
THE EXISTENCE OF HILBERT-PÓLYA OPERATOR

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ABSTRACT

Hilbert-Pólya conjecture is proved by constructing Hilbert-Pólya operator, the self-adjoint operator where its eigenvalues are exactly the imaginary parts of zeros of Riemann zeta function on the critical line. Hence, the Riemann hypothesis is true.

Keywords Hilbert-Pólya conjecture, Self-adjoint operator, Riemann zeta function, The Riemann hypothesis.

1 Introduction

At the crossroads of number theory and spectral theory lies one of the most creative ideas in modern mathematics: the Pólya-Hilbert Conjecture, an alternative approach to prove the Riemann hypothesis. Inspired by David Hilbert and George Pólya, this conjecture suggests that the imaginary parts of non-trivial zeros on the critical line of the Riemann zeta function can be taken as the eigenvalues of a self-adjoint operator known as Hilbert-Pólya operator, acting on some Hilbert space [1]. Therefore, offering a potential pathway to proving the long-standing and enigmatic Riemann Hypothesis, this is an unsolved problem in mathematics and much effort has been expended using various approaches to prove or disprove such conjecture.

In this paper, we prove Hilbert-Pólya conjecture by constructing Hilbert-Pólya operator, a self-adjoint operator denoted by L in $\mathcal{L}_2(\mathbb{R}) \times \mathcal{L}_2(\mathbb{R})$, where the eigenvalues are exactly the imaginary parts of zeros of Riemann zeta function on the critical line. As a starting point, we construct an operator denoted by H in $\mathcal{L}_2(\mathbb{R})$, we force the eigenvalues to be the positive imaginary parts of zeros of Riemann zeta function on the critical line by creating some kind of parameters called "Algorithmic parameter", such that we apply supersymmetric quantum mechanics for shape invariance potentials besides to invoking the notion of multivalued operator and selections, in order to establish the associated spectrum and the associated eigenfunctions of the operator H . The operator H is proved essentially self-adjoint hence the existence of a self-adjoint extension denoted by $\overline{H}$, that is used to construct the Hilbert-Pólya operator L which is a matrix of operators starting from $\overline{H}$. The paper is organized as follows: in section 2 we present the main tools to construct the eigenvalues and the eigenfunctions of the operator H , namely, the one dimensional supersymmetric quantum mechanics with its application for the case of shape invariance potentials, the notion of multivalued operator and selection. In section 3 we prove the existence of a self-adjoint operator denoted by H where the eigenvalues are exactly the positive imaginary parts of zeros of Riemann zeta function on the critical line. In section 4, we prove the existence of Hilbert-Pólya operator L , by constructing a matrix of operators starting from the extension of the essentially self-adjoint operator H , and we show that the eigenvalues of L coincide exactly with the imaginary parts of zeros of Riemann zeta function on the critical line.

2 Methods

2.1 The one dimensional supersymmetric quantum mechanics (SUSYQM)

We consider the following one dimensional Hamiltonian H_- [2, 3]

$$H_- = K^\dagger K = -\frac{d^2}{dx^2} + v_-(x), \quad (1)$$

in which the operators K and $K^\dagger$ are defined by

$$K = \frac{d}{dx} + W(x), \quad K^\dagger = -\frac{d}{dx} + W(x), \quad (2)$$

where W is a real function, it is known as the superpotential.

W generates the first partner potential v_- as

$$v_-(x) = W^2(x) - W'(x). \quad (3)$$

The supersymmetric partner H_+ of (1) is constructed by changing the order of the operators in (2) to have

$$H_+ = K K^\dagger = -\frac{d^2}{dx^2} + v_+(x). \quad (4)$$

W generates the second partner potential associated to H_+ as

$$v_+(x, s) = W^2(x) + W'(x). \quad (5)$$

Assuming the existence of an eigenvalue E_i and an eigenfunction ψ_i such that H_- satisfies [4]

$$H_- \psi_i = K^\dagger K \psi_i = E_i \psi_i, \quad (6)$$

clearly, we can have

$$H_+ K \psi_i = (K K^\dagger) K \psi_i = K (K^\dagger K) \psi_i = E_i K \psi_i, \quad (7)$$

From Eq. (7), by putting $\psi_i^+ = K \psi_i$, the pair (E_i, ψ_i^+) is an eigensolution to $H_+ \psi_i^+ = E_i \psi_i^+$.

On the other hand, if there is an eigenvalue E_j and an eigenfunction ψ_j in which we have

$$H_+ \psi_j = K K^\dagger \psi_j = E_j \psi_j. \quad (8)$$

Then, we obtain

$$H_- K^\dagger \psi_j = (K^\dagger K) K^\dagger \psi_j = K^\dagger (K K^\dagger) \psi_j = E_j K^\dagger \psi_j. \quad (9)$$

From Eq. (9), by putting $\psi_j^- = K^\dagger \psi_j$, the pair (E_j, ψ_j^-) is an eigensolution to $H_- \psi_j^- = E_j \psi_j^-$.

