Notes on Perfect Numbers

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Abstract

A set of relations between perfect numbers, then some properties of this relations and how they behave, next, a geometric interpretation, a function, the way this function works, an algorithm to find Perfect Numbers and finally the limits of two specific functions related to this algorithm.

Introduction.

A Perfect Number is an integer number such that its value is equal to the sum of its proper divisors[1]. The first seven Perfect Numbers are:
6, 28, 496, 8128, 33550336, 8589869056, 137438691328. In this paper we use the terms Perfect Number=Pf, Superperfect Number=Sp, Mersenne Prime=Mp, Mersenne Exponent=Me. The first Pf will be the 28 and we will call it Pf_1, 496 will be Pf_2, etc.

Relation between two consecutive Perfect Numbers.

Assuming that all Perfect Numbers have the form: $2n^4 - n^2 = Pf_k$
Then:
$[(2 \cdot (n_1)^2) - 1](n_1)^2 = Pf_1$.
$[(2 \cdot (n_2)^2) - 1](n_2)^2 = Pf_2$.

$[(2 \cdot (n_k)^2) - 1](n_k)^2 = Pf_k$.

Exists a relation (r) of the form:
$\frac{n_k}{n_{k-1}}$.
for every Pf_k and Pf_{k-1}.

This is:

$\sqrt{\frac{Sp_k}{Sp_{k-1}}}$

For example, the relation between 28 and 496 is equal to: $\sqrt{\frac{16}{4}} = 2$

Table 1: Relation between consecutive Perfect Numbers.

Pf 1	**Pf 2**	**Relation (r)**
28	496	2
496	8128	2
8128	33550336	8
33550336	8589869056	4
8589869056	137438/691328	2
137438/691328	2305843008139952128	64

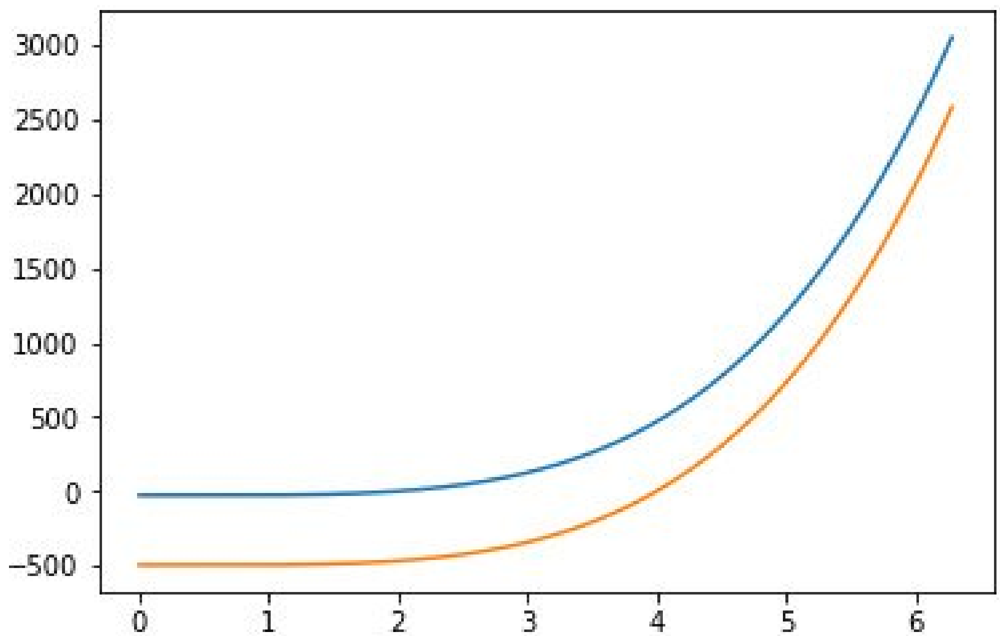


Figure 1: (r) of 28 and 496

Relation between two non-consecutive Perfect Numbers.

Given two Perfect Numbers Pf_1 and Pf_2 and their respective Mersenne Primes Mp_1 and Mp_2:

$$\begin{bmatrix} Pf_1 & Mp_1 \\ Pf_2 & Mp_2 \end{bmatrix}$$

Exists a relation (r) of the form:

$$r = \frac{\sqrt{Pf_1 * Mp_1 * Pf2 * Mp_2}}{Pf_1 * Mp_2}$$

This means that the relation (r) between two non-consecutive Perfect Numbers is equal to: $r_n * r_{n+1} * r_{n+2} * ...r_k$

For example, the relation between 8589869056 and 28 is equal to:
$r = \frac{\sqrt{28*7*8589869056*131071}}{28*131071}$=128=2*2*8*4

Table 2: Relation between non-consecutive Perfect Numbers.

Perfect Number	**28**	**496**	**8128**	**33550336**	**8589869056**
28	1	2	4	32	128
496	2	1	2	16	64
8128	4	2	1	8	32
33550336	32	16	8	1	4
8589869056	128	64	32	4	1

Relation between number 28 and upper Perfect Numbers.

Given two Perfect Numbers Pf_1 and Pf_2 and their respective Mersenne Primes Mp_1 and Mp_2.
Exists a relation (r) such that: r= $2^{\frac{Me_{Pf2}-3}{2}}$, where Pf_1=28 and Me_{Pf2} is the Mersenne Exponent of the Mersenne Prime of the upper Perfect Number Pf_2.

For example: $Pf_1 = 28$ and $Pf_2 = 2305843008139952128$.
$Mp_1 = 7$ and $Mp_2 = 2147483647$ and $Me_2 = 31$.

We have:
$r = \frac{\sqrt{28*7*2305843008139952128*2147483647}}{28*2147483647} = 16384$

This is equal to the product of the relations between this two Perfect Numbers, this is:
r=2*2*8*4*2*64=16384.

And this is:
r= $2^{\frac{Me_{Pf2}-3}{2}} = 2^{\frac{31-3}{2}} = 2^{14} = 16384$.

Table 3: Relation between 28 and upper Perfect Numbers.

Pf 1	**Pf 2**	**Relation (r)**	$2^{\frac{Me-3}{2}}$
28	496	2	2^1
496	8128	2	2^2
8128	33550336	8	2^5
33550336	8589869056	4	2^7
8589869056	137438/691328	2	2^8
137438691328	2305843008139952128	64	2^{14}

Geometrical interpretation of the relation between Perfect Numbers.

