

Cohomological Detection of Good Reduction on Tame Moduli Stacks: Nearby Cycles, $N = 0$, and a Stacky Néron–Ogg–Shafarevich Criterion

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Abstract

Assuming purity and tame monodromy of nearby cycles on algebraic stacks and the existence of smooth local slice models of abelian or K3 type, we establish a cohomological criterion for detecting good reduction in families parametrized by tame moduli stacks. Under these hypotheses, trivial monodromy on ℓ -adic cohomology implies the integral extendability of points with finite stabilizers, generalizing the Néron–Ogg–Shafarevich criterion from abelian varieties to stacks with linearly reductive inertia. The theory yields density results for good-reduction loci, clarifies the role of weight–monodromy and nearby-cycle purity in extension problems, and provides explicit examples and counterexamples illustrating the necessity of the hypotheses. Applications include elliptic curves with level structure, abelian surfaces with complex multiplication, and tame quotient stacks $[\mathbb{A}^1/\mu_n]$ with $(n, p) = 1$, linking cohomological purity to arithmetic integrality and moduli-theoretic extension properties.

Keywords. *good reduction, moduli stacks, tame stabilizers, nearby cycles, vanishing cycles, weight–monodromy, monodromy $N = 0$, stacky Néron mapping property, slice method (abelian/K3), Dirichlet density, Shimura/adelic compatibility, L -factors.*

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Contents

1 Introduction	2
2 Background and Preliminaries	4
2.1 Good Reduction of Arithmetic Schemes	4
2.2 Moduli Stacks: Definitions and Basic Properties	5
2.3 Cohomological Invariants and Their Arithmetic Interpretation	7

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3	Foundational Constructions	12
3.1	Moduli-Theoretic Encoding of Good Reduction	12
3.2	Cohomology of Associated Stacks	22
3.3	Comparison with Classical Néron Models	25
4	Main Results	27
4.1	Theorem Statements	28
4.2	Further Structural Results	32
5	Applications	33
5.1	Cohomological Interpretation of Good Reduction	33
5.2	Connections with Modular Curves and Shimura Varieties	33
5.3	Arithmetic Schemes over Local and Global Fields	34
6	Explicit Examples and Computations	35
6.1	Case Study: Elliptic Curves with Good Reduction	35
6.2	Higher-Dimensional Abelian Varieties	37
6.3	Counterexamples Illustrating Necessity of Hypotheses	40
7	Global Perspective	41
7.1	From Cohomology to Moduli	42
7.2	From Local to Global	42
7.3	Interplay Between Good Reduction and Arithmetic Invariants	44
8	Conclusion and Outlook	45

1 Introduction

The notion of *good reduction* has long served as a central invariant in arithmetic geometry, governing the behaviour of varieties and their cohomology under specialization. Throughout this work, all main results are obtained *under Assumption 3.13 (purity of nearby cycles on stacks)* together with the *slice hypothesis* (Assumption 3.12); see Theorem 3.16 for the full detection statement. Classically, one studies smooth proper schemes X/K over a discretely valued field and asks whether X extends to a smooth proper model over $\mathcal{O}_K$ (Definition 2.1). For abelian varieties, the Néron–Ogg–Shafarevich criterion (Lemma 2.3) provides a cohomological test: trivial inertia action on ℓ -adic cohomology is equivalent to the existence of good reduction. This illustrates the guiding theme that geometric and cohomological criteria are deeply intertwined.

The traditional perspective, however, is limited to schemes. In the presence of automorphisms or families parameterized by moduli problems, one is naturally led to stacks. Algebraic stacks introduce subtleties: flatness and smoothness may fail globally (Counterexample 2.8), stabilizers influence extension problems, and cohomological descent requires refinement (Lemma 2.9 and proposition 2.14). A systematic framework for detecting good reduction inside moduli stacks has so far remained incomplete.

Theorem 1.1 (Main detection theorem). *Let K be a discretely valued field with ring of integers $\mathcal{O}_K$ and residue characteristic $p \geq 0$. Let $\mathcal{M}$ be a tame algebraic stack of finite type over $\mathcal{O}_K$ with affine diagonal and finite linearly reductive stabilizers. Assume:*

- (i) (**Purity**) *Assumption 3.13 holds for $\mathcal{M}$, i.e. for every $\ell \neq p$ the nearby-cycles complex $R\Psi(\mathbb{Q}_\ell)$ on $\mathcal{M}_s$ is pure of weight 0 (contrast the wild failure in Example 6.12);*
- (ii) (**Cohomological inertia**) *for all i , the inertia group I_K acts unipotently with trivial monodromy $N = 0$ on $H_{\text{ét}}^i(\mathcal{M}_K, \mathbb{Q}_\ell)$ for $\ell \neq p$;*
- (iii) (**Slice**) *Assumption 3.12 holds at the given point after possibly a finite unramified extension of K .*

Then every K -point $x_\eta: \text{Spec } K \rightarrow \mathcal{M}_\eta$ with finite stabilizer extends, after a finite unramified extension of K , to a morphism $\text{Spec } \mathcal{O}_K \rightarrow \mathcal{M}$.

Remark 1.2. A detailed version and proof are given in Theorem 3.16; the abbreviated statement above is for convenient reference.

New perspective.

In this work we extend the philosophy of Néron models and cohomological detection to moduli stacks with tame, linearly reductive stabilizers. Our central innovation is a moduli-theoretic detection theorem (Theorem 3.16 and remark 3.17): under the Assumption 3.13, the purity of nearby cycles together with trivial monodromy ($N = 0$) ensures that every K -point with finite stabilizer extends integrally. This generalizes the classical Néron–Ogg–Shafarevich criterion from abelian varieties to stacks, and establishes a bridge between weight–monodromy phenomena and extension properties (Proposition 3.6, definition 3.29, and theorem 3.31).

Outline of main results. The paper is organized around a recurring pattern: *Theorem $\Rightarrow$ Consequence $\Rightarrow$ Example $\Rightarrow$ Counterexample.*

- In Section 3 we set up moduli-theoretic encodings (Section 3.1, definition 3.1, and lemma 3.3) and compare with classical Néron models (Section 3.3).
- In Section 4 we establish density results (Theorem 4.2 and remark 4.3), a local vanishing-cycles criterion (Theorem 4.9 and remark 4.10), and structural openness (Proposition 4.15), illustrated by CM abelian varieties (Example 4.7) and toric quotients (Example 4.17), together with counterexamples showing necessity of linear reductivity (Counterexamples 4.8 and 4.18).
- In Section 5 we connect to cohomological invariants (Definition 5.1, theorem 5.2, and remark 5.3), modular and Shimura varieties (Proposition 5.6 and corollary 5.7), and adelic local–global compatibility (Theorem 5.11 and remark 5.12).
- In Section 6 explicit computations confirm the theory: elliptic curves with integral j (Proposition 6.1 and example 6.2), ordinary abelian varieties (Proposition 6.6 and example 6.7), and counterexamples from wild ramification and non-separated diagonals (Counterexamples 6.5, 6.11 and 6.14).
- Finally, Section 7 formulates bridges to motivic and adelic contexts: functoriality of cohomological footprints (Proposition 7.2), global L -functions (Theorem 7.9 and remark 7.11), and invariant alignment

criteria ([Theorem 7.15](#) and [remark 7.18](#)). The conclusion ([Section 8](#) and [construction 8.1](#)) synthesizes these into a unified “schematic footprint” principle.

This roadmap positions the paper as a synthesis: classical good reduction criteria, stack-theoretic extension properties, and cohomological purity are shown to be facets of a single arithmetic principle. (See [Theorem 3.16](#) for a numbered summary of the main detection result.)

Novelty Statement. *New:* a moduli-theoretic, **stack-level** NOS-type detection criterion ([Theorem 3.16](#)) under [Assumption 3.13](#) (purity) and the slice hypothesis ([Assumption 3.12](#)); together with a density theorem ([Theorem 4.2](#)) derived via tame slices and Chebotarev, and explicit counterexamples demonstrating the necessity of linear-reductivity, separated diagonal, and purity.

2 Background and Preliminaries

Throughout this section we fix a discretely valued field K with ring of integers $\mathcal{O}_K$, uniformizer π , and residue field k of characteristic $p \geq 0$. We write $\eta = \operatorname{Spec} K$ and $s = \operatorname{Spec} k$. All schemes and stacks are assumed locally of finite type over $\mathcal{O}_K$ unless otherwise specified. Notation and standing conventions introduced here will remain in force for the remainder of the paper.

2.1 Good Reduction of Arithmetic Schemes

Definition 2.1 (Good reduction). Let X be a smooth, proper scheme over K . We say that X has *good reduction* over $\mathcal{O}_K$ if there exists a smooth, proper scheme $\mathcal{X}$ over $\mathcal{O}_K$ such that $\mathcal{X}_\eta \cong X$.

Remark 2.2 (Necessity of properness). Properness in [Definition 2.1](#) ensures that extension of morphisms across the special fiber is governed by valuative criteria. For non-proper families one must instead work with Néron models, cf. [\[3\]](#).

Lemma 2.3 (Néron–Ogg–Shafarevich criterion for abelian varieties). *Let K be a discretely valued field with ring of integers $\mathcal{O}_K$, residue characteristic $p \geq 0$, and separable closure $\overline{K}$. Let A/K be an abelian variety. Then A has good reduction over $\mathcal{O}_K$ if and only if, for some (equivalently, for all) primes $\ell \neq p$, the I_K -action on $H_{\text{ét}}^1(A_{\overline{K}}, \mathbf{Q}_\ell)$ is trivial.*

Proof. ($\Rightarrow$) If A extends to an abelian scheme $\mathcal{A}/\mathcal{O}_K$ with smooth special fiber, smooth proper base change gives canonical isomorphisms

$$H_{\text{ét}}^1(A_{\overline{K}}, \mathbf{Q}_\ell) \simeq H_{\text{ét}}^1(\mathcal{A}_{\overline{s}}, \mathbf{Q}_\ell),$$

functorial in the trait; the right side carries trivial I_K -action, hence so does the left.

($\Leftarrow$) Assume I_K acts trivially on $H_{\text{ét}}^1(A_{\overline{K}}, \mathbf{Q}_\ell)$ for some $\ell \neq p$. Equivalently, the ℓ -adic Tate module $T_\ell(A)$ is unramified. Let $\mathcal{N}/\mathcal{O}_K$ be the Néron model of A . The connected special fiber $\mathcal{N}_s^0$ fits into a Chevalley extension

$$0 \longrightarrow T \longrightarrow \mathcal{N}_s^0 \longrightarrow B \longrightarrow 0,$$

with T a torus and B an abelian variety. By the monodromy–weight description of I_K on $T_\ell(A)$ (or, concretely, Raynaud’s criterion in the Néron model), nontrivial toric rank forces nontrivial unipotent

monodromy on $T_\ell(A)$; since the I_K -action is trivial by hypothesis, the toric rank is zero, hence $T = 0$ and $\mathcal{N}_s^0$ is abelian. Therefore $\mathcal{N}$ is an abelian scheme, so A has good reduction.

Finally, the parenthetical “equivalently all ℓ ” follows because good reduction is an ℓ -independent property: if it holds for one ℓ , it holds for every $\ell \neq p$ by the first implication above. $\square$

$$\begin{array}{ccc}
& H_{\text{ét}}^1(A_{\overline{K}}, \mathbf{Q}_\ell) & \\
\text{monodromy } N \swarrow & & \searrow \text{sp} \\
0 & \xrightarrow{(\text{= } N = 0 \text{ and trivial } I_K)} & H^1(R\Psi(\mathbf{Q}_\ell)) \\
& & \downarrow \text{smooth base change} \\
& & H_{\text{ét}}^1(\mathcal{A}_{\overline{s}}, \mathbf{Q}_\ell)
\end{array}$$

Figure 1: Néron–Ogg–Shafarevich bridge: trivial I_K on $H^1 \iff$ good reduction of A .

Notation 2.4 (Galois inertia vs. stack inertia). Let $G_K = \text{Gal}(\overline{K}/K)$ with inertia subgroup $I_K \subset G_K$. For an algebraic stack M over $\text{Spec } \mathcal{O}_K$ with finite inertia, write

$$\mathcal{I}_M := M \times_{M \times_{\mathcal{O}_K} M} M$$

for the inertia stack and, for a geometric point $x \in M(\overline{K})$, denote the stabilizer group scheme by $G_x := \text{Aut}_M(x)$ over $\overline{K}$.

For $\ell \neq p$, we distinguish two actions on ℓ -adic cohomology:

- (*Galois inertia*) the natural I_K -action on $H_{\text{ét}}^i(M_{\overline{K}}, \mathbf{Q}_\ell)$ induced functorially by base change along $\text{Spec } \overline{K} \rightarrow \text{Spec } K$ (SGA7 II, Exp. IX).
- (*Stack inertia*) the action coming from stabilizers: the constant groups $G_x(\overline{K})$ act on the cohomology computed via a simplicial presentation of M ; see [8] for the six-operations formalism.

These are a priori distinct. After a finite *unramified* extension of K that makes all residual stabilizers finite *étale and constant*, the I_K -action and the stack-inertia action *commute* on $H_{\text{ét}}^i(M_{\overline{K}}, \mathbf{Q}_\ell)$ (Lemma 2.11). Throughout, “ I_K acts on $H_{\text{ét}}^i(M_K, \mathbf{Q}_\ell)$ ” refers to *Galois* inertia unless explicitly stated otherwise.

2.2 Moduli Stacks: Definitions and Basic Properties

Definition 2.5 (Algebraic stack with good reduction). An algebraic stack $\mathcal{M}$ of finite type over $\mathcal{O}_K$ is said to have *good reduction* if it is flat, its diagonal is affine, and its special fiber $\mathcal{M}_s$ is smooth over k .

Proposition 2.6 (Basic properties). *Let $\mathcal{M}$ be an algebraic stack with affine stabilizers, locally of finite presentation over $\mathcal{O}_K$. Then:*

1. $\mathcal{M}$ admits a smooth cover by a scheme $U \rightarrow \mathcal{M}$ with U of finite type over $\mathcal{O}_K$.
2. If $\mathcal{M}$ has good reduction in the sense of Definition 2.5, then so does any smooth cover U .

3. Conversely, if some smooth cover U has good reduction, then $\mathcal{M}$ has good reduction.

Proof. This is standard; see e.g. [6] and the Stacks Project (Tags 06D0, 0BX4, 02L6) for smooth atlases and the descent/ascent of flatness and smoothness.

(1) *Smooth presentation by a scheme.* By Artin's criteria and algebraization, an algebraic stack locally of finite presentation with affine stabilizers admits a smooth surjective morphism $U \rightarrow \mathcal{M}$ from a scheme U locally of finite type over $\mathcal{O}_K$ (see Laumon–Moret-Bailly and the Stacks Project for existence of smooth atlases). After shrinking U if necessary, we may take U of finite type over $\mathcal{O}_K$ and with $U \rightarrow \mathcal{M}$ smooth and surjective.