Therefore, the partner Hamiltonians H_- and H_+ share the same set of eigenvalues and eigenfunctions.

From (1), the solution of the equation $K\psi = 0$ is given explicitly by

$$\psi_0^-(x) = e^{-\int W(\tau) d\tau}, \quad (10)$$

as a result, the pair $(0, \psi_0^-)$ is always a solution to $H_- \psi_0^- = 0$ [2].

Briefly, all the spectrum of the Hamiltonian H_+ (resp. H_-) can be obtained by knowing all the pairs (E, ψ) that satisfy (6) (resp. (8)).

2.2 The One dimensional SUSYQM for shape invariance potentials

In this part, we show that using SUSYQM, we can construct explicitly a set of solutions (E, ψ) for a special class of potentials called Shape Invariance potentials, where the partner potentials have the same form but for different parameters [2] .

For a given Hamiltonian $H_-(s_0)$ where s_0 stands for a set of parameters. If the partner potential satisfy an integrability condition called Shape Invariance condition i.e. [5, 6, 7]

$$v_+(x, s_0) = v_-(x, s_1) + Q(s_0), \quad (11)$$

then, the original potential $v_-(\cdot, s_0)$ is called shape invariance potential.

The expression in (11) is equivalent to

$$H_+(s_0) = H_-(s_1) + Q(s_0), \quad (12)$$

where $Q(s_0)$ is a constant term.

From (10), the function

$$\psi(x) = e^{-\int W(\tau, s_1) d\tau}, \quad (13)$$

satisfies the following equation

$$H_-(s_1)\psi(x) = 0. \quad (14)$$

For (12) and (14) together, this equation is hold

$$H_+(s_0) e^{-\int W(\tau, s_1) d\tau} = Q(s_0) e^{-\int W(\tau, s_1) d\tau}. \quad (15)$$

Using Eqs. (8) and (9), we can have the function

$$\psi_1^-(x) = K^\dagger(x, s_0) e^{-\int W(\tau, s_1) d\tau}, \quad (16)$$

and the scalar $E_1^- = Q(s_0)$ a solution to

$$H_-(s_0)\psi_1^-(x) = E_1^- \psi_1^-(x). \quad (17)$$

Again, we construct the partner of $H_-(s_1)$ and if it satisfies the shape invariance condition (11) we write

$$H_+(s_1) = H_-(s_2) + Q(s_1). \quad (18)$$

According to (10), we have the equation

$$H_-(s_2)e^{-\int W(\tau, a_2) d\tau} = 0, \quad (19)$$

therefore, from Eqs. (18) and (19), we get

$$H_+(s_1)e^{-\int W(\tau, a_2) d\tau} = Q(s_1)e^{-\int W(\tau, a_2) d\tau}. \quad (20)$$

Applying Eqs. (8) and (9), gives the function

$$\psi_2^-(x) = K^\dagger(x, s_0)K^\dagger(x, s_1)e^{-\int W(\tau, a_2) d\tau}, \quad (21)$$

as a solution for

$$H_-(s_0)\psi_2^-(x) = E_2^- \psi_2^-(x), \quad (22)$$

where $E_2^- = Q(s_0) + Q(s_1)$.

By repeating the same process, we construct the n^{th} partner Hamiltonian for $n \geq 1$ in which all the partners satisfy the shape invariance condition (11) where the n^{th} Hamiltonian is written as

$$H_+(s_{n-1}) = H_-(s_n) + Q(s_{n-1}), \quad (23)$$

then, using the expression (10) we have

$$H_-(s_n)e^{-\int W(\tau, a_n) d\tau} = 0. \quad (24)$$

As a result, with the help of Eqs. (8) and (9), the pair (E_n, ψ_n^-) satisfies [5]

$$H_-(s_0)\psi_n^-(x) = E_n^- \psi_n^-(x), \quad (25)$$

such that

$$E_n^- = \sum_{k=1}^n Q(s_k), \quad (26)$$

$$\psi_n^-(x) = K^\dagger(x, s_0)K^\dagger(x, s_1) \cdots K^\dagger(x, s_{n-2})K^\dagger(x, s_{n-1})e^{-\int W(\tau, a_n) d\tau}. \quad (27)$$

2.3 Definition of multivalued operator

Definition 1 Let E be a Banach space and $\mathcal{P}(E)$ is the set of all subset of E .

We call a multivalued operator A the operator defined by

$$A = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{pmatrix} : E \rightarrow \mathcal{P}(E),$$

where $(A_i)_{i=1}^n$ are single-valued operators that associates to every element $x \in E$ a subset $F(x) \subset E$ of at most n element.

Definition 2 We call a domain of a multivalued operator A : $\text{Dom}(A)$, the set of elements $x \in E$ such that $F(x) \neq \emptyset$.

Definition 3 We call a selection a single-valued operator A_i where $i \in \{1, 2, \dots, n\}$ such that for an element x it associates a single image $A_i x$ where $A_i x \in F(x)$.