Assuming that all Perfect Numbers have the form $(2n^2 - 1)n^2$.
This is $2n^4 - n^2$.
When solving for $2n^4 - n^2 = Pf_n$.
We obtain four roots, two Complex of the form:
$i\sqrt{\frac{Mp_n}{2}}$ and $-i\sqrt{\frac{Mp_n}{2}}$
And two Real roots running on the x axis such that:
Given two Perfect Numbers Pf_1 and Pf_2 equaled to the polynomial $2n^4 - n^2$, the relation between their Real roots is equal to the relation (r).
$2n^4 - n^2 - Pf_1 = x_1$
$2n^4 - n^2 - Pf_2 = x_2$
Then:
$r = \frac{x_2}{x_1}$

***Also is possible to use the polynomial $2n^4 + 8n^3 + 11n^2 + 6n + 1$. The only difference is that all the Complex roots will have Real part (-1).

Example:
Given two Perfect Numbers $Pf_1 = 28$ and $Pf_2 = 496$.
Solving for:
$2n^4 - n^2 = 28$ and $2n^4 - n^2 = 496$.
We get their roots on the xy axis.

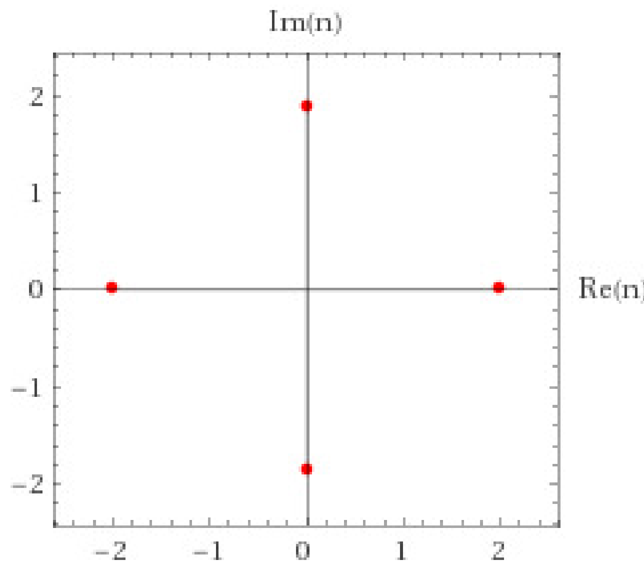


Figure 2: $2n^4 - n^2 = 28$

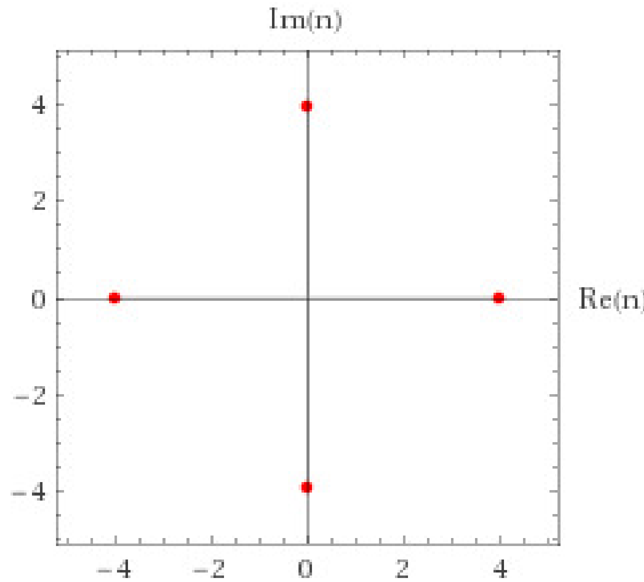


Figure 3: $2n^4 - n^2 = 496$

The Real roots of 496 are [4,-4] and the Real roots of 28 are [-2,2], or the lenght between two points are 8 and 4, in any case, the relation (r) is equal to 2.
The Complex roots are $\sqrt{\frac{7}{2}}$ and $-\sqrt{\frac{7}{2}}$ in the case of $2n^4 - n^2 = 28$ and $\sqrt{\frac{31}{2}}$ and $-\sqrt{\frac{31}{2}}$ in the case of $2n^4 - n^2 = 496$.
The figure they form apparently tend to be a perfect square (as we will see in the next section), but this never happens because the area of this figures is equal to $Mp_n + \frac{1}{2}$.

Graphic of Perfect Numbers.

$\log_{10}(2n^4 - n^2 = Pf_n) \approx \log_{10}\sqrt{\frac{Mp_n}{2}}$

Table 4: log-log.

$2n^4 - n^2 = Pf_n$	$\sqrt{\frac{Mp_n}{2}}$
.301029995664	.272034022175
.602059991328	.595165849085
.903089986992	.901386862646
1.80617997398	1.80615346513
2.40823996531	2.4082383086
2.70926996098	2.7092695468
4.51544993496	4.51544993486
9.03089986992	9.03089986992

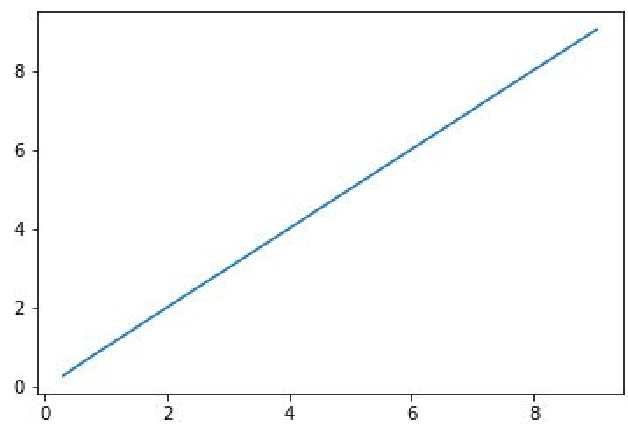


Figure 4: log-log

Binary Representation of Perfect Numbers.

The binary representation of a Perfect Number is a concatenation of (n) consecutive digits (1) and (n-1) consecutive digits (0).

Table 5: Binary Representation of Perfect Numbers.

Perfect Number.	**Binary Representation.**
28	11100
496	111110000
8128	1111111000000
33550336	1111111111111000000000000
8589869056	111111111111111110000000000000000

The function $y = 2^{3x-2} - 2^{x-1}$ and its Binary Representation.[2]

The binary representation of the function $y = 2^{3x-2} - 2^{x-1}$ is a concatenation of (n) consecutive digits (1) and $\frac{n-1}{2}$ digits (0).