(2) *Descent of good reduction to smooth covers.* Assume $\mathcal{M}$ has good reduction in the sense of Definition 2.5: it is flat over $\mathcal{O}_K$ with affine diagonal and smooth special fiber $\mathcal{M}_s/k$. Let $U \rightarrow \mathcal{M}$ be smooth surjective with U a scheme of finite type over $\mathcal{O}_K$.

- Flatness: flatness is stable under base change, hence U is flat over $\mathcal{O}_K$.
- Smooth special fiber: the base change $U_s \rightarrow \mathcal{M}_s$ is smooth and surjective; since $\mathcal{M}_s$ is smooth over k and composition of smooth morphisms is smooth, it follows that U_s is smooth over k .

Thus U has good reduction.

(3) *Ascent of good reduction from one smooth cover.* Suppose there exists a smooth surjective morphism $U \rightarrow \mathcal{M}$ with U a scheme of finite type over $\mathcal{O}_K$ and U has good reduction, i.e. U is flat over $\mathcal{O}_K$ and U_s is smooth over k .

- Flatness of $\mathcal{M}/\mathcal{O}_K$: flatness is fpqc-local on the target and descends along smooth surjective morphisms; hence $\mathcal{M}$ is flat over $\mathcal{O}_K$.
- Smoothness of the special fiber: smoothness is local on the target for the smooth topology and ascends along smooth surjective morphisms. Since $U_s \rightarrow \mathcal{M}_s$ is smooth surjective and U_s is smooth over k , it follows that $\mathcal{M}_s$ is smooth over k .

By hypothesis $\mathcal{M}$ has affine stabilizers, hence affine diagonal; therefore $\mathcal{M}$ has good reduction in the sense of Definition 2.5. $\square$

$$\begin{array}{ccc}
 U & \xrightarrow{\text{smooth surj.}} & \mathcal{M} \\
 \downarrow & & \downarrow \\
 \text{Spec}(\mathcal{O}_K) & \xlongequal{\quad} & \text{Spec}(\mathcal{O}_K) \\
 \\
 U_s & \xrightarrow{\text{smooth surj.}} & \mathcal{M}_s \\
 \downarrow \text{smooth}/k & \swarrow \text{smooth}/k & \\
 \text{Spec}(k) & &
 \end{array}$$

Figure 2: Good reduction is smooth-local on the source and target: U good $\iff \mathcal{M}$ good via a smooth cover.

Example 2.7 (Moduli of elliptic curves). Let $\mathcal{M}_{1,1}$ denote the Deligne–Mumford stack of elliptic curves. It is smooth over $\mathbb{Z}[1/6]$ (while ramification occurs at 2, 3). In the sense of Definition 2.1—which is *relative to a DVR* O_K —this means: for any DVR O_K with residue characteristic $p \geq 5$, the base-change $\mathcal{M}_{1,1} \times_{\mathrm{Spec} \mathbb{Z}} \mathrm{Spec} O_K$ has good reduction (flat over O_K with smooth special fiber).

Equivalently, one may pass to a level structure that kills automorphisms: for $N \geq 3$ with $p \nmid N$, the stack $\mathcal{M}_{1,1}[N]$ is a *scheme* smooth over $\mathbb{Z}[1/N]$, so $\mathcal{M}_{1,1}[N] \times \mathrm{Spec} O_K$ has good reduction for $p \nmid N$; hence $\mathcal{M}_{1,1}$ has good reduction after inverting 6 in the relative (traitwise) sense.

Counterexample 2.8 (Failure without flatness). Consider the stack quotient $[\mathrm{Spec}(\mathcal{O}_K[x]/(\pi x))/\mathbb{G}_m]$. Its special fiber has embedded associated points and is not flat over $\mathcal{O}_K$; hence this stack fails to have good reduction in the sense of Definition 2.5.

$$\begin{array}{ccccc}
\mathcal{M}_\eta & \xhookrightarrow{\text{open immersion}} & \mathcal{M} & \xleftarrow{\text{closed immersion}} & \mathcal{M}_s \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec} K & \xhookrightarrow{\text{open immersion}} & \mathrm{Spec} \mathcal{O}_K & \xleftarrow{\text{closed immersion}} & \mathrm{Spec} k
\end{array}$$

Figure 3: Specialization diagram of a moduli stack over a discrete valuation ring.

2.3 Cohomological Invariants and Their Arithmetic Interpretation

Lemma 2.9 (Smooth base change for stacks). *Let $\mathcal{M}$ be an algebraic stack, locally of finite type and smooth over $\mathcal{O}_K$. For any prime $\ell \neq p$ and any field extension $K \subset L$ (in particular $L = \overline{K}$), the natural base-change map*

$$H_{\mathrm{\acute{e}t}}^i((\mathcal{M}_K) \times_{\mathrm{Spec} K} \mathrm{Spec} L, \mathbb{Q}_\ell) \xrightarrow{\sim} H_{\mathrm{\acute{e}t}}^i(\mathcal{M}_L, \mathbb{Q}_\ell)$$

is an isomorphism, functorial in L/K and compatible with the action of G_L (transported from G_K via restriction). Equivalently, writing $f : \mathcal{M} \rightarrow \mathrm{Spec} \mathcal{O}_K$ and $\eta = \mathrm{Spec} K$, for any geometric generic point $\bar{\eta} \rightarrow \mathrm{Spec} \mathcal{O}_K$ one has the canonical identification

$$H_{\mathrm{\acute{e}t}}^i(\mathcal{M}_{\bar{\eta}}, \mathbb{Q}_\ell) \cong (R^i f_{\eta*} \mathbb{Q}_\ell)_{\bar{\eta}}.$$

Proof. Step 1: Reduction to a strict simplicial presentation. Choose a smooth surjection from a scheme (or algebraic space) $U \rightarrow \mathcal{M}$ and let $U_\bullet$ be its Čech nerve. On the lisse-étale site, cohomological descent gives a canonical quasi-isomorphism

$$R\Gamma(\mathcal{M}_\eta, \mathbb{Q}_\ell) \simeq R\Gamma(\mathrm{Tot}(U_{\bullet, \eta}, \mathrm{\acute{e}t}), \mathbb{Q}_\ell),$$

functorial in $(\mathcal{M} \rightarrow \mathrm{Spec} \mathcal{O}_K)$ and compatible with the six operations (pullback, proper/smooth pushforward, tensor, internal Hom). We use that cohomological descent holds for strict simplicial hypercovers on Artin stacks and is compatible with the six-functor formalism; (see [8]).

Step 2: Fiber-stalk and smooth base-change on the terms U_n . For each n , the structural morphism $f_n : U_n \rightarrow \mathrm{Spec} \mathcal{O}_K$ is smooth (as a base change of f). For torsion coefficients, [7, Exp. XVI, Th. 1.2]

identifies the geometric fiber cohomology with the stalks of $R^i f_{n,\eta*}$ and shows compatibility with smooth base change; passing to ℓ -adic coefficients via the usual limit formalism yields

$$H_{\text{ét}}^i((U_n)_\eta \times_{\text{Spec } K} \text{Spec } L, \mathbb{Q}_\ell) \xrightarrow{\sim} H_{\text{ét}}^i((U_n)_L, \mathbb{Q}_\ell)$$

for every field extension L/K (in particular $L = \overline{K}$). Equivalently, $H_{\text{ét}}^i((U_n)_{\bar{\eta}}, \mathbb{Q}_\ell) \cong (R^i f_{n,\eta*} \mathbb{Q}_\ell)_{\bar{\eta}}$.

Step 3: Glueing along faces/degeneracies (descent). The isomorphisms in Step 2 are natural in n and compatible with pullbacks along face/degeneracy maps. Totalizing over the simplicial object and invoking Step 1 yields the desired base-change isomorphism on $\mathcal{M}$:

$$H_{\text{ét}}^i(\mathcal{M}_\eta \times_{\text{Spec } K} \text{Spec } L, \mathbb{Q}_\ell) \xrightarrow{\sim} H_{\text{ét}}^i(\mathcal{M}_L, \mathbb{Q}_\ell).$$

The fiber-stalk identification follows from the same argument with $L = \kappa(\bar{\eta})$.

Step 4: Galois and stabilizers. Functoriality in L/K gives compatibility with the natural restriction $G_L \hookrightarrow G_K$. Under the hypothesis of [Lemma 2.11](#) (finite linearly reductive inertia made constant after a finite unramified extension), the I_K -action on $H_{\text{ét}}^i(\mathcal{M}_{\bar{\eta}}, \mathbb{Q}_\ell)$ commutes with residual stabilizers; this compatibility is preserved termwise on $U_\bullet$ and hence after descent. $\square$

$$\begin{array}{ccccc} U_{n,L} & \xrightarrow{\pi_n} & U_n & \xleftarrow[s_j]{\partial_i} & U_{m,L} \\ \downarrow & \lrcorner & \downarrow & & \downarrow \\ \text{Spec } L & \longrightarrow & \text{Spec } K & \longleftarrow & \text{Spec } L \end{array}$$

Figure 4: Termwise base change along L/K on the Čech nerve $U_\bullet \rightarrow \mathcal{M}$. Each square is cartesian; smooth base change on every U_n and descent along faces/degeneracies glue to give the canonical comparison $H_{\text{ét}}^i(\mathcal{M}_K, \mathbb{Q}_\ell) \xrightarrow{\sim} H_{\text{ét}}^i(\mathcal{M}_L, \mathbb{Q}_\ell)$.

Remark 2.10 (Scope and role in the paper). (1) No properness is used: we only need smoothness of $\mathcal{M}/\mathcal{O}_K$, descent on the lisse-étale site, and the smooth base-change formalism on each U_n . (2) This lemma is invoked wherever you transfer I_K -(un)ramifiedness from slices or atlases to the stack (e.g. in [Section 2.3](#) and the proofs feeding into [Theorem 3.16](#) and [Theorem 4.9](#)). (3) Together with [Lemma 2.11](#) and the weight-monodromy formalism ([Proposition 2.14](#)), it justifies checking $N = 0$ and specialization isomorphisms on quotient charts/slices without loss of information.

Lemma 2.11 (Galois vs. stack inertia commute). *Let K be a complete discretely valued field with ring of integers $\mathcal{O}_K$ and residue field k ; fix a separable closure $\overline{K}$ and inertia subgroup $I_K \subset \text{Gal}(\overline{K}/K)$. Let $\mathcal{M}/\text{Spec } \mathcal{O}_K$ be an Artin stack, locally of finite type with affine diagonal, whose inertia is finite and linearly reductive. Then, after a finite unramified extension K'/K there exists an étale cover by a quotient stack*

$$\mathcal{M}_{K'}|_U \simeq [U/G]$$

with U an algebraic space over K' and G a finite constant group. For any $i \geq 0$ and any $\ell \neq \text{char}(k)$, the natural actions of I_K (Galois inertia) and of G (stack inertia) on $H_{\text{ét}}^i(\mathcal{M}_{\overline{K}}, \mathbb{Q}_\ell)$ commute.

Explicit use: *we first pass to a finite unramified extension K'/K so that the finite linearly reductive stabilizer becomes a constant finite group scheme; see [19, Main Thm.]. All subsequent functorialities are taken over K' and descend back to K .*

Proof. (1) *Unramified base change makes stabilizers constant.* By the local structure theorems for stacks with finite linearly reductive inertia, étale locally on the coarse space and after a finite unramified extension K'/K , there is an atlas on which $\mathcal{M}_{K'}|_U \simeq [U/G]$ with G a finite *constant* group scheme over K' ; see [19, Main Thm.] (tame stacks are étale locally quotients by linearly reductive finite group schemes).¹

(2) *Čech nerve / descent setup on the lisse-étale site.* Choose a smooth presentation $U \rightarrow \mathcal{M}_{K'}$ and write $U_\bullet$ for its Čech nerve. For $\Lambda = \mathbf{Q}_\ell$, the lisse-étale derived category on stacks satisfies cohomological descent for strict simplicial hypercovers by unbounded complexes; see [8, Prop. 2.3.11]. Hence

$$R\Gamma(\mathcal{M}_{\overline{K}}, \Lambda) \simeq R\Gamma(\mathrm{Tot}(U_{\bullet, \overline{K}, \acute{e}t}), \Lambda),$$

functorially in the morphism $\mathcal{M} \rightarrow \mathrm{Spec} \mathcal{O}_K$; compare [8, §4.3].

(3) *Define the two actions and functoriality.* The I_K -action on $R\Psi$ (and hence on $H_{\acute{e}t}^i(\mathcal{M}_{\overline{K}}, \Lambda)$) is the standard monodromy action from [13, Exp. XIII, §1.3] via the vanishing/nearby topoi. On the other hand, the stack inertia action is induced by the G -action on $U_\bullet$ when $\mathcal{M} \simeq [U/G]$; the six-operations on stacks are G -equivariant and compatible with descent by [8, §4.3].

(4) *Commutation after K'/K unramified.* Over K' the group G is constant, hence acts trivially on the trait and therefore on the complex computing $R\Gamma$. The I_K -action factors through the nearby/vanishing topoi and is functorial in morphisms of traits; by [13, Exp. XIII, (1.3) and §2] it commutes with pullbacks and proper pushforwards. Pushing this along the descent equivalence in (2) gives that the two actions commute on $R\Gamma$ and hence on H^i . For the precise diagrammatic compatibilities needed in (2)–(4), see [8, Props. 5.3.2–5.3.3].

Therefore I_K and G commute on $H_{\acute{e}t}^i(\mathcal{M}_{\overline{K}}, \Lambda)$, as claimed. $\square$

Remark 2.12 (Scope and hypotheses). The commutation uses that after an unramified base change the stabilizer group is *constant* and hence acts independently of the trait; this follows from the local-structure theorems for tame stacks ([19, Main Thm.], [21, Thm. 12.1]). Consequently, the Galois inertia action on $R\Psi$ ([13, Exp. XIII, §1.3]) and the stack inertia action (via the G -action on a quotient chart) pass through cohomological descent and base-change compatibilities ([8, Prop. 2.3.11, Props. 5.3.2–5.3.3]). Without the “constant after unramified base change” hypothesis, commutation may fail in positive characteristic for non-constant stabilizers.

Lemma 2.13 (Proper hypercover reduction for stacks). *Let M be a proper algebraic stack over $\mathrm{Spec} \mathcal{O}_K$, smooth over K , with finite inertia. Then there exists a strict simplicial proper hypercover $U_\bullet \rightarrow M$ by schemes (or algebraic spaces) over $\mathcal{O}_K$ such that, for any $\ell \neq p$ and $\Lambda = \mathbf{Q}_\ell$,*

$$R\Gamma(M_\eta, \Lambda) \simeq R\Gamma(\mathrm{Tot}(U_{\bullet, \eta}, \acute{e}t), \Lambda), \quad R\Psi_M(\Lambda) \simeq \mathrm{Tot}(R\Psi_{U_\bullet}(\Lambda))$$

in the derived category on M_s , and the six operations $(f_!, f_, f^*, f^!, \otimes, \mathrm{Hom})$ as well as nearby and vanishing cycles are compatible with descent along $U_\bullet$.*

¹Any of [21, Thm. 12.1] or [6, Th. 10.1] would also suffice.

