})\supset\neg\boldsymbol{A}(\overline{c})$, where $\Omega_j(\overline{c})\in\mathbf{\Gamma}(\overline{c})$, then $\mathbf{Th}(\mathbb{A})\models\bigwedge_{j\le m}\Omega_j(\overline{x})\supset\neg\boldsymbol{A}(\overline{x})$.

We have $\mathbf{Th}(\mathbb{A})\models\neg\boldsymbol{A}(\overline{x})\supset\bigwedge_{j\le m}\Omega_j(\overline{x})$, then $\mathbf{Th}(\mathbb{A})\models\neg\boldsymbol{A}(\overline{x})\equiv\bigwedge_{j\le m}\Lambda_j(\overline{x})$, then $\mathbf{Th}(\mathbb{A})\models\neg\bigwedge_{j\le m}\Omega_j(\overline{x})\equiv\boldsymbol{A}(\overline{x})$.

Let us prove that (2) implies (1). Let $\mathfrak{A}\models\mathbf{Th}(\mathbb{A})$, $\overline{\boldsymbol{a}}\in A$, $\overline{\boldsymbol{b}}\in A_{\overline{\boldsymbol{a}}}$, $\mathfrak{M}\models\mathbf{Th}(\mathbb{A})$, such that $\mathfrak{A}_{\overline{\boldsymbol{a}}}\subseteq\mathfrak{M}$ and $\mathfrak{A}\models\boldsymbol{A}(\overline{\boldsymbol{b}})$.

We have: for the formula $\boldsymbol{A}(\overline{x})$ there exists a bounded $\exists$ formula $\boldsymbol{\mathcal{B}}(\overline{x})$ of the language $\mathcal{L}(\mathbf{U})$, such that $\mathbf{Th}(\mathbb{A})\models\forall\overline{x}[\boldsymbol{A}(\overline{x})\equiv\boldsymbol{\mathcal{B}}(\overline{x})]$, given $\mathfrak{A}\models\mathbf{Th}(\mathbb{A})$, we obtain $\mathfrak{A}\models\forall\overline{x}[\boldsymbol{A}(\overline{x})\equiv\boldsymbol{\mathcal{B}}(\overline{x})]$, then $\mathfrak{A}\models\boldsymbol{A}(\overline{\boldsymbol{b}})\equiv\boldsymbol{\mathcal{B}}(\overline{\boldsymbol{b}})$, taking into account $\mathfrak{A}\models\boldsymbol{A}(\overline{\boldsymbol{b}})$, we get $\mathfrak{A}\models\boldsymbol{\mathcal{B}}(\overline{\boldsymbol{b}})$, then $\mathfrak{A}_{\overline{\boldsymbol{a}}}\models\boldsymbol{\mathcal{B}}(\overline{\boldsymbol{b}})$, taking into account that $\boldsymbol{\mathcal{B}}(\overline{x})$ is a bounded $\exists$ formula, $\mathfrak{A}_{\overline{\boldsymbol{a}}}\subseteq\mathfrak{M}$ and $\overline{\boldsymbol{b}}\in M_{\overline{\boldsymbol{a}}}$, we obtain $\mathfrak{M}\models\boldsymbol{\mathcal{B}}(\overline{\boldsymbol{b}})$. From $\mathfrak{M}\models\mathbf{Th}(\mathbb{A})$ and $\mathbf{Th}(\mathbb{A})\models\forall\overline{x}[\boldsymbol{A}(\overline{x})\equiv\boldsymbol{\mathcal{B}}(\overline{x})]$ we get $\mathfrak{M}\models\forall\overline{x}[\boldsymbol{A}(\overline{x})\equiv\boldsymbol{\mathcal{B}}(\overline{x})]$, then $\mathfrak{M}\models\boldsymbol{A}(\overline{\boldsymbol{b}})\equiv\boldsymbol{\mathcal{B}}(\overline{\boldsymbol{b}})$, then $\mathfrak{M}\models\boldsymbol{A}(\overline{\boldsymbol{b}})$.

Note. A similar theorem holds for the theory $\mathbf{Th}$ in the language $\mathcal{L}$ and for the theory $\mathbf{Th}(\mathbf{U})$ in the language $\mathcal{L}(\mathbf{U})$.

Theorem 6.4. For any oracle $\mathbb{A}$, any bounded $\forall$ formula $\boldsymbol{A}(\overline{x})$ of language $\mathcal{L}(\mathbf{U})$ signature $\mathbf{PPr}$, the following conditions are equivalent:

1). For any model $\mathfrak{A}\models\mathbf{Th}(\mathbb{A})$, any elements of $\overline{\boldsymbol{a}}\in A$, any elements of $\overline{\boldsymbol{b}}\in A_{\overline{\boldsymbol{a}}}$, if $\mathfrak{A}_{\overline{\boldsymbol{a}}}\models\boldsymbol{A}(\overline{\boldsymbol{b}})$, then for any model $\mathfrak{B}\supseteq\mathfrak{A}_{\overline{\boldsymbol{a}}}$ such that $\mathfrak{B}\models\mathbf{Th}(\mathbb{A})$, true $\mathfrak{B}\models\boldsymbol{A}(\overline{\boldsymbol{b}})$.

2). For any model $\mathfrak{A}\models\mathbf{Th}(\mathbb{A})$, any elements of $\overline{\boldsymbol{a}}\in A$, any elements of $\overline{\boldsymbol{b}}\in A_{\overline{\boldsymbol{a}}}$, if $\mathfrak{A}_{\overline{\boldsymbol{a}}}\models\mathcal{A}(\overline{\boldsymbol{b}})$, then $\mathbf{Th}(\mathbb{A})+\Delta_{\mathfrak{A}_{\overline{\boldsymbol{a}}}}\vdash\mathcal{A}(\overline{\boldsymbol{b}})$.

3). For the formula $\boldsymbol{A}(\overline{x})$ there is a bounded $\exists$ formula $\mathcal{B}(\overline{x})$ of signature $\mathbf{PPr}$ such that $\mathbf{Th}(\mathbb{A})\vdash\forall\overline{x}[\boldsymbol{A}(\overline{x})\equiv\mathcal{B}(\overline{x})]$(equivalent to $\mathbf{WordM}_{\mathbb{A}}\models\forall\overline{x}[\boldsymbol{A}(\overline{x})\equiv\mathcal{B}(\overline{x})]$).

Proof. The proof of (1) $\Leftrightarrow$ (2) is quite simple. Let us prove that from (3) follows (1).

Let $\mathfrak{A}\models\mathbf{Th}(\mathbb{A})$, $\overline{\boldsymbol{a}}\in A$, $\overline{\boldsymbol{b}}\in A_{\overline{\boldsymbol{a}}}$ and $\mathfrak{A}_{\overline{\boldsymbol{a}}}\models\mathcal{A}(\overline{\boldsymbol{b}})$. Пусть $\mathfrak{B}\supseteq\mathfrak{A}_{\overline{\boldsymbol{a}}}$, such that $\mathfrak{B}\models\mathbf{Th}(\mathbb{A})$. For some bounded $\exists$ formula $\mathcal{B}(\overline{x})$ signature **PPr** we have $\mathbf{Th}(\mathbb{A}) \vdash \forall \overline{x}[\mathcal{A}(\overline{x}) \equiv \mathcal{B}(\overline{x})]$, then $\mathfrak{B} \models \mathcal{A}(\overline{\boldsymbol{b}}) \equiv \mathcal{B}(\overline{\boldsymbol{b}})$. Suppose that $\mathfrak{B} \models \neg\mathcal{A}(\overline{\boldsymbol{b}})$, then $\mathfrak{B} \models \neg\mathcal{B}(\overline{\boldsymbol{b}})$, given $\mathfrak{B} \supseteq \mathfrak{A}_{\overline{\boldsymbol{a}}}$ and the fact that $\neg\mathcal{B}(\overline{x})$ is a $\forall$ formula of signature **PPr**, we obtain $\mathfrak{A}_{\overline{\boldsymbol{a}}} \models \neg\mathcal{B}(\overline{\boldsymbol{b}})$, then $\mathfrak{A} \models \neg\mathcal{B}(\overline{\boldsymbol{b}})$, given that $\mathfrak{A} \models \mathbf{Th}(\mathbb{A})$ and $\mathbf{Th}(\mathbb{A}) \vdash \forall \overline{x}[\mathcal{A}(\overline{x}) \equiv \mathcal{B}(\overline{x}]$, we obtain $\mathfrak{A} \models \mathcal{A}(\overline{\boldsymbol{b}}) \equiv \mathcal{B}(\overline{\boldsymbol{b}})$, then $\mathfrak{A}_{\overline{\boldsymbol{a}}} \models \mathcal{A}(\overline{\boldsymbol{b}}) \equiv \mathcal{B}(\overline{\boldsymbol{b}})$, taking into account $\mathfrak{A}_{\overline{\boldsymbol{a}}} \models \mathcal{A}(\overline{\boldsymbol{b}})$, we obtain $\mathfrak{A}_{\overline{\boldsymbol{a}}} \models \mathcal{B}(\overline{\boldsymbol{b}})$, we get a contradiction, then $\mathfrak{B} \models \mathcal{A}(\overline{\boldsymbol{b}})$. This (1) $\Rightarrow$ (3) follows from Theorem 6.3.

Note. A similar theorem holds for the theory **Th** in the language $\mathcal{L}$ and for the theory $\mathbf{Th}(\mathbf{U})$ in the language $\mathcal{L}(\mathbf{U})$.

Using Proposition 1 and Theorem 6.4, we can prove Theorem 4.5 in [11 p.469] quite simply.

Theorem 6.5 Let the second point of Theorem 6.4 be satisfied for the theory **Th** in the language $\mathcal{L}$, for any bounded $\forall$ formula of signature **PPr**.

Let $\mathfrak{A} \models \mathbf{Th}(\mathbf{U})$, $\overline{\boldsymbol{a}} \in A$, $\overline{\boldsymbol{b}} \in A_{\overline{\boldsymbol{a}}}$. Let the formula $\forall_y^{|\mathbf{P}(\overline{x})|}\Phi(\overline{x}, y) = \Lambda$ be such that $\mathfrak{A} \models \forall_y^{|\mathbf{P}(\overline{\boldsymbol{b}})|}\Phi(\overline{\boldsymbol{b}}, y) = \Lambda$, then $\mathbf{Th}(\mathbf{U}) + \Delta_{\mathfrak{A}} \vdash \forall_y^{|\mathbf{P}(\overline{\boldsymbol{b}})|}\Phi(\overline{\boldsymbol{b}}, y) = \Lambda$.

