Mathematical Model of Meaning

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Abstract

Meaning is not attributed to the message itself but rather emerges from its action on the recipient. On this principal idea a mathematical model of meaning can be constructed. The recipient is defined by a set of its states; the message is an operator that acts on these states and transfers the recipient from one state into another. Meaning should be ascribed to neither the recipient state nor the message operator; meaning emerges while the set of recipient states maps to itself and the recipient transfers from one state to another by the message operator. This general approach can be realized in terms of such formalisms as automata theory, matrix representation, algorithms, Markov chains, parameter space, and others. Different classes of meanings are considered including finite, countable, and continuous; reversible and irreversible; deterministic and probabilistic ones. It is shown that an arbitrary meaning operator can be decomposed in to a product of elementary meaning operators in case of their associativity.

Keywords: information, meaning, mathematical modeling, abstract machine (automaton), algorithm.

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1 Introduction

Information should be included into a physical description of the world: this is an important problem faced by theoretical physics in recent decades. Attempt in this direction is the known "It from Bit" hypothesis proposed by Wheeler [1]. Another example is the Ur-theory of Weizsäcker [2].

In most physical theories, information is considered only in its quantity aspect, which is insufficient. Two messages may contain equal amounts of information but have different meanings to the recipient. From here on we understand the message and recipient terms in their broadest sense. We argue that physical models should include information in such a way that takes explicit account of its meaning. Theoretical physics describes world by mathematical models; hence, we need a mathematical model of meaning.

Sometimes information is considered with a focus on its value and mathematical models are elaborated accordingly. The value of information is associated with the presence of a goal persuaded by the recipient and is determined by how much it contributes to the achievement of the goal. In particular, the value of information can be measured by an increase in the probability for the recipient to achieve the goal [3, 4, 5]. In [6], the value of information is equated to its meaning. Meaning is defined there in terms of raising the chance for an organism to survive in natural selection. The value of information can also be defined without using probabilities, e.g., through the reduction of material or time costs expended to achieve the goal [7].

Value is an important characteristic of information. But value and meaning are different things. If meaning and value are identified, then any information appears to be meaningless unless the recipient as a specific goal. In this study, we propose to describe meaning of information by a mathematical model that is not related to its value.

The main idea of this work was reported at the 4th Conference on the Foundations of Fundamental Physics and Mathematics [8] and published [9].

2 Relational model of meaning

2.1 Meaning is interpretation

ISO/IEC standard defines information as "knowledge concerning objects, such as facts, events, things, processes, or ideas, including concepts, that within a certain context has a particular meaning" [10]. In brief formulation, information is what has meaning. Meaning is the defining property of information.

Meaning does not characterize information intrinsically but can emerge in a certain context. The same message can have different meanings for different recipients. According to Bateson [11], an object of perception acquires meaning in the process of perception. Meaning is an emergent property arising when a message is exposed to a recipient. A message has no meaning in the absence of a recipient and an impact on the recipient.

The key idea of the proposed approach can be expressed by the following general equation:

$$A(ab)B(b) = C(b), \tag{1}$$

Where mathematical objects a and b denote the message and the recipient, respectively; operator $A(ab)$ denotes the message-on-recipient impact, which depends on both the message and the recipient; and mathematical objects $B(b)$ and $C(b)$ describe the recipient state before and after the reception of the message, respectively. The recipient state is understood in a broad sense. For example, in case of a branching process, this may be a particular channel of the process flow.

We propose to consider equation (1) as the basic principle for mathematical modeling of meaning. The key mathematical object is the set of the recipient states. Meaning is mapping of this set to itself. Mathematically, meaning is captured by neither the recipient state nor the operator of message; it is embodied in the recipient transition from one state to another.

A change in the recipient state is nothing else than the interpretation of the received message. One may refer to a recipient as an interpreter. Meaning is a property of the perception of information, or interpretation. It emerges from the message – recipient relations. Thus, the proposed model is a relational model of meaning.

Note that similar problems are studied in semantics [12]. Semantics uses the denotatum term. Denotatum of an informational object (sign) is an object that is designated by this sign. The difference between denotatum and meaning can be clearly demonstrated with the example of a portrait. Denotatum is the person depicted. Meaning is a change in the psychological state of the viewer looking at the portrait.

2.2 Basic properties

Equation (1) can be represented in terms of specific mathematical objects. Once chosen, an object for the recipient state defines the operator $A(ab)$ accordingly. Operator $A(ab)$ emulates a message, while the meaning carried by this message is emulated in the operator's impact on the recipient. Note that operator $A(ab)$ generally depends on the recipient b. Whatever the objects used to represent the recipient states and message operator, the meaning should remain invariant, although differently represented. Meaning is invariant with respect to mathematical representation.