Proof. Step 1: Construction of a proper hypercover. Choose a smooth presentation $U_0 \rightarrow M$ by a scheme proper over $\mathcal{O}_K$ (possible since M is proper and has finite inertia, hence admits such atlases by the results of Laumon–Moret-Bailly). Form its Čech nerve

$$U_\bullet = U_0 \rightrightarrows U_0 \times_M U_0 \rightrightarrows U_0 \times_M U_0 \times_M U_0 \longrightarrow \cdots,$$

which is a strict simplicial object in proper $\mathcal{O}_K$ -schemes. By replacing it with a proper hypercover refinement if necessary (using cohomological descent for hypercovers of Artin stacks), we obtain the desired $U_\bullet \rightarrow M$ that is termwise proper and smooth over $\mathcal{O}_K$.

Step 2: Cohomological descent on the lisse-étale site. The complex $R\Gamma(M_\eta, \Lambda)$ is computed by the totalization of the cosimplicial complex $R\Gamma(U_\bullet, \eta, \Lambda)$. Indeed, Laszlo–Olsson [8, Prop. 2.3.11, §4.3] prove that for a strict simplicial hypercover of an Artin stack by algebraic spaces (resp. schemes), the natural augmentation morphism $R\Gamma(M_\eta, \Lambda) \rightarrow R\Gamma(\text{Tot}(U_\bullet, \eta), \Lambda)$ is a quasi-isomorphism, functorial in M and compatible with the six-functor formalism. Hence

$$R\Gamma(M_\eta, \Lambda) \simeq R\Gamma(\text{Tot}(U_\bullet, \eta, \text{ét}), \Lambda).$$

Step 3: Nearby cycles and totalization. For each n , U_n is proper over $\mathcal{O}_K$, so $R\Psi_{U_n}(\Lambda)$ is defined and carries its canonical G_K -action. The descent formalism yields an equivalence in the derived category on M_s

$$R\Psi_M(\Lambda) \simeq \text{Tot}(R\Psi_{U_\bullet}(\Lambda)),$$

where the totalization is taken with respect to the simplicial structure maps. This follows again from [8, Props. 5.3.2–5.3.3], which guarantee compatibility of nearby/vanishing cycles with simplicial descent and with the six operations.

Step 4: Compatibility of the six operations. Each functor in the six-operations formalism commutes with strict simplicial descent: if $f: M \rightarrow \text{Spec } \mathcal{O}_K$ and $f_n: U_n \rightarrow \text{Spec } \mathcal{O}_K$, then $f_* R\Psi_M(\Lambda) \simeq \text{Tot}(f_{n,*} R\Psi_{U_n}(\Lambda))$, and similarly for $f_!$, f^* , $f^!$, $\otimes$, $\mathcal{H}om$. This is crucial in later sections when weight–monodromy and vanishing-cycle functorialities are traced through $U_\bullet$.

Step 5: Functoriality and independence of choices. Any two proper hypercovers dominate a common refinement; cohomological descent is invariant under such refinement, so the resulting complexes and identifications are canonical in the derived category of M_s . $\square$

$$\begin{array}{ccccccc}
& \xrightarrow{\partial_0} & & \xrightarrow{\partial_0} & & \xrightarrow{f_0} & \\
U_2 & \xrightarrow{\partial_1} & U_1 & \xrightarrow{\partial_1} & U_0 & \longrightarrow & M \\
& \xrightarrow{\partial_2} & & \xrightarrow{\partial_1} & & & \\
f_2 \downarrow & & f_1 \downarrow & & \downarrow & & \downarrow \\
\text{Spec } \mathcal{O}_K & \xlongequal{\quad} & \text{Spec } \mathcal{O}_K & \xlongequal{\quad} & \text{Spec } \mathcal{O}_K & \xlongequal{\quad} & \text{Spec } \mathcal{O}_K
\end{array}$$

Figure 5: Strict simplicial proper hypercover $U_\bullet \rightarrow M$: Cohomological descent and nearby-cycle totalization commute with the face and degeneracy maps.

N.B. [Lemma 2.13](#) is the technical hinge between the geometric input (properness, finite inertia) and the cohomological machinery used in [Proposition 3.26](#) and [Theorem 3.16](#). It legitimizes transferring all weight–monodromy, purity, and specialization arguments from a stack M to a simplicial diagram of proper schemes $U_\bullet$, where classical SGA-style results apply. This reduction is repeatedly invoked:

- in [Proposition 2.14](#) to build the weight–monodromy spectral sequence;
- in [Theorem 3.16](#) to compare $R\Psi_M$ with the termwise $R\Psi_{U_n}$;
- and in Section 4 when passing to slices and proving [Theorem 4.9](#).

Hence the lemma provides the “cohomological descent bridge” cementing the six-operations formalism on stacks with classical proper-scheme theorems.

Proposition 2.14 (Weight–monodromy spectral sequence). *Let $\mathcal{M}$ be a proper algebraic stack over $\mathcal{O}_K$, smooth over K . Then for each $\ell \neq p$, the spectral sequence*

$$E_1^{a,b} = H^{a+b}(\mathrm{Gr}_{-a}^W R\Psi(\mathbb{Q}_\ell)) \Rightarrow H^{a+b}(\mathcal{M}_{\bar{\eta}}, \mathbb{Q}_\ell)$$

is functorial in $\mathcal{M}$ and compatible with the action of G_K .

Proof. Pick a strict simplicial proper hypercover $U_\bullet \rightarrow M$ by schemes as in [Lemma 2.13](#). By cohomological descent on the lisse-étale site and compatibility of the six operations with descent,

$$R\Gamma(M_\eta, \mathbb{Q}_\ell) \simeq R\Gamma(\mathrm{Tot}(U_\bullet, \eta), \mathbb{Q}_\ell), \quad R\Psi_M(\mathbb{Q}_\ell) \simeq \mathrm{Tot}(R\Psi_{U_\bullet}(\mathbb{Q}_\ell)),$$

and these identifications are functorial in M ; see [8, Prop. 2.3.11, §4.3] for cohomological descent and functoriality, and [8, Props. 5.3.2–5.3.3] for the compatibilities of nearby/vanishing cycles with simplicial descent and the six functor formalism.

For each n , the scheme U_n is proper and smooth over K , so the classical weight–monodromy spectral sequence of [13, Exp. IX, Th. 4.3.5] holds *levelwise*:

$$E_1^{a,b}(U_n) = H^{a+b}(\mathrm{Gr}_{-a}^W R\Psi_{U_n}(\mathbb{Q}_\ell)) \Longrightarrow H^{a+b}(U_{n,\eta}, \mathbb{Q}_\ell),$$

and it is G_K -equivariant [13, Exp. IX, §5.1]. Passing to the total complex along the simplicial object and using the compatibilities from [8, §4.3; Props. 5.3.2–5.3.3] yields the stated spectral sequence for M , functorial in M and compatible with the G_K -action. $\square$

Definition 2.15 (Cohomological good reduction). A stack M has cohomological good reduction at p if the *Galois* inertia I_K acts unipotently with trivial monodromy $N = 0$ on $H_{\text{ét}}^i(M_K, \mathbb{Q}_\ell)$ for all i and all $\ell \neq p$.

Remark 2.16 (Terminology warning). Unless stated otherwise, “inertia action” in §§2–5 refers to the *local Galois* inertia $I_K \subset G_K$, not to the stack inertia $\mathcal{I}_M$. Where stabilizers intervene, we work after the unramified base change of [Lemma 2.11](#), so the two actions commute.

Remark 2.17 (Comparison with geometric good reduction). Geometric good reduction (Definition 2.5) implies cohomological good reduction by smooth base change, but the converse may fail without further hypotheses ([4]). This tension motivates our later theorems connecting the two notions.

Example 2.18 (Abelian varieties in moduli). If $\mathcal{M}$ is the moduli stack of principally polarized abelian varieties, then by Faltings’ theorem [10] cohomological good reduction for the universal abelian scheme implies the existence of smooth integral models with good reduction after finite unramified base change.

Counterexample 2.19 (Wild ramification). Consider X/K a curve of genus $g \geq 2$ with potentially good, but not semistable, reduction at p . Then the inertia action on $H_{\text{ét}}^1(X_{\overline{K}}, \mathbb{Q}_\ell)$ has nontrivial unipotent part, so X fails to have cohomological good reduction, even though it attains stable reduction after ramified extension.

3 Foundational Constructions

This section encodes geometric good reduction (Definitions 2.1 and 2.5) in a moduli-theoretic language, sets up the cohomological functors that will be used later (Definition 2.15, lemma 2.9, and proposition 2.14), and compares the stack-theoretic framework with the classical Néron perspective (Lemma 2.3). We retain the standing hypotheses and notation of Section 2, in particular K , $\mathcal{O}_K$, k , and the inertia group I_K from Notation 2.4.

3.1 Moduli-Theoretic Encoding of Good Reduction

Definition 3.1 (Good-reduction locus in a moduli stack). Let $\mathcal{M}$ be an algebraic stack locally of finite type over $\mathcal{O}_K$ with affine diagonal. Define the *good-reduction locus* $\mathcal{M}^{\text{gr}} \subset \mathcal{M}$ to be the full substack of objects $\mathcal{X} \rightarrow \text{Spec } R$ for which the structural morphism $\mathcal{M} \rightarrow \text{Spec } \mathcal{O}_K$ is smooth at the image of the closed point of $\text{Spec } R$ and whose automorphism group scheme over the closed fiber is linearly reductive.

Remark 3.2 (Link to Definition 2.5). If $\mathcal{M}$ is flat over $\mathcal{O}_K$ with smooth special fiber (i.e. $\mathcal{M}$ has good reduction in the sense of Definition 2.5), then $\mathcal{M}^{\text{gr}} = \mathcal{M}$. Conversely, $\mathcal{M}^{\text{gr}}$ describes the locus where geometric good reduction holds objectwise; this distinction is essential for stacks with mixed reduction behavior.

Lemma 3.3 (Openness of $\mathcal{M}^{\text{gr}}$ in the tame case). *Let $\mathcal{M}$ be an algebraic stack locally of finite type over $\mathcal{O}_K$ with finite linearly reductive inertia and affine diagonal. Then the good-reduction locus $\mathcal{M}^{\text{gr}}$ of Definition 2.5 is an open substack of $\mathcal{M}$.*

Proof. We argue in two layers corresponding to the geometric and stack-inertial components.

(1) Smoothness locus.— For any morphism $f: \mathcal{M} \rightarrow \text{Spec } \mathcal{O}_K$, the set of points where f is smooth is open and stable under smooth base change. In particular, if $U \rightarrow \mathcal{M}$ is a smooth presentation by a scheme, the smoothness locus

$$U^{\text{sm}} := \{u \in U \mid f \circ u \text{ is smooth over } \mathcal{O}_K\}$$

is open in U , and the image of U^{sm} in $\mathcal{M}$ is open since smooth morphisms are universally open. This part is purely geometric and uses only Artin’s representability and standard openness of the smooth locus (EGA IV₂, Th. 11.12).

(2) Linear-reductivity locus.— The second condition in [Definition 2.5](#) concerns the stabilizer groups. For algebraic stacks with finite linearly reductive inertia, the property that

$$x \longmapsto G_x \text{ is linearly reductive over the residue field of } x$$

is an open condition on the underlying topological space. This follows from the local-structure theorem for tame stacks ([\[19, 11, 6, 9\]](#)): étale-locally on $\mathcal{M}$ there exists a chart

$$[U/G] \longrightarrow \mathcal{M}$$

with G a finite flat linearly reductive group scheme. Linear reductivity of fibers of G is open in flat families by standard deformation theory—indeed, it amounts to the vanishing of $\text{Ext}_G^1(V, W)$ for all simple G -modules (V, W) , a property stable under specialization since the relevant cohomology sheaves are coherent and upper-semicontinuous.

(3) Descent to the stack.— Intersecting the two open loci in U and descending along the smooth surjection $U \rightarrow \mathcal{M}$ gives an open substack

$$\mathcal{M}^{\text{gr}} = \mathcal{M}^{\text{sm}} \cap \mathcal{M}^{\text{lr}} \subset \mathcal{M}.$$

Because the intersection of open substacks is open, $\mathcal{M}^{\text{gr}}$ is open in $\mathcal{M}$. □

Remark 3.4. This lemma is used to transfer the verification of purity and vanishing-monodromy from local slices to the global moduli stack. It ensures that the locus of geometric good reduction is open, permitting Zariski-local propagation of integral models.

It ensures that once purity of nearby cycles and the vanishing of monodromy ([Theorem 3.16](#)) are verified locally, the locus of good-reduction points propagates openly through the moduli stack (note that such a tame slice may not exist for wild stabilizers; cf. [Example 6.12](#)). Consequently, the cohomological conditions (*purity* + $N = 0$) yield not only existence but also Zariski-local persistence of integral models—crucial when passing from a local slice to global density arguments ([Theorems 4.2](#) and [4.9](#)).

Geometrically, the lemma identifies the following factorization diagram that recurs throughout [Sections 3](#) and [4](#):

$$\begin{array}{ccccc} U^{\text{gr}} & \xhookrightarrow{\text{closed immersion}} & U & \xrightarrow{\quad} & \mathcal{M} \\ \downarrow q & \lrcorner & \downarrow q & & \downarrow f \\ \mathcal{M}^{\text{gr}} & \xhookrightarrow{\text{closed immersion}} & \mathcal{M} & \xrightarrow{f} & \text{Spec } \mathcal{O}_K \end{array}$$

Figure 6: Good-reduction square: the closed immersion $\mathcal{M}^{\text{gr}} \hookrightarrow \mathcal{M}$ and its pullback along a smooth presentation $U \rightarrow \mathcal{M}$. Each vertical map q is a smooth surjection, and both squares are cartesian.

Here q is a smooth presentation, and the open immersion of good-reduction loci is preserved both on the presentation and on the stack. This diagram embodies the principle that *good reduction descends*

and ascends through smooth covers, bridging the scheme-level and stack-level geometries that underlie the six-functor descent in [Lemma 2.13](#).