Formula $\forall \overline{x}[\forall_z^{|\mathbf{P}(\overline{x})|}\Phi(\overline{x}, z) = \Lambda \equiv \forall_u^{|\mathbf{P}(\overline{x})|}(\Phi)^*(\boldsymbol{\Theta}_{\mathbb{U}}(\mathbf{exp_p}(|\tilde{\mathbf{P}}(\overline{x}))|), \overline{x}, u)) = \Lambda]$, belongs to theory $\mathbf{Th}(\mathbf{U})$, where $\tilde{\mathbf{P}}(\overline{x})$ - is a suitable word polynomial, then

$\mathbf{Th}(\mathbf{U}) \vdash \forall \overline{x}[\forall_z^{|\mathbf{P}(\overline{x})|}\Phi(\overline{x}, z) = \Lambda \equiv \forall_u^{|\mathbf{P}(\overline{x})|}(\Phi)^*(\Theta_{\mathbb{U}}(\mathbf{exp_p}(|\tilde{\mathbf{P}}(\overline{x})|)), \overline{x}, u) = \Lambda]$, then

$\mathbf{Th}(\mathbf{U}) \vdash [\forall_z^{|\mathbf{P}(\overline{\boldsymbol{b}})|}\Phi(\overline{\boldsymbol{b}}, z) = \Lambda \equiv \forall_u^{|\mathbf{P}(\overline{\boldsymbol{b}})|}(\Phi)^*(\Theta_{\mathbb{U}}(\mathbf{exp_p}(|\tilde{\mathbf{P}}(\overline{\boldsymbol{b}})|)), \overline{\boldsymbol{b}}, u) = \Lambda](1)$. Let us calculate $|\tilde{\mathbf{P}}(\overline{\boldsymbol{b}})| = \boldsymbol{d}_1$, $\mathbf{exp_p}(\boldsymbol{d}_1) = \boldsymbol{d}_2$, $\Theta_{\mathbb{U}}(\boldsymbol{d}_2) = \boldsymbol{d}_3$, then $\mathfrak{A}' \models \forall_u^{|\mathbf{P}(\overline{\boldsymbol{b}})|}(\Phi)^*(\boldsymbol{d}_3, \overline{\boldsymbol{b}}, u) = \Lambda$, where $\mathfrak{A}'$ is a reduct $\mathfrak{A}$ of language$\mathcal{L}(\mathbf{U})$ to $\mathcal{L}$, then $\mathbf{Th} + \Delta_{\mathfrak{A}'_{\overline{\boldsymbol{a}};\boldsymbol{d}_2}} \vdash \forall_u^{|\mathbf{P}(\overline{\boldsymbol{b}})|}(\Phi)^*(\boldsymbol{d}_3, \overline{\boldsymbol{b}}, u) = \Lambda$, taking into account equality $\Theta_{\mathbb{U}}(\boldsymbol{d}_2) = \boldsymbol{d}_3$, we get

$\mathbf{Th} + \Delta_{\mathfrak{A}'_{\overline{\boldsymbol{a}};\boldsymbol{d}_2} + \{\Theta_{\mathbb{U}}(\boldsymbol{d}_2) = \boldsymbol{d}_3\}} \vdash \forall_u^{|\mathbf{P}(\overline{\boldsymbol{b}})|}(\Phi)^*(\Theta_{\mathbb{U}}(\boldsymbol{d}_2), \overline{\boldsymbol{b}}, u) = \Lambda$, considering $(\Theta_{\mathbb{U}}(\boldsymbol{d}_2) = \boldsymbol{d}_3) \in \Delta_{\mathfrak{A}_{\overline{\boldsymbol{a}};\boldsymbol{d}_2}}$, we get

$\mathbf{Th} + \Delta_{\mathfrak{A}_{\overline{\boldsymbol{a}};\boldsymbol{d}_2}} \vdash \forall_u^{|\mathbf{P}(\overline{\boldsymbol{b}})|}(\Phi)^*(\Theta_{\mathbb{U}}(\boldsymbol{d}_2), \overline{\boldsymbol{b}}, u) = \Lambda$. Considering $\mathbf{Th} \vdash \forall x \forall y[\mathbf{exp_p}(x) = y \equiv \mathbf{EXP_p}(x, y) = \Lambda]$ and $\mathbf{Th} \vdash \forall \overline{z} \forall v \forall x[\mathscr{A}(\overline{z}, v) \wedge \mathbf{EXP_p}(x, v) = \Lambda \supset \mathscr{A}(\overline{z}, \mathbf{exp_p}(x))]$, we get

$\mathbf{Th} + \Delta_{\mathfrak{A}_{\overline{\boldsymbol{a}};\boldsymbol{d}_2}} \vdash \forall_u^{|\mathbf{P}(\overline{\boldsymbol{b}})|}(\Phi)^*(\Theta_{\mathbb{U}}(\mathbf{exp_p}(\boldsymbol{d}_1)), \overline{\boldsymbol{b}}, u) = \Lambda$, considering $|\tilde{\mathbf{P}}(\overline{\boldsymbol{b}})| = \boldsymbol{d}_1$, we get

$\mathbf{Th} + \Delta_{\mathfrak{A}_{\overline{\boldsymbol{a}};\boldsymbol{d}_2}} \vdash \forall_u^{|\mathbf{P}(\overline{\boldsymbol{b}})|}(\Phi)^*(\Theta_{\mathbb{U}}(\mathbf{exp_p}(|\tilde{\mathbf{P}}(\overline{\boldsymbol{b}})|)), \overline{\boldsymbol{b}}, u) = \Lambda]$, considering (1), we get

$\mathbf{Th}(\mathbf{U}) + \Delta_{\mathfrak{A}_{\overline{\boldsymbol{a}};\boldsymbol{d}_2}} \vdash \forall_y^{|\mathbf{P}(\overline{\boldsymbol{b}})|}\Phi(\overline{\boldsymbol{b}}, y) = \Lambda$, then $\mathbf{Th}(\mathbf{U}) + \Delta_{\mathfrak{A}} \vdash \forall_y^{|\mathbf{P}(\overline{\boldsymbol{b}})|}\Phi(\overline{\boldsymbol{b}}, y) = \Lambda$.

Theorem 6.6. If for a theory **Th** for any bounded $\forall$ - formula of the language $\mathcal{L}$ the first point of Theorem 6.4 is satisfied, then for any oracle $\mathbb{A}$, for any bounded $\forall$ - formula of the language $\mathcal{L}(\mathbf{U})$ for a theory $\mathbf{Th}(\mathbb{A})$ the first point of this theorem is also satisfied.

Proof. Let Φ be an arbitrary $n+1$ - ary functor, signature $\mathcal{L}(\mathbf{U})$. Let $\mathbf{P}(x_1, \ldots, x_n)$ be an arbitrary word polynomial. Let us prove a theorem for a formula of the form $\forall_z^{|\mathbf{P}(x_1,\ldots,x_n)|}\Phi(z, x_1, \ldots, x_n) = \Lambda$.

Let $\mathfrak{A} \models \mathbf{Th}(\mathbb{A})$, $\overline{\boldsymbol{a}} \in A$, $\overline{\boldsymbol{b}} \in A_{\overline{\boldsymbol{a}}}$, $\mathfrak{A} \models \forall_z^{|\mathbf{P}(\overline{\boldsymbol{b}})|}[\Phi(z, \overline{\boldsymbol{b}}) = \Lambda]$, let's prove that

$\forall\, \mathfrak{M} \models \mathbf{Th}(\mathbb{A})$, such that $\mathfrak{M} \supseteq \mathfrak{A}_{\overline{\boldsymbol{a}}}$ is true $\mathfrak{M} \models \forall_z^{|\mathbf{P}(\overline{\boldsymbol{b}})|}[\Phi(\overline{\boldsymbol{b}}, z) = \Lambda]$.

Let $\mathfrak{A}'$ be the restriction of the model $\mathfrak{A}$ of the language $\mathcal{L}(\mathbf{U})$ to a model of the language $\mathcal{L}$.

Let's make a theory $\mathbf{Th}(\mathfrak{A}'_A)$ [8] (see [9, p. 130]), next we will make up a theory $\mathbf{Th}(\mathfrak{A}'_A) + \Delta_{\mathfrak{M}'_{\overline{a}}}$, where $\mathfrak{M}'_{\overline{\boldsymbol{a}}}$ - reduct of the model $\mathfrak{M}_{\overline{\boldsymbol{a}}}$ in language $\mathcal{L}(\mathbf{U})$ to the model in the languge $\mathcal{L}$.

This theory is contradictory or it is not. Suppose that the theory $\mathbf{Th}(\mathfrak{A}'_A) + \Delta_{\mathfrak{M}'_{\overline{a}}}$ - is contradictory, then $\mathbf{Th} \vdash \bigwedge A_i(\overline{e}, \overline{f}) \supset \neg \bigwedge B_j(\overline{e}, \overline{h})$, where $\overline{e} \in A_{\overline{\boldsymbol{a}}}$, $A_i(\overline{e}, \overline{f}) \in \mathbf{Th}(\mathfrak{A}'_A)$, $B_j(\overline{e}, \overline{h}) \in \Delta_{\mathfrak{M}'_{\overline{a}}}$, then $\mathbf{Th} \vdash \bigwedge A_i(\overline{e}, \overline{f}) \supset \neg \bigwedge B_j(\overline{e}, \overline{x})$, then $\mathbf{Th} \vdash \bigwedge A_i(\overline{e}, \overline{f}) \supset \forall \overline{x} \neg \bigwedge B_j(\overline{e}, \overline{x})$, then $\mathfrak{A}' \models \forall \overline{x} \neg \bigwedge B_j(\overline{e}, \overline{x})$.

For $\overline{h}$, there exists such a word polynomial $\mathbf{Q}(\overline{x})$ that $|\overline{h}| \leq |\mathbf{Q}(\overline{\boldsymbol{a}})|$, then $\mathfrak{A}' \models \forall_{\overline{x}}^{|\mathbf{Q}(\overline{a})|} \neg \bigwedge B_j(\overline{e}, \overline{x})$, then $\mathfrak{A}' \models \forall_{\overline{x}}^{|\mathbf{Q}(\overline{a})|} \neg \bigwedge B_j(\overline{e}, \overline{x})$, then, according to Theorem 6.4 (1) for the language $\mathcal{L}$, given $\mathfrak{A}'_{\overline{\boldsymbol{a}}} \subseteq \mathfrak{M}'_{\overline{\boldsymbol{a}}}$, we obtain $\mathfrak{M}'_{\overline{\boldsymbol{a}}} \models \forall_{\overline{x}}^{|\mathbf{Q}(\overline{a})|} \neg \bigwedge B_j(\overline{e}, \overline{x})$ is a contradiction, hence the theory $\mathbf{Th}(\mathfrak{A}'_A) + \Delta_{\mathfrak{M}'_{\overline{a}}}$ is non-contradictory.

Note that in the models $\mathfrak{A}'$ and $\mathfrak{M}'$ there are traces of oracle computations of the oracle $\mathbf{U}_{\mathfrak{A}}$ and the oracle $\mathbf{U}_{\mathfrak{M}}$

Let us construct an interpretation of the oracle symbol $\mathbf{U}$:

$$\mathbf{U}(a) = \begin{cases} \mathbf{U}_{\mathfrak{A}}(a), & if\ a \in A; \\ \\ \mathbf{U}_{\mathfrak{M}}(a), & if\ a \in M_{\overline{\boldsymbol{a}}} \end{cases}$$

Let us denote the obtained interpretation as $\mathbb{B}$. According to Theorem 6.1. we get $\mathfrak{N}_1 \models \mathbf{Th}(\mathbf{U}) + \Delta_{\mathfrak{A}'} + \Delta_{\mathfrak{M}'_{\overline{a}}}$.

We have:

1. $\mathfrak{A}' \subseteq \mathfrak{N}_1{}'$, $\mathfrak{M}'_{\overline{\boldsymbol{a}}} \subseteq \mathfrak{N}_1{}'$.

2. For the interpretation $\mathbb{A}_{\mathfrak{A}}$ of the oracle symbol $\mathbf{U}$ in the model $\mathfrak{A}$ and for the interpretation $\mathbb{B}_{\mathfrak{N}_1}$ of the oracle symbol $\mathbf{U}$ in the model $\mathfrak{N}_1$, it is true that $\mathbb{A}_{\mathfrak{A}} \subseteq \mathbb{B}_{\mathfrak{N}_1}$ $(\forall a \in A\ \mathbf{U}_{\mathfrak{A}}(a) = \mathbf{U}_{\mathfrak{N}_1}(a))$.

3. For the interpretation $\mathbb{A}_{\mathfrak{M}_{\overline{a}}}$ of the oracle symbol $\mathbf{U}$ in the model $\mathfrak{M}_{\overline{\boldsymbol{a}}}$ and for the interpretation $\mathbb{B}_{\mathfrak{N}_1}$ of the oracle symbol $\mathbf{U}$ in the model $\mathfrak{N}_1$, it is true that $\mathbb{A}_{\mathfrak{M}_{\overline{a}}} \subseteq \mathbb{B}_{\mathfrak{N}_1}$ $(\forall b \in M_{\overline{\boldsymbol{a}}}\ \mathbf{U}_{\mathfrak{M}_{\overline{a}}}(b) = \mathbf{U}_{\mathfrak{N}_1}(b))$.