3 The existence of an operator with eigenvalues that are exactly the positive imaginary parts of zeros of Riemann zeta function on the critical line

Let be $(\eta_i)_{i \geq 1}$ the positive imaginary parts of Riemann zeta function zeros on the critical line where $0 < \eta_1 < \eta_2 < \dots < \eta_n < \dots$ [8].

We consider the following operator acting on $\mathcal{L}_2(\mathbb{R})$ as

$$H = H_0 + \eta_1 I, \quad (28)$$

such that the operator H_0 is defined as follows

$$H_0 = -\frac{d^2}{dx^2} + \mathcal{A}^2 - \mathcal{B} + (\mathcal{A}\mathcal{B} - \mathcal{A})x + \left(\frac{\mathcal{A}^2}{2} + \mathcal{B}^2 - 3\mathcal{C}\right)x^2 + (\mathcal{A}\mathcal{C} + \mathcal{A}\mathcal{B})x^3 + \left(\frac{\mathcal{A}^2}{4} + 2\mathcal{B}\mathcal{C}\right)x^4 + 2\mathcal{A}\mathcal{C}x^5 + \mathcal{C}^2x^6 \quad (29)$$

where $\mathcal{A}$ is a real number and $\mathcal{C}$ is a positive number, while $\mathcal{B}$ behaves as an algorithm inside the operator, where the outcome is a real number, it change its values according to the following algorithm

initialization the outcome is $\mathcal{B}$ and it is initialized to take a real value and $k = 1$.

if the sign in front of the parameter $\mathcal{A}$ of the coefficient of "x" which is $(\mathcal{A}\mathcal{B} - \mathcal{A})$ is minus (-), then, the value of the outcome of $\mathcal{B}$ and k are unchanged.

else

if the sign in front of the parameter $\mathcal{A}$ of the coefficient of "x" which is $(\mathcal{A}\mathcal{B} + \mathcal{A})$ is plus (+), then $\mathcal{B}$ updates its outcome according to the value of k , where

if $k=1$ then the following changes in the coefficients happen:

- The new outcome of the algorithm in the coefficient of x becomes $\mathcal{B} \leftarrow \eta_2 - \eta_1 - \mathcal{B}$, and k increases by +1 (i.e. $k \leftarrow k + 1$).
- The sign in front of the parameter $\mathcal{A}$ of the coefficient of "x" which is $(\mathcal{A}\mathcal{B} + \mathcal{A})$ is changed to (-).
- The outcome of the $\mathcal{B}$ that appears in the other coefficients of the non-constants term updates the outcome by the new outcome of $\mathcal{B}$ in the coefficient of x besides to increase k by +1 for this coefficients.

else

if $k \geq 2$ then the following changes in the coefficients happen:

- The new outcome of the algorithm in the coefficient of x becomes $\mathcal{B} \leftarrow \eta_{k+1} - \eta_1 - 2 \sum_{i=1}^{k-1} \mathcal{B}_i - \mathcal{B}_0$, where $\sum_{i=1}^{k-1} \mathcal{B}_i$ refers to the sum of the successive previous outcomes from 1 to $k - 1$ and $\mathcal{B}_0$ is the value of the first outcome . with k increases by +1 (i.e. $k \leftarrow k + 1$)
- The sign in front of the parameter $\mathcal{A}$ of the coefficient of "x" which is $(\mathcal{A}\mathcal{B} + \mathcal{A})$ is changed to (-).
- The outcome of $\mathcal{B}$ that appears in the other coefficients of the non-constant terms updates the outcome by the new outcome of $\mathcal{B}$ in the coefficient of x and besides to increase k by +1 for this coefficients.

then, according to the nature of the parameter $\mathcal{B}$, the outcomes are indexed by integers, such that we can write

$$H_0 = -\frac{d^2}{dx^2} + \mathcal{A}^2 - \mathcal{B}_0 + (\mathcal{A}\mathcal{B}_0 - \mathcal{A})x + \left(\frac{\mathcal{A}^2}{2} + \mathcal{B}_0^2 - 3\mathcal{C}\right)x^2 + (\mathcal{A}\mathcal{C} + \mathcal{A}\mathcal{B}_0)x^3 + \left(\frac{\mathcal{A}^2}{4} + 2\mathcal{B}_0\mathcal{C}\right)x^4 + 2\mathcal{A}\mathcal{C}x^5 + \mathcal{C}^2x^6, \quad (30)$$

such that $\mathcal{B}_0$ is the initialization.

3.1 The eigenvalues and the eigenfunctions of H_0

The aim of this part is to establish the the eigenvalues of H_0 with the associated eigenfunctions, by combining between supersymmetric quantum mechanics, the notion of multivalued operator and selection.