Construction 3.5 (Trait test and extension). Let $T = \operatorname{Spec} R$ be a trait with fraction field L and residue field a finite extension of k . Given a morphism $x_\eta : \operatorname{Spec} L \rightarrow \mathcal{M}$ with finite automorphism group, choose a smooth presentation $U \rightarrow \mathcal{M}$ and lift x_η to U_L . Properness (if available) or the valuative criterion relative to inertia (tame, linearly reductive) provides an extension $x : T \rightarrow \mathcal{M}$ if and only if the obstruction in the inertia cohomology vanishes; cf. [\[3, 9\]](#). This criterion will be coupled with cohomology in [Section 3.2](#).

Proposition 3.6 (Stacky Néron mapping property: uniqueness; existence under a smooth proper atlas). *Assume $\mathcal{M}$ is an algebraic stack locally of finite type over $\mathcal{O}_K$ with finite linearly reductive inertia and separated diagonal. Let S be a smooth $\mathcal{O}_K$ -scheme with generic fiber S_η and let $f_\eta : S_\eta \rightarrow \mathcal{M}_\eta$ have image in $\mathcal{M}_{\text{gr}, \eta}$.*

(Uniqueness) *There is **at most one** extension $f : S \rightarrow \mathcal{M}$ up to unique 2-isomorphism.*

(Existence under a smooth proper atlas) *If, in addition, there exists a **smooth and proper** surjective atlas $q : U \rightarrow \mathcal{M}$ with U a scheme, then such an extension f **exists**.*

(Traitwise existence) *Without the atlas hypothesis, for every height-one point $t \in S$ with DVR $\mathcal{O}_{S,t}$, the restriction $f_\eta|_{\operatorname{Spec} K(S)_t}$ extends uniquely to $\operatorname{Spec} \mathcal{O}_{S,t} \rightarrow \mathcal{M}$.*

Proof. Step 1. Reduction via smooth presentation. Choose a smooth surjective atlas $q : U \rightarrow \mathcal{M}$ by a scheme, with Čech nerve $U_\bullet$. By the openness of the good reduction locus ([Lemma 3.3](#)), $U^{\text{gr}} := U \times_{\mathcal{M}} \mathcal{M}^{\text{gr}}$ is open in U , and this construction is compatible with the simplicial structure on $U_\bullet$.

Step 2. Uniqueness (valuative criterion for separated diagonal). Let $f_1, f_2 : S \rightarrow \mathcal{M}$ be two putative extensions of f_η . Since $\Delta_{\mathcal{M}}$ is representable and separated, the stack of isomorphisms

$$\underline{\operatorname{Isom}}_{\mathcal{M}}(f_1, f_2) := S \times_{(f_1, f_2), \mathcal{M} \times \mathcal{M}, \Delta_{\mathcal{M}}} \mathcal{M}$$

is represented by a separated algebraic space over S . The given identification of $f_{1, \eta}$ and $f_{2, \eta}$ provides a section over S_η . Because S is smooth (hence normal) over $\mathcal{O}_K$ and the target is separated, this section extends uniquely over S by the valuative criterion for separated morphisms of algebraic stacks. Therefore f_1 and f_2 are uniquely 2-isomorphic, and any automorphism of an extension is trivial.

Step 3 (Existence under a smooth proper atlas). Assume there exists a smooth and proper atlas $q : U \rightarrow \mathcal{M}$. Lifting f_η to $u_\eta : S_\eta \rightarrow U_\eta$ and using properness of $U/\mathcal{O}_K$, the valuative criterion yields a unique $u : S \rightarrow U$ with image in U_{gr} . Since $U \times_{\mathcal{M}} U \rightrightarrows U$ is proper and smooth over S , the two pullbacks of u agree by proper descent, so u descends to the desired $f : S \rightarrow \mathcal{M}$.

Step 4. Vanishing of obstructions in the tame case. In the tame case (finite linearly reductive inertia), obstruction classes to effectivity of compatible formal liftings lie in H^2 of the inertia group, which vanish for linearly reductive groups. Hence the extension process is unobstructed, and descent along $U_\bullet$ is homotopically effective in degrees ≥ 2 . This recovers the classical unobstructedness familiar from Néron models, now in the stacky setting.

Step 5. Compatibility with the good-reduction locus. Because $\mathcal{M}^{\text{gr}}$ is open and smooth base change preserves it, the extended morphism f lands in $\mathcal{M}^{\text{gr}}$ automatically after possibly shrinking S along its

open smooth locus. No new singularities appear in codimension one when extending across DVRs in the proper case. Thus the extension respects the good-reduction condition.

This completes the proof of uniqueness (always) and existence (in the proper case). $\square$

Remark 3.7 (Necessity of hypotheses; failure already for $\mathbb{P}^1$). Even for a proper target, global existence over S can fail without a smooth proper atlas or properness of S : take $\mathcal{M} = \mathbb{P}_{O_K}^1$ and S a smooth (non-proper) O_K -scheme; a rational map $S_\eta \dashrightarrow \mathbb{P}_K^1$ with indeterminacy along a vertical divisor does not extend to $S \rightarrow \mathbb{P}_{O_K}^1$. This shows the original “existence for proper $\mathcal{M}$ ” claim is too strong without extra hypotheses, while the traitwise extension holds by the valuative criterion.

Remark 3.8 (Conceptual and structural role). This proposition provides the geometric counterpart of the cohomological detection theorem ([Theorem 3.16](#)), converting cohomological conditions into integral extensions via the stacky Néron mapping property.

The latter identifies the good-reduction locus cohomologically (via $R\Psi$ and $N = 0$), while [Proposition 3.6](#) furnishes the *mapping property* that converts that detection into an actual integral extension of morphisms. Together with the openness of $\mathcal{M}^{\text{gr}}$, this ensures that cohomological good reduction extends Zariski-open in moduli, bridging local inertia criteria and global Néron-type phenomena.

$$\begin{array}{ccc}
 S_\eta & \xrightarrow{f_\eta} & \mathcal{M}_\eta \\
 \downarrow \hookrightarrow & & \downarrow \hookrightarrow \\
 S & \begin{array}{c} \xrightarrow{f_1} \\ \xrightarrow{f_2} \end{array} & \mathcal{M}
 \end{array}$$

(The diagram shows a commutative square with a 2-isomorphism between the bottom arrows f_1 and f_2 .)

Figure 7: **Uniqueness of the extension.** Any two extensions f_1, f_2 of f_η coincide up to a unique 2-isomorphism, by the valuative criterion applied to the separated diagonal $\Delta_{\mathcal{M}}$.

- Remark 3.9* (Cross-checks with earlier constructions). 1. The traitwise extension and descent mechanism here is the higher-dimensional reformulation of the *trait test* ([Construction 3.5](#)).
2. The openness of $\mathcal{M}^{\text{gr}}$ ([Lemma 3.3](#)) ensures local effectivity of extensions and compatibility with smooth atlases.
3. This proposition underlies the “stacky Néron envelope” construction in later sections, ensuring its uniqueness and functoriality once the stabilizers are tame.

Remark 3.10 (Compatibility with [Proposition 2.6](#)). When $\mathcal{M}$ admits a *smooth proper* cover $U \rightarrow \mathcal{M}$ by a scheme, [Proposition 3.6](#) reduces to the scheme-theoretic Néron mapping property on U and descent along $U \times_{\mathcal{M}} U \rightrightarrows U$.

Lemma 3.11 (Quotient-chart purity lemma). *Let $M = [X/G]$ where $|G|$ is prime to p and $X/\mathcal{O}_K$ is proper and smooth in a neighbourhood of the given point. Then for every $\ell \neq p$ there is a canonical isomorphism*

of complexes on the special fibre

$$R\Psi_M(\mathbb{Q}_\ell) \simeq (R\Psi_X(\mathbb{Q}_\ell))^G,$$

and this complex is pure of weight 0.

Proof. Since $|G|$ is invertible on $\mathcal{O}_K$, the quotient morphism $\pi : X \rightarrow [X/G]$ is finite and the six-operations formalism for stacks ([8, Props. 5.3.2–5.3.3 in Parts I–II]) yields

$$R\Psi_{[X/G]}(\mathbb{Q}_\ell) \simeq (R\pi_* R\Psi_X(\mathbb{Q}_\ell))^G \simeq (R\Psi_X(\mathbb{Q}_\ell))^G.$$

Taking G -invariants is an idempotent direct summand when $|G|$ is invertible, so purity of $R\Psi_X(\mathbb{Q}_\ell)$ (coming from smoothness of $X/\mathcal{O}_K$) implies purity of $R\Psi_M(\mathbb{Q}_\ell)$ of weight 0. $\square$

Assumption 3.12 (Slice of abelian/K3-type near a point). *Let M be an algebraic stack over $\mathrm{Spec} \mathcal{O}_K$ and $x_\eta : \mathrm{Spec} K \rightarrow M_\eta$ a K -point with finite automorphism group. After a finite unramified extension of K , there exists an étale slice V through x_η with V an algebraic space and a proper smooth morphism $g : Y \rightarrow V$ with geometrically connected fibers such that one of the following holds:*

(A) *Abelian type.*

g is an abelian scheme (e.g. the universal abelian family on the slice), and the cohomology used in Item (2) of Theorem 3.16 is the direct summand of $R^1 g_{\eta} \mathbf{Q}_\ell$ discussed in Section 5.1. In this case, unramified (equivalently, trivial inertia with $N = 0$) action on $H_{\text{ét}}^1$ implies (potential) good reduction by the Néron–Ogg–Shafarevich criterion ([15, Ch. VII], [3, Ch. 1–3]).*

(B) *K3 type.*

The geometric fibers of g are K3 surfaces and the relevant cohomology is $R^2 g_{\eta} \mathbf{Q}_\ell$ (cf. Section 5.1). For $p \geq 5$, unramified or crystalline $H_{\text{ét}}^2$ implies (potential) good reduction by the known K3 criteria ([16, Th. 1.1], [17, Th. 1.1]).*

More generally, it suffices that there exists a proper smooth morphism $g : Y \rightarrow V$ for which the implication

$$\text{“unramified } I_K\text{-action on } R^i g_{\eta*} \mathbf{Q}_\ell \Rightarrow \text{(potential) good reduction of the fiber”}$$

is established in the literature.

New result (moduli-theoretic Néron–Ogg–Shafarevich). The following theorem is the central mechanism that turns cohomological constraints into geometric extension on stacks. It is not a formal corollary of Lemma 2.3 because stabilizers need not be trivial and one works on a stack rather than a scheme.

Assumption 3.13. *Standing Assumption (Purity of nearby cycles on stacks). Throughout Sections 3–5, for an algebraic stack $M/\mathrm{Spec} \mathcal{O}_K$ with affine diagonal and finite linearly reductive inertia, and for each $\ell \neq p$, we assume that the nearby-cycles complex $R\Psi(\mathbf{Q}_\ell)$ on the special fiber M_s is pure of weight 0.*

This is known in standard settings (e.g. strict semistability/tame monodromy for schemes, and along étale charts) via the six-operations formalism for stacks and weight/monodromy theory [8, 13, 14]; see also Saito’s work on weight spectral sequences [18].

In full generality for stacks with arbitrary finite linearly reductive stabilizers, this remains conjectural; we use it here as a hypothesis.

Remark 3.14 (Verifiability of [Assumption 3.13](#)). In the situations actually used in [Section 3–Section 5](#), the purity requirement on $R\Psi(\mathbb{Q}_\ell)$ is verified by standard results:

(i) *Smooth reduction.* If $M/\mathrm{Spec} \mathcal{O}_K$ is smooth near the point under consideration (equivalently $R\Phi(\mathbb{Q}_\ell) = 0$), then $R\Psi(\mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell$ and is pure of weight 0 by the nearby–vanishing formalism ([\[13, Exp. XIII\]](#)) together with the six–operations on stacks ([\[8\]](#)); stability of purity under the operations used here follows from [\[14, Th. 1.6.1\]](#).

(ii) *Tame quotients of smooth families.* If $M = [X/G]$ with $|G|$ prime to p and $X/\mathcal{O}_K$ proper and smooth in a neighbourhood of the point, then

$$R\Psi_M(\mathbb{Q}_\ell) \simeq (R\Psi_X(\mathbb{Q}_\ell))^G$$

by functoriality and finite pushforward in the stack formalism ([\[8, I–III\]](#)); since taking G –invariants is a direct summand when $|G|$ is invertible, purity of weight 0 is preserved (cf. [\[14, Cor. 1.3.9\]](#); [\[13, Exp. XIII–XIV\]](#)) for the functoriality of monodromy) (failure in the wild α_p setting is illustrated in [Example 6.12](#)).

(iii) *Slice models of abelian/K3 type.* In the slice setting of [Assumption 3.12](#) and [Lemma 3.18](#), when there exists a proper smooth morphism $g: Y \rightarrow V$ on the slice (as in the abelian and K3 cases used below), the complex $R\Psi_V(\mathbb{Q}_\ell)$ is pure of weight 0 ([\[13, Exp. XIII\]](#)); functoriality and strict simplicial descent transfer purity from V back to M ([\[8, I §4, II §5, III §4\]](#)). For the semistable framework and the associated weight filtration on $R\Psi$, see Saito’s weight spectral sequence [\[18\]](#).

These are precisely the instances invoked later (e.g. [Theorems 3.16](#) and [4.2](#)), so [Assumption 3.13](#) is verifiable in our applications.

Lemma 3.15 (Tame slice existence after unramified base change). *After a finite unramified base change on K , there exists a tame étale slice $V \rightarrow M$ passing through the given point, such that the stabilizer on V is a constant finite linearly reductive group scheme. Purity of nearby cycles and the condition $N = 0$ descend to V , and specialization/base change commute.*

Proof. By the local-structure theorem for tame stacks with linearly reductive inertia ([\[19, Main Thm.\]](#)), after a finite unramified extension of K one obtains an étale neighbourhood $V \rightarrow M$ in which the stabilizer becomes constant and finite. The six-functor formalism on stacks ensures that purity of nearby cycles and vanishing of monodromy are stable under such base change ([\[8, Props. 5.3.2–5.3.3 I,II\]](#)); hence $R\Psi(Q_\ell)$ and the condition $N = 0$ descend to V , and specialization/base change commute. This slice is precisely the one used implicitly in the proof of [Theorem 3.16](#). $\square$

Theorem 3.16 (Cohomological detection of good reduction on a moduli stack; under [Assumption 3.13](#)). *Let M be an algebraic stack of finite type over $\mathcal{O}_K$ with affine diagonal and finite linearly reductive stabilizers. Suppose:*

1. (Purity) [Assumption 3.13](#) holds for M ; equivalently, for every $\ell \neq p$, the nearby-cycles complex $R\Psi(\mathbb{Q}_\ell)$ on M_s is pure of weight 0;

2. for all i , the I_K -action on $H_{\text{ét}}^i(M_K, \mathbf{Q}_\ell)$ is unipotent with trivial monodromy $N = 0$;
3. (Slice hypothesis) [Assumption 3.12](#) holds at the given K -point $x_\eta : \text{Spec } K \rightarrow M_\eta$.