Table 6: Binary Representation of $y = 2^{3x-2} - 2^{x-1}$

$y = 2^{3x-2} - 2^{x-1}$	**Binary Representation.**
14	1110
124	1111100
1016	1111111000
8176	1111111110000
65504	1111111111100000

Algorithm to find Perfect Numbers, Mersenne Primes and Mersenne Exponents.

Given the function $y = 2^{3x-2} - 2^{x-1}$, if (y) have at maximum two distinct odd factors, F_1 and F_2, then:
$F_1 = 1$ and F_2 is a Mersenne Prime (Mp) also $\frac{y^2}{Mp} = Pf$.
The number of digits (1) of the binary representation of (y) is equal to the Mersenne Exponent (Me).

Limits of the function $f(x) = 2^{3x-2} - 2^{x-1}$

$$\lim_{x\to\infty} \frac{f(x+1)}{f(x)} \approx 8$$

$$\lim_{x\to\infty} \frac{f(x+1)}{f(x)} \approx 8 + \frac{6}{2^{2x-1} - 1}$$

In particular, if $f(x)$ have a Mersenne Prime (Mp_k) as factor, then:

$$\lim_{x\to\infty} \frac{f(x+1)}{f(x)} \approx 8 + \frac{6}{Mp_k}$$

Table 7: Limits of $f(x) = 2^{3x-2} - 2^{x-1}$ for $x \geq 2$

$f(x) = 2^{3x-2} - 2^{x-1}$	$2^{2x-1} - 1$	$\lim_{x\to\infty} \frac{f(x+1)}{f(x)}$	$8 + \frac{6}{2^{2x-1}-1}$
14	7	8.85714285	$8 + \frac{6}{7}$
124	31	8.1935483871	$8 + \frac{6}{31}$
1016	127	8.0472440944	$8 + \frac{6}{127}$
8176	511	8.01174168297	$8 + \frac{6}{511}$
65504	2047	8.00293111871	$8 + \frac{6}{2047}$

Notice that when the function $f(x) = 2^{2x-1} - 1 = Mp_k$ then the values of $f(x) = \frac{2^{3x-2} - 2^{x-1}}{2^{2x-1}-1}$ are equal to the positive Real roots of the equation $2n^4 - n^2 = Pf_k$ which is the form of a Perfect Number that we assumed.

Expanding Real Roots.

Given the equation of a circle:

$r^2 = (x-h)^2 + (y-k)^2.$

and given the numbers:

$p = n^2 + (n+1)^2$ where p is prime.

we have:

$r^2 = (x-n)^2 + (y-(n+1))^2$

The graphic of the equation $x^2 + y^2 + Ax + By + C = 0$ is a circumference, a point or have no points at all.

When is a circumference, the center is at: $(\frac{-A}{2}, \frac{-B}{2})$

and the radius is $r = \frac{1}{2}\sqrt{A^2 + B^2 - 4C}$ [3]

or $r = \frac{1}{2}\sqrt{(2n)^2 + 2(n+1)^2 - 4C}$

These type of equations are prime numbers generators, and when we search for prime numbers, we can express them as :

primes of the form $2n^2 - p$ where $p = n^2 + (n+1)^2$.

or

solutions of the equation:

$\sqrt{C+x} = n\sqrt{2}$

when C is prime

$\sqrt{p+x} = n\sqrt{2}$

For example:
$n = 9; (n+1) = 10; p = 181$

Table 8: Solutions of Equation $\sqrt{181+x} = n\sqrt{2}$ for n ≥ 0

$\sqrt{181+x} = n\sqrt{2}$	**(x) Solution.**
$\sqrt{181+x} = 0\sqrt{2}$	-181
$\sqrt{181+x} = 1\sqrt{2}$	-179
$\sqrt{181+x} = 2\sqrt{2}$	-173
$\sqrt{181+x} = 3\sqrt{2}$	-163
$\sqrt{181+x} = 4\sqrt{2}$	-149
$\sqrt{181+x} = 5\sqrt{2}$	-131
$\sqrt{181+x} = 6\sqrt{2}$	-109
$\sqrt{181+x} = 7\sqrt{2}$	-83
$\sqrt{181+x} = 8\sqrt{2}$	-53
$\sqrt{181+x} = 9\sqrt{2}$	**-19**
$\sqrt{181+x} = 10\sqrt{2}$	**+19**
$\sqrt{181+x} = 11\sqrt{2}$	+ 61

Notice that when n=9 and n=10, this is n+(n+1)=19, the solutions jump from negative to positive, this is important as we will see later.

Now if we have the equation $\sqrt{p+x} = n\sqrt{2}$ then p is a Mersenne Prime so $p = Mp_k$ and n are the positive Real roots of the equation $2n^4 - n^2 = Pf_k$. We have:

Table 9: Solutions of Equation $\sqrt{Mp_k+x} = n\sqrt{2}$

$\sqrt{Mp_k+x} = n\sqrt{2}$	**(x) Solution.**
$\sqrt{7+x} = 2\sqrt{2}$	1
$\sqrt{31+x} = 4\sqrt{2}$	1
$\sqrt{127+x} = 8\sqrt{2}$	1
$\sqrt{8191+x} = 64\sqrt{2}$	1
$\sqrt{131071+x} = 256\sqrt{2}$	1
$\sqrt{524287+x} = 512\sqrt{2}$	1
$\sqrt{2147483647+x} = 32768\sqrt{2}$	1

This is

$$\sqrt{Mp_k+1} = \sqrt{\frac{Mp_k+1}{2}}\sqrt{2}$$

As we noticed before, when we have values n and $(n+1)$ the solutions jump from negative to positive so in this case the positive solutions are equal to 1 and the negative solutions are given in the next table.

Table 10: Solutions of Equation $\sqrt{Mp_k + x} = (n-1)\sqrt{2}$

$\sqrt{Mp_k + x} = (n-1)\sqrt{2}$	**(x) Solution.**
$\sqrt{7 + x} = 1\sqrt{2}$	-5
$\sqrt{31 + x} = 3\sqrt{2}$	-13
$\sqrt{127 + x} = 7\sqrt{2}$	-29
$\sqrt{8191 + x} = 63\sqrt{2}$	-253
$\sqrt{131071 + x} = 255\sqrt{2}$	-1021
$\sqrt{524287 + x} = 511\sqrt{2}$	-2045
$\sqrt{2147483647 + x} = 32767\sqrt{2}$	-131069

The binary representation of these solutions is a concatenation of (d) digits (1) plus (01), this number $(d+1)$ or the total number of digits (1), represents the (x) of the function $f(x) = 2^{2x-1} - 1$.