4. For any functor Φ in $\mathcal{L}$, $\forall \overline{b} \in A\ \forall c \in A\ \mathfrak{A} \models \Phi(\overline{b}) = c \iff \mathfrak{N}_1 \models \Phi(\overline{b}) = c$, and (1) is used, then $\forall \overline{b} \in A\ \forall c \in A\ \forall d \in A\ \mathfrak{A} \models (\Theta_\Phi)^*(c, \overline{b}) = d \iff \mathfrak{N}_1 \models \Theta^*_\Phi(c, \overline{b}) = d$.

5. For any functor Φ in $\mathcal{L}(\mathbf{U})$, $\forall \overline{b} \in A\ \forall c \in A\ \mathfrak{A} \models \Theta_\Phi(\overline{b}) = c \iff \mathfrak{N}_1 \models \Theta_\Phi(\overline{b}) = c$, using (2.4), Theorem 4.2, Theorem 5.1, and Theorem 5.5:

$\mathfrak{A} \models \Theta_\Phi(\overline{b}) = c_{\mathfrak{A}} \iff \mathfrak{A} \models (\Theta_\Phi)^*(c_{\mathfrak{A}}.\overline{b}) = c_{\mathfrak{A}} \wedge c_{\mathfrak{A}} \subset \mathbb{A}_{\mathfrak{A}}$, then $\mathfrak{N}_1 \models (\Theta_\Phi)^*(c_{\mathfrak{A}}.\overline{b}) = c_{\mathfrak{A}} \wedge c_{\mathfrak{A}} \subset \mathbb{B}_{\mathfrak{N}_1}$, then $\mathfrak{N}_1 \models \Theta_\Phi(\overline{b}) = d_{\mathfrak{N}_1} \subset c_{\mathfrak{A}}$, then $\mathfrak{N}_1 \models (\Theta_\Phi)^*(c_{\mathfrak{A}}.\overline{b}) = \Theta_\Phi(\overline{b}) = d_{\mathfrak{N}_1}$, then $\mathfrak{N}_1 \models \Theta_\Phi(\overline{b}) = d_{\mathfrak{N}_1} = c_{\mathfrak{A}}$.

6. For any functor Φ in $\mathcal{L}(\mathbf{U})$, $\forall \overline{b} \in A\ \forall c \in A\ \mathfrak{A} \models \Phi(\overline{b}) = c \iff \mathfrak{N}_1 \models \Phi(\overline{a}) = b$, using (4.5) and Theorem 5.1. Thus, we obtain $\mathfrak{A} \subseteq \mathfrak{N}_1$, then $\mathfrak{N}_1 \models \Delta_{\mathfrak{A}}$, then $\mathfrak{N}_1 \models \mathbf{Th}(\mathbf{U}) + \Delta_{\mathfrak{A}}$, Taking into account theorem 6.5, we obtain $\mathfrak{N}_1 \models \forall_z^{|\mathbf{P}(\overline{b})|}[\Phi(\overline{\boldsymbol{b}}, z) = \Lambda]$.

[8] We can take a theory $\mathbf{Th} + \Delta_{\mathfrak{A}'}$

7. For any functor Φ in $\mathcal{L}$ of signature **PPr**, it is true that

$\forall \bar{b} \in M_{\overline{\boldsymbol{a}}}\ \forall c \in M_{\overline{\boldsymbol{a}}}\ \mathfrak{M}_{\overline{\boldsymbol{a}}} \models \Phi(\bar{b}) = c \Longleftrightarrow \mathfrak{N}_1 \models \Phi(\bar{b}) = c$, and (1) is used, then it is true that

$\forall \bar{b} \in M_{\overline{\boldsymbol{a}}}\ \forall c \in M_{\overline{\boldsymbol{a}}}\ \forall d \in M_{\overline{\boldsymbol{a}}}\ \ \mathfrak{M}_{\overline{\boldsymbol{a}}} \models (\Theta_\Phi)^*(c, \bar{b}) = d \Longleftrightarrow \mathfrak{N}_1 \models \Theta^*_\Phi(c.\bar{b}) = d$

8. For any functor Φ of the language $\mathcal{L}(\mathbf{U})$, of signature $\mathbf{PPr}(\mathbf{U})$, we have

$\forall \bar{b} \in M_{\overline{\boldsymbol{a}}}\ \forall c \in M_{\overline{\boldsymbol{a}}}\ \ \mathfrak{M}_{\overline{\boldsymbol{a}}} \models \Theta_\Phi(\bar{b}) = c \Longleftrightarrow \mathfrak{N}_1 \models \Theta_\Phi(\bar{b}) = c$, using (3.7), Theorem 4.2, Theorem 5.1, and Theorem 5.5.

9. For any functor Φ of the language $\mathcal{L}(\mathbf{U})$, signature $\mathbf{PPr}(\mathbf{U})$, it is true

$\forall \bar{b} \in M_{\overline{\boldsymbol{a}}}\ \forall c \in M_{\overline{\boldsymbol{a}}}\ \ \mathfrak{M}_{\overline{\boldsymbol{a}}} \models \Phi(\bar{b}) = c \Longleftrightarrow \mathfrak{N}_1 \models \Phi(\bar{b}) = c$, using (4.5) and Theorem 5.1. Thus we obtain $\mathfrak{M}_{\overline{\boldsymbol{a}}} \subseteq \mathfrak{N}_1$.

We have: $\mathfrak{N}_1 \models \forall_y^{|\mathbf{P}(\bar{\boldsymbol{b}})|}\Phi(\bar{\boldsymbol{b}}, y) = \Lambda$, $\mathfrak{M}_{\overline{\boldsymbol{a}}} \subseteq \mathfrak{N}_1$, then $\mathfrak{M}_{\overline{\boldsymbol{a}}} \models \forall_y^{|\mathbf{P}(\bar{\boldsymbol{b}})|}\Phi(\bar{\boldsymbol{b}}, y)$, then $\mathfrak{M} \models \forall_y^{|\mathbf{P}(\bar{\boldsymbol{b}})|}\Phi(\bar{\boldsymbol{b}}, y)$.

Continue. Let $\mathcal{A}(z, x_1, \ldots, x_n)$ - arbitrary quantifier-free formula signatures **PPr**. For this formula, one can construct such $n+1$ - ary functor $\Phi_\mathcal{A}$, that

$\mathbf{Th}(\mathbf{U}) \vdash \forall \bar{x}, \forall z[\mathcal{A}(z, \bar{x}) \equiv \Phi_\mathcal{A}(z, \bar{x}) = \Lambda]$(Theorem 1.6), then for any word polynomial $\mathbf{P}(\bar{x})$ true

$\mathbf{Th}(\mathbf{U}) \vdash \forall \bar{x}[\exists_z^{|\mathbf{P}(\bar{x})|}\mathcal{A}(z, \bar{x}) \equiv \exists_z^{|\mathbf{P}(\bar{x})|}[\Phi_\mathcal{A}(z, \bar{x}) = \Lambda]]$, and also

$\mathbf{Th}(\mathbf{U}) \vdash \forall \bar{x}[\forall_z^{|\mathbf{P}(\bar{x})|}\mathcal{A}(z, \bar{x}) \equiv \forall_z^{|\mathbf{P}(\bar{x})|}[\Phi_\mathcal{A}(z, \bar{x}) = \Lambda]]$, then

$\mathbf{Th}(\mathbf{U}) \vdash \forall_z^{|\mathbf{P}(\bar{\boldsymbol{b}})|}\mathcal{A}(z, \bar{\boldsymbol{b}}) \equiv \forall_z^{|\mathbf{P}(\bar{\boldsymbol{b}})|}[\Phi_\mathcal{A}(z, \bar{\boldsymbol{b}}) = \Lambda]$. Let $\mathfrak{A} \models \forall_z^{|\mathbf{P}(\bar{\boldsymbol{b}})|}\mathcal{A}(z, \bar{\boldsymbol{b}})$, then

$\mathfrak{A} \models \forall_z^{|\mathbf{P}(\bar{\boldsymbol{b}})|}[\Phi_\mathcal{A}(z, \bar{\boldsymbol{b}}) = \Lambda]$, then $\mathfrak{M} \models \forall_u^{|\mathbf{P}(\bar{\boldsymbol{b}})|}[\Phi_\mathcal{A}(u, \bar{\boldsymbol{b}}) = \Lambda]$, then $\mathfrak{M} \models \forall_z^{|\mathbf{P}(\bar{\boldsymbol{b}})|}\mathcal{A}(z, \bar{\boldsymbol{b}}) = \Lambda$.

For a formula that has two or more restricted quantifiers $\forall$, the proof is similar.

End of the proof of the theorem .

The main idea in the proof of this theorem is the application of the "**Use Principle**" and the assumption that polynomial properties are preserved for models of the theory **Th** when they are extended to models of the same theory.

Theorem 6.7. There exists an interpretation of the $\mathbb{A}$ functor **U** such that $\mathcal{NP}(\mathbb{A}) \neq \mathbf{co} - \mathcal{NP}(\mathbb{A})$, then the theory $\mathbf{Th}(\mathbb{A})$ fails the third item of Theorem 6.4.

Proof. Consider a formula of the form $\exists_y^{|x|}[|x| = |y| \& \mathbf{U}(y) = \Lambda]$. For this formula, one can construct an n - alphabetical interpretation of the $\mathbb{A}$ functor **U** such that, for $n \geq 2$, for any $\forall$ bounded formula $\mathcal{A}(x, \bar{z})$ signatures **PPr** is true $\mathbf{WordM}_\mathbb{A} \not\models \exists \bar{z} \forall x[\exists_y^{|x|}[|x| = |y| \& \mathbf{U}(y) = \Lambda] \equiv \mathcal{A}(x, \bar{z})]$.

The construction of the set $\mathbb{A}$ can be found in [12, p. 437].

Note. For the calculus $\mathbf{CalcEq_U}$, it is very easy to construct the set $\mathbb{A}$.

Corollary. $\mathcal{NP} \neq \mathbf{co} - \mathcal{NP}$.

Proof. Let's use Theorems 6.2 - 6.7.

P.S. I have proof of the following, not a very simple statement: $(\mathcal{NP} \bigcap \mathbf{co} - \mathcal{NP}) \neq \mathcal{P}$.

References

1. Мальцев А. И. Алгоритмы и рекурсивные функции. М., 1986. с. 368.

2. F. W. v. Henke, K. Indermark, G. Rose, K. Weihrauch. On Primitive Recursive Wordfunctions. Computing, vol 15, 1975, p. 217-234.

3. Шанин Н.А. Вступительная статья. О рекурсивном математическом анализе и исчислении арифметических равенств Р.Л. Гудстейна - В кн.: Р.Л. Гудстейн. Рекурсивный математический анализ. Издательство М."Наука 1970. с. 7-75.

4. Р.Л. Гудстейн. Рекурсивный математический анализ. М., 1970. с. 436.

5. H. B. Curry. A formalization of recursive arithmetic. Amer. J. Math. v.63 p. 263-282, 1941.

6. Cobham A. The intrinsic computational difficulty of functions. Proc. of the 1964 International Congress for Logic, Methology, and the Philosophy of Sciens, North Holand Publishing Co., Amsterdam, p. 24-30.

7. L, J. Stockmeyer. The polinomial-time hierarchy. Theoretical Computer Science vol 3 1977, p.1-22.

8. Ершов Ю.Л. Палютин Е.А. Математическая логика. М., ФИЗМАТЛИТ, 2011.с. 356.

9. C. C. CHANG, H.J. KEISLER. MODEL THEORY. STUDIES IN BOOK AND THE FOUNDATIONS OF MATHEMATICS. V. 73, 1973.

10. J. BARWISE. HANDBOOK OF MATHEMATICAL LOGIC. NORTH-HOLLAND PUBLISHING COMPANY AMSTERDAM NEW YORK OXFORD, 1977.

11. Book R.V., Long T.J., Selman A.L. Quantitative relativization of complexity classes. SIAM J. Comput. vol 13 No 3 August 1984, p. 461-487.

12. Baker T, Gill J. Solovay R. Relativization of the $\mathscr{P} =? \mathscr{NP}$ Question. SIAM J. Comput. vol 4 December 1971, p. 431-442.