Then x_η extends, after a finite unramified extension of K , to a section over $\text{Spec } \mathcal{O}_K$. Equivalently, x_η lands in M_η^{gr} and admits integral extension.

Remark 3.17. For schemes, the criterion reduces to [Lemma 2.3](#). For stacks, existing literature gives deformation/extension under tame hypotheses, and weight–monodromy for cohomology on stacks, but an explicit equivalence as in the theorem—linking *purity of nearby cycles plus $N = 0$* to the *existence of integral extensions for all finite-stabilizer points*—does not appear to be formulated in this generality; the proof couples weight–monodromy on stacks with a stacky Néron mapping argument ([Proposition 3.6](#)) and a reduction to an algebraic space near the point via linear reductivity.

Proof of Theorem 3.16. Choose a smooth cover $U \rightarrow \mathcal{M}$ with U a scheme and let $x_\eta : \text{Spec } K \rightarrow \mathcal{M}_\eta$ be given with finite automorphism group. After a finite unramified extension of K , the stabilizer becomes constant linearly reductive, hence étale, and one may replace a neighborhood of x_η by an étale slice which is an algebraic space (cf. the tame slice theorem for stacks with linearly reductive inertia [[19](#), Thm. 2.5]; see also [[20](#), Thm. 12.1] for the existence of étale slices after finite unramified base change).

By [Lemma 2.11](#), after this unramified base change the I_K -action on $H_{\text{ét}}^i$ commutes with the residual stabilizer action on the slice. Hence the unipotence and $N = 0$ condition in (2) can be checked on the slice independently of stabilizers and is compatible with nearby cycles.

Lemma 3.18 (Slice invariance of nearby cycles and monodromy). *Let $M/\text{Spec } \mathcal{O}_K$ be as in [Theorem 3.16](#) and let $x_\eta : \text{Spec } K \rightarrow M_\eta$ have finite stabilizer. After a finite unramified extension K'/K , there exists a tame étale slice $V \rightarrow M$ with V an algebraic space. Then:*

1. The formation of nearby cycles is preserved on the slice:

$$R\Psi_M(\mathbf{Q}_\ell)|_{V_s} \simeq R\Psi_V(\mathbf{Q}_\ell),$$

functorially in a neighbourhood of x_η .

2. The $I_{K'}$ -action on $H_{\text{ét}}^i(M_{K'}, \mathbf{Q}_\ell)$ restricts to the $I_{K'}$ -action on $H_{\text{ét}}^i(V_{K'}, \mathbf{Q}_\ell)$, and this action commutes with the residual (constant) stabilizer on the slice. In particular, “ I_K unipotent with $N = 0$ ” on $H_{\text{ét}}^i(M_K, \mathbf{Q}_\ell)$ holds near x_η if and only if it holds on $H_{\text{ét}}^i(V_K, \mathbf{Q}_\ell)$.

Proof. By [Lemma 2.11](#), after an unramified extension K'/K the residual stabilizer becomes a *constant* finite group and the $I_{K'}$ -action on cohomology commutes with the stack-inertia action. Choose a strict simplicial proper hypercover adapted to the slice. Cohomological descent on the lisse-étale site identifies

$$R\Gamma(M_{K'}, \mathbf{Q}_\ell) \simeq R\Gamma(\text{Tot}(U_{\bullet, K'}, \text{ét}), \mathbf{Q}_\ell), \quad R\Psi_M(\mathbf{Q}_\ell) \simeq \text{Tot}(R\Psi_{U_\bullet}(\mathbf{Q}_\ell)),$$

and these identifications are *functorial* and compatible with the six operations; see [[8](#), Prop. 2.3.11, §4.3; Props. 5.3.2–5.3.3]. The monodromy action of $I_{K'}$ is defined via the nearby/vanishing topos and commutes with pullbacks and proper pushforwards ([[13](#), Exp. XIII, §1.3 and §2]). Restricting along the tame étale

slice $V \rightarrow M$ gives the asserted equivalences and the commutation with residual stabilizers. Hence the $N = 0$ and purity conditions are equivalent on M near x_η and on V . $\square$

Assumptions (1)–(2) descend to the slice, since $R\Psi$ and inertia actions are invariant under such local equivalences ([8]). By (2) and smooth base change, the weight–monodromy spectral sequence (Proposition 2.14) degenerates with $N = 0$, so the G_K -action on the cohomology is unramified.

Clarification. Recall that Proposition 2.14 requires the target stack to be *proper* over $\mathcal{O}_K$. Accordingly, the reference to the weight–monodromy spectral sequence here is made only in the situations where the local slice or atlas is proper, so that Lemma 2.13 applies and the spectral sequence of [4, Exp. IX] is valid. For nonproper stacks, the argument uses this step merely formally, reducing to proper neighborhoods or slices when needed.

Passing to an étale slice V near x_η (available by linear reductivity), assumptions (1)–(2) descend to V ; see Lemma 3.24. Now invoke Assumption 3.12:

Case (A) (abelian type). On the slice we have a proper smooth abelian scheme $g : Y \rightarrow V$. Unramifiedness of $H_{\text{ét}}^1$ for the fiber over x_η implies potential good reduction by the Néron–Ogg–Shafarevich criterion (Lemma 2.3). Hence the point extends over $\mathcal{O}_K$ on V , and therefore on M by Proposition 3.6.

Case (B) (K3 type). On the slice we have a proper smooth family of K3 surfaces $g : Y \rightarrow V$. For $p \geq 5$ (and under the standard hypotheses in the cited K3 good reduction criteria), unramifiedness (equivalently, crystallinity) of the G_K -representation on $H_{\text{ét}}^2$ of the fiber over x_η implies potential good reduction; applying this to Y/V yields the integral extension on the slice, and then on M by Proposition 3.6.

This completes the proof. $\square$

Corollary 3.19 (Potential good reduction under finite monodromy). *Under the hypotheses of Theorem 3.16 for every K -point with finite stabilizer (in particular, Assumption 3.12), if I_K acts through a finite quotient on $H_{\text{ét}}^*(M_K, \mathbf{Q}_\ell)$, then any K -point with finite automorphism group attains good reduction after a finite (tamely) ramified extension of K .*

Scope. Applications of this corollary occur only in contexts where Assumption 3.13 (purity of nearby cycles) and the slice hypothesis (Assumption 3.12) hold. See Remark 3.20 for the precise limitations.

Proof (expanded). Step 1: From finite monodromy to $N = 0$ and trivial wild inertia. By Grothendieck’s monodromy theorem, the I_K -action on $H_{\text{ét}}^*(M_K, \mathbf{Q}_\ell)$ is quasi-unipotent. If the image is finite, the unipotent part must be trivial; hence the logarithm of monodromy N vanishes and the wild inertia subgroup (pro- p) acts trivially. Thus only a finite *tame* (prime-to- p) part of I_K may remain.

Step 2: Killing the residual tame action by a finite tame extension. Because the residual I_K -image is finite of order prime to p , there exists a finite tamely ramified extension K'/K such that $I_{K'}$ acts trivially on $H_{\text{ét}}^*(M_{K'}, \mathbf{Q}_\ell)$.

Step 3: Compatibility with stabilizers and passage to a slice. Let $x_\eta : \text{Spec } K \rightarrow M_\eta$ be a K -point with finite automorphism group. After a finite *unramified* extension (which we absorb into K'), the residual stabilizer becomes a constant finite linearly reductive group, and the Galois inertia action commutes with the stack-inertia action on cohomology (cf. the “Galois vs. stack inertia commute” lemma used earlier). Consequently, the triviality of the $I_{K'}$ -action on $H_{\text{ét}}^*(M_{K'}, \mathbf{Q}_\ell)$ is reflected on any tame étale slice $V \rightarrow M$ through the image of x_η (available by Assumption 3.12).

Step 4: Invoke the detection theorem on the slice (abelian/K3 type). By [Assumption 3.13](#), nearby cycles on the special fiber are pure of weight 0, and by Step 2 the $I_{K'}$ -action on cohomology is trivial, hence unipotent with $N = 0$. The hypotheses of [Theorem 3.16](#) are therefore satisfied on the slice V (abelian or K3 type by [Assumption 3.12](#)). Applying [Theorem 3.16](#) yields integral extendability of $x_{\eta'} : \operatorname{Spec} K' \rightarrow M_{\eta}$ across $\operatorname{Spec} \mathcal{O}_{K'}$ (potential good reduction).

Step 5: Return to M . The extension on the slice lifts to M by the stacky Néron mapping property (uniqueness up to unique 2-isomorphism follows from separated/affine diagonal as used earlier) (uniqueness relies on separated/affine diagonal; cf. [Counterexample 3.33](#)). Thus, after the finite tamely ramified extension K'/K , the given K -point with finite automorphism group has good reduction. $\square$

$$\begin{array}{ccc}
 \operatorname{Spec} K & \xrightarrow{x_{\eta}} & M_{\eta} \\
 \downarrow & & \downarrow \\
 \operatorname{Spec} K' & \xrightarrow{x_{\eta'}} & M_{\eta} \\
 \downarrow & & \downarrow \\
 \operatorname{Spec} \mathcal{O}_{K'} & \dashrightarrow_{x'} & M
 \end{array}$$

Figure 8: **Finite monodromy and integral extension.** Finite monodromy implies $N = 0$ and trivial wild inertia. A finite tame extension K'/K annihilates the residual tame action, and [Theorem 3.16](#) provides the integral extension $x' : \operatorname{Spec} \mathcal{O}_{K'} \rightarrow M$.

Remark 3.20 (Bridge, Scope, and Limitations). This corollary and its proof rest on three interlocking pillars—group-theoretic, cohomological, and geometric—which together linking the algebraic and arithmetic components of the argument.

(1) *Group-theoretic input.* The only additional group-theoretic ingredient beyond [Theorem 3.16](#) is the implication

$$\text{finite } I_K\text{-image} \Rightarrow N = 0, \quad \text{wild inertia acts trivially,}$$

so that the residual finite image is of order prime to p and hence killed by a finite *tame* extension. This explains the parenthetical “tamely” in [Corollary 3.19](#) and ensures compatibility with the unipotent–semisimple decomposition in the monodromy theorem.

(2) *Purity and slice hypotheses.* Purity of nearby cycles ([Assumption 3.13](#)) and the slice condition ([Assumption 3.12](#)) guarantee that the cohomological criterion of [Theorem 3.16](#) applies on a neighbourhood of the point and descends to $\mathcal{M}$ via the stacky Néron mapping property. The purity assumption is *established* for strictly semistable or tame morphisms of schemes and along étale charts (cf. [\[13, 8\]](#)), and is *expected* more generally for stacks with finite linearly reductive inertia. In practice, all applications below occur in settings where this purity is known or verifiable slice-wise—see [Example 3.28](#) (tame quotient stacks satisfying [Assumption 3.13](#)) and [Section 6.1–Section 6.2](#) (elliptic and abelian loci, cf. [Proposition 6.1](#), [Proposition 6.6](#)).

(3) *Bridge to arithmetic.* When $\mathcal{M}$ carries a universal family—such as the moduli of abelian varieties or K3 surfaces—[Theorem 3.16](#) identifies the local ℓ -adic Galois representations on the cohomology of the

geometric fibers as *unramified at p* . This provides the expected arithmetic consequence: the local Euler factors of the associated L -function coincide with the characteristic polynomials of Frobenius on $H_{\text{ét}}^i$, establishing the link between geometric good reduction and arithmetic unramifiedness.

(4) *Scope and limitations.* Outside the abelian/K3 slice hypothesis, the implication

$$\text{“unramified cohomology with } N = 0\text{”} \Rightarrow \text{(potential) good reduction}$$

fails even for smooth schemes: one can have unramified cohomology while the fiber lacks good reduction (cf. [Proposition 2.14](#) and [section 6.3](#)). All uses of [Theorem 3.16](#) and [corollary 3.19](#) below are confined to loci satisfying [Assumption 3.12](#), ensuring that the argument remains valid without unproven generalities.

Finally, the uniqueness of the integral extension up to unique 2-isomorphism follows from the separated/affine diagonal arguments already established; no new hypotheses are introduced here.

Example 3.21 (Explicit instance on $\mathcal{M}_{1,1}$). Let $\mathcal{M} = \mathcal{M}_{1,1}$ over $\mathbb{Z}[1/6]$ ([Example 2.7](#)). Here $R\Psi(\mathbb{Q}_\ell)$ is pure of weight 0 on the good fibers and inertia acts trivially on $H_{\text{ét}}^i(\mathcal{M}_{\overline{K}}, \mathbb{Q}_\ell)$ away from 2, 3 (cf. [\[12\]](#)). Thus any elliptic curve E/K with $j(E)$ integral at $p \geq 5$ lies in $\mathcal{M}^{\text{gr}}$ and extends to a smooth model; this recovers the classical criterion but through the stack.

Counterexample 3.22 (Necessity of linear reductivity: explicit small- p models). Work in equal characteristic $p > 0$ with $K = k((t))$ and $\mathcal{O}_K = k[[t]]$, and fix $\ell \neq p$.

(a) *An α_p -quotient that destroys étale slices.* Set

$$X = \text{Spec}(\mathcal{O}_K[z]/(z^p - t)) \xrightarrow{f} \text{Spec } \mathcal{O}_K.$$

There is an α_p -action on X by translation $z \mapsto z + \varepsilon$ (coaction $z \mapsto z \otimes 1 + 1 \otimes \varepsilon$ with $\varepsilon^p = 0$), which preserves the equation since

$$(z + \varepsilon)^p - t = z^p - t + \varepsilon^p = 0 \quad \text{in} \quad \mathcal{O}_K[z]/(z^p - t) \otimes_k k[\varepsilon]/(\varepsilon^p).$$

On the special fiber $X_s = \text{Spec } k[z]/(z^p)$ the closed point has stabilizer α_p ; hence the quotient stack $[X/\alpha_p]$ is *not tame*. In particular, Luna-type étale slices fail (linear reductivity of the stabilizer is required for such slices), so one cannot reduce to an algebraic-space chart near this point. Moreover f is not smooth (Jacobian criterion: $\partial(z^p - t)/\partial z = 0$ in char. p), hence $R\Phi_{X/\mathcal{O}_K}(\mathbb{Q}_\ell) \neq 0$; consequently the purity+ $N = 0$ detection mechanism cannot apply in this non-linearly-reductive setting. This makes the use of linear reductivity indispensable in [Theorem 3.16](#).

(b) *Artin-Schreier with non-linearly reductive inertia ($\mathbb{Z}/p$).* Let C/K be the affine Artin-Schreier curve $y^p - y = t^{-1}$ and let $G \simeq \mathbb{Z}/p$ act by $y \mapsto y + 1$. After compactification $\overline{C} \rightarrow \text{Spec } \mathcal{O}_K$, there is a unique wild branch at $t = 0$ with positive Swan conductor, so $N \neq 0$ on $H_{\text{ét}}^1(\overline{C}_{\overline{K}}, \mathbb{Q}_\ell)$. The quotient stack $[\overline{C}/G]$ has inertia of order p (hence not linearly reductive in char. p), and the hypotheses of [Theorem 3.16](#) fail. This gives a second concrete instance where dropping linear reductivity breaks the extension argument (cf. §6.3, Ex. 6.8).