3.1.1 The first eigenpairs of H_0 :

Using the following first order operators that are the hermitian conjugate of each other

$$Y = \frac{d}{dx} + W(x), \quad Y^\dagger = -\frac{d}{dx} + W(x), \quad (31)$$

where

$$W(x) = \mathcal{A} + \mathcal{B}_0 x + \frac{\mathcal{A}}{2} x^2 + \mathcal{C} x^3, \quad (32)$$

we can write

$$H_0 = Y^\dagger(\mathcal{A}, \mathcal{B}_0, \mathcal{C}) Y(\mathcal{A}, \mathcal{B}_0, \mathcal{C}), \quad (33)$$

then, H_0 is a positive operator on his domain then its eigenvalues are positive, hence, the operator H is a definite positive operator and his eigenvalues are strictly positive, and since the function q defined below tends to $+\infty$ when $|x| \rightarrow +\infty$ then the spectrum of H_0 is purely discrete [9]

$$q(x) = \mathcal{A}^2 - \mathcal{B}_0 + (\mathcal{A}\mathcal{B}_0 - \mathcal{A})x + \left(\frac{\mathcal{A}^2}{2} + \mathcal{B}_0^2 - 3\mathcal{C}\right)x^2 + (\mathcal{A}\mathcal{C} + \mathcal{A}\mathcal{B}_0)x^3 + \left(\frac{\mathcal{A}^2}{4} + 2\mathcal{B}_0\mathcal{C}\right)x^4 + 2\mathcal{A}\mathcal{C}x^5 + \mathcal{C}^2x^6. \quad (34)$$

As a result from the expression (10), the following function

$$\psi_1(x) = e^{-\int W(\tau, \mathcal{A}, \mathcal{B}_0, \mathcal{C}) d\tau} = e^{-\mathcal{A}x - \frac{\mathcal{B}_0}{2}x^2 - \frac{\mathcal{A}}{6}x^3 - \frac{\mathcal{C}}{4}x^4} \in \mathcal{L}_2(\mathbb{R}), \quad (35)$$

is the first eigenfunction of H_0 which is associated to the first eigenvalue $E_1 = 0$, where $\mathcal{B}_0$ represents the initialization of the algorithm.

3.1.2 The second eigenpairs of H_0 :

To construct the second eigenfunction and eigenvalue, we alter the order of the operators $Y^\dagger(x, \mathcal{A}, \mathcal{B}_0, \mathcal{C})$ and $Y(x, \mathcal{A}, \mathcal{B}_0, \mathcal{C})$ in (33), and by taking into account the nature of $\mathcal{B}$, we get

$$Y(\mathcal{A}, \mathcal{B}_0, \mathcal{C}) Y^\dagger(\mathcal{A}, \mathcal{B}_0, \mathcal{C}) = -\frac{d^2}{dx^2} + \mathcal{A}^2 + \mathcal{B}_0 + (\mathcal{A}\mathcal{B}_1 - \mathcal{A})x + \left(\frac{\mathcal{A}^2}{2} + \mathcal{B}_1^2 + 3\mathcal{C}\right)x^2 + (\mathcal{A}\mathcal{C} + \mathcal{A}\mathcal{B}_1)x^3 + \left(\frac{\mathcal{A}^2}{4} + 2\mathcal{B}_1\mathcal{C}\right)x^4 + 2\mathcal{A}\mathcal{C}x^5 + \mathcal{C}^2x^6, \quad (36)$$

such that $\mathcal{B}_1 = \eta_2 - \eta_1 - \mathcal{B}_0$.

By comparing between (30) and (36) one can obtain that this supersymmetric partners are similar in their non-constant terms with some transformations that manifest, where the following transformations are appearing

$$\mathcal{B}_0 \rightarrow \mathcal{B}_1, \quad (37)$$

$$\mathcal{C} \rightarrow \mathcal{C}, \quad (38)$$

$$\mathcal{C} \rightarrow -\mathcal{C}. \quad (39)$$

The transformation (39) is rejected, hence to create the supersymmetric partner of (30) that respects the shape invariance condition, then, The supersymmetric partner (36) is set as multivalued operator with respects to translations (37) and (38) as follows

$$Y_{(\mathcal{A}, \mathcal{B}_0, \mathcal{C})} Y_{(\mathcal{A}, \mathcal{B}_0, \mathcal{C})}^\dagger = \begin{cases} \bullet -\frac{d^2}{dx^2} + \mathcal{A}^2 + \mathcal{B}_0 + (\mathcal{A}\mathcal{B}_1 - \mathcal{A})x + \left(\frac{\mathcal{A}^2}{2} + \mathcal{B}_1^2 + 3\mathcal{C}\right)x^2 + (\mathcal{A}\mathcal{C} + \mathcal{A}\mathcal{B}_1)x^3 + \left(\frac{\mathcal{A}^2}{4} + 2\mathcal{B}_1\mathcal{C}\right)x^4 + 2\mathcal{A}\mathcal{C}x^5 + \mathcal{C}^2x^6, \\ \bullet Y_{(\mathcal{A}, \mathcal{B}_1, \mathcal{C})}^\dagger Y_{(\mathcal{A}, \mathcal{B}_1, \mathcal{C})} + \mathcal{B}_1 + \mathcal{B}_0. \end{cases} \quad (40)$$