Table 11: Binary Representation for (x) solutions.

(x) Solution	**Binary Representation**	$2^{2x-1} - 1$	Mersenne Prime
5	101	$2^{(2\cdot 2)-1} - 1$	7
13	1101	$2^{(2\cdot 3)-1} - 1$	31
29	11101	$2^{(2\cdot 4)-1} - 1$	127
253	11111101	$2^{(2\cdot 7)-1} - 1$	8191
1021	1111111101	$2^{(2\cdot 9)-1} - 1$	131071
2045	11111111101	$2^{(2\cdot 10)-1} - 1$	524287
131069	11111111111111101	$2^{(2\cdot 16)-1} - 1$	2147483647

So the parabola $y = 2x^2 - 1$ is in direct relation with the Mersenne Primes of the form $2^{2x-1} - 1$.

Limits of the function $f(x) = 2^{2x-1} - 1$

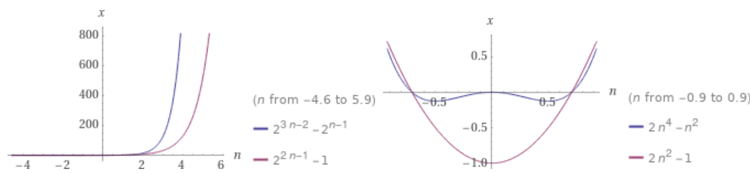
$$\lim_{x\to\infty} \frac{f(x+1)}{f(x)} \approx 4 + \frac{3}{2^{2x-1} - 1}$$

$$\lim_{x\to\infty} 2^{3x-2} - 2^{x-1} \div \lim_{x\to\infty} 2^{2x-1} - 1 \approx 2$$

Euler: $(2^{n-1})(2^n - 1) = Pf_k$

From Euler. If N is an even perfect number, then N can be written in the form $N = 2^{n-1}(2^n - 1)$, where $2^n - 1$ is prime.[4]

$$f(x) = \frac{2^{3x-2} - 2^{x-1}}{2^{2x-1} - 1} = 2^{x-1}$$



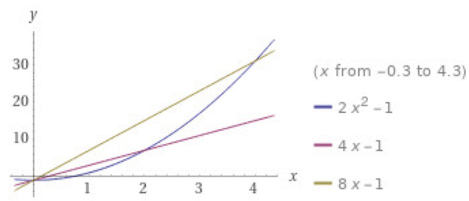
(a) Real root = $\frac{1}{2}$ (b) Real root = $\frac{1}{\sqrt{2}}$

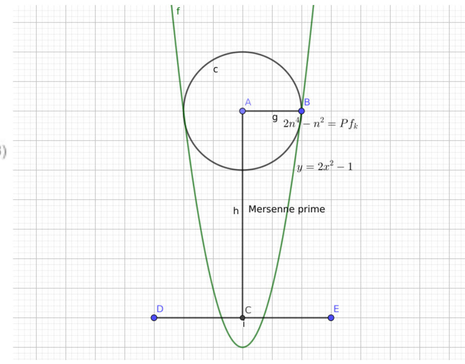
Figure 5

To obtain a given Perfect Number, we square the positive Real root of the equation $2n^4 - n^2 = Pf_k$ and multiply by the given Mersenne Prime (MP_k) to obtain the Perfect Number (Pf_k).
On the other hand, we square the value of $f(x) = 2^{3x-2} - 2^{x-1}$ (such that $f(x) = 2^{2x-1} - 1$ is a Mersenne Prime) and divide by the given Mersenne Prime (MP_k) to obtain the Perfect Number (Pf_k). (as we did in the algorithm, $\frac{y^2}{Mp_k} = Pf_k$).

Geometric Representation of Perfect Numbers.

We can represent Perfect Numbers in two different forms, firstly, as a family of parallel parabolas of the form $2n^2 - Mp_k$ with respective circles with center at the point (0,1), the radius of the circles is equal to the positive Real roots of the equation $2n^4 - n^2 = Pf_k$, the other form is using the parabola $2n^2 - 1$ with a family of circles with center at the points (0,Mp_k), the radius of the circles is equal to the positive Real roots of the equation $2n^4 - n^2 = Pf_k$[5]
In this last case, exists a family of straight lines of the form y = Cx-1, where C is also the diameter of the circle, the relation of the values of C, fulfil the condition of the relation between Perfect Numbers. (See, Relation between number 28 and upper Perfect Numbers.)





Perfect Numbers, Pythagorean Triples[6] and Fibonacci Boxes[7]

Given the matrix:
$A = \begin{vmatrix} n & 1 \\ n+1 & 2n+1 \end{vmatrix} \rightarrow |A| = 2n^2 - 1 = Mp_k \rightarrow |A| \cdot n^2 = PF_k$

Where n represents the positive Real solutions of the system $\begin{Bmatrix} 2n^4 - n^2 = Pf_k \\ 2n^2 - 1 = Mp_k \end{Bmatrix}$ and is the radius of the circle of the previous figure. Clearly the relations between Perfect Numbers are the relations between the different radii of the circles that fit on the parabola.