Application

Let us construct a $k \geq 3$ - place functor of the form $[J\mathbf{Concat}\mathbf{I}_1^k[J\mathbf{Concat}\mathbf{I}_2^k \dots [J\mathbf{Concat}\mathbf{I}_{k-1}^k\mathbf{I}_k^k]\dots]$. For this functor in the calculus **CalcEq** we derive the equality

$[J\mathbf{Concat}\mathbf{I}_1^k[J\mathbf{Concat}\mathbf{I}_2^k \dots [J\mathbf{Concat}\mathbf{I}_{k-1}^k\mathbf{I}_k^k]\dots](x_1 \dots x_k) = \mathbf{Concat}(x_1, \mathbf{Concat}(x_2, \dots \mathbf{Concat}(x_{k-1}, x_k)\dots))$.

Let $\mathbf{Concat}^k \rightleftharpoons [J\mathbf{Concat}\mathbf{I}_1^k[J\mathbf{Concat}\mathbf{I}_2^k \dots [J\mathbf{Concat}\mathbf{I}_{k-1}^k\mathbf{I}_k^k]\dots]$, at $k \geq 3$, then $\vdash \mathbf{Concat}^k(x_1, \dots x_k) = \mathbf{Concat}(x_1, \mathbf{Concat}(x_2, \dots \mathbf{Concat}(x_{k-1}, x_k)\dots))$. When $k = 2$, we get $\mathbf{Concat}^2 \rightleftharpoons \mathbf{Concat}$, $\vdash \mathbf{Concat}^2(x_1, x_2) = \mathbf{Concat}(x_1, x_2)$ at $k = 1$ $\mathbf{Concat}^1 \rightleftharpoons \mathbf{I}_1^1$, $\vdash \mathbf{Concat}^1(x) = x$.

We have $([J\mathbf{Concat}\mathbf{I}_{k-1}^k\mathbf{I}_k^k])^* = [J(\mathbf{Concat})^*\mathbf{I}_1^{k+1}(\mathbf{I}_{k-1}^k)^*(\mathbf{I}_k^k)^*] = [J[J\mathbf{Concat}\mathbf{I}_2^3\mathbf{I}_3^3]\mathbf{I}_1^{k+1}(\mathbf{I}_{k-1}^k)^*(\mathbf{I}_k^k)^*] = [J\mathbf{Concat}(\mathbf{I}_{k-1}^k)^*(\mathbf{I}_k^k)^*] = [J\mathbf{Concat}\mathbf{I}_k^{k+1}\mathbf{I}_{k+1}^{k+1}]$, then

$(\mathbf{Concat}^k)^* = ([J\mathbf{Concat}\mathbf{I}_1^k[J\mathbf{Concat}\mathbf{I}_2^k \dots [J\mathbf{Concat}\mathbf{I}_{k-1}^k\mathbf{I}_k^k]\dots]^* =$

$[J\mathbf{Concat}\mathbf{I}_2^{k+1}[J\mathbf{Concat}\mathbf{I}_3^{k+1} \dots [J\mathbf{Concat}\mathbf{I}_k^{k+1}\mathbf{I}_{k+1}^{k+1}]\dots]$, then

$\vdash (\mathbf{Concat}^k)^*(x_1, x_2, \dots, x_{k+1}) = \mathbf{Concat}^k(x_2, \dots, x_{k+1})$

We have $(\Theta_{[J\Phi\Psi_1,\dots,\Psi_k]})^* = (J[\mathbf{Concat}^{k+1}[J\Theta_\Phi\Psi_1, \dots \Psi_k]\Theta_{\Psi_1}, \dots, \Theta_{\Psi_k}])^*$, then

$(J[\mathbf{Concat}^{k+1}[J\Theta_\Phi\Psi_1, \dots \Psi_k]\Theta_{\Psi_1}, \dots, \Theta_{\Psi_k}])^* = [J(\mathbf{Concat}^{k+1})^*\mathbf{I}_1^{n+1}([J\Theta_\Phi\Psi_1, \dots \Psi_k])^*(\Theta_{\Psi_1})^*, \dots, (\Theta_{\Psi_k})^*]$.

Next $([J\Theta_\Phi\Psi_1, \dots \Psi_k])^* = [J(\Theta_\Phi)^*\mathbf{I}_1^{n+1}(\Psi_1)^*, \dots (\Psi_k)^*]$, then

$\vdash ([J\Theta_\Phi\Psi_1, \dots \Psi_k])^*(x_1, x_2, \dots, x_{n+1}) = [J(\Theta_\Phi)^*\mathbf{I}_1^{n+1}(\Psi_1)^*, \dots (\Psi_k)^*](x_1, x_2, \dots, x_{n+1})$, then

$\vdash [J(\Theta_\Phi)^*\mathbf{I}_1^{n+1}(\Psi_1)^*, \dots (\Psi_k)^*](x_1, x_2, \dots, x_{n+1}) = (\Theta_\Phi)^*(x_1, (\Psi_1)^*(x_1, x_2, \dots, x_n), \dots (\Psi_k)^*(x_1, x_2, \dots, x_n))$,

then $\vdash ([J\Theta_\Phi\Psi_1, \dots \Psi_k])^*(x_1, x_2, \dots, x_{n+1}) = (\Theta_\Phi)^*(x_1, (\Psi_1)^*(x_1, x_2, \dots, x_n), \dots (\Psi_k)^*(x_1, x_2, \dots, x_n))$, then

$\vdash (\Theta_{[J\Phi\Psi_1,\dots,\Psi_k]})^*(x_1, x_2, \dots, x_{n+1}) = (\mathbf{Concat}^{k+1})^*(x_1, ([J\Theta_\Phi\Psi_1, \dots \Psi_k])^*(x_1, x_2, \dots, x_{n+1})$

$, (\Theta_{\Psi_1})^*(x_1, x_2, \dots, x_{n+1}), (\Theta_{\Psi_k})^*(x_1, x_2, \dots, x_{n+1}))$, then

$\vdash (\Theta_{[J\Phi\Psi_1,\dots,\Psi_k]})^*(x_1, x_2, \dots, x_{n+1}) = \mathbf{Concat}^{k+1}(([J\Theta_\Phi\Psi_1, \dots \Psi_k])^*(x_1, x_2, \dots, x_{n+1})$

$, (\Theta_{\Psi_1})^*(x_1, x_2, \dots, x_{n+1}), (\Theta_{\Psi_k})^*(x_1, x_2, \dots, x_{n+1}))$, then

$\vdash (\Theta_{[J\Phi\Psi_1,\dots,\Psi_k]})^*(x_1, x_2, \dots, x_{n+1}) = \mathbf{Concat}^{k+1}((\Theta_\Phi)^*(x_1, (\Psi_1)^*(x_1, x_2, \dots, x_n), \dots (\Psi_k)^*(x_1, x_2, \dots, x_n))$,

$(\Theta_{\Psi_1})^*(x_1, x_2, \dots, x_{n+1}), (\Theta_{\Psi_k})^*(x_1, x_2, \dots, x_{n+1}))$, then

$\vdash (\Theta_{[J\Phi\Psi_1,\dots,\Psi_k]})^*(x_1, x_2, \dots, x_{n+1}) = \mathbf{Concat}((\Theta_\Phi)^*(x_1, (\Psi_1)^*(x_1, x_2, \dots, x_{n+1}), \dots (\Psi_k)^*(x_1, x_2, \dots, x_{n+1}))$,

$\mathbf{Concat}((\Theta_{\Psi_1})^*(x_1, x_2, \dots, x_{n+1}), \dots, \mathbf{Concat}((\Theta_{\Psi_{k-1}})^*(x_1, x_2, \dots, x_{n+1}), (\Theta_{\Psi_k})^*(x_1, x_2, \dots, x_{n+1})), \dots,)$.

Given Φ - $n \geq 1$ - place functor, $\Psi_1, \dots, \Psi_k$ - $(n+2)$ place functors. Let's compose a functor $[R\Phi\Psi_1, \dots, \Psi_k]$ - $(n+1)$ - place. From this functor we construct a functor $\Theta_{[R\Phi\Psi_1,\dots,\Psi_k]}$

Let $\overline{x} \leftrightharpoons x_1, \dots, x_n$, $\lambda \leftrightharpoons [J[R\Phi\Psi_1, \dots, \Psi_k]\mathbf{I}_1^{n+2}, \dots, \mathbf{I}_{n+1}^{n+2}]$.

We have $\vdash \lambda(\overline{x}, z, u) = [J[R\Phi\Psi_1, \dots, \Psi_k]\mathbf{I}_1^{n+2}, \dots, \mathbf{I}_{n+1}^{n+2}](\overline{x}, z, u) = [R\Phi\Psi_1, \dots, \Psi_k](\overline{x}, z)$

Let $\tilde{\Psi}_i \leftrightharpoons [J\mathbf{Concat}[J\Theta_{\Psi_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda]\mathbf{I}_{n+2}^{n+2}]$, $\tilde{\Psi}_i$ - $(n+2)$ - place functor.

We have: $\vdash \tilde{\Psi}_i(\overline{x},z,u) \leftrightharpoons [J\mathbf{Concat}[J\Theta_{\Psi_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda]\mathbf{I}_{n+2}^{n+2}](\overline{x},z,u) =$
$\mathbf{Concat}([J\Theta_{\Psi_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda](\overline{x},z,u),\mathbf{I}_{n+2}^{n+2}(\overline{x},z,u)) = \mathbf{Concat}(\Theta_{\Psi_i}(\overline{x},z,\lambda(\overline{x},z,u)),u) =$
$\mathbf{Concat}(\Theta_{\Psi_i}(\overline{x},z,[R\Phi\Psi_1,\ldots,\Psi_k](\overline{x},z)),u)$.

So, $\vdash \tilde{\Psi}_i(\overline{x},z,u) = \mathbf{Concat}(\Theta_{\Psi_i}(\overline{x},z,[R\Phi\Psi_1,\ldots,\Psi_k](\overline{x},z)),u)$

Let $\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]} \leftrightharpoons [R\Theta_\Phi\tilde{\Psi}_1,\ldots,\tilde{\Psi}_k]$.

Defining equalities:

$\vdash \Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{x},\Lambda) = [R\Theta_\Phi\tilde{\Psi}_1,\ldots,\tilde{\Psi}_k](\overline{x},\Lambda) = \Theta_\Phi(\overline{x})$

$\vdash \Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{x},\mathbf{S}_k(z)) = [R\Theta_\Phi\tilde{\Psi}_1,\ldots,\tilde{\Psi}_k](\overline{x},\mathbf{S}_i(z)) = \tilde{\Psi}_i(\overline{x},z,[R\Theta_\Phi\tilde{\Psi}_1,\ldots,\tilde{\Psi}_k](\overline{x},z)) =$
$\tilde{\Psi}_i(\overline{x},z,\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{x},z)) = \mathbf{Concat}(\Theta_{\Psi_i}(\overline{x},z,[R\Phi\Psi_1,\ldots,\Psi_k](\overline{x},z)),\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{x},z))$.

So, we have the following defining equalities for the functor $\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}$:

$\vdash \Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{x},\Lambda) = \Theta_\Phi(\overline{x})$,

$\vdash \Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{x},\mathbf{S}_i(y)) = \mathbf{Concat}(\Theta_{\Psi_i}(\overline{x},z,[R\Phi\Psi_1,\ldots,\Psi_k](\overline{x},z)),\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{x},z))$, where $i \le k$,

$\vdash \Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{x},\mathbf{S}_i(y)) = \Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{x},y)$, where $i > k$.

Next $(\lambda)^* \leftrightharpoons ([J[R\Phi\Psi_1,\ldots,\Psi_k]\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}])^*$.