At this stage to establish the second eigenfunction associated to H_0 , we make a selection, where we select the factorized operator with the shape invariance condition (12) : $Y_{(\mathcal{A}, \mathcal{B}_1, \mathcal{C})}^\dagger Y_{(\mathcal{A}, \mathcal{B}_1, \mathcal{C})} + \mathcal{B}_1 + \mathcal{B}_0$ from (40), where

$$Q_1 = \mathcal{B}_1 + \mathcal{B}_0 = \eta_2 - \eta_1, \quad (41)$$

then, according to supersymmetric quantum mechanics and the formulas (26) and (27), the expression of the second eigenfunction of H_0 is given by

$$\psi_2(x) = Y_{(\mathcal{A}, \mathcal{B}_0, \mathcal{C})}^\dagger e^{-\int W(\tau, \mathcal{A}, \mathcal{B}_1, \mathcal{C}) d\tau}, \quad (42)$$

for the second eigenvalue $E_2 = Q_1 = \eta_2 - \eta_1$, and after some calculations, we get

$$\psi_2(x) = \left(2\mathcal{A} + (\mathcal{B}_0 + \mathcal{B}_1)x + \mathcal{A}x^2 + 2\mathcal{C}x^3\right) e^{-\mathcal{A}x - \frac{\mathcal{B}_1}{2}x^2 - \frac{\mathcal{A}}{6}x^3 - \frac{\mathcal{C}}{4}x^4} \in \mathcal{L}_2(\mathbb{R}). \quad (43)$$

3.1.3 The third eigenpairs of H_0 :

Here, to establish the third eigenfunction associated to H_0 , we construct the supersymmetric partner of the selected factorized operator with the shape invariance condition (12) : $Y_{(\mathcal{A}, \mathcal{B}_1, \mathcal{C})}^\dagger Y_{(\mathcal{A}, \mathcal{B}_1, \mathcal{C})} + \mathcal{B}_1 + \mathcal{B}_0$ from the multivalued operator (40), so, by altering the order of the operators $Y_{(\mathcal{A}, \mathcal{B}_1, \mathcal{C})}^\dagger$ and $Y_{(\mathcal{A}, \mathcal{B}_1, \mathcal{C})}$ we get

$$Y_{(\mathcal{A}, \mathcal{B}_1, \mathcal{C})} Y_{(\mathcal{A}, \mathcal{B}_1, \mathcal{C})}^\dagger = -\frac{d^2}{dx^2} + \mathcal{A}^2 + \mathcal{B}_1 + (\mathcal{A}\mathcal{B}_2 - \mathcal{A})x + \left(\frac{\mathcal{A}^2}{2} + \mathcal{B}_2^2 + 3\mathcal{C}\right)x^2 + (\mathcal{A}\mathcal{C} + \mathcal{A}\mathcal{B}_2)x^3 + \left(\frac{\mathcal{A}^2}{4} + 2\mathcal{B}_2\mathcal{C}\right)x^4 + 2\mathcal{A}\mathcal{C}x^5 + \mathcal{C}^2x^6, \quad (44)$$

such that $\mathcal{B}_2 = \eta_3 - \eta_1 - 2\mathcal{B}_1 - \mathcal{B}_0$.

Another time, one can obtain this transformations that manifest between $Y_{(\mathcal{A}, \mathcal{B}_1, \mathcal{C})} Y_{(\mathcal{A}, \mathcal{B}_1, \mathcal{C})}^\dagger$ and $Y_{(\mathcal{A}, \mathcal{B}_1, \mathcal{C})}^\dagger Y_{(\mathcal{A}, \mathcal{B}_1, \mathcal{C})}$ as follows

$$\mathcal{B}_1 \rightarrow \mathcal{B}_2, \quad (45)$$

$$\mathcal{C} \rightarrow \mathcal{C}, \quad (46)$$

$$\mathcal{C} \rightarrow -\mathcal{C}. \quad (47)$$

The transformation (47) is rejected. Hence, to create the supersymmetric partner that satisfies the shape invariance condition, the operator (44) is set as a multivalued operator that respects the transformations (45) and (46) as follows

$$Y_{(\mathcal{A}, \mathcal{B}_1, \mathcal{C})} Y_{(\mathcal{A}, \mathcal{B}_1, \mathcal{C})}^\dagger = \begin{cases} \bullet -\frac{d^2}{dx^2} + \mathcal{A}^2 + \mathcal{B}_1 + (\mathcal{A}\mathcal{B}_2 - \mathcal{A})x + \left(\frac{\mathcal{A}^2}{2} + \mathcal{B}_2^2 + 3\mathcal{C}\right)x^2 + (\mathcal{A}\mathcal{C} + \mathcal{A}\mathcal{B}_2)x^3 + \left(\frac{\mathcal{A}^2}{4} + 2\mathcal{B}_2\mathcal{C}\right)x^4 + 2\mathcal{A}\mathcal{C}x^5 + \mathcal{C}^2x^6, \\ \bullet Y_{(\mathcal{A}, \mathcal{B}_2, \mathcal{C})}^\dagger Y_{(\mathcal{A}, \mathcal{B}_2, \mathcal{C})} + \mathcal{B}_2 + \mathcal{B}_1. \end{cases} \quad (48)$$