Let (a,b,c) be a primitive pythagorean triple where:

$a = \sqrt{b+c} = 2n+1$

$b = 2n(n{+}1) = \sqrt{\int_1^{2n+1} x^3 - x \, dx}$

$c = 2n(n{+}1){+}1 = \sqrt{\int_1^{2n+1} x^3 - x \, dx}{+}1$

If $n = 2 \cdot 2^{\frac{M_e - 3}{2}}$, we can write: $\begin{vmatrix} 2 \cdot 2^{\frac{M_e-3}{2}} & 1 \\ 2 \cdot 2^{\frac{M_e-3}{2}} + 1 & 2^2 \cdot 2^{\frac{M_e-3}{2}} + 1 \end{vmatrix}$

(being M_e the Mersenne exponent).
So, we can represent every Perfect Number as a matrix. Here we have a few examples and their respective primitive pythagorean triples:

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}}_{(5,12,13)} \underbrace{\begin{bmatrix} 4 & 1 \\ 5 & 9 \end{bmatrix}}_{(9,40,41)} \underbrace{\begin{bmatrix} 8 & 1 \\ 9 & 17 \end{bmatrix}}_{(17,144,145)} \underbrace{\begin{bmatrix} 64 & 1 \\ 65 & 129 \end{bmatrix}}_{(129,8320,8321)} \underbrace{\begin{bmatrix} 256 & 1 \\ 257 & 513 \end{bmatrix}}_{(513,131584,131585)} \underbrace{\begin{bmatrix} 512 & 1 \\ 513 & 1025 \end{bmatrix}}_{(1025,525312,525313)}$$

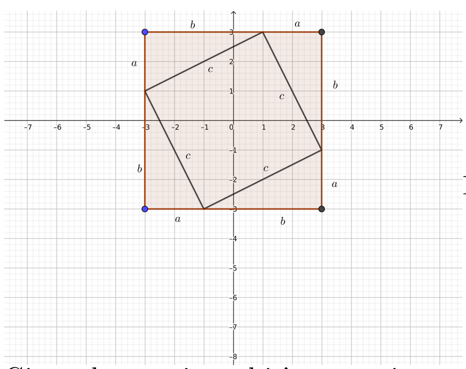
All these primitive pythagorean triples (whose first term have the form $2^n + 1$) will be obtained from the recursive multiplication of the H. Lee Price matrix:

For example:

$$\begin{bmatrix} 2 & 1 & -1 \\ -2 & 2 & 2 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 & 12 & 13 \end{bmatrix} \rightarrow \frac{13-12}{5} = \frac{1}{5}; \frac{13-5}{12} = \frac{2}{3} \rightarrow \begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} \rightarrow |A| = 7$$

Note that if the short leg (5 in this case) is on the y axis, we should multiply it until it reaches the perimeter of the circle and then we add the radius of the circle (2) and the sum will be the Mersenne prime (7), this must happen in every matrix, so we have: $(2n + 1)(n - 1) + n = 2n^2 - 1$.

Perfect Numbers and The Pythagoras Theorem.



Pythagoras Theorem.

Given the matrix and it's respective pythagorean triple

$$\underbrace{\begin{bmatrix} n & 1 \\ n+1 & 2n+1 \end{bmatrix}}_{\text{(a,b,c)}} \rightarrow b - a = Mp_k$$

As every Perfect Number have the form $Sp_k \cdot Mp_k = Pf_k$, $\left\{ \begin{matrix} a = 2n + 1 \\ c = (2n + 1)n + (n + 1) = 2n^2 + 2n + 1 \\ b = c - 1 = 2n^2 + 2n \end{matrix} \right\}$

$Sp_k \cdot Mp_k = n^2 \cdot (b - a) = n^2 \cdot (2n^2 + 2n - 2n - 1) = n^2 \cdot (2n^2 - 1) = 2n^4 - n^2 = Pf_k$

$\frac{n}{n+1}$ and $\frac{1}{2n+1}$ represent 'slopes', as the derivative of the parabola $2n^2 - 1$ is equal to $4n$, we solve $4n = \frac{n}{n+1}$ and $4n = \frac{1}{2n+1}$, so we get two points, the difference between them is a number $\frac{Mp_k}{t}$ where t is an integer such that $\frac{a \cdot b}{t} = r' \left| \frac{r'_k}{r_{k-1}} = r \right.$ being r the relation between Perfect Numbers.

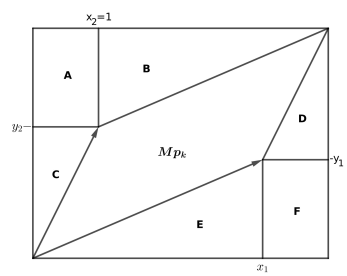
For example, we get r_1.

$$\underbrace{\begin{bmatrix}2 & 1\\3 & 5\end{bmatrix}}_{(5,12,13)} \to \frac{1}{6}-\frac{1}{20}=\frac{7}{60}\to\frac{5\cdot 12}{60}=1;\ \underbrace{\begin{bmatrix}4 & 1\\5 & 9\end{bmatrix}}_{(9,40,41)}\to\frac{1}{5}-\frac{1}{36}=\frac{31}{180}\to\frac{9\cdot 40}{180}=2\to r_1=\frac{2}{1}=2$$

So we can represent relations between Perfect Numbers as fractions.

If every Perfect Number has a representation as a pythagorean triple and every Perfect Number is equal to $\sum_{1}^{b-a} j_i$ we can write $\frac{(a+b)^2}{\sum_a^b j_i} = \frac{a+b}{n^2} = \frac{a+b}{Sp_k}$ being n^2 the Superperfect Sp_k. As the area of the pythagorean triple is: $(n)(n+1)(2n+1)$, then $(a+b)^2 = 4n^4+16n^3+20n^2+4n+1$.
From $\frac{a+b}{n^2}$ for every Perfect number, we have: $\frac{17}{4}, \frac{49}{16}, \frac{161}{64}, \frac{8449}{4096}, \frac{132097}{65536}, \frac{526337}{262144},,,,$ notice that $\frac{(a+b)_k}{(n_k)^2} = \frac{(a+b)_k}{Sp_k} \approx 2$, but we don't know if the Supeperfect's are infinite.
Now, from this same sequence, we take $\frac{(a+b)_k}{(a+b)_{k-1}}$, so we have: $\frac{23}{7}, \frac{1207}{23}, \frac{18871}{1207}, \frac{75191}{18871},$ as for every Superperfect number corresponds a different $(a+b)_k$, this sequence suggest that Superperfect Numbers are infinite.
Note that $\frac{(a+b)_k}{(a+b)_{k-1}} = \frac{p}{q}\Big| p = \frac{a+b}{7}$, so every $(a+b)_k$ runs on the function $y = 7x$.

Perfect Numbers and Determinants.