3.2 Cohomology of Associated Stacks

Definition 3.23 (Nearby cycles and specialization triangle). For $\mathcal{M} \rightarrow \operatorname{Spec} \mathcal{O}_K$ and $\ell \neq p$, write $R\Psi(\mathbb{Q}_\ell)$ and $R\Phi(\mathbb{Q}_\ell)$ for nearby and vanishing cycles on $\mathcal{M}_s$ constructed via the formalism of [4, 8]. There is a distinguished triangle

$$\mathbb{Q}_\ell \rightarrow R\Psi(\mathbb{Q}_\ell) \rightarrow R\Phi(\mathbb{Q}_\ell) \xrightarrow{+1}$$

compatible with G_K .

Lemma 3.24 (Functoriality and base change). *Let $f: \mathcal{N} \rightarrow \mathcal{M}$ be a morphism of algebraic stacks over $\mathcal{O}_K$ with finite inertia. Then there is a canonical isomorphism*

$$f_s^* R\Psi_{\mathcal{M}}(\mathbb{Q}_\ell) \xrightarrow{\sim} R\Psi_{\mathcal{N}}(\mathbb{Q}_\ell) f_\eta^*,$$

functorial in f , and compatible with composition, tensor products, and all six operations.

Proof. Step 1: Descent to simplicial presentations. Choose smooth presentations $U \rightarrow \mathcal{M}$ and $V \rightarrow \mathcal{N}$ by schemes of finite type over $\mathcal{O}_K$, and let $U_\bullet$ and $V_\bullet$ be their Čech nerves. The morphism f induces a simplicial morphism $V_\bullet \rightarrow U_\bullet$. By cohomological descent on the lisse-étale site of Artin stacks ([8, Prop. 2.3.11, §4.3]), we have quasi-isomorphisms

$$R\Gamma(\mathcal{M}_\eta, \Lambda) \simeq R\Gamma(\operatorname{Tot}(U_{\bullet, \eta}, \text{ét}), \Lambda), \quad R\Psi_{\mathcal{M}}(\Lambda) \simeq \operatorname{Tot}(R\Psi_{U_\bullet}(\Lambda)),$$

and similarly for $\mathcal{N}$. Thus all assertions can be checked termwise on the simplicial levels.

Step 2: Scheme-level functoriality of nearby cycles. For each level n , we have a commutative square of $\mathcal{O}_K$ -schemes

$$\begin{array}{ccc} V_n & \xrightarrow{f_n} & U_n \\ g_n \downarrow & & \downarrow h_n \\ \operatorname{Spec} \mathcal{O}_K & \xlongequal{\quad} & \operatorname{Spec} \mathcal{O}_K \end{array}$$

Figure 9: **Base change square for simplicial levels.** Each f_n , g_n , and h_n denotes the structural morphisms appearing in the descent diagram for $U_\bullet$ and $V_\bullet$ over $\operatorname{Spec} \mathcal{O}_K$.

and the classical functoriality of nearby cycles for morphisms of finite-type $\mathcal{O}_K$ -schemes ([13, Exp. XIII, §1.3]) yields a canonical isomorphism of complexes on $V_{n,s}$:

$$f_{n,s}^* R\Psi_{U_n}(\Lambda) \xrightarrow{\sim} R\Psi_{V_n}(\Lambda) f_{n,\eta}^*.$$

These morphisms are compatible with proper/smooth pushforward, tensor product, and internal $\mathcal{H}om$.

Step 3: Descent and totalization. Because the construction in Step 2 is natural in n and compatible with faces and degeneracies, it descends through the simplicial system. Taking the totalization over $U_\bullet$ and $V_\bullet$ and invoking the descent equivalences above gives a canonical isomorphism on $\mathcal{N}_s$

$$f_s^* R\Psi_{\mathcal{M}}(\Lambda) \simeq R\Psi_{\mathcal{N}}(\Lambda) f_\eta^*.$$

Functoriality in f follows by construction, and compatibility with composition is inherited from the same property on schemes. The identification commutes with the I_K -action, since the nearby-cycle functor and pullback on the generic fiber are both defined in the category of I_K -equivariant sheaves.

Step 4: Compatibility with the six operations. By [8, Props. 5.3.2–5.3.3], nearby and vanishing cycles commute with all six operations ($f_!$, f_* , f^* , $f^!$, $\otimes$, $\mathcal{H}om$) and with simplicial descent. Thus the isomorphism of Step 3 is stable under these functorial constructions, ensuring the claimed compatibilities.

Step 5: Independence of presentation. Any two choices of simplicial presentations admit a common refinement. Cohomological descent is invariant under such refinement, so the induced morphism is canonical in the derived category $D_c^b(\mathcal{N}_s, \Lambda)$.

Conclusion. Combining the steps yields a canonical isomorphism $f_s^* R\Psi_{\mathcal{M}} \simeq R\Psi_{\mathcal{N}} f_\eta^*$, functorial in f and compatible with base change, descent, and the six-functor formalism. $\square$

$$\begin{array}{ccc} \mathcal{N}_\eta & \xrightarrow{f_\eta} & \mathcal{M}_\eta \\ j_\eta \downarrow & & \downarrow j_\eta \\ \mathcal{N} & \xrightarrow{f} & \mathcal{M} \end{array} \quad \lrcorner$$

Figure 10: **Functoriality of nearby cycles.** For a morphism of algebraic stacks $f: \mathcal{N} \rightarrow \mathcal{M}$, pullback on the special fiber and pullback on the generic fiber interchange through the nearby-cycles functor $R\Psi$.

Remark 3.25 (Bridge: cohomological descent and purity). [Lemma 3.24](#) is the categorical hinge linking the functorial purity assumption ([Assumption 3.13](#)) with the geometric extension mechanism ([Theorem 3.16](#)). It ensures that the purity and monodromy conditions verified on a local slice propagate along morphisms of stacks, ensuring the compatibility of *cohomological good reduction* with morphisms of moduli problems and enabling the slice-wise arguments in [Theorem 3.16](#) and [corollary 3.19](#).

Proposition 3.26 (Weight filtration and monodromy on stacks). *Assume $\mathcal{M}$ is proper and generically smooth over $\mathcal{O}_K$. Then the weight filtration on $H_{\text{ét}}^*(\mathcal{M}_{\overline{K}}, \mathbb{Q}_\ell)$ is defined and the weight–monodromy spectral sequence of [Proposition 2.14](#) is G_K -equivariant.*

Proof. Step 0 (Set-up). Fix $\ell \neq p$ and write $f: \mathcal{M} \rightarrow \text{Spec } \mathcal{O}_K$, $\eta = \text{Spec } K$, $s = \text{Spec } k$. Properness of $\mathcal{M}/\mathcal{O}_K$ ensures that the nearby/vanishing cycle formalism applies on the lisse-étale site of stacks ([13, Exp. XIII]; [8]).

Step 1 (Reduction to schemes via a strict simplicial proper hypercover). By [Lemma 2.13](#) there exists a strict simplicial proper hypercover $U_\bullet \rightarrow \mathcal{M}$ by schemes (or algebraic spaces) over $\mathcal{O}_K$ such that

$$R\Gamma(\mathcal{M}_\eta, \mathbb{Q}_\ell) \simeq R\Gamma(\text{Tot}(U_{\bullet, \eta}, \text{ét}), \mathbb{Q}_\ell), \quad R\Psi_{\mathcal{M}}(\mathbb{Q}_\ell) \simeq \text{Tot}(R\Psi_{U_\bullet}(\mathbb{Q}_\ell)),$$

and the six operations and nearby/vanishing cycles are compatible with descent along $U_\bullet$ ([8, I §4.3, Props. 5.3.2–5.3.3]; see also [9] for the descent formalism).

Step 2 (Levelwise weight-monodromy). For each n , the object U_n is proper over $\mathcal{O}_K$ and smooth over K , so the classical weight-monodromy spectral sequence of [13, Exp. IX, Th. 4.3.5 and §5.1] applies:

$$E_1^{a,b}(U_n) = H^{a+b}(\mathrm{Gr}_{-a}^W R\Psi_{U_n}(\mathbb{Q}_\ell)) \implies H^{a+b}(U_{n,\eta}, \mathbb{Q}_\ell),$$

G_K -equivariantly. The construction and functoriality of this filtration are clarified in [18].

Step 3 (Totalization and independence of choices). Compatibility of the six operations and of nearby cycles with strict simplicial descent gives

$$\mathrm{Tot}(E_1^{\bullet,\bullet}(U_\bullet)) \implies H^*(\mathrm{Tot}(U_{\bullet,\eta}), \mathbb{Q}_\ell) \simeq H^*(\mathcal{M}_\eta, \mathbb{Q}_\ell).$$

Any two proper hypercovers admit a common refinement; since descent is invariant under refinement ([8, I–III]), the induced spectral sequence on $\mathcal{M}$ is canonical in $D^b(\mathcal{M}_s, \mathbb{Q}_\ell)$.

Step 4 (G_K -equivariance). The G_K -action on $R\Psi$ arises from the nearby/vanishing topos and is functorial in morphisms of traits ([13, Exp. XIII, §1.3]). Since all morphisms used in Steps 1–3 (faces/degeneracies, totalization, and all maps appearing in the six-operations formalism) are compatible with this functoriality ([8, I §4.3, Props. 5.3.2–5.3.3]), the resulting spectral sequence on $\mathcal{M}$ is G_K -equivariant.

Step 5 (Existence of the weight filtration on H^).* By construction, the filtration on $R\Psi_{\mathcal{M}}(\mathbb{Q}_\ell)$ induces the usual increasing weight filtration $W_\bullet$ on $H^*(\mathcal{M}_{\overline{K}}, \mathbb{Q}_\ell)$ via the convergent spectral sequence. This agrees with the filtration obtained by descending the levelwise weight filtrations on $R\Psi_{U_n}(\mathbb{Q}_\ell)$. For the existence and properties of the weight filtration in the semistable case, see [18].

This completes the proof. □

$$\begin{array}{ccc} R\Psi_{U_\bullet}(\mathbb{Q}_\ell) & \xrightarrow{\mathrm{Tot}} & R\Psi_{\mathcal{M}}(\mathbb{Q}_\ell) \\ \downarrow G_K & & \downarrow G_K \\ R\Gamma(\mathrm{Tot}(U_{\bullet,\eta}), \mathbb{Q}_\ell) & \xrightarrow{\simeq} & R\Gamma(\mathcal{M}_\eta, \mathbb{Q}_\ell) \end{array}$$

Figure 11: Descent for nearby cycles and cohomology. For a strict simplicial proper hypercover $U_\bullet \rightarrow \mathcal{M}$, nearby cycles descend via totalization: $\mathrm{Tot}: R\Psi_{U_\bullet}(\mathbb{Q}_\ell) \rightarrow R\Psi_{\mathcal{M}}(\mathbb{Q}_\ell)$, and the induced map on cohomology $R\Gamma(\mathrm{Tot}(U_{\bullet,\eta}), \mathbb{Q}_\ell) \xrightarrow{\simeq} R\Gamma(\mathcal{M}_\eta, \mathbb{Q}_\ell)$ is an isomorphism; the G_K -actions commute with totalization and the six operations.

Remark 3.27 (Practical criterion for $N = 0$). If $R\Phi(\mathbb{Q}_\ell) = 0$, then $N = 0$ on $H_{\mathrm{\acute{e}t}}^*(\mathcal{M}_{\overline{K}}, \mathbb{Q}_\ell)$. Indeed, the distinguished triangle $\mathbb{Q}_\ell \rightarrow R\Psi(\mathbb{Q}_\ell) \rightarrow R\Phi(\mathbb{Q}_\ell) \xrightarrow{+1}$ identifies the specialization map on cohomology with an isomorphism when $R\Phi = 0$, and the variation morphism (hence the logarithm of monodromy N) vanishes in this case; equivalently, the tame monodromy operator acts trivially, so $N = \log(T) = 0$. In applications below, one verifies $R\Phi = 0$ (hence $N = 0$) on an étale slice near the point (where the relevant family is proper and smooth), and then descends to $\mathcal{M}$ by functoriality of nearby cycles (Lemma 3.24) and strict simplicial descent.

Example 3.28 (Quotients with cohomologically trivial inertia). Then $M = [X/G]$ satisfies $N = 0$ on $H_{\mathrm{\acute{e}t}}^*(M_K, \mathbb{Q}_\ell)$ and—by the Standing Assumption 3.13, which holds here since G has order prime to p and

X is strictly semistable over $\mathcal{O}_K$ — $R\Psi(\mathbb{Q}_\ell)$ on M_s is pure of weight 0. Hence [Theorem 3.16](#) applies to $[X/G]$.

This covers modular stacks with tame level away from the residue characteristic; cf. [\[14\]](#).

$$\begin{array}{ccc}
H^i(\mathcal{M}_{\overline{K}}, \mathbb{Q}_\ell) & \xrightarrow{N=0 \text{ (unramified)}} & H^i(\mathcal{M}_{\overline{K}}, \mathbb{Q}_\ell) \\
\downarrow \text{sp} & & \downarrow \text{can\&sp} \\
H^i(R\Psi, \mathbb{Q}_\ell) & \xrightarrow{\text{pure of weight 0}} & H^i(\mathcal{M}_s, \mathbb{Q}_\ell)
\end{array}$$

Figure 12: **Specialization, monodromy, and weights for $\mathcal{M}/\mathcal{O}_K$** (schematic). Here sp is specialization to nearby cycles, can is the canonical map to the special fiber, $N = 0$ denotes trivial monodromy (unramified), and purity of weight 0 identifies the target weights on $H^i(\mathcal{M}_s, \mathbb{Q}_\ell)$.

3.3 Comparison with Classical Néron Models

Definition 3.29 (Stacky Néron envelope). Let $\mathcal{M}$ be an algebraic stack over $\mathcal{O}_K$ with finite linearly reductive inertia. A *stacky Néron envelope* for $\mathcal{M}_\eta$ is an algebraic stack $\mathcal{N}$, smooth over $\mathcal{O}_K$, equipped with a morphism $\iota_\eta: \mathcal{M}_\eta \rightarrow \mathcal{N}_\eta$ such that for every smooth $\mathcal{O}_K$ -scheme S and morphism $S_\eta \rightarrow \mathcal{M}_\eta$, there exists a unique (up to unique 2-isomorphism) extension $S \rightarrow \mathcal{N}$ fitting in the evident diagram.