From (12) and (48) we have

$$Q_2 = \mathcal{B}_2 + \mathcal{B}_1. \quad (49)$$

To establish the eigenfunction, we make selections, by selecting the factorized operators from (40) and from (48) that satisfy the shape invariance condition, then, according to supersymmetric quantum mechanics and the expression (27) the third eigenfunction of H_0 is given by

$$\psi_3(x) = Y_{(\mathcal{A}, \mathcal{B}_0, \mathcal{C})}^\dagger Y_{(\mathcal{A}, \mathcal{B}_1, \mathcal{C})}^\dagger e^{-\int W(\tau, \mathcal{A}, \mathcal{B}_2, \mathcal{C}) d\tau}, \quad (50)$$

then, adding (41) to (49), according to the expression (26), we get the third eigenvalue of H_0 as follows

$$E_3 = \eta_3 - \eta_1, \quad (51)$$

and after calculations, we get

$$\begin{aligned} \psi_3(x) = & \left(2\mathcal{A}^2 + \mathcal{A} - \mathcal{B}_1 - \mathcal{B}_2 + (\mathcal{A}(\mathcal{B}_1 + 3\mathcal{B}_2 - 2) + \mathcal{B}_0)x + (\mathcal{B}_2(\mathcal{B}_1 + \mathcal{B}_2) \right. \\ & + \mathcal{A}\left(2\mathcal{A} + \frac{1}{2}\right) - 6\mathcal{C})x^2 + \left(\frac{\mathcal{A}}{2}(\mathcal{B}_1 + 3\mathcal{B}_2) + \mathcal{C}(4\mathcal{A} + 1)\right)x^3 + \left(\frac{\mathcal{A}^2}{2} + \right. \\ & \left. \mathcal{C}(\mathcal{B}_1 + 3\mathcal{B}_2)\right)x^4 + 2\mathcal{A}\mathcal{C}x^5 + 2\mathcal{C}^2x^6 \Big) e^{-\mathcal{A}x - \frac{\mathcal{B}_2}{2}x^2 - \frac{\mathcal{A}}{6}x^3 - \frac{\mathcal{C}}{4}x^4} \in \mathcal{L}_2(\mathbb{R}). \end{aligned} \quad (52)$$

3.1.4 The n^{th} eigenpair of H_0 for $n \geq 4$:

By keeping repeating the same procedure of constructing a succession of multivalued operators, by altering the order of the factorized operators that satisfy the shape invariance condition to get the supersymmetric partner, and to have the shape invariance condition (11) from the new supersymmetric partner, this last is set as a multivalued operator. After that, selections are made to establish the eigenfunction, through selecting the factorized operators that satisfy the shape

invariance condition from this succession of multivalued operators, such that for the $(n-1)^{th}$ time and according to the nature of $\mathcal{B}$, we get the $(n-1)^{th}$ supersymmetric partner of H_0 as

$$Y_{(\mathcal{A}, \mathcal{B}_{n-2}, \mathcal{C})} Y_{(\mathcal{A}, \mathcal{B}_{n-2}, \mathcal{C})}^\dagger = \begin{cases} \bullet -\frac{d^2}{dx^2} + \mathcal{A}^2 + \mathcal{B}_{n-2} + (\mathcal{A}\mathcal{B}_{n-1} - \mathcal{A})x + \left(\frac{\mathcal{A}^2}{2} + \mathcal{B}_{n-1}^2 + 3\mathcal{C}\right) \\ x^2 + (\mathcal{A}\mathcal{C} + \mathcal{A}\mathcal{B}_{n-1})x^3 + \left(\frac{\mathcal{A}^2}{4} + 2\mathcal{B}_{n-1}\mathcal{C}\right)x^4 + 2\mathcal{A}\mathcal{C}x^5 + \mathcal{C}^2x^6, \\ \bullet Y_{(\mathcal{A}, \mathcal{B}_{n-1}, \mathcal{C})}^\dagger Y_{(\mathcal{A}, \mathcal{B}_{n-1}, \mathcal{C})} + \mathcal{B}_{n-1} + \mathcal{B}_{n-2}. \end{cases} \quad (53)$$

such that

$$\begin{aligned} \mathcal{B}_{n-2} &= \eta_{n-1} - \eta_1 - 2 \sum_{k=1}^{n-3} \mathcal{B}_k - \mathcal{B}_0, \\ \mathcal{B}_{n-1} &= \eta_n - \eta_1 - 2 \sum_{k=1}^{n-2} \mathcal{B}_k - \mathcal{B}_0, \end{aligned}$$

and from (12) with (53) we have

$$Q_{n-1} = \mathcal{B}_{n-1} + \mathcal{B}_{n-2}, \quad (54)$$

and this translations are going to appear

$$\mathcal{B}_{n-2} \rightarrow \mathcal{B}_{n-1}, \quad (55)$$

$$\mathcal{C} \rightarrow \mathcal{C}, \quad (56)$$

$$\mathcal{C} \rightarrow -\mathcal{C}, \quad (57)$$

such that the transformation (57) is always rejected.