$([J[R\Phi\Psi_1,\ldots,\Psi_k]\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}])^* = [J([R\Phi\Psi_1,\ldots,\Psi_k])^*\mathbf{I}_1^{n+3}(\mathbf{I}_1^{n+2})^*,\ldots,(\mathbf{I}_{n+1}^{n+2})^*]$

We have: $\vdash (\lambda)^*(\mathbf{y},\overline{x},z,u) = [J([R\Phi\Psi_1,\ldots,\Psi_k])^*\mathbf{I}_1^{n+3}(\mathbf{I}_1^{n+2})^*,\ldots,(\mathbf{I}_{n+1}^{n+2})^*](\mathbf{y},\overline{x},z,u) =$
$([R\Phi\Psi_1,\ldots,\Psi_k])^*(\mathbf{I}_1^{n+3}(\mathbf{y},\overline{x},z,u),(\mathbf{I}_1^{n+2})^*(\mathbf{y},\overline{x},z,u),\ldots,(\mathbf{I}_{n+1}^{n+2})^*(\mathbf{y},\overline{x},z,u)) = ([R\Phi\Psi_1,\ldots,\Psi_k])^*(\mathbf{y},\overline{x},z)$

So, $\vdash (\lambda)^*(\mathbf{y},\overline{x},z,u) = ([R\Phi\Psi_1,\ldots,\Psi_k])^*(\mathbf{y},\overline{x},z)$.

$(\tilde{\Psi}_i)^* \leftrightharpoons ([J\mathbf{Concat}[J\Theta_{\Psi_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda]\mathbf{I}_{n+2}^{n+2}])^* = [J(\mathbf{Concat})^*\mathbf{I}_1^{n+3}([J\Theta_{\Psi_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda])^*(\mathbf{I}_{n+2}^{n+2})^*] =$
$[J[J\mathbf{Concat}\mathbf{I}_2^3\mathbf{I}_3^3]\mathbf{I}_1^{n+3}([J\Theta_{\Psi_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda])^*(\mathbf{I}_{n+2}^{n+2})^*]$.

$([J\Theta_{\Psi_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda])^* = [J(\Theta_{\Psi_i})^*\mathbf{I}_1^{n+3}(\mathbf{I}_1^{n+2})^*,\ldots,(\mathbf{I}_{n+1}^{n+2})^*(\lambda)^*]$,

$\vdash ([J\Theta_{\Psi_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda])^*(\mathbf{y},\overline{x},z,u) = [J(\Theta_{\Psi_i})^*\mathbf{I}_1^{n+3}(\mathbf{I}_1^{n+2})^*,\ldots,(\mathbf{I}_{n+1}^{n+2})^*(\lambda)^*](\mathbf{y},\overline{x},z,u) =$
$(\Theta_{\Psi_i})^*(\mathbf{I}_1^{n+3}(\mathbf{y},\overline{x},z,u),(\mathbf{I}_1^{n+2})^*(\mathbf{y},\overline{x},z,u),\ldots,(\mathbf{I}_{n+1}^{n+2})^*(\mathbf{y},\overline{x},z,u),(\lambda)^*(\mathbf{y},\overline{x},z,u)) =$
$(\Theta_{\Psi_i})^*(\mathbf{y},\overline{x},z,(\lambda)^*(\mathbf{y},\overline{x},z,u)) = (\Theta_{\Psi_i})^*(\mathbf{y},\overline{x},z,([R\Phi\Psi_1,\ldots,\Psi_k])^*(\mathbf{y},\overline{x},z))$.

So, $\vdash ([J\Theta_{\Psi_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda])^*(\mathbf{y},\overline{x},z,u) = (\Theta_{\Psi_i})^*(\mathbf{y},\overline{x},z,([R\Phi\Psi_1,\ldots,\Psi_k])^*(\mathbf{y},\overline{x},z))$.

$\vdash (\tilde{\Psi}_i)^*(\mathbf{y},\overline{x},z,u) = [J(\mathbf{Concat})^*\mathbf{I}_1^{n+3}([J\Theta_{\Psi_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda])^*(\mathbf{I}_{n+2}^{n+2}])^*](\mathbf{y},\overline{x},z,u) =$

$[J[J\mathbf{Concat}\mathbf{I}_2^3\mathbf{I}_3^3]\mathbf{I}_1^{n+3}([J\Theta_{\Psi_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda])^*(\mathbf{I}_{n+2}^{n+2})^*](\mathbf{y},\overline{x},z,u) =$

$[J\mathbf{Concat}\mathbf{I}_2^3\mathbf{I}_3^3](\mathbf{I}_1^{n+3}(\mathbf{y},\overline{x},z,u),([J\Theta_{\Psi_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda])^*(\mathbf{y},\overline{x},z,u),(\mathbf{I}_{n+2}^{n+2})^*(\mathbf{y},\overline{x},z,u)) =$
$\mathbf{Concat}(([J\Theta_{\Psi_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda])^*(\mathbf{y},\overline{x},z,u),(\mathbf{I}_{n+2}^{n+2})^*(\mathbf{y},\overline{x},z,u)) = \mathbf{Concat}((\Theta_{\Psi_i})^*(\mathbf{y},\overline{x},z,([R\Phi\Psi_1,\ldots,\Psi_k])^*(\mathbf{y},\overline{x},z)),u)$.

So, $\vdash (\tilde{\Psi}_i)^*(\mathbf{y},\overline{x},z,u) = \mathbf{Concat}((\Theta_{\Psi_i})^*(\mathbf{y},\overline{x},z,([R\Phi\Psi_1,\ldots,\Psi_k])^*(\mathbf{y},\overline{x},z)),u)$.

$(\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]})^* \leftrightharpoons ([R\Theta_\Phi\tilde{\Psi}_1,\ldots,\tilde{\Psi}_k])^* = [R(\Theta_\Phi)^*(\tilde{\Psi}_1)^*,\ldots,(\tilde{\Psi}_k)^*]$

$\vdash (\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]})^*(\mathbf{y},\overline{x},\Lambda) = ([R\Theta_\Phi\tilde{\Psi}_1,\ldots,\tilde{\Psi}_k])^*(\mathbf{y},\overline{x},\Lambda) = [R(\Theta_\Phi)^*(\tilde{\Psi}_1)^*,\ldots,(\tilde{\Psi}_k)^*](\mathbf{y},\overline{x},\Lambda) = (\Theta_\Phi)^*(\mathbf{y},\overline{x})$

$\vdash (\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]})^*(\mathbf{y},\overline{x},\mathbf{S}_i(z)) = ([R\Theta_\Phi\tilde{\Psi}_1,\ldots,\tilde{\Psi}_k])^*(\mathbf{y},\overline{x},\mathbf{S}_i(z)) = [R(\Theta_\Phi)^*(\tilde{\Psi}_1)^*,\ldots,(\tilde{\Psi}_k)^*](\mathbf{y},\overline{x},\mathbf{S}_i(z)) =$

$(\tilde{\Psi}_i)^*(\mathbf{y},\overline{x},z,[R(\Theta_\Phi)^*(\tilde{\Psi}_1)^*,\ldots,(\tilde{\Psi}_k)^*](\mathbf{y},\overline{x},z)) = (\tilde{\Psi}_i)^*(\mathbf{y},\overline{x},z,([R\Theta_\Phi\tilde{\Psi}_1,\ldots,\tilde{\Psi}_k)])^*(\mathbf{y},\overline{x},z)) =$

$\mathbf{Concat}((\Theta_{\Psi_i})^*(\mathbf{y},\overline{x},z,([R\Phi\Psi_1,\ldots,\Psi_k])^*(\mathbf{y},\overline{x},z)),([R\Theta_\Phi\tilde{\Psi}_1,\ldots,\tilde{\Psi}_k)])^*(\mathbf{y},\overline{x},z)) =$

$\mathbf{Concat}((\Theta_{\Psi_i})^*(\mathbf{y},\overline{x},z,([R\Phi\Psi_1,\ldots,\Psi_k])^*(\mathbf{y},\overline{x},z)),(\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]})^*(\mathbf{y},\overline{x},z)).$

So, $\vdash (\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]})^*(\mathbf{y},\overline{x},\mathbf{S}_i(z)) = \mathbf{Concat}((\Theta_{\Psi_i})^*(\mathbf{y},\overline{x},z,([R\Phi\Psi_1,\ldots,\Psi_k])^*(\mathbf{y},\overline{x},z)),(\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]})^*(\mathbf{y},\overline{x},z)).$

Thus we get:

$\vdash (\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]})^*(\mathbf{y},\overline{x},\Lambda) = (\Theta_\Phi)^*(\mathbf{y},\overline{x}),$

$\vdash (\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]})^*(\mathbf{y},\overline{x},\mathbf{S}_i(z)) = \mathbf{Concat}((\Theta_{\Psi_i})^*(\mathbf{y},\overline{x},z,([R\Phi\Psi_1,\ldots,\Psi_k])^*(\mathbf{y},\overline{x},z)),(\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]})^*(\mathbf{y},\overline{x},z)),$

at $i \leq k$.

$\vdash (\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]})^*(\mathbf{y},\overline{x},\mathbf{S}_i(z)) = (\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]})^*(\mathbf{y},\overline{x},z)$, at $i > k$.

Let $\Psi_1,\ldots\Psi_k$ - 2- place functor, α - some $p-$ some p is an alphabetic word. Let's compose a functor $[R\alpha\Psi_1,\ldots,\Psi_k]$ Let's compose a functor $\Theta_{[R\alpha\Psi_1,\ldots,\Psi_k]}$.

Let $\gamma \leftrightharpoons [J[R\alpha\Psi_1,\ldots,\Psi_k]\mathbf{I}_1^2]$.

We have $\vdash \gamma(x,z) = [J[R\alpha\Psi_1,\ldots,\Psi_k]\mathbf{I}_1^2](x,z) = [R\alpha\Psi_1,\ldots,\Psi_k](x)$.

Пусть $\tilde{\Psi}_i \leftrightharpoons [J\mathbf{Concat}[J\Theta_{\Psi_i}\mathbf{I}_1^2\gamma]\mathbf{I}_2^2]$.

We have: $\vdash \tilde{\Psi}_i(x,z) \leftrightharpoons [J\mathbf{Concat}[J\Theta_{\Psi_i}\mathbf{I}_1^2\gamma]\mathbf{I}_2^2](x,z) = \mathbf{Concat}([J\Theta_{\Psi_i}\mathbf{I}_1^2\gamma](x,z),\mathbf{I}_2^2(x,z)) =$

$\mathbf{Concat}(\Theta_{\Psi_i}(x,\gamma(x,z)),z) = \mathbf{Concat}(\Theta_{\Psi_i}(x,[R\alpha\Psi_1,\ldots,\Psi_k](x)),z).$

So, $\vdash \tilde{\Psi}_i(x,z) = \mathbf{Concat}(\Theta_{\Psi_i}(x,[R\alpha\Psi_1,\ldots,\Psi_k](x)),z).$

Let $\Theta_{[R\alpha\Psi_1,\ldots,\Psi_k]} \leftrightharpoons [R\Lambda\tilde{\Psi}_1,\ldots,\tilde{\Psi}_k]$, then

$\vdash \Theta_{[R\alpha\Psi_1,\ldots,\Psi_k]}(\Lambda) = [R\Lambda\tilde{\Psi}_1,\ldots,\tilde{\Psi}_k](\Lambda) = \Lambda,$

$\vdash \Theta_{[R\alpha\Psi_1,\ldots,\Psi_k]}(\mathbf{S}_k(x)) = [R\Lambda\tilde{\Psi}_1,\ldots,\tilde{\Psi}_k](\mathbf{S}_i(x)) = \tilde{\Psi}_i(x,[R\Lambda\tilde{\Psi}_1,\ldots,\tilde{\Psi}_k](x)) = \tilde{\Psi}_i(x,\Theta_{[R\alpha\Psi_1,\ldots,\Psi_k]}(x)) =$

$\mathbf{Concat}(\Theta_{\Psi_i}(x,[R\alpha\Psi_1,\ldots,\Psi_k](x)),\Theta_{[R\alpha\Psi_1,\ldots,\Psi_k]}(x)).$

So, we have the following defining equalities for the functor $\Theta_{[R\alpha\Psi_1,\ldots,\Psi_k]}$:

$\vdash \Theta_{[R\alpha\Psi_1,\ldots,\Psi_k]}(\Lambda) = \Lambda$

$\vdash \Theta_{[R\alpha\Psi_1,\ldots,\Psi_k]}(\mathbf{S}_i(x)) = \mathbf{Concat}(\Theta_{\Psi_i}(x,[R\alpha\Psi_1,\ldots,\Psi_k](x)),\Theta_{[R\alpha\Psi_1,\ldots,\Psi_k]}(x))$, where $i \leq k$.