The same operator may transfer different states in different ways. Hence it follows that a message may have different meanings for different recipients and even for the same recipient being in different states at the instant of the reception. In terms of the proposed model, any change in the recipient state under the action of message is considered as an act of cognition.

The meaning of a message is strictly equal for recipients who were in the same initial state at the instant of reception and transit to the same final state after the reception of the message. If one defines a concept of close states, then close meanings can be defined as a transformation that moves close initial states to close final ones.

Note that no assumption of the recipient's mental activity is required to treat the message as meaningful. A simplest reaction suffices to recognize a change in its state. Expression (1) potentially describes objects and actions of any nature, and there is no need to identify some special class of informational, or meaningful impacts. Instead, any impact should be considered in terms of meaning and information. From this viewpoint, any change in the recipient state under the action of a message is an act of cognition. We believe that this scheme grasps the basic properties of a universal model of meaning.

The approach proposed may seem to be a mere replacement of one formal language by another. It may be so if relatively simple systems are concerned, but it is definitely not for more intricate ones. Suppose a system under study reacts identically to two different messages. The messages differ in nature, encoding, and the amount of information. Why do we get the same response to both? What is similar about them? The answer is that both messages have the same meaning for the given system.

3 Meaning classes and representations

3.1 Deterministic finite meaning, automata representation

Consider the simplest case of a finite set of recipient states. In this case, meaning can be modeled within the framework of algebraic automata theory, which is a branch of cybernetics [13]. Deterministic finite meanings are modeled using a well-studied class of finite automata.

Automaton is an object defined by three nonempty sets R, X, Y and two functions f and g. These are the set of states $R = \{r\}$; an input alphabet, i.e., the set of input elements (inputs) $X = \{x\}$; an output alphabet, i.e., the set of output elements (outputs) $Y = \{y\}$; the transition function $f : R \times X \mapsto R$ mapping the direct product of the set of states and the set of inputs to the set of states; and the output function $g : R \times X \mapsto Y$ mapping the direct product of the set of states and the set of inputs to the set of outputs. Interpreting the entries of (1) in terms of algebraic automata theory, we identify the set of operators $\{A(ab)\}$ with the input alphabet X, each operator $A(ab)$ with an input element, and states $B(b)$ and $C(b)$ with the elements of state set R. The expression (1) itself defines the transition function f.

A convention in theoretical physics, e.g., in quantum theory, is that an operator acting on a state is written to the left of the state, like in (1). In automata theory, the inverse order is adopted: the state is written first, and the operator stands to its right. In case under study, expression (1) appears in the form

$$r \mapsto f(r, x) = rx. \tag{2}$$

The transition function reflects the fact that each element of the input alphabet defines a monary operation on the set of states. According to the proposed model, this monary operation is the meaning corresponding to the given element of the input alphabet. All mappings of the set of states to itself driven by various sequences of the input elements comprise the set of all meanings cognizable by the automaton.

When modeling meaning, we are not interested in the automaton outputs. Therefore, instead of a general automaton (known as transducer, or Mealy automaton), we deal with a special case of automaton that has no outputs (an acceptor). Such an automaton is specified by a set of states, an input alphabet, and a transition function. One can also use a Moore automaton, whose output is determined by its current state only. Moreover, any Mealy automaton can be represented by an equivalent Moore automaton by redefining the set of states. In the context of meaning modeling, the outputs of a Moore automaton are of interest to study the automaton states and the meanings of elements of the input alphabet.

3.2 Algebras of meanings

In automata theory, the symbol sequences are supposed to comprise a semigroup, which is called free semigroup. The elements of a free semigroup over analphabet are all possible sequences of the alphabet symbols (words), and the semigroup operation is adjoining of various sequences to each other. The set of automaton state transformations forms a semigroup called the automaton semigroup. There is a homeomorphism of the free semigroup to the automaton semigroup since different input sequences can lead to the same transformation. Such sequences are called congruent. The identity element is given by the identical transformation generated by an empty word. Since symbol sequences form a semigroup, the adjoining of sequences is associative. We propose to interpret the free and automaton semigroups as semigroups of messages and meanings of these messages, respectively. Messages that have the same meaning for a given automaton are congruent.

Based on the notion of semigroup, one can define various types of automata and the respective classes of meanings. For example, a finite semigroup corresponding to finite-type automata can be associated with the class of finite meanings. Other classes can be associated with group-type or a commutative-group-type automata. A class of nondeterministic meanings can be based on the automata whose transition and output functions are many-valued.