Determinant.[10]

From de diagram, we have $\begin{bmatrix}x_1 & x_2\\ y_1 & y_2\end{bmatrix}$ which turns to $\begin{bmatrix}n & 1\\ n+1 & 2n+1\end{bmatrix}$
The distance (d) between vectors is described by the equation $y = \frac{Mp_k}{n-1} - \frac{nx}{n-1}$
we have the next identities: $\begin{Bmatrix} d^2 = (n-1)^2+n^2 \\ d^4 = (2n-1)^2+(d^2-1)^2 \\ Mp_k - (d^2-1) = 2n-1 \end{Bmatrix}$
Where $Mp_k = x_1\cdot y_2 - y_1\cdot x_2 = 2n^2-1$

Since $d=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}=\sqrt{2n^2-2n+1}\to d^2=2n^2-2n+1$
From the second identity, we have the Primitive Pythagorean Triple:
$(2n^2-2n+1)^2=(2n-1)^2+(2n^2-2n)^2$

Note that $d^2=2n^2-2n+1=(2^{n_k}-1)^2+(2^{n_k})^2$ where (n_k) runs on the function $f(n)=2^n-1$.

Example:
$\begin{bmatrix}512 & 1\\ 513 & 1025\end{bmatrix}$ the distance between vectors $[512,513]$ and $[1,1025]=\sqrt{523265}$

we have:
$d^2=(n-1)^2+n^2\to 523265=511^2+512^2$
$d^4=(2n-1)^2+(2n^2-2n)^2\to 523265^2=1023^2+523264^2$

$(2^{n_k}-1)^2+(2^{n_k})^2=(2^9-1)^2+(2^9)^2=523265$
This means:
If $Mp_k=2^{Me_k}-1\to d^2=\left(2^{\frac{Me_k-1}{2}}-1\right)^2+\left(2^{\frac{Me_k-1}{2}}\right)^2$

$Mp_k-(d^2-1)=524287-523264=1023$
$\to[2^{19}-1-(d^2-1)=2^{10}-1]\to[2^{Me_k}-1-(d^2-1)=2^{\frac{Me_k+1}{2}}-1]$

Families of Primitive Pythagorean Triples.
We have the identity:

$$\underbrace{\begin{bmatrix}n & 1\\ n+1 & 2n+1\end{bmatrix}}_{\text{(a,b,c)}}\to b-a=Mp_k$$

In the diagram, when we expand coordinates, (x_1+x_2) becomes the newest x_1 and (y_1+y_2) becomes the newest y_1. This is:
$$\begin{bmatrix}x_1 & x_2\\ y_1 & y_2\end{bmatrix}\to\begin{bmatrix}x_{1(k+1)}=x_{1k}+x_{2k} & x_{2(k+1)}=y_{1(k+1)}-x_{1(k+1)}\\ y_{1(k+1)}=y_{1k}+y_{2k} & y_{2(k+1)}=x_{1(k+1)}+y_{1(k+1)}\end{bmatrix}$$

$$\underbrace{\begin{bmatrix}n & 1\\ n+1 & 2n+1\end{bmatrix}}_{\text{(a,b,c)}}\to\underbrace{\begin{bmatrix}n+1 & 2n+1\\ 3n+2 & 4n+3\end{bmatrix}}_{\text{(a,b,c)}}\to\underbrace{\begin{bmatrix}3n+2 & 4n+3\\ 7n+5 & 10n+7\end{bmatrix}}_{\text{(a,b,c)}}\to\underbrace{\begin{bmatrix}7n+5 & 10n+7\\ 17n+12 & 24n+17\end{bmatrix}}_{\text{(a,b,c)}}\to\cdots$$

When (n) is the positive real root of the equations: $2n^4-n^2=Pf_k$ or $2n^2-1=Mp_k$ the determinant of all these matrices is equal to the same Mersenne Prime and all of them produce a different Primitive Pythagorean Triple where the difference 'b-a' is also the given Mersenne Prime, (n) has the same value for all matrices. Example:

$$\underbrace{\begin{bmatrix}2 & 1\\ 3 & 5\end{bmatrix}}_{(5,12,13)}\underbrace{\begin{bmatrix}3 & 5\\ 8 & 11\end{bmatrix}}_{(48,55,73)}\underbrace{\begin{bmatrix}8 & 11\\ 19 & 27\end{bmatrix}}_{(297,304,425)}\underbrace{\begin{bmatrix}19 & 27\\ 46 & 65\end{bmatrix}}_{(1748,1755,2477)}\underbrace{\begin{bmatrix}46 & 65\\ 111 & 157\end{bmatrix}}_{(10205,10212,14437)}\underbrace{\begin{bmatrix}111 & 157\\ 268 & 379\end{bmatrix}}_{(59496,59503,84145)}$$

All these determinants are equal to 7, and all the differences between 'b' and 'a' are also 7, the value of (n) is 2. The angle between vectors tend to cero.
To build them, we can do this:

$$\underbrace{\begin{bmatrix} r & t \\ f & g \end{bmatrix}}_{\text{(a,b,c)}} \rightarrow \underbrace{\begin{bmatrix} r & t \\ \uparrow & \uparrow \\ f & g \end{bmatrix}}_{\text{(a,b,c)}} \rightarrow \begin{matrix} f = t + r \\ g = r + f \end{matrix} \rightarrow \begin{Bmatrix} (g \cdot t) = a \vee b \\ 2(f \cdot r) = b \vee a \\ (g \cdot r) + (f \cdot t) = c \end{Bmatrix}$$

The silver Ratio.
Given A and B such that $A > B$ then $\frac{2A+B}{A} = \frac{A}{B}$, in this case;
$\lim\limits_{k \to \infty} \frac{\|\text{Vector}_{(k)}\|}{\|\text{Vector}_{(k-1)}\|} \approx 1 + \sqrt{2}$

So, we have:
$B = \begin{bmatrix} n & 1 \\ n+1 & 2n+1 \end{bmatrix}$ $A = \begin{bmatrix} n+1 & 2n+1 \\ 3n+2 & 4n+3 \end{bmatrix}$

$\|B\| = \sqrt{(n+1)^2 + (3n+2)^2} = \sqrt{10n^2 + 14n + 5}$

$\|A\| = \sqrt{(3n+2)^2 + (7n+5)^2} = \sqrt{58n^2 + 82n + 29}$

$\frac{2\sqrt{58n^2+82n+29}+\sqrt{10n^2+14n+5}}{\sqrt{58n^2+82n+29}} = \frac{\sqrt{58n^2+82n+29}}{\sqrt{10n^2+14n+5}} \rightarrow \frac{2\sqrt{425}+\sqrt{73}}{\sqrt{425}} = \frac{\sqrt{425}}{\sqrt{73}}$

for n=2; $2\sqrt{73}\sqrt{425} + 73 \approx 425$
for n=4; $2\sqrt{221}\sqrt{1285} + 221 \approx 1285$
for n=8; $2\sqrt{757}\sqrt{4397} + 757 \approx 4397$