To establish the n^{th} eigenfunction of H_0 , like previously, we make selections, such that we select the factorized operators from every partner that is set already as a multivalued operator, then according to (27) the expression of the eigenfunction is given by

$$\psi_n(x) = Y_{(\mathcal{A}, \mathcal{B}_0, \mathcal{C})}^\dagger Y_{(\mathcal{A}, \mathcal{B}_1, \mathcal{C})}^\dagger \cdots Y_{(\mathcal{A}, \mathcal{B}_{n-3}, \mathcal{C})}^\dagger Y_{(\mathcal{A}, \mathcal{B}_{n-2}, \mathcal{C})}^\dagger e^{-\int W(\tau, \mathcal{A}, \mathcal{B}_{n-1}, \mathcal{C}) d\tau}, \quad (58)$$

where

$$e^{-\int W(\tau, \mathcal{A}, \mathcal{B}_{n-1}, \mathcal{C}) d\tau} = e^{-\mathcal{A}x - \frac{\mathcal{B}_{n-1}}{2}x^2 - \frac{\mathcal{A}}{6}x^3 - \frac{\mathcal{C}}{4}x^4} \quad (59)$$

and

$$Y^\dagger = -\frac{d}{dx} + \mathcal{A} + \mathcal{B}_i x + \frac{\mathcal{A}}{2}x^2 + \mathcal{C}x^3, \quad i = 0, 1, 2, \dots, n-2 \quad (60)$$

such that $\mathcal{B}_0$ represents the initialization, with

$$\mathcal{B}_1 = \eta_2 - \eta_1 - \mathcal{B}_0, \quad (61)$$

$$\mathcal{B}_i = \eta_{i+1} - \eta_1 - 2 \sum_{k=1}^{i-1} \mathcal{B}_k - \mathcal{B}_0, \quad i = 2, 3, \dots, n-1. \quad (62)$$

The expression (58) will produce a polynomial multiplied by the term $e^{-\mathcal{A}x - \frac{\mathcal{B}_{n-1}}{2}x^2 - \frac{\mathcal{A}}{6}x^3 - \frac{\mathcal{C}}{4}x^4}$, then the function $\psi_n \in \mathcal{L}_2(\mathbb{R})$.

According to the formula (26), the n^{th} eigenvalue of H_0 is given by

$$E_n = \sum_{k=1}^n Q_k = \eta_n - \eta_1. \quad (63)$$

Finally, the eigenvalues of the operator H_0 are given by

$$E_i = \eta_i - \eta_1, \quad i = 4, 5, 6, \dots \quad (64)$$

with the associated eigenfunctions given in (58).

As a result, the operator H given in (28) and the operator H_0 share the same set of eigenfunctions, and the spectrum of the operator H is a translation of the spectrum of H_0 by $+\eta_1$, hence, the eigenvalues of H are given explicitly by

$$E'_i = \eta_i, \quad i = 1, 2, 3, \dots \quad (65)$$

such that η_i represents the i^{th} positive imaginary parts of zeta function zeros on the critical line while the associated eigenfunctions are given in (35), (43), (52) and (58).

4 The existence of Hilbert-Pólya operator

4.1 The essentially self-adjointness of H

In this part we establish that the operator H is essentially self-adjoint on the space $C_0^\infty(\mathbb{R})$.

We denote the domain of H by $\text{Dom}(H)$, we have also that

$$C_0^\infty(\mathbb{R}) \subset \text{Dom}(H) \subset \mathcal{L}_2(\mathbb{R}) \quad (66)$$

such that

$$\overline{C_0^\infty(\mathbb{R})} = \mathcal{L}_2(\mathbb{R}), \quad (67)$$

besides to the fact that potential function is a polynomial q defined in (34) is a polynomial of degree 6 which is bounded from below, then, the operator H is essentially self-adjoint [10], therefore it exist only one self-adjoint extension of H [11] which is its closure that we denote by $\overline{H}$ where it is equal to the adjoint of H .

Since the operators in (31) are the hermitian conjugate of each other, then the expression of H and $\overline{H}$ is the same, then, depending on the formulation expanded in section 2 from the references [2, 3], it is a natural result that applying supersymmetric quantum for shape invariance potentials will produce the whole discrete spectrum with the associated eigenfunctions, i.e. the discrete spectrum is fully determined through supersymmetric quantum mechanics for shape invariance potentials and since the set of functions $(\psi_n)_{n \geq 0}$ are in $\mathcal{L}_2(\mathbb{R})$, so that the eigenspace of each eigenvalue can not be empty, as a result, the operators H and $\overline{H}$ will surely share the same eigenvalues i.e. the discrete spectrum of both of them coincide.