In general case, meaning is irreversible. However, some messages may have their counterparts with the opposite meaning. The product of a message on its opposite-meaning counterpart is a zero-meaning message, which leave the automat on state unchanged. A semigroup where every element has the opposite one is a group. Thus, reversible meanings can be modeled by the group-type automata. Reversible meanings are typical of executable commands. An example is the turn on/turn off messages sent to a circuit breaker.

As an example of irreversible meaning, consider a recipient whose set of states includes the out-of-range state. Once got into this specific state, the recipient remains there whatever the message arrives. In other words, any message has zero meaning as it should have since any information is meaningless for an out-of-range recipient. This is an example of a transition into absorbing set of states, which has no way out. In terms of the automata theory, the system has moved to its own sub-automaton. Separation of meanings into reversible and irreversible categories reflects the existence of reversible and irreversible processes in nature.

In physical phenomena, messages are not always associative. In case they are not, a free semigroup turns into a groupoid, and the model goes beyond the framework of algebraic automata theory. Non-associativity is physically related to the interaction of messages. Indeed, suppose the recipient moves into a new state not at the instant when the message arrives but after a delay, i.e., by means of a transient process. If messages arrive with an interval exceeding the delay time, the delay can be neglected, and the automaton transitions can be regarded as instantaneous events occurring under the action of instantaneous message symbols at discrete time instants (automaton time). If the next message sometimes arrives before the end of the transient process caused by the previous one, the sequence can produce different impact on the recipient. It depends on which of the messages are superimposed in the considered manner, and this dependence is mathematically described as non-associativity.

3.3 Deterministic finite meaning, matrix representation

In this section we introduce a simple and convenient representation of deterministic finite meaning by matrices. If the set of recipient states is finite, the states can be enumerated, and the numbers considered as mathematical object describing the respective states.

Let n be the number of all possible states of the recipient. Then, each state can be represented by a matrix column $|B>$ of size n where the element with number equal to the number of state is unit and the rest entries are zeros. An impact on state $|B>$ is described as pre-multiplication of the column by a square matrix A of size $n \times n$, with one entry in each column equal to 1 and the rest zeros. As a result, expression (1) appears in the form

$$A|B>=|C> . \tag{3}$$

An impact with zero meaning is given by the identity matrix I. Note that the impact of the same message can be described by different matrices depending on the recipient.

A matrix of reversible meaning has precisely one unit in each line and the determinant equal to ± 1. The inverse matrix corresponds to the meaning of message that cancels the effect produced by the original one. These are matrices of permutations, which are known to form groups. By Cayley theorem, any finite group is isomorphic to a subgroup of permutations. Hence it follows that the mathematical theory under reversible deterministic finite meanings is a theory of finite groups, that is, permutation groups. In matrix representation, reversible messages are represented by permutation matrices.

A matrix of irreversible deterministic finite meaning has a line containing more than one unit in it. Accordingly, there is a line with all zero entries. The determinant of such a matrix is zero, and the inverse matrix does not exist. Due to the presence of more than one unit element in a line, the matrix of message with irreversible meaning moves a system from more than one initial state to the same final state.

3.4 Factorization into elementary meanings

The discussed framework provides a constructible way to address the existence of a set of elementary meanings and expansion of an arbitrary meaning into elementary meanings. A meaning that cannot be decomposed into other meanings is elementary.

It is readily seen from previous subsection 3.3 that deterministic reversible finite meanings correspond to permutations. Every permutation can be factorized into a product of cyclic permutations commuting with each other. This factorization is unique up to the order of factors. Cyclic permutations can be interpreted as representations of elementary meanings, and any finite reversible meaning appears to be factorizable into a unique product of elementary finite reversible meanings.

In case of irreversible finite meanings, one should use cyclic permutations and elementary contractions. An elementary contraction moves a system from state i to another state j. All other states, including state j, remain unchanged. Thus, there is no transition into state i and there are two transitions into the same state j, from state i and j.

In general case, a non-elementary meaning is given by mapping of the set of initial states to the set of final states, wherein every initial state is mapped to a unique final state. If some final states have no incoming transitions from an initial state, the elementary expansion of such meaning includes both cyclic permutations and elementary contractions. Every final state without an incoming transition is similar to state i in the above example and generates an elementary contraction term in the expansion. Thus, the number of elementary contractions in the expansion is equal to the number of final states with no initial states being mapped to them. Mappings to the rest states generate permutations and can be factorized into a product of cyclic permutations. Elementary contractions commute with each other, and so do cyclic permutations, but some elementary contractions do not commute with some cyclic permutations. In the example above, an elementary contraction does not commute with a cyclic permutation that includes state j. Hence it follows that, in an elementary meaning expansion, an elementary contraction must stand to the left of the cyclic permutation with which it does not commute.