We can build this array for $n = 2$:

$$\frac{\sqrt{425}}{\sqrt{73}} \approx (1+\sqrt{2})^1$$

$$\frac{\sqrt{425}}{\sqrt{73}} \cdot \frac{\sqrt{2477}}{\sqrt{425}} \approx \left(1 + \sqrt{2}\right)^2$$

$$\frac{\sqrt{425}}{\sqrt{73}} \cdot \frac{\sqrt{2477}}{\sqrt{425}} \cdot \frac{\sqrt{14437}}{\sqrt{2477}} \approx \left(1 + \sqrt{2}\right)^3$$

$$\frac{\sqrt{425}}{\sqrt{73}} \cdot \frac{\sqrt{2477}}{\sqrt{425}} \cdot \frac{\sqrt{14437}}{\sqrt{2477}} \cdot \frac{\sqrt{84145}}{\sqrt{14437}} \approx \left(1 + \sqrt{2}\right)^4$$

$$\frac{\sqrt{425}}{\sqrt{73}} \cdot \frac{\sqrt{2477}}{\sqrt{425}} \cdot \frac{\sqrt{14437}}{\sqrt{2477}} \cdot \frac{\sqrt{84145}}{\sqrt{14437}} \cdot \frac{\sqrt{490433}}{\sqrt{84145}} \approx \left(1 + \sqrt{2}\right)^5$$

$$\frac{\sqrt{425}}{\sqrt{73}} \cdot \frac{\sqrt{2477}}{\sqrt{425}} \cdot \frac{\sqrt{14437}}{\sqrt{2477}} \cdot \frac{\sqrt{84145}}{\sqrt{14437}} \cdot \frac{\sqrt{490433}}{\sqrt{84145}} \cdots \approx \left(1 + \sqrt{2}\right)^n$$

From Pell Numbers.
$P_k = 2P_{(k-1)} + p_{(k-2)}$ or $\|\text{Vector}_k\| = 2\|\text{Vector}_{(k-1)}\| + \|\text{Vector}_{(k-2)}\|$
Given:

$\sqrt{73}, \sqrt{425}, \sqrt{2477}, \sqrt{14437}, \sqrt{84145}, \sqrt{490433}, \ldots$

Then:

$\sqrt{2477} \approx 2\sqrt{425}+\sqrt{73}$

$\sqrt{14437} \approx 2\sqrt{2477}+\sqrt{425}$

The polynomials related to these vectors (with value n=2) are:
$\sqrt{10n^2 + 14n + 5} \approx \sqrt{73}$
$\sqrt{58n^2 + 82n + 29} \approx \sqrt{425}$
$\sqrt{338n^2 + 478n + 169} \approx \sqrt{2477}$
$\sqrt{1970n^2 + 2786n + 985} \approx \sqrt{14437}$
$\cdots\cdots$
As they are recursive, we can find a general form with initial value $W = 5$:

$$\begin{array}{cc} & F_k = 2W;\; G_k = 2\sqrt{2W^2 - 1} \\ (F_k)n^2 + (G_k)n + W_k & W_{(k+1)} = F_k + G_k + W_k \\ (F_{(k+1)})n^2 + (G_{(k+1)})n + W_{(k+1)} \rightarrow & F_{(k+1)} = 2W_{(k+1)};\; G_{(k+1)} = 2\sqrt{(2W_{(k+1)})^2 - 1} \\ (F_{(k+2)})n^2 + (G_{(k+2)})n + W_{(k+2)} & G_{(k+1)} = F_{(k+1)} + (W_{(k+1)} - W_k) \\ & F_{(k+2)} = 4G_{(k+1)} + F_k \end{array}$$

The matrices that produce the Primitive Pythagorean Triples and the polynomials related to the vectors are always the same, the only thing that changes is the value on (n), for example, the Artemas Martin PPT[11] where the difference 'b-a=1' and $2n^2 - 1 = 1$

$$\underbrace{\begin{bmatrix}1 & 1\\ 2 & 3\end{bmatrix}}_{(3,4,5)} \underbrace{\begin{bmatrix}2 & 3\\ 5 & 7\end{bmatrix}}_{(20,21,29)} \underbrace{\begin{bmatrix}5 & 7\\ 12 & 17\end{bmatrix}}_{(119,120,169)} \underbrace{\begin{bmatrix}12 & 17\\ 29 & 41\end{bmatrix}}_{(696,697,985)} \underbrace{\begin{bmatrix}29 & 41\\ 70 & 99\end{bmatrix}}_{(4059,4060,5741)} \underbrace{\begin{bmatrix}70 & 99\\ 169 & 239\end{bmatrix}}_{(23660,23661,33461)}$$

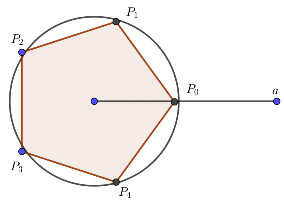
Note that in this case, as $W_{(k+1)} = F_k + G_k + W_k$ the value of the polynomials and therefore the $\|\text{vectors}\|$ is equal to $\sqrt{W_{(k+n)}}$. This is:

$$\frac{\sqrt{29}}{\sqrt{5}} \approx \left(1 + \sqrt{2}\right)^1$$

$$\frac{\sqrt{29}}{\sqrt{5}} \cdot \frac{\sqrt{169}}{\sqrt{29}} \approx \left(1 + \sqrt{2}\right)^2$$

$$\frac{\sqrt{29}}{\sqrt{5}} \cdot \frac{\sqrt{169}}{\sqrt{29}} \cdot \frac{\sqrt{985}}{\sqrt{169}} \approx \left(1 + \sqrt{2}\right)^3$$

Perfect Numbers and The Cotes Theorem.



Cotes Theorem.

The Cotes Theorem, performed by the English Mathematician Roger Cotes, states that "Given a regular polygon inscribed in a circle and given a point *(a)* fixed at the x *axis*, then, the product of the distances between *(a)* and all the vertices is equal to $r^N - a^N$ if the point *(a)* is inside the circle or $a^N - r^N$ if the point *(a)* is outside the circle".[8] If we take a unit circle and *(a)* at the point $(2, 0)$, we have $a^N - r^N = 2^N - 1$ and if N is a prime p we have $2^p - 1$ which is a Mersenne Number, if $p = Me_k$ it is a Mersenne Prime.