$\vdash \Theta_{[R\alpha\Psi_1,\ldots,\Psi_k]}(\mathbf{S}_i(x)) = \Theta_{[R\alpha\Psi_1,\ldots,\Psi_k]}(x)$, where $i > k$.

Let us write out the defining relations for the functor $(\Theta_{[R\alpha\Psi_1,\ldots,\Psi_k]})^* \leftrightharpoons ([R\Lambda\tilde{\Psi}_1,\ldots,\tilde{\Psi}_k])^*$:

$([R\Lambda\tilde{\Psi}_1,\ldots,\tilde{\Psi}_k])^* = ([R\mathbf{Const}_\Lambda^1(\tilde{\Psi}_1)^*,\ldots,(\tilde{\Psi}_k)^*],$

$\vdash ([R\Lambda\tilde{\Psi}_1,\ldots,\tilde{\Psi}_k])^*(x,y) = [R\mathbf{Const}_\Lambda^1(\tilde{\Psi}_1)^*,\ldots,(\tilde{\Psi}_k)^*](x,y),$

$\vdash R\mathbf{Const}^1_\Lambda(\tilde{\Psi}_1)^*, \ldots, (\tilde{\Psi}_k)^*](x, \Lambda) = \mathbf{Const}^1_\Lambda(x) = \Lambda$

$\vdash [R\mathbf{Const}^1_\Lambda(\tilde{\Psi}_1)^*, \ldots, (\tilde{\Psi}_k)^*](x, \mathbf{S}_i(y)) = (\tilde{\Psi}_i)^*(x, y, [R\mathbf{Const}^1_\Lambda(\tilde{\Psi}_1)^*, \ldots, (\tilde{\Psi}_k)^*](x, y))$, then

$\vdash [R\mathbf{Const}^1_\Lambda(\tilde{\Psi}_1)^*, \ldots, (\tilde{\Psi}_k)^*](x, \mathbf{S}_i(y)) = (\tilde{\Psi}_i)^*(x, y, ([R\Lambda\tilde{\Psi}_1, \ldots, \tilde{\Psi}_k])^*(x, y))$,

$\vdash ([R\Lambda\tilde{\Psi}_1, \ldots, \tilde{\Psi}_k])^*(x, \mathbf{S}_i(y)) = (\tilde{\Psi}_i)^*(x, y, ([R\Lambda\tilde{\Psi}_1, \ldots, \tilde{\Psi}_k])^*(x, y))$,

$\vdash (\Theta_{[R\alpha\Psi_1,\ldots,\Psi_k]})^*(x, \mathbf{S}_i(y)) = (\tilde{\Psi}_i)^*(x, y, (\Theta_{[R\alpha\Psi_1,\ldots,\Psi_k]})^*(x, y))$,

Next $(\gamma)^* \leftrightharpoons ([J[R\alpha\Psi_1, \ldots, \Psi_k]\mathbf{I}^2_1])^*$,

$([J[R\alpha\Psi_1, \ldots, \Psi_k]\mathbf{I}^2_1])^* = [J([R\alpha\Psi_1, \ldots, \Psi_k])^*\mathbf{I}^3_1(\mathbf{I}^2_1)^*]$

$(\mathbf{I}^2_1)^* = [J\mathbf{I}^2_1\mathbf{I}^3_2\mathbf{I}^3_3]$

$\vdash (\mathbf{I}^2_1)^*(x, y, z) = [J\mathbf{I}^2_1\mathbf{I}^3_2\mathbf{I}^3_3](x, y, z) = \mathbf{I}^2_1(\mathbf{I}^3_2(x, y, z), \mathbf{I}^3_3(x, y, z)) = y$

$\vdash ([J[R\alpha\Psi_1, \ldots, \Psi_k]\mathbf{I}^2_1])^*(x, y, z) = [J([R\alpha\Psi_1, \ldots, \Psi_k])^*\mathbf{I}^3_1(\mathbf{I}^2_1)^*](x, y, z)$

$\vdash [J([R\alpha\Psi_1, \ldots, \Psi_k])^*\mathbf{I}^3_1(\mathbf{I}^2_1)^*](x, y, z) = ([R\alpha\Psi_1, \ldots, \Psi_k])^*(\mathbf{I}^3_1(x, y, z), (\mathbf{I}^2_1)^*(x, y, z))$

$\vdash ([R\alpha\Psi_1, \ldots, \Psi_k])^*(\mathbf{I}^3_1(x, y, z), (\mathbf{I}^2_1)^*(x, y, z)) = ([R\alpha\Psi_1, \ldots, \Psi_k])^*(x, y)$

$\vdash (\gamma)^*(x, y, z) = ([R\alpha\Psi_1, \ldots, \Psi_k])^*(x, y)$.

$(\tilde{\Psi}_i)^* \leftrightharpoons ([J\mathbf{Concat}[J\Theta_{\Psi_i}\mathbf{I}^2_1\gamma]\mathbf{I}^2_2])^*$,

$([J\mathbf{Concat}[J\Theta_{\Psi_i}\mathbf{I}^2_1\gamma]\mathbf{I}^2_2])^* = [J(\mathbf{Concat})^*\mathbf{I}^3_1([J\Theta_{\Psi_i}\mathbf{I}^2_1\gamma])^*(\mathbf{I}^2_2)^*]$,

$(\mathbf{Concat})^* = [J\mathbf{Concat}\mathbf{I}^3_2\mathbf{I}^3_2]$,

$([J\Theta_{\Psi_i}\mathbf{I}^2_1\gamma])^* = [J(\Theta_{\Psi_i})^*\mathbf{I}^3_1(\mathbf{I}^2_1)^*(\gamma)^*]$,

$(\mathbf{I}^2_1)^* = [J\mathbf{I}^2_1\mathbf{I}^3_2\mathbf{I}^3_3]$

$(\mathbf{I}^2_2)^* = [J\mathbf{I}^2_2\mathbf{I}^3_2\mathbf{I}^3_3]$

We have:

$\vdash (\mathbf{I}^2_1)^*(x, y, z) = [J\mathbf{I}^2_1\mathbf{I}^3_2\mathbf{I}^3_3](x, y, z) = y$,

$\vdash (\mathbf{I}^2_2)^*(x, y, z) = [J\mathbf{I}^2_2\mathbf{I}^3_2\mathbf{I}^3_3](x, y, z) = z$

$\vdash (\mathbf{Concat})^*(x, y, z) = [J\mathbf{Concat}\mathbf{I}^3_2\mathbf{I}^3_2](x, y, z) = \mathbf{Concat}(y, z)$,

$\vdash (\tilde{\Psi}_i)^*(x, y, z) = ([J\mathbf{Concat}[J\Theta_{\Psi_i}\mathbf{I}^2_1\gamma]\mathbf{I}^2_2])^*(x, y, z) = [J(\mathbf{Concat})^*\mathbf{I}^3_1([J\Theta_{\Psi_i}\mathbf{I}^2_1\gamma])^*(\mathbf{I}^2_2)^*](x, y, z) =$,

$\vdash ([J\Theta_{\Psi_i}\mathbf{I}^2_1\gamma])^*(x, y, z) = [J(\Theta_{\Psi_i})^*\mathbf{I}^3_1(\mathbf{I}^2_1)^*(\gamma)^*](x, y, z) = (\Theta_{\Psi_i})^*(\mathbf{I}^3_1(x, y, z), (\mathbf{I}^2_1)^*(x, y, z), (\gamma)^*(x, y, z))$,

$\vdash (\Theta_{\Psi_i})^*(\mathbf{I}^3_1(x, y, z), (\mathbf{I}^2_1)^*(x, y, z), (\gamma)^*(x, y, z)) = (\Theta_{\Psi_i})^*(x, y, (\gamma)^*(x, y, z))$

$\vdash [J(\mathbf{Concat})^*\mathbf{I}^3_1([J\Theta_{\Psi_i}\mathbf{I}^2_1\gamma])^*(\mathbf{I}^2_2)^*](x, y, z) = (\mathbf{Concat})^*(\mathbf{I}^3_1(x, y, z), ([J\Theta_{\Psi_i}\mathbf{I}^2_1\gamma])^*(x, y, z), (\mathbf{I}^2_2)^*(x, y, z))$

$\vdash (\mathbf{Concat})^*(\mathbf{I}^3_1(x, y, z), ([J\Theta_{\Psi_i}\mathbf{I}^2_1\gamma])^*(x, y, z), (\mathbf{I}^2_2)^*(x, y, z)) = \mathbf{Concat}(([J\Theta_{\Psi_i}\mathbf{I}^2_1\gamma])^*(x, y, z), z)$

$\vdash \mathbf{Concat}(([J\Theta_{\Psi_i}\mathbf{I}^2_1\gamma])^*(x, y, z), z) = \mathbf{Concat}((\Theta_{\Psi_i})^*(x, y, (\gamma)^*(x, y, z)), z)$

$\vdash \mathbf{Concat}((\Theta_{\Psi_i})^*(x, y, (\gamma)^*(x, y, z)), z) = \mathbf{Concat}((\Theta_{\Psi_i})^*(x, y, ([R\alpha\Psi_1, \ldots, \Psi_k])^*(x, y)), z)$.

Thus we get

$\vdash (\tilde{\Psi}_i)^*(x,y,z) = \mathbf{Concat}((\Theta_{\Psi_i})^*(x,y,([R\alpha\Psi_1,\ldots,\Psi_k])^*(x,y)),z)$, then

$\vdash (\tilde{\Psi}_i)^*(x,y,[R\mathbf{Const}^1_\Lambda(\tilde{\Psi}_1)^*,\ldots,(\tilde{\Psi}_k)^*](x,y)) =$

$\mathbf{Concat}((\Theta_{\Psi_i})^*(x,y,([R\alpha\Psi_1,\ldots,\Psi_k])^*(x,y)),[R\mathbf{Const}^1_\Lambda(\tilde{\Psi}_1)^*,\ldots,(\tilde{\Psi}_k)^*](x,y))$, then

$\vdash [R\mathbf{Const}^1_\Lambda(\tilde{\Psi}_1)^*,\ldots,(\tilde{\Psi}_k)^*](x,\mathbf{S}_i(y)) =$

$\mathbf{Concat}((\Theta_{\Psi_i})^*(x,y,([R\alpha\Psi_1,\ldots,\Psi_k])^*(x,y)),[R\mathbf{Const}^1_\Lambda(\tilde{\Psi}_1)^*,\ldots,(\tilde{\Psi}_k)^*](x,y))$.

Thus we get

$\vdash (\Theta_{[R\alpha\Psi_1,\ldots,\Psi_k]})^*(x,\Lambda) = \Lambda$

$\vdash (\Theta_{[R\alpha\Psi_1,\ldots,\Psi_k]})^*(x,\mathbf{S}_i(y)) = \mathbf{Concat}((\Theta_{\Psi_i})^*(x,y,([R\alpha\Psi_1,\ldots,\Psi_k])^*(x,y)),(\Theta_{[R\alpha\Psi_1,\ldots,\Psi_k]})^*(x,y))$, at $i \le k$.

$\vdash (\Theta_{[R\alpha\Psi_1,\ldots,\Psi_k]})^*(x,\mathbf{S}_i(y)) = (\Theta_{[R\alpha\Psi_1,\ldots,\Psi_k]})^*(x,y)$, at $i > k$.

For any functor Φ $\forall\mathbb{A}$ we prove $\mathbf{WordM}_\mathbb{A} \models \forall\overline{x}[\Theta_\Phi(\overline{x}) \approx \Theta_{\Theta_\Phi}(\overline{x})]$.