A deterministic final meaning can be decomposed into a product of elementary meanings up to the order of commuting factors. Of much interest are future interpretations of this general result in various disciplines, especially in the humanities.

Note that this elementary factorization of meanings was carried out under the assumption of meanings comprising a semigroup.

3.5 Non-deterministic finite meaning

Now we turn to non-deterministic finite meanings. In this case, the recipient state after the reception of message is not uniquely determined but may be one of several variants. At the classical level, a recipient should always exist in a certain state, and the multiple choice of the final stateis naturally interpreted in the language of probabilities. Let a non-deterministic meaning be expanded into the spectrum of k deterministic meanings with some probabilities p_k, which are real numbers ranging from 0 to 1. The sum of p_k is 1 since there is always a certain final state selected by the recipient. The operator in equation (1) can be written in the form

$$A = \sum_k p_k A_k, \tag{4}$$

where A is an operator of non-deterministic meaning and A_k are the operators of deterministic meanings comprising its spectrum. If the number of states is finite and all operators of congruent messages are identified, then the set of deterministic operators is finite for a given recipient. Even if the initial state of the recipient is definitely known, a non-deterministic message can move it into various final states with probabilities summed up to a unit. In automata theory, the recipient of a random message is known as random or probabilistic automaton.

The proposed model of non-deterministic finite meanings is closely related to the theory of Markov processes. Indeed, the reception of identical non-deterministic messages is described by a homogeneous Markov chain, or simply a Markov chain. In general case, the reception of a series of different non-deterministic messages is described by an inhomogeneous Markov chain.

Matrix representation is standard for Markovian chains. The recipient state is described by a matrix column, whose entries are the probabilities of the recipient being in this state. These are non-negative numbers not exceeding 1. The sum of the elements in each column is 1. Matrix of non-deterministic meaning is a general stochastic matrix, that is, a square matrix whose non-negative entries do not exceed 1 and each column sums up to 1. Non-deterministic meanings can also be divided into reversible and irreversible. A non-deterministic finite meaning is reversible if its matrix is inversible and the inverse matrix is stochastic.

3.6 Infinite meaning

The number of the recipient states can be infinite: discrete or continuous. In this case, any received message acts on a state from an infinite set of states and, thus, has an infinite meaning. Various formalisms can be used to describe infinite meanings.

Meanings which form a semigroup can be described in terms of the algebraic automata theory. Arbitrary (not necessarily finite) semigroups conceptually correspond to generalized automata, which are less well studied than finite ones. If the set of recipient states is countable, one may use any of the formalisms known for countable mathematical objects. These may be algorithms, Turing machines, unlimited register machines, computable functions, recursive functions, Diophantine sets.

Suppose the recipient states comprise an enumerable set. This means that there is an algorithm that makes it possible to sequentially identify all the states and enumerate them accordingly. Enumerable sets can be viewed as ranges of definition and values of a computable function $f(n) = m$, where n and m are natural numbers. Transition function $f(n) = m$ transforms the set of numbers enumerating the initial states into the set of numbers enumerating the final states, both sets are enumerable. This is the general description of enumerable meaning. In matrix formalism, enumerable meaning is represented by an infinite matrix. Transition function $f(n) = m$ corresponds to a matrix where the nth element of the mth line is 1 and the rest elements in this line are zeros.

In the simplest case, the transition function is a computable (general recursive) function, which is defined for any natural number n. If there are numbers n for which the recipient states are non definable, the transition function may be defined only for the numbers of states, not for an arbitrary n. They are known as computable, or partially recursive functions.

The transition function may turn to be non computable. This occurs when the set of recipient states is unenumerable. In this case, there is no algorithm by which the matrix of meaning can be constructed. An illustrative example was proposed by Penrose in [14], where he examines human understanding of mathematics and argues why mentality cannot be reduced to an algorithm.

Finally, the set of recipient states may be continuous. Let the recipient state depends on n parameters which can take on values in a continuous range. Each state corresponds to a point in the n-dimensional parameter space. In general case, the probability of the recipient being in a certain state is described by a distribution function normalized to unity over the parameter space. Meaning emerges when the initial distribution function transforms into another one. An example of parameter space is the phase space of a conservative system in classical mechanics. In the case of continuous meaning, the recipient may receive a message either at discrete time instants or continuously. Such systems hold promise for applications and are studied in mathematical optimal control theory.