4.2 The construction of an operator L , a matrix of operators

Let consider the following operator

$$L : \mathcal{L}_2(\mathbb{R}) \times \mathcal{L}_2(\mathbb{R}) \rightarrow \mathcal{L}_2(\mathbb{R}) \times \mathcal{L}_2(\mathbb{R}), \quad (68)$$

$$L = \begin{pmatrix} \overline{H} & 0 \\ 0 & -\overline{H} \end{pmatrix} \quad (69)$$

such that for every element

$$\begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{L}_2(\mathbb{R}) \times \mathcal{L}_2(\mathbb{R}), \quad (70)$$

we have that

$$L \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \overline{H}f \\ -\overline{H}g \end{pmatrix}. \quad (71)$$

Since $\mathcal{L}_2(\mathbb{R})$ is a Hilbert space, then, the space $\mathcal{L}_2(\mathbb{R}) \times \mathcal{L}_2(\mathbb{R})$ is a Hilbert space and the inner product is defined for every two elements $\begin{pmatrix} f1 \\ g1 \end{pmatrix}$ and $\begin{pmatrix} f2 \\ g2 \end{pmatrix}$ as

$$\left\langle \begin{pmatrix} f1 \\ g1 \end{pmatrix}, \begin{pmatrix} f2 \\ g2 \end{pmatrix} \right\rangle_{\mathcal{L}_2(\mathbb{R}) \times \mathcal{L}_2(\mathbb{R})} = \langle f1, f2 \rangle_{\mathcal{L}_2(\mathbb{R})} + \langle g1, g2 \rangle_{\mathcal{L}_2(\mathbb{R})} \quad (72)$$

The operator $-\overline{H}$ is a self-adjoint operator on $\mathcal{L}_2(\mathbb{R})$ since $\overline{H}$ is a self-adjoint on $\mathcal{L}_2(\mathbb{R})$, as a result the operator L is also a self-adjoint operator on $\mathcal{L}_2(\mathbb{R}) \times \mathcal{L}_2(\mathbb{R})$.

4.3 The eigenpairs of L

According to part (3.1.1), then, the operator $\overline{H}$ is definite positive, hence the operator $-\overline{H}$ is definite negative at the other side, as a result, the spectrum of $\overline{H}$ is strictly positive while the spectrum of $-\overline{H}$ is strictly negative, so that they do not share an eigenvalue, besides to this, the value 0 can not be an eigenvalue for one of them.

Let η_n be an eigenvalue of $\overline{H}$ with the eigenfunction ψ_n from section 3, so one can write for $n \in \mathbb{N}^*$

$$\overline{H}\psi_n = \eta_n\psi_n, \quad (73)$$

then, from (73) one can have that

$$-\overline{H}\psi_n = -\eta_n\psi_n, \quad (74)$$

therefore, the spectrum of $-\overline{H}$ can be deduced starting from the spectrum of $\overline{H}$, such that for every eigenvalue η_n of $\overline{H}$, then $-\eta_n$ is an eigenvalue for $-\overline{H}$ where they share the same eigenfunction ψ_n .

Since the operators $\overline{H}$ and $-\overline{H}$ do not share an eigenvalue, then there is no real value λ and a element $\begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{L}_2(\mathbb{R}) \times \mathcal{L}_2(\mathbb{R})$ where $\begin{pmatrix} f \\ g \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ almost everywhere that verifies

$$L \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \overline{H}f \\ -\overline{H}g \end{pmatrix} = \lambda \begin{pmatrix} f \\ g \end{pmatrix} \quad (75)$$

hence the eigenpairs of L must be written as

$$L \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} \overline{H}f \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} f \\ 0 \end{pmatrix} \quad (76)$$

or

$$L \begin{pmatrix} 0 \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ -\overline{H}g \end{pmatrix} = -\lambda \begin{pmatrix} 0 \\ g \end{pmatrix}, \quad (77)$$

where λ represents an eigenvalue of the operator $\overline{H}$ and f is the associated eigenfunction.

By returning then identification with section 3, the eigenpairs of the operator L are the following

$$L \begin{pmatrix} \psi_n \\ 0 \end{pmatrix} = \begin{pmatrix} \overline{H}\psi_n \\ 0 \end{pmatrix} = \eta_n \begin{pmatrix} \psi_n \\ 0 \end{pmatrix} \quad (78)$$

and

$$L \begin{pmatrix} 0 \\ \psi_n \end{pmatrix} = \begin{pmatrix} 0 \\ -\overline{H}\psi_n \end{pmatrix} = -\eta_n \begin{pmatrix} 0 \\ \psi_n \end{pmatrix}, \quad (79)$$

for $n \in \mathbb{N}^*$ where the set $(\eta_n)_{n \geq 1}$ represents the positive imaginary parts of zeta function zeros on the critical line and the functions ψ_n are given in section (3). Hence, the set $\{(\eta_n)_{n \geq 1} \cup (-\eta_n)_{n \geq 1}\}$ which is the set of imaginary parts of the zeros of zeta function on the critical line, represents the eigenvalues of the operator L .

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