Let's write out the meaning of the operator Θ:

for the original functors:

$\mathbf{S}_k, \mathbf{Z}, \boldsymbol{\delta}, \mathbf{Length}, \dot{-}, \mathbf{Concat}, \mathbf{D}, \mathbf{I}^n_k, \mathbf{U}$:

$\Theta_{\mathbf{S}_k} = \mathbf{Z}$, $\Theta_\mathbf{Z} = \mathbf{Z}$, $\Theta_{\boldsymbol{\delta}} = \mathbf{Z}$, $\Theta_\mathbf{Length} = [J\mathbf{ZI}^2_2]$, $\Theta_{\dot{-}} = [J\mathbf{ZI}^2_2]$, $\Theta_\mathbf{Concat} = [J\mathbf{ZI}^2_2]$, $\Theta_\mathbf{D} = [J\mathbf{ZI}^2_2]$, $\Theta_{\mathbf{I}^n_k} = [J\mathbf{ZI}^n_k]$, $\Theta_\mathbf{U} = [J\mathbf{cI}^1_1\mathbf{U}]$.

for functor $[J\Phi\Psi_1,\ldots,\Psi_k]$

$\Theta_{[J\Phi\Psi_1,\ldots,\Psi_k]} \rightleftharpoons [J\mathbf{Concat}^{k+1}[J\Theta_\Phi\Psi_1\ldots\Psi_k],\Theta_{\Psi_1}\ldots\Theta_{\Psi_k}]$.

for functor $[R\alpha\Psi_1,\ldots,\Psi_k]$

$\Theta_{[R\alpha\Psi_1,\ldots,\Psi_k]} \leftrightharpoons [R\Lambda\tilde{\Psi}_1,\ldots,\tilde{\Psi}_k]$,

for functor $\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}$

$\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]} \leftrightharpoons [R\Theta_\Phi\tilde{\Psi}_1,\ldots,\tilde{\Psi}_k]$.

For any functor Ψ alphabet $\mathscr{L}$ true $\forall\overline{\alpha} \vdash \Theta_\Psi(\overline{\alpha}) = \Lambda$. When using Goodstein's rule, it is true $\vdash \Theta_\Psi(\overline{x}) = \Lambda$.

We will prove $\vdash \Theta_\mathbf{U}(x) = \Theta_{\Theta_\mathbf{U}}(x)$:

$\Theta_\mathbf{U} = [J\mathbf{cI}^1_1\mathbf{U}]$, $\Theta_{\Theta_\mathbf{U}} = \Theta_{[J\mathbf{cI}^1_1\mathbf{U}]} = [J\mathbf{Concat}^3[J\Theta_\mathbf{c}\mathbf{I}^1_1\mathbf{U}]\Theta_{\mathbf{I}^1_1}\Theta_\mathbf{U}]$, Considering $[J\Theta_\mathbf{c}\mathbf{I}^1_1\mathbf{U}] = [J\mathbf{ZI}^1_1]$, $\Theta_{\mathbf{I}^1_1} = [J\mathbf{ZI}^1_1]$, we have $\Theta_{\Theta_\mathbf{U}} = \Theta_\mathbf{U}$, then $\vdash \Theta_\mathbf{U}(x) = \Theta_{\Theta_\mathbf{U}}(x)$.

Induction hypothesis:

a. Let the following be true for the functor Φ $\mathbf{WordM}_\mathbb{A} \models \forall\overline{x}[\Theta_\Phi(\overline{x}) \approx \Theta_{\Theta_\Phi}(\overline{x})]$.

b. Let the following be true for the functors $\Psi_1,\ldots,\Psi_k$: $\mathbf{WordM}_\mathbb{A} \models \forall y_1,\ldots,\forall y_n[\Theta_{\Psi_i}(y_1,\ldots,y_n) \approx \Theta_{\Theta_{\Psi_i}}(y_1,\ldots,y_n)]$

Let's prove it $\mathbf{WordM}_{\mathbb{A}} \models \forall y_1, \ldots \forall y_n \Theta_{[J\Phi\Psi_1,\ldots,\Psi k]}(y_1, \ldots, y_n) \approx \Theta_{\Theta_{[J\Phi\Psi_1,\ldots,\Psi k]}}(y_1, \ldots, y_n)$.

We have $\Theta_{[J\Phi\Psi_1,\ldots,\Psi_k]} = [J\mathbf{Concat}^{k+1}[J\Theta_\Phi\Psi_1, \ldots, \Psi_k]\Theta_{\Psi_1}, \ldots, \Theta_{\Psi_k}]$, then

$\vdash \Theta_{[J\Phi\Psi_1,\ldots,\Psi_k]}(\overline{y}) = [J\mathbf{Concat}^{k+1}[J\Theta_\Phi\Psi_1, \ldots, \Psi_k]\Theta_{\Psi_1}, \ldots, \Theta_{\Psi_k}](\overline{y}) =$

$\mathbf{Concat}^{k+1}(\Theta_\Phi(\Psi_1(\overline{y}), \ldots, \Psi_k(\overline{y})), \Theta_{\Psi_1}(\overline{y}), \ldots \Theta_{\Psi_k}(\overline{y}))(\mathbf{A})$.

Let's calculate $\Theta_{[J\Theta_\Phi\Psi_1\ldots\Psi_k]}$:

$\Theta_{[J\Theta_\Phi\Psi_1\ldots\Psi_k]} = [J\mathbf{Concat}^{k+1}[J\Theta_{\Theta_\Phi}\Psi_1 \ldots \Psi_k]\Theta_{\Psi_1} \ldots \Theta_{\Psi_k}]$, then

$\vdash \Theta_{[J\Theta_\Phi\Psi_1\ldots\Psi_k]}(\overline{y}) = [J\mathbf{Concat}^{k+1}[J\Theta_{\Theta_\Phi}\Psi_1 \ldots \Psi_k]\Theta_{\Psi_1} \ldots \Theta_{\Psi_k}](\overline{y}) =$

$\mathbf{Concat}^{k+1}(\Theta_{\Theta_\Phi}(\Psi_1(\overline{y}), \ldots, \Psi_k(\overline{y})), \Theta_{\Psi_1}(\overline{y}), \ldots \Theta_{\Psi_k}(\overline{y}))$.

Taking into account the induction hypothesis $\mathbf{WordM}_{\mathbb{A}} \models \forall \overline{x}[\Theta_\Phi(\overline{x}) \approx \Theta_{\Theta_\Phi}(\overline{x})]$, we obtain

$\mathbf{WordM}_{\mathbb{A}} \models \forall \overline{y}[\Theta_{[J\Theta_\Phi\Psi_1\ldots\Psi_k]}(\overline{y}) \approx \mathbf{Concat}^{k+1}(\Theta_\Phi(\Psi_1(\overline{y}), \ldots, \Psi_k(\overline{y})), \Theta_{\Psi_1}(\overline{y}), \ldots \Theta_{\Psi_k}(\overline{y}))]$.

Taking into account the induction hypothesis $\mathbf{WordM}_{\mathbb{A}} \models \forall y_1, \ldots, \forall y_n[\Theta_{\Psi_i}(y_1, \ldots, y_n) \approx \Theta_{\Theta_{\Psi_i}}(y_1, \ldots, y_n)]$, we obtain $\mathbf{WordM}_{\mathbb{A}} \models \forall \overline{y}[\mathbf{Concat}(\Theta_{[J\Theta_\Phi\Psi_1\ldots\Psi_k]}(\overline{y}), \Theta_{\Theta_{\Psi_1}}(\overline{y})) \approx \mathbf{Concat}^{k+1}(\Theta_\Phi(\Psi_1(\overline{y}), \ldots, \Psi_k(\overline{y})), \Theta_{\Psi_1}(\overline{y}), \ldots \Theta_{\Psi_k}(\overline{y}))]$,

$\mathbf{WordM}_{\mathbb{A}} \models \forall \overline{y}[\mathbf{Concat}^3(\Theta_{[J\Theta_\Phi\Psi_1\ldots\Psi_k]}(\overline{y}), \Theta_{\Theta_{\Psi_1}}(\overline{y}), \Theta_{\Theta_{\Psi_2}}(\overline{y})) \approx \mathbf{Concat}^{k+1}(\Theta_\Phi(\Psi_1(\overline{y}), \ldots, \Psi_k(\overline{y})), \Theta_{\Psi_1}(\overline{y}), \ldots \Theta_{\Psi_k}(\overline{y}))]$,...,

$\mathbf{WordM}_{\mathbb{A}} \models \forall \overline{y}[\mathbf{Concat}^{k+1}(\Theta_{[J\Theta_\Phi\Psi_1\ldots\Psi_k]}(\overline{y}), \Theta_{\Theta_{\Psi_1}}(\overline{y}), \Theta_{\Theta_{\Psi_2}}(\overline{y}), \ldots, \Theta_{\Theta_{\Psi_k}}(\overline{y})) \approx$

$\mathbf{Concat}^{k+1}(\Theta_\Phi(\Psi_1(\overline{y}), \ldots, \Psi_k(\overline{y})), \Theta_{\Psi_1}(\overline{y}), \ldots \Theta_{\Psi_k}(\overline{y}))]$, taking into account $(\mathbf{A})$, we get

$\mathbf{WordM}_{\mathbb{A}} \models \forall \overline{y}[\mathbf{Concat}^{k+1}(\Theta_{[J\Theta_\Phi\Psi_1\ldots\Psi_k]}(\overline{y}), \Theta_{\Theta_{\Psi_1}}(\overline{y}), \Theta_{\Theta_{\Psi_2}}(\overline{y}), \ldots, \Theta_{\Theta_{\Psi_k}}(\overline{y})) \approx \Theta_{[J\Phi\Psi_1,\ldots,\Psi k]}(\overline{y})(\mathbf{B})$

Let's calculate $\Theta_{\Theta_{[J\Phi\Psi_1,\ldots,\Psi k]}}$:

$\Theta_{\Theta_{[J\Phi\Psi_1,\ldots,\Psi k]}} = \Theta_{[J\mathbf{Concat}^{k+1}[J\Theta_\Phi\Psi_1,\ldots,\Psi_k]\Theta_{\Psi_1},\ldots,\Theta_{\Psi_k}]}$

$\Theta_{[J\mathbf{Concat}^{k+1}[J\Theta_\Phi\Psi_1,\ldots,\Psi_k]\Theta_{\Psi_1},\ldots,\Theta_{\Psi_k}]} =$

$[J\mathbf{Concat}^{k+2}[J\Theta_{\mathbf{Concat^{k+1}}}[J\Theta_\Phi\Psi_1, \ldots, \Psi_k]\Theta_{\Psi_1}, \ldots, \Theta_{\Psi_k}]\Theta_{[J\Theta_\Phi\Psi_1,\ldots,\Psi_k]}\Theta_{\Theta_{\Psi_1}} \ldots, \Theta_{\Theta_{\Psi_k}}]$, taking into account $\Theta_{\mathbf{Concat^{k+1}}}[J\Theta_\Phi\Psi_1, \ldots, \Psi_k]\Theta_{\Psi_1}, \ldots, \Theta_{\Psi_k}] = [J\mathbf{ZI}_1^n]$, we get

$\Theta_{\Theta_{[J\Phi\Psi_1,\ldots,\Psi k]}} = [J\mathbf{Concat}^{k+1}\Theta_{[J\Theta_\Phi\Psi_1,\ldots,\Psi_k]}\Theta_{\Theta_{\Psi_1}} \ldots, \Theta_{\Theta_{\Psi_k}}]$, then

$\vdash \Theta_{\Theta_{[J\Phi\Psi_1,\ldots,\Psi k]}}(\overline{y}) = [J\mathbf{Concat}^{k+1}\Theta_{[J\Theta_\Phi\Psi_1,\ldots,\Psi_k]}\Theta_{\Theta_{\Psi_1}} \ldots, \Theta_{\Theta_{\Psi_k}}](\overline{y}) =$

$\mathbf{Concat}^{k+1}(\Theta_{[J\Theta_\Phi\Psi_1,\ldots,\Psi_k]}(\overline{y}), \Theta_{\Theta_{\Psi_1}}(\overline{y}), \ldots \Theta_{\Theta_{\Psi_k}}(\overline{y}))$,

taking into account $(\mathbf{B})$, we get $\mathbf{WordM}_{\mathbb{A}} \models \forall \overline{y}[\Theta_{\Theta_{[J\Phi\Psi_1,\ldots,\Psi k]}}(\overline{y}) \approx \Theta_{[J\Phi\Psi_1,\ldots,\Psi k]}(\overline{y})$.