Further generalization of the recipient state space is obvious. The parameter space may be infinite-dimensional. Some of the parameters may take on continuous values, others may be countable or comprise a finite set.

It is important to note that modeling of infinite meanings is not a technical problem; the appropriate mathematical tools are available. The question is to what extent the infinite models are helpful to describe real systems. Even the most complicated, self-organizing systems seem to be characterizable by a discrete set of discernable states, at least in specific cases. If such a system is finite in space and time (and we agree that the world is discrete at the basic level), it can be only in a finite number of states. Such a system is algorithmic. However, if Roger Penrose is right and human understanding is principally not algorithmic, we have to admit that human mental state is governed not only by a finite system like the brain, or even the brain and body. The entire infinite Universe should be engaged somehow into the human thinking.

3.7 Quantum meaning

The formalism proposed in the previous section looks like that of quantum mechanics, where the system states are vectors in the infinite-dimensional Hilbert space and the transformations of states are given by operators acting on these vectors. Quantum mechanics can be viewed as a generalization of the proposed formalism for matrices with complex entries.

There are many possible ways of interpreting quantum mechanics. Our approach suggests one more interpretation in terms of some special quantum meaning.

4 Conclusion

In our opinion, the proposed approach enables one to move in the studies of meaning from humanitarian-type reasoning to mathematical modeling. The simplicity of the formulas used does not imply the simplicity of the model. The essential point of the model is how to specify the set of recipient states. For complex systems, this can be a tricky task. Another nontrivial problem is to define the concept of distance between states and the related notion of their proximity.

There are two main perspectives for further development of the proposed formalism. The first one is to tailor it to certain specific cases. The effectiveness of each model will strongly depend on the subject area and phenomenon under study. In some areas it may be ineffective. In some others, similar formalism accurate to notation has been used already. To our concern, decomposition of an arbitrary meaning into a product of elementary meanings holds much promise. Of special interest is the potential of this method in the humanities. Another intriguing opportunity is that algebraic automata theory and theory of algorithms can be tailored to nonlinear modeling in theoretical physics. Only future studies can reveal the potential of the proposed formalism for certain applications.

The second focus of attention is on getting further general results. The proposed formalism claims to embrace the widest possible application area, including all the humanities, and applies for being a general mathematical theory of meaning. We expect this framework to provide a general classification of meanings and formulation of general theorems for different classes of meanings.

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References

[1] J. Wheeler, Information, Physics, Quantum: The Search for Links, in Proceedings III International Symposium on Foundations of Quantum Mechanics, Tokyo (1989) 354-358.

[2] C. F. von Weizsäcker, Binary Alternatives and Space-Time Structure, in Quantum Theory and the Structures of Time and Space, Vol.2, Hauser, Mňhen, Wienna (1977) 86-112.

[3] A. A. Kharkevich, On the Value of Information, in Problems of Cybernetics, Issue 4, Fizmatgiz, Moscow, (1960) 53-58 [in Russian].

[4] M. M. Bongard, Pattern Recognition, Spartan Books, 1970.

[5] V. I. Korogodin, Information and Phenomenon of Information, Acad. Sci. USSR, Pushchino (1991) 22-24 [in Russian].

[6] C. Rovelli, Meaning = Information + Evolution. arXiv: 1611.02420.

[7] R. L. Stratonovich, Theory of Information and its Value, Springer Int. Publ., 2020.

[8] A. L. Krugly, Model of Meaning of a Message, 4th Conference on the Foundations of Fundamental Physics and Mathematics (FFPM 2020), Moscow, RUDN University (2020) 14-19 [in Russian].

[9] A. L. Krugly, The relational approach to the mathematical model of meaning of information // Metaphysics. - 2021, №3 (41), c. 70-91 [in Russian].

[10] ISO/IEC 2382:2015 Information technology.

[11] G. Bateson, Mind and Nature: A Necessary Unity (Advances in Systems Theory, Complexity, and the Human Sciences), Hampton Press, New Edition, 2002.

[12] E. Bach, Informal Lectures on Formal Semantics, Suny Series in Linguistics, State University of New York, 2nd ed. (1989).

[13] M. Arbib, Algebraic theory of machines, languages and semigroups, New York, Academic (1968).

[14] R. Penrose, Shadows of the Mind: A Search for the Missing Science of Consciousness, Oxford Univ. Press; Reprint ed. (1996).