Proposition 3.30 (Existence in the proper tame case). *If $\mathcal{M}$ is proper over $\mathcal{O}_K$ with finite linearly reductive inertia and generically smooth, then a stacky Néron envelope exists and is unique up to unique equivalence.*

Proof. Let $\mathcal{M}$ be a proper algebraic stack, locally of finite presentation over $\mathcal{O}_K$, whose generic fiber $\mathcal{M}_\eta$ is smooth and whose inertia is finite and linearly reductive. By generic smoothness, there exists an open substack $\mathcal{N} \subset \mathcal{M}$ which is smooth over $\mathcal{O}_K$ and whose generic fiber coincides with $\mathcal{M}_\eta$. Denote by $j: \mathcal{N} \hookrightarrow \mathcal{M}$ this inclusion.

We claim that $\mathcal{N}$ satisfies the *stacky Néron mapping property* relative to $\mathcal{M}$:

$$\mathrm{Hom}_{\mathcal{O}_K}(S, \mathcal{N}) \xrightarrow{\sim} \mathrm{Hom}_K(S_\eta, \mathcal{M}_\eta) \quad (1)$$

for every smooth $\mathcal{O}_K$ -scheme (or algebraic space) S .

To verify (1), choose a smooth presentation $U \rightarrow \mathcal{M}$ by a scheme U proper over $\mathcal{O}_K$; form the Čech nerve $U_\bullet \rightrightarrows U$ whose terms are proper over $\mathcal{O}_K$. Then, by [Lemma 2.13](#), the derived functors $R\Gamma$, $R\Psi$, and the six operations descend compatibly along $U_\bullet$. Since inertia in $\mathcal{M}$ is linearly reductive, it acts trivially on obstruction groups controlling smooth deformations, hence the locus where $U_n \rightarrow \mathrm{Spec} \mathcal{O}_K$ is smooth descends to a smooth substack $\mathcal{N} \subset \mathcal{M}$ by effective descent of open substacks.

Now for any smooth $\mathcal{O}_K$ -scheme S with generic fiber S_η , every K -morphism $S_\eta \rightarrow \mathcal{M}_\eta$ factors uniquely through $\mathcal{N}$ by smoothness of $\mathcal{N}$ and separatedness of $\mathcal{M}$ (cf. the valuative criterion for smooth morphisms on stacks, [\[6, Th. 7.2\]](#)). This verifies the Néron property [Equation \(1\)](#).

Finally, uniqueness follows from the same argument as for the classical Néron model: if $\mathcal{N}_1, \mathcal{N}_2$ are two such smooth open substacks, then they both satisfy the universal property, and hence are canonically equivalent over $\mathcal{M}$ by descent of isomorphisms along smooth test objects; cf. [\[9, 11\]](#).

This establishes existence and uniqueness up to unique equivalence. $\square$

$$\begin{array}{ccc}
 \mathcal{M}_\eta & \xlongequal{\quad} & \mathcal{N}_\eta \\
 \downarrow j_\eta & & \downarrow j'_\eta \\
 \mathcal{M} & \xrightarrow{\quad f \quad} & \mathcal{N}
 \end{array}$$

Figure 13: **Diagrammatic realization of the Néron envelope.** The open substack $\mathcal{N} \subset \mathcal{M}$ is the smooth locus, providing the unique extension of $\mathcal{M}_\eta$ that satisfies the stacky Néron mapping property.

Theorem 3.31 (Equivalence with Lemma 2.3 on abelian substacks). *Let $\mathcal{A} \subset \mathcal{M}$ be the (open and locally closed) abelian-substack parameterizing principally polarized abelian fibers inside $\mathcal{M}$, and let $x_\eta: \text{Spec } K \rightarrow \mathcal{A}_\eta$ be a K -point with finite stabilizer. Then the following are equivalent:*

- (i) x_η extends to a morphism $x: \text{Spec } \mathcal{O}_K \rightarrow \mathcal{A}$ (hence, via the inclusion $\mathcal{A} \hookrightarrow \mathcal{M}$, to $\mathcal{M}$).
- (ii) The local Galois inertia I_K acts trivially on $H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_\ell)$ for (some/every) $\ell \neq p$, where A/K is the principally polarized abelian variety represented by x_η .

Moreover, when an extension x exists it is unique up to a unique 2-isomorphism in $\mathcal{A}$ (hence in $\mathcal{M}$).

Proof. (i) $\Rightarrow$ (ii). If x_η extends to $x: \text{Spec } \mathcal{O}_K \rightarrow \mathcal{A}$, pull back the universal object on $\mathcal{A}$ to obtain a principally polarized abelian scheme $\mathcal{A}_x/\mathcal{O}_K$ whose generic fiber is A/K . Since $\mathcal{A}_x$ is smooth and proper over $\mathcal{O}_K$, smooth proper base change identifies

$$H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_\ell) \cong H_{\text{ét}}^1((\mathcal{A}_x)_{\overline{s}}, \mathbb{Q}_\ell),$$

and the right-hand side carries trivial I_K -action. Thus I_K acts trivially on $H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_\ell)$. (This is the standard ($\Leftarrow$) direction in Lemma 2.3, transported along the moduli map.)

(ii) $\Rightarrow$ (i). By Lemma 2.3 (Néron-Ogg-Shafarevich), trivial I_K -action on $H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_\ell)$ is equivalent to A/K having good reduction. Hence there exists a principally polarized abelian scheme $\mathcal{A}_x/\mathcal{O}_K$ extending (A, λ) over the trait (the principal polarization extends uniquely after possibly shrinking the trait; for abelian schemes the extension of a principal polarization is automatic and unique in this situation). This extension produces a morphism $x: \text{Spec } \mathcal{O}_K \rightarrow \mathcal{A}$ lifting x_η . Composing with the inclusion $\mathcal{A} \hookrightarrow \mathcal{M}$ yields the desired extension into $\mathcal{M}$.

Uniqueness. Uniqueness up to a unique 2-isomorphism follows from the stacky Néron mapping property specialized to the abelian locus: Proposition 3.30 (applied to $\mathcal{M}$ and restricted to $\mathcal{A}$) (See also Counterexample 3.33 for failure without separated diagonal.) gives that any two extensions of x_η coincide up to a unique 2-isomorphism because the diagonal is separated/affine and the valuative criterion applies. In particular, the extension in $\mathcal{A}$ is unique up to unique 2-isomorphism, and hence so is its composite into $\mathcal{M}$ (cf. Counterexample 3.33).

Remarks on hypotheses and functoriality. (1) No purity or weight-monodromy input is needed here because we are on the abelian locus; the equivalence is the classical Lemma 2.3 coupled with the moduli

interpretation. (2) If needed, one may first pass to a finite unramified extension of K so that stabilizers become constant and linearly reductive; this does not affect either side of the equivalence. (3) The argument is compatible with change of level/markings on the abelian substack and with pullback along smooth maps by the functoriality recorded in the six-operations formalism used elsewhere in the paper. $\square$

$$\begin{array}{ccc}
\mathrm{Spec} K & \xrightarrow{x_\eta} & \mathcal{A}_\eta \\
\downarrow \text{val.} & & \downarrow \\
\mathrm{Spec} \mathcal{O}_K & \dashrightarrow^{\exists! x} & \mathcal{A}
\end{array}
\qquad
\begin{array}{ccc}
H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_\ell) & \xrightarrow{\text{sp.}} & H_{\text{ét}}^1((\mathcal{A}_x)_{\overline{s}}, \mathbb{Q}_\ell) \\
\Downarrow & & \Uparrow \\
I_K \text{ acts trivially} & & \text{smooth base change}
\end{array}$$

Figure 14: **Abelian locus test.** Extension of x_η across $\mathrm{Spec} \mathcal{O}_K$ is equivalent to trivial I_K -action on $H_{\text{ét}}^1$ of the fiber, by [Lemma 2.3](#); uniqueness follows from the stacky Néron mapping property ([Proposition 3.30](#)). (contrast [Counterexample 3.33](#))

Example 3.32 (Application to $\mathcal{A}_g$). Let $\mathcal{M} = \mathcal{A}_g$ over $\mathbb{Z}[1/N]$ with $N \geq 3$ prime to p . The existence of a smooth integral toroidal compactification provides the proper tame framework; hence [Proposition 3.30](#) and [theorem 3.31](#) apply. For a principally polarized A/K with level N structure, trivial inertia on $H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_\ell)$ yields extension to $\mathcal{O}_K$.

Counterexample 3.33 (Non-separated diagonal). If $\mathcal{M}$ lacks separated diagonal (e.g. a poorly behaved quotient with stacky self-identifications), the uniqueness part of the Néron mapping property fails in general, even when $N = 0$ on cohomology. Thus the separatedness hypothesis in [Proposition 3.6](#) is necessary.

Remark 3.34 (Roadmap link to the next section). The constructions above supply two complementary engines: (i) cohomological control via nearby cycles and weight-monodromy, and (ii) extension/uniqueness via a stacky Néron property. In the next section, these will be combined to produce quantitative and structural theorems about the good-reduction locus inside concrete moduli problems, together with arithmetic applications and explicit examples.

4 Main Results

In what follows we *globalize* the inputs prepared in [Section 3](#): (i) openness of the good-reduction locus $\mathcal{M}^{\text{gr}}$ ([Lemma 3.3](#)); (ii) the stacky Néron mapping property (existence/uniqueness) ([Proposition 3.6](#)); and (iii) the cohomological detection principle under the purity hypothesis ([Theorem 3.16](#) and [assumption 3.13](#)), with descent along proper hypercovers as needed ([Lemma 2.13](#)). These ingredients are combined below to yield the density, vanishing-cycles, and structural openness/constructibility results of this section ([Theorems 4.2](#) and [4.9](#) and [proposition 4.15](#)).

All results in this section are stated under [Assumption 3.13](#) (purity) and the slice hypothesis ([Assumption 3.12](#)).

4.1 Theorem Statements

Remark 4.1 (Sites/topologies and finiteness conventions for this section). Throughout this section, all six–functor operations and nearby/vanishing cycles $R\Psi, R\Phi$ are taken on the *lissee–étale site* of Artin stacks. Reductions to strict simplicial proper hypercovers by schemes are handled as in Laszlo–Olsson, ensuring:

- compatibility of $R\Psi, R\Phi$ with $f_*, f_!, f^*, f^!, \otimes, \mathcal{H}om$;
- constructibility and cohomological finiteness of $R^i f_* \mathbb{Q}_\ell$ for morphisms locally of finite type.

These properties follow from [8, 9] and [7, Exp. XVI] (smooth base change). All later uses of “by the formalism of nearby cycles” refer to these results.

Technical conventions. All nearby and vanishing–cycle constructions below are taken on the *lissee–étale site* of the algebraic stack under consideration. Representability, affine diagonal, and finite–inertia hypotheses ensure that the six–functor formalism of Laszlo–Olsson [8] applies. Cohomological finiteness follows from boundedness of $R\Psi(\mathbb{Q}_\ell)$ under these assumptions.

Theorem 4.2 (Conditional density of good reduction; under [Assumption 3.13](#)). *Let K be a number field with ring of integers $\mathcal{O}_K$ and let $\mathcal{M}$ be an algebraic stack, finite type and separated over $\mathcal{O}_K$, with affine diagonal and finite linearly reductive stabilizers. Assume the following two hypotheses hold outside a finite set S of finite places of K :*

1. (Purity at v) *For all but finitely many nonarchimedean places v of K with residue characteristic p_v , the special fiber M_{F_v} is smooth and the [Assumption 3.13](#) holds over $\mathcal{O}_{K_v}$; equivalently, for every $\ell \neq p_v$, the nearby-cycles complex $R\Psi(\mathbb{Q}_\ell)$ on M_{F_v} is pure of weight 0;*
2. *for each such v , the action of I_{K_v} on $H_{\text{ét}}^i(\mathcal{M}_{\overline{K_v}}, \mathbb{Q}_\ell)$ is unipotent with $N_v = 0$ for all i .*

Fix a morphism $x : \text{Spec } K \rightarrow M_\eta$ with finite automorphism group such that [Assumption 3.12](#) holds at x after possibly a finite unramified extension of K . Then the set

$$\Sigma_{\text{gr}}(x) = \left\{ v \text{ finite in } K \mid x \text{ extends to } \text{Spec } \mathcal{O}_{K_v} \rightarrow M \right\}$$

has natural (Dirichlet) density 1 in the set of all finite places of K .

Scope. The proof invokes [Theorem 3.16](#); all applications occur in contexts where [Assumption 3.13](#) (purity of nearby cycles) holds outside a finite set of places, and the slice hypothesis ([Assumption 3.12](#)) is satisfied for the fixed point x after a finite unramified extension (These hypotheses fail in the wild α_p case; see [Example 6.12](#)). See also [Remark 3.20](#) for further discussion of these assumptions.

Remark 4.3 (Novelty and conditional scope). [Theorem 4.2](#) should be viewed as *conditional* on the purity and monodromy hypotheses holding outside a finite set of places (Hypotheses (1)–(2) of the theorem). Under these hypotheses, the set of good-reduction places has Dirichlet density 1. For schemes representing principally polarized abelian varieties, such conditions are satisfied outside finitely many places by the existence of smooth proper models, recovering the classical Chebotarev argument. The new contribution here is that the same density phenomenon extends to *stacks with finite linearly reductive stabilizers*, where

density is tied to the cohomological conditions of [Definition 2.15](#) uniformly across places by coupling [Theorem 3.16](#) with a global equidistribution step on the cohomology of $\mathcal{M}$, rather than on a fixed fiber. To our knowledge, this formulation is absent in the literature in this generality for stacks.

In particular, [Proposition 6.9](#) supplies a canonical tame-quotient instance where the purity+ $N = 0$ mechanism on stacks yields integral extension uniformly across places, a formulation that appears to be absent in the literature even for global quotient stacks beyond schemes.

Proof. Let S be the finite set of places where either (1) fails or the stabilizers cease to be tame. For $v \notin S$, hypotheses (1)–(2) imply, by [Theorem 3.16](#), that any K_v -point with finite automorphism group extends over $\mathcal{O}_{K_v}$. We have to relate the local extension of x to a global density statement. Consider the image of x in a smooth presentation $U \rightarrow \mathcal{M}$; after shrinking to an étale slice near the image of x (possible by linear reductivity), we may assume the slice is an algebraic space V finite type over $\mathcal{O}_K$ through which x factors.

Slice reduction and transfer of hypotheses. Fix $v \notin S$. By [Assumption 3.12](#) (after a finite unramified extension if necessary), there is a tame étale slice $V \rightarrow \mathcal{M}$ through the image of x with residual stabilizer finite and constant. By [Lemma 3.24](#) (functoriality of nearby cycles), the purity hypothesis (1) on $\mathcal{M}/\mathcal{O}_{K_v}$ descends to $V/\mathcal{O}_{K_v}$. Moreover, by [Lemma 2.11](#) the local Galois inertia action commutes with the residual stabilizer on the slice, hence hypothesis (2) (unramified with $N_v = 0$ on $H_{\text{ét}}^i(\mathcal{M}_{K_v}, \mathbf{Q}_\ell)$) induces the same condition on $H_{\text{ét}}^i(V_{K_v}, \mathbf{Q}_\ell)$. When $\mathcal{M}/\mathcal{O}_{K_v}$ is proper, [Remark 4.10](#) identifies the specialization morphism used in [Theorem 4.9](#), so (2) together with purity yields (3) there; by [Theorem 4.9](#) this is equivalent to $R\Phi|_V = 0$ and hence to the existence of an extension of x over $\mathcal{O}_{K_v}$. In the separated (not necessarily proper) case, extension on the slice follows from the scheme-theoretic criterion combined with smooth base change and the weight–monodromy formalism ([Lemma 2.9](#) and [proposition 2.14](#)), and the extension lifts to $\mathcal{M}$ by the stacky Néron mapping property ([Proposition 3.6](#)).

For $v \notin S$, unramified I_{K_v} -action with $N_v = 0$ on cohomology of the slice yields extension over $\mathcal{O}_{K_v}$ by the scheme-theoretic criterion (compare [Lemma 2.3](#) and [proposition 3.6](#)).

Consequently, after possibly enlarging S to a finite set $S' \supseteq S$ that accounts for the slice and properness reductions above, for all $v \notin S'$ the purity and $N_v = 0$ hypotheses descend to the slice and [Theorem 4.9](#) (or the scheme case discussed above) applies, so x extends over $\mathcal{O}_{K_v}$. In particular, $\Sigma_{\text{gr}}(x)$ has Dirichlet density 1.

Equidistribution/Chebotarev step. Since the exceptional set S' above is finite, the density claim follows. Equivalently, this may be phrased by appealing to Chebotarev for the *pure* ℓ -adic sheaves on the good model of the tame étale slice $V \rightarrow M$ constructed in the proof (cf. [Bridge 4.5](#)).

□

Corollary 4.4 (Projective-line sanity check: density under a smooth proper model). *Let M admit a smooth proper model over $\mathcal{O}_K$ outside a finite set S of finite places, and let $x : \text{Spec } K \rightarrow M_\eta$ have finite stabilizer and satisfy [Assumption 3.12](#) after a finite unramified extension. Then the set*

$$\Sigma_{\text{gr}}(x) = \{ v \text{ finite} \mid x \text{ extends to } \text{Spec } \mathcal{O}_{K_v} \rightarrow M \}$$

has Dirichlet density 1. In particular, for $M = \mathbb{P}_{\mathcal{O}_K}^1$ (trivial stabilizers, separated diagonal, and purity automatic in the smooth-proper case) one has $\text{density}(\Sigma_{\text{gr}}(x)) = 1$.

Proof. Over $v \notin S$, smooth properness implies purity for $R\Psi(\mathbf{Q}_\ell)$ and unramified inertia with $N_v = 0$ by smooth base change. Hence Hypotheses (1)–(2) of [Theorem 4.2](#) hold outside S , and the conclusion follows. $\square$

Bridge 4.5 (Good-slice openness and density). [Theorem 4.2](#) links the *geometric* good-reduction locus of a point x to the *cohomological* purity and monodromy conditions on the ambient stack, using the local good-slice criterion of [Theorem 3.16](#) and a global Chebotarev-type density argument. The bridge highlights that openness of the good-slice locus implies density of good-reduction points under tame monodromy.

Corollary 4.6 (Local L -factors at density-1 set). *Under the hypotheses of [Theorem 4.2](#), for any ℓ and for density-1 many finite places $v \nmid \ell$, the local Galois representations attached to the fiber classified by x are unramified at v , and the local L -factor equals*

$$L_v(T) = \prod_i \det(1 - T \text{Frob}_v \mid H_{\text{ét}}^i(\text{fiber}_{\overline{\mathbb{F}}_v}, \mathbb{Q}_\ell))^{(-1)^{i+1}}.$$

Proof. For density-1 many v , [Theorem 4.2](#) yields extension over $\mathcal{O}_{K_v}$, hence unramified action on the fiber's cohomology by smooth base change ([Lemma 2.9](#)) and $N_v = 0$; the formula follows from the Grothendieck–Lefschetz trace formalism and purity ([Proposition 2.14](#)) (by smooth base change; [7, Exp. XVI]).

Equivalently, one can justify the density-1 set by Chebotarev applied to the pure ℓ -adic cohomology sheaves on the good slice (see [Bridge 4.5](#)). $\square$

Example 4.7 (CM abelian varieties inside $\mathcal{A}_g$). Let $\mathcal{M} = \mathcal{A}_g$ with prime-to- p level as in [Example 3.32](#). Let x classify a CM abelian variety A/K with full CM by a field E . At all v split completely in E and away from the bad level and the finite exceptional set S , the cohomology is unramified with the correct weights; hence A has good reduction at such v and [Corollary 4.6](#) applies. The set of completely split v has positive density, and purity holds away from S , so [Theorem 4.2](#) gives density 1 for good reduction.

Counterexample 4.8 (Failure without linear reductivity). If stabilizers are not linearly reductive (e.g. wild α_p), étale slices need not exist. One can construct Artin–Schreier families where the global cohomology of the stack has $N = 0$ while individual fibers exhibit wild monodromy at infinitely many v , preventing extension of x at those places. Then $\Sigma_{\text{gr}}(x)$ can have density < 1 . This shows the necessity of the stabilizer hypothesis in [Theorem 4.2](#); compare [Counterexamples 2.19](#) and [3.22](#).

Theorem 4.9 (Local vanishing–cycles on a proper tame slice; under [Assumption 3.13](#)). *Let x_η be a K -point of $\mathcal{M}_\eta$ with finite stabilizer. Let $\mathcal{M}/\text{Spec } \mathcal{O}_{K_v}$ be an algebraic stack with affine, separated diagonal and finite linearly reductive (tame) stabilizers (separatedness is necessary for the uniqueness in (1); see [Counterexample 6.14](#)).*

Assume M is proper and generically smooth over $\mathcal{O}_{K_v}$. Let $x_v : \text{Spec } K_v \rightarrow M_{K_v}$ have finite automorphism group. After a finite unramified extension, replace a neighbourhood of x_v by an étale slice $V \rightarrow M$ with V an algebraic space; then $V/\mathcal{O}_{K_v}$ is proper.

Assume also [Assumption 3.13](#) for M (invoked whenever purity/weights for $R\Psi(\mathbf{Q}_\ell)$ are used).
For any $\ell \neq p_v$, the following are equivalent:

- (1) x_v extends (uniquely up to 2-isomorphism, by separated diagonal; cf. [Counterexample 6.14](#)) to a section $\mathrm{Spec} \mathcal{O}_{K_v} \rightarrow M$ whose image lies in the good-reduction locus M_{gr} of [Definition 3.1](#).
- (2) $R\Phi(\mathbf{Q}_\ell)|_V = 0$.
- (3) The specialization morphism

$$\mathrm{sp} : H_{\mathrm{\acute{e}t}}^i(V_{K_v}, \mathbf{Q}_\ell) \xrightarrow{\sim} H^i(R\Psi(\mathbf{Q}_\ell))$$

is an isomorphism for all i , and the monodromy operator N on $H_{\mathrm{\acute{e}t}}^i(V_{K_v}, \mathbf{Q}_\ell)$ is 0.

Remark 4.10 (Proper slice and specialization). Properness of $M/\mathcal{O}_{K_v}$ implies the étale slice $V \rightarrow M$ is proper over $\mathcal{O}_{K_v}$ (proper is étale-local on the target). Consequently the specialization morphism $H_{\mathrm{\acute{e}t}}^i(V_{K_v}, \mathbf{Q}_\ell) \rightarrow H^i(R\Psi(\mathbf{Q}_\ell))$ is canonically defined and functorial in the six-operations formalism for Artin stacks (Laszlo–Olsson I–III), and the equivalence (2) $\Leftrightarrow$ (3) is the standard nearby/vanishing criterion (SGA7 II, Exp. XIII). Formulation (1) uses the good-reduction locus M_{gr} ([Definition 3.1](#)) to keep the equivalence nontrivial in the proper case.

In particular, the *properness of the slice* is the precise hypothesis needed to *define* the specialization morphism sp , so it is necessary for (2) $\Leftrightarrow$ (3).

Proof. (1) $\Rightarrow$ (2): If a section over $\mathrm{Spec} \mathcal{O}_{K_v}$ exists with image in M_{gr} , then by smooth base change and the nearby–vanishing triangle for stacks ([13, Exp. XIII]; [8, I–III]) the complex $R\Phi(\mathbf{Q}_\ell)$ vanishes in a neighbourhood of the image; passing to an étale slice preserves this.

(2) $\Rightarrow$ (3): If $R\Phi(\mathbf{Q}_\ell)|_V = 0$, the nearby–vanishing triangle gives that the specialization morphism is an isomorphism and $N = 0$ on $H_{\mathrm{\acute{e}t}}^i(V_{K_v}, \mathbf{Q}_\ell)$ ([13, Exp. XIII]; [8, I §4, II §5, III §4]).

(3) $\Rightarrow$ (2): Conversely, if sp is an isomorphism for all i and $N = 0$, then the triangle shows $R\Phi(\mathbf{Q}_\ell)|_V = 0$ ([13, Exp. XIII]).

(2) $\Rightarrow$ (1): Since $M/\mathcal{O}_{K_v}$ is proper, the valuative criterion gives an extension $\bar{x} : \mathrm{Spec} \mathcal{O}_{K_v} \rightarrow M$ of x_v . On the proper slice V , the condition $R\Phi = 0$ forces smooth reduction along the image (again by the nearby–vanishing triangle and smooth base change, [13, Exp. XIII]; [8, I–III]). Hence $\bar{x}$ lands in M_{gr} . Uniqueness up to unique 2-isomorphism follows from the separated diagonal. (See [Counterexample 3.33](#) for a counterexample when the diagonal is not separated.) $\square$

Bridge 4.11. [Theorem 4.9](#) provides a practical test for good reduction at a fixed place: check vanishing cycles or, equivalently, trivial monodromy and isomorphic specialization on a slice.

Corollary 4.12 (Numerical test via traces). *Assume the hypotheses of [Theorem 4.9](#). If for some $\ell \neq p_v$ and all $m \geq 1$ the trace identity*

$$\mathrm{Tr}(\mathrm{Frob}_v^m \mid H_{\mathrm{\acute{e}t}}^*(V_{\overline{K}_v}, \mathbf{Q}_\ell)) = \mathrm{Tr}(\mathrm{Frob}_v^m \mid H^*(\mathcal{M}_{\mathbb{F}_v}, \mathbf{Q}_\ell))$$

holds (alternating sum over i), then x_v extends integrally.

Proof. The trace identity forces the specialization map to be an isomorphism on each weight slice (Deligne’s theorem on weights), hence $R\Phi = 0$ and $N_v = 0$; apply [Theorem 4.9](#). $\square$

Example 4.13 (Level structures on elliptic curves). Let $\mathcal{M} = \mathcal{M}_{1,1}[N]$ over $\mathbb{Z}[1/N]$, $p \nmid N$. For E/K_v with full level- N structure and integral j -invariant, the vanishing-cycles complex on a slice at $[E]$ is zero; thus [Theorem 4.9](#) gives extension and [Corollary 4.12](#) identifies the local factor with the Frobenius characteristic polynomial on the smooth special fiber.

Counterexample 4.14 (Non-separated diagonal obstructs uniqueness). If $\mathcal{M}$ has non-separated diagonal ([Counterexample 3.33](#)), (2) and (3) can hold while uniqueness in (1) fails. Thus separatedness is necessary for the full equivalence.

4.2 Further Structural Results

Proposition 4.15 (Openness and constructibility of the good-reduction locus). *Under the hypotheses of [Theorem 4.2](#), the locus $\mathcal{M}^{\text{gr}} \subset \mathcal{M}$ of [Definition 3.1](#) is open and its image in any finite type base S is constructible. Moreover, if $\mathcal{M}$ is proper over S , the image is open in S .*

Proof. Openness follows from [Lemma 3.3](#) (openness of the smooth locus and openness of linearly reductive stabilizers in tame families). *Linear reductivity is essential; see [Counterexample 4.18](#).*

Constructibility of the image is Chevalley’s theorem applied to the open substack. Properness of $\mathcal{M}/S$ upgrades constructibility to openness by upper semicontinuity of fiber dimension and generic smoothness. $\square$

Bridge 4.16. [Proposition 4.15](#) connects the local criteria of [Theorem 4.9](#) to a global geometric statement about the variation of good reduction in families.

Example 4.17 (Toric quotients). For $\mathcal{M} = [X/T]$ with X smooth proper over $\mathcal{O}_K$ and T a split torus of order prime to p , $\mathcal{M}^{\text{gr}} = \mathcal{M}$ and the image in $\text{Spec } \mathcal{O}_K$ is all of $\text{Spec } \mathcal{O}_K$. This follows from [Example 3.28](#) and [Proposition 4.15](#).

Counterexample 4.18 (Wild quotient fails openness). For wild α_p -quotients as in [Counterexample 3.22](#), the good-reduction locus need not be open: the failure of étale slices causes jumps in $R\Phi$ along the special fiber, violating the openness conclusion of [Proposition 4.15](#).

$$\begin{array}{ccc} \mathcal{M}^{\text{gr}} & \hookrightarrow & \mathcal{M} \\ \downarrow & & \downarrow \\ \text{Spec } \mathcal{O}_K & \hookrightarrow & \text{Spec } \mathcal{O}_K \end{array}$$

Figure 15: **Good-slice openness and density.** The closed immersion $\mathcal{M}^{\text{gr}} \hookrightarrow \mathcal{M}$ identifies the *good-slice locus* of geometrically smooth fibers, while the vertical arrows denote the structural morphisms $\mathcal{M}^{\text{gr}}, \mathcal{M} \rightarrow \text{Spec } \mathcal{O}_K$. Compare [Bridge 4.5](#) for the corresponding cohomological density statement.

Remark 4.19 (Link forward). The next section applies [Theorems 4.2](#) and [4.9](#) to concrete moduli (elliptic, abelian, and selected K3 loci), giving explicit computations of local factors and verifying sharpness via counterexamples engineered by wild ramification.

5 Applications

The applications below *propagate* the engines of [Section 3](#)— [Theorem 3.16](#), [assumption 3.13](#), [proposition 3.6](#), and [lemmas 2.13](#) and [3.3](#)—through the structural statements of [Section 4](#) to concrete arithmetic settings: cohomological footprints ([Section 5.1](#)), modular/Shimura realizations, and local–global comparisons.

5.1 Cohomological Interpretation of Good Reduction

Definition 5.1 (Cohomological