In the figure we have a pentagon so, the Mersenne exponent is $Me = 5$ and the product of the distances from point a is equal to: $d_0 \cdot d_1 \cdot d_2 \cdot d_3 \cdot d_4 = 2^5 - 1$. From the "distance formula between two points", we have:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x - \cos(\theta))^2 + (0 - \sin(\theta))^2} = \sqrt{x^2 - 2x\cos(\theta) + 1}$$

The distances from point a to all the vertices are: $d_0 = 1$, $d_1 = d_4$, $d_2 = d_3$ $\theta = \left(\frac{2\pi}{N}\right) n = \frac{2\pi n}{Me_k}$. So:

$$d_1 = d_4 \rightarrow d_1 \cdot d_4 = 1 + x^2 - 2x\cos\left(\frac{2\pi \cdot 1}{5}\right)$$

$$d_2 = d_3 \rightarrow d_2 \cdot d_3 = 1 + x^2 - 2x\cos\left(\frac{2\pi \cdot 2}{5}\right)$$

Since the Mersenne Exponents(Me_k) are always odd, we have:

$$(x-1)\prod_{n=1}^{\frac{Me_k-1}{2}} \left\{1+x^2-2x\cos\left(\frac{2\pi n}{Me_k}\right)\right\} = Mp_k$$

Since $x = a = 2$ then:

$$(2-1)\prod_{n=1}^{\frac{Me_k-1}{2}} \left\{1+4-4\cos\left(\frac{2\pi n}{Me_k}\right)\right\} = \prod_{n=1}^{\frac{Me_k-1}{2}} \left\{5-4\cos\left(\frac{2\pi n}{Me_k}\right)\right\} = Mp_k$$

For example:

$$\left[5-4\cos\left(\frac{2\pi}{3}\right)\right]=7$$

$$\left[5-4\cos\left(\frac{2\pi}{5}\right)\right]\cdot\left[5-4\cos\left(\frac{4\pi}{5}\right)\right]=31$$

$$\left[5-4\cos\left(\frac{2\pi}{7}\right)\right]\cdot\left[5-4\cos\left(\frac{4\pi}{7}\right)\right]\cdot\left[5-4\cos\left(\frac{6\pi}{7}\right)\right]=127$$

and so on.

Thus, we can find polynomials for every Perfect Number, this is:

$$Pf_k=28\rightarrow\left[x-5+4\cos\left(\frac{2\pi}{3}\right)\right]=x-7$$

$$Pf_k=496\rightarrow\left[x-5+4\cos\left(\frac{2\pi}{5}\right)\right]\cdot\left[x-5+4\cos\left(\frac{4\pi}{5}\right)\right]=x^2-12x+31$$

$$Pf_k=8128\rightarrow x^3-17x^2+87x-127$$
$$Pf_k=33550336\rightarrow x^6-32x^5+405x^4-2568x^3+8491x^2-13656x+8191$$

Note that if we use Degrees instead of Radians, we find that:
$\cos(496°)=\cos(33550336°)=\cos(8589869056°)=\cos(2658455991569831744654692615953842176°)$

On the other hand:
$\cos(28°)=-cos(8128°)=cos(137438691328°)=-\cos(2305843008139952128°)$

Perfect Numbers as a Parallelepiped.

As every Perfect Number have the form $Sp_k\cdot Mp_k=Pf_k$, $\left\{\begin{array}{c}a=2n+1\\ c=(2n+1)n+(n+1)=2n^2+2n+1\\ b=c-1=2n^2+2n\end{array}\right\}$
$Sp_k\cdot Mp_k=n^2\cdot(b-a)=n^2\cdot(2n^2+2n-2n-1)=n^2\cdot(2n^2-1)=2n^4-n^2=Pf_k$

($Sp_k=$ *Superperfect Number*, $Mp_k=$ *Mersenne Prime*, $Pf_k=$ *Perfect Number*)

$$\textit{Superperfect Number}=\frac{\textit{Perfect number}}{\textit{Mersenne prime}}=\frac{2n^4-n^2}{2n^2-1}=n^2\ \forall n\neq\left\{-\frac{1}{\sqrt{2}}\ \textit{and}\ \frac{1}{\sqrt{2}}\right\}$$

we can represent the whole system the next way:
Mersenne Prime $=2n^2-1=$ *Parallelogram.*
Perfect Number $=2n^4-n^2=$ *Parallelepiped.*
SuperPerfect Number $=n^2=$ *Height.*

So, given $'n'$ we have:

$$n \Rightarrow \underbrace{\begin{bmatrix} n & 1 \\ n+1 & 2n+1 \end{bmatrix}}_{Parallelogram} \Rightarrow \underbrace{\begin{bmatrix} 1 & 2n+1 & n \\ n & n+1 & n^2 \\ n+1 & 1 & n \end{bmatrix}}_{Parallelepiped}$$

Example:

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \Rightarrow |A| = 7 \Rightarrow A' = \begin{bmatrix} 1 & 5 & 2 \\ 2 & 3 & 4 \\ 3 & 1 & 2 \end{bmatrix} \Rightarrow |A'| = 28$$

$$A = \begin{bmatrix} 4 & 1 \\ 5 & 9 \end{bmatrix} \Rightarrow |A| = 31 \Rightarrow A' = \begin{bmatrix} 1 & 9 & 4 \\ 4 & 5 & 16 \\ 5 & 1 & 4 \end{bmatrix} \Rightarrow |A'| = 496$$

$$A = \begin{bmatrix} 8 & 1 \\ 9 & 17 \end{bmatrix} \Rightarrow |A| = 127 \Rightarrow A' = \begin{bmatrix} 1 & 17 & 8 \\ 8 & 9 & 64 \\ 9 & 1 & 8 \end{bmatrix} \Rightarrow |A'| = 8128$$

$$A = \begin{bmatrix} 64 & 1 \\ 65 & 129 \end{bmatrix} \Rightarrow |A| = 8191 \Rightarrow A' = \begin{bmatrix} 1 & 129 & 64 \\ 64 & 65 & 4096 \\ 65 & 1 & 64 \end{bmatrix} \Rightarrow |A'| = 33550336$$

and so on.

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