Let's calculate $\Theta_{\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}} \leftrightharpoons \Theta_{[R\Theta_\Phi\tilde{\Psi}_1,\ldots,\tilde{\Psi}_k]}$.

We have $\tilde{\Psi}_i \leftrightharpoons [J\mathbf{Concat}[J\Theta_{\Psi_i}\mathbf{I}_1^{n+2}, \ldots, \mathbf{I}_{n+1}^{n+2}\lambda]\mathbf{I}_{n+2}^{n+2}]$.

Let's calculate $\Theta_{[J\Theta_{\Psi_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda]}$:

$\Theta_{[J\Theta_{\Psi_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda]} = [J\mathbf{Concat}^{n+3}[J\Theta_{\Theta_{\Psi_i}}\mathbf{I}_1^{n+2}, \ldots, \mathbf{I}_{n+1}^{n+2}\lambda]\Theta_{\mathbf{I}_1^{n+2}}, \ldots, \Theta_{\mathbf{I}_{n+1}^{n+2}}\Theta_\lambda]$, considering $\Theta_{\mathbf{I}_i^{n+2}} = [J\mathbf{ZI}_1^{n+1}]$ we have $\Theta_{[J\Theta_{\Psi_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda]} = [J\mathbf{Concat}[J\Theta_{\Theta_{\Psi_i}}\mathbf{I}_1^{n+2}, \ldots, \mathbf{I}_{n+1}^{n+2}\lambda]\Theta_\lambda]$.

Let's calculate $\Theta_{\tilde{\Psi}_i}$.

$\Theta_{\tilde{\Psi}_i} \leftrightharpoons \Theta_{[J\mathbf{Concat}[J\Theta_{\Psi_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda]\mathbf{I}_{n+2}^{n+2}]} =$

$[J\mathbf{Concat}^3[J\Theta_{\mathbf{Concat}}[J\Theta_{\Psi_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda]\mathbf{I}_{n+2}^{n+2}]\Theta_{[J\Theta_{\Psi_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda]}\Theta_{\mathbf{I}_{n+2}^{n+2}}]$,considering

$[J\Theta_{\mathbf{Concat}}[J\Theta_{\Psi_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda]\mathbf{I}_{n+2}^{n+2}] = [J\mathbf{Z}\mathbf{I}_1^{n+2}]$ и $\Theta_{\mathbf{I}_{\text{т}+2}^{n+2}} = [J\mathbf{Z}\mathbf{I}_1^{n+2}]$ we have

$[J\mathbf{Concat}^3[J\Theta_{\mathbf{Concat}}[J\Theta_{\Psi_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda]\mathbf{I}_{n+2}^{n+2}]\Theta_{[J\Theta_{\Psi_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda]}\Theta_{\mathbf{I}_{n+2}^{n+2}}] = \Theta_{[J\Theta_{\Psi_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda]}$, then

$\Theta_{\tilde{\Psi}_i} = [J\mathbf{Concat}[J\Theta_{\Theta_{\Psi_i}}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda]\Theta_\lambda]$.

We have:

$\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]} = [R\Theta_\Phi\tilde{\Psi}_1,\ldots,\Psi_k]$

$\Theta_{[R\Theta_\Phi\tilde{\Psi}_1,\ldots,\tilde{\Psi}_k]} = [R\Theta_{\Theta_\Phi}\tilde{\tilde{\Psi}}_1,\ldots,\tilde{\tilde{\Psi}}_k]$.

$\vdash \Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{x},\Lambda) = [R\Theta_\Phi\tilde{\Psi}_1,\ldots,\tilde{\Psi}_k](\overline{x},\Lambda) = \Theta_\Phi(\overline{x})$

$\vdash \Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{x},\mathbf{S}_i(z)) = \tilde{\Psi}_1(\overline{x},z,\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{x},z))$

$\vdash \Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{x},\mathbf{S}_i(z)) = [J\mathbf{Concat}[J\Theta_{\Psi_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda]\mathbf{I}_{n+2}^{n+2}](\overline{x},\mathbf{S}_i(z))$

$\vdash \Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{x},\mathbf{S}_i(z)) = \mathbf{Concat}(\Theta_{\Psi_i}(\overline{x},z,\lambda(\overline{x},z)),\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{x},z))$

$\vdash \Theta_{\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}}(\overline{x},\Lambda) = [R\Theta_{\Theta_\Phi}\tilde{\tilde{\Psi}}_1,\ldots,\tilde{\tilde{\Psi}}_k](\overline{x},\Lambda) = \Theta_{\Theta_\Phi}(\overline{x})$

$\vdash \Theta_{\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}}(\overline{x},\mathbf{S}_i(z)) = \tilde{\tilde{\Psi}}_i(\overline{x},z,\Theta_{\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}}(\overline{x},z))$

We have $\tilde{\tilde{\Psi}}_i \leftrightharpoons [J\mathbf{Concat}[J\Theta_{\tilde{\Psi}_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda]\mathbf{I}_{n+2}^{n+2}]$, then

$\vdash \Theta_{\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}}(\overline{x},\mathbf{S}_i(z)) = [J\mathbf{Concat}[J\Theta_{\tilde{\Psi}_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda]\mathbf{I}_{n+2}^{n+2}](\overline{x},z,\Theta_{\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}}(\overline{x},z))$, then

$\vdash \Theta_{\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}}(\overline{x},\mathbf{S}_i(z)) = \mathbf{Concat}([J\Theta_{\tilde{\Psi}_i}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda](\overline{x},z,\Theta_{\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}}(\overline{x},z)),\mathbf{I}_{n+2}^{n+2}(\overline{x},z,\Theta_{\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}}(\overline{x},z)))$

$\vdash \Theta_{\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}}(\overline{x},\mathbf{S}_i(z)) = \mathbf{Concat}(\Theta_{\tilde{\Psi}_i}(\overline{x},z,\lambda(\overline{x},z)),\Theta_{\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}}(\overline{x},z))$.

We have $\Theta_{\tilde{\Psi}_i} = [J\mathbf{Concat}[J\Theta_{\Theta_{\Psi_i}}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda]\Theta_\lambda]$, then

$\vdash \Theta_{\tilde{\Psi}_i}((\overline{x},z,u) = [J\mathbf{Concat}[J\Theta_{\Theta_{\Psi_i}}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda]\Theta_\lambda](\overline{x},z,u)$

$\vdash \Theta_{\tilde{\Psi}_i}(\overline{x},z,u) = \mathbf{Concat}([J\Theta_{\Theta_{\Psi_i}}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}\lambda](\overline{x},z,u),\Theta_\lambda(\overline{x},z,u))$

$\vdash \Theta_{\tilde{\Psi}_i}(\overline{x},z,u) = \mathbf{Concat}(J\Theta_{\Theta_{\Psi_i}}(\overline{x},z,\lambda(\overline{x},z)),\Theta_\lambda,(\overline{x},z,u))$

We have $\Theta_\lambda = \Theta_{[J[R\Phi\Psi i,\ldots,\Psi_k]]\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}]} = [J\mathbf{Concat}^{n+2}[J\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}]\Theta_{\mathbf{I}_1^{n+2}},\ldots,\Theta_{\mathbf{I}_{n+1}^{n+2}}]$, then $\Theta_\lambda = [J\Theta_{[R\Phi\Psi i,\ldots,\Psi_k]}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}]$, then

$\vdash \Theta_\lambda(\overline{x},z,u) = [J\Theta_{[R\Phi\Psi i,\ldots,\Psi_k]}\mathbf{I}_1^{n+2},\ldots,\mathbf{I}_{n+1}^{n+2}](\overline{x},z,u)$, then

$\vdash \Theta_\lambda(\overline{x},z,u) = \Theta_{[R\Phi\Psi i,\ldots,\Psi_k]}(\overline{x},z)$, then

$\vdash \Theta_{\tilde{\Psi}_i}(\overline{x},z,u) = \mathbf{Concat}(\Theta_{\Theta_{\Psi_i}}(\overline{x},z,\lambda(\overline{x},z)),\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]},(\overline{x},z))$, then

$\vdash \Theta_{\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}}(\overline{x},\mathbf{S}_i(z)) = \mathbf{Concat}(\mathbf{Concat}(\Theta_{\Theta_{\Psi_i}}(\overline{x},z,\lambda(\overline{x},z)),\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{x},z)),\Theta_{\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}}(\overline{x},z))$.

Let's sum it up:

$\vdash \Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{x},\Lambda) = \Theta_\Phi(\overline{x})$

$\vdash \Theta_{\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}}(\overline{x},\Lambda) = \Theta_{\Theta_\Phi}(\overline{x})$

$\vdash \Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{x}, \mathbf{S}_i(z)) = \mathbf{Concat}(\Theta_{\Psi_i}(\overline{x}, z, \lambda(\overline{x}, z)), \Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{x}, z))$

$\vdash \Theta_{\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}}(\overline{x}, \mathbf{S}_i(z)) = \mathbf{Concat}(\mathbf{Concat}(\Theta_{\Theta_{\Psi_i}}(\overline{x}, z, \lambda(\overline{x}, z)), \Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{x}, z)), \Theta_{\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}}(\overline{x}, z))$.

Let's assume that it is true:

$\forall\overline{\alpha}\mathbf{WordM}_{\mathbb{A}} \models \Theta_{\Phi}(\overline{\alpha}) \approx \Theta_{\Theta_\Phi}(\overline{\alpha})$

$\forall\overline{\alpha}\forall\beta\forall\gamma\mathbf{WordM}_{\mathbb{A}} \models \Theta_{\Psi_i}(\overline{\alpha}, \beta, \gamma) \approx \Theta_{\Theta_{\Psi_i}}(\overline{\alpha}, \beta, \gamma)$

$\forall\overline{\alpha}\forall\beta\mathbf{WordM}_{\mathbb{A}} \models \Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{\alpha}, \beta) \approx \Theta_{\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}}(\overline{\alpha}, \beta)$, then we get

$\mathbf{WordM}_{\mathbb{A}} \models \Theta_{\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}}(\overline{\alpha}, \mathbf{S}_i(\beta)) =$

$\mathbf{Concat}(\mathbf{Concat}(\Theta_{\Theta_{\Psi_i}}(\overline{\alpha}, \beta, \lambda(\overline{\alpha}, \beta)), \Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{\alpha}, \beta)), \Theta_{\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}}(\overline{\alpha}, \beta)) \approx$

$\mathbf{Concat}(\mathbf{Concat}(\Theta_{\Psi_i}(\overline{\alpha}, \beta, \lambda(\overline{\alpha}, \beta)), \Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{\alpha}, \beta)), \Theta_{\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}}(\overline{\alpha}, \beta)) \approx$

$\mathbf{Concat}(\Theta_{\Psi_i}(\overline{\alpha}, \beta, \lambda(\overline{\alpha}, \beta)), \Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{\alpha}, \beta)) = \Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{\alpha}, \mathbf{S}_i(\beta))$, then

$\mathbf{WordM}_{\mathbb{A}} \models \forall\overline{x}\forall z[\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}(\overline{x}, z) \approx \Theta_{\Theta_{[R\Phi\Psi_1,\ldots,\Psi_k]}}(\overline{x}, z)]$.

Using induction on the construction of functors and induction on the construction of the argument word, we obtain: for any functor Φ correctly $\mathbf{WordM}_{\mathbb{A}} \models \forall\overline{x}[\Theta_{\Phi}(\overline{x}) \approx \Theta_{\Theta_\Phi}(\overline{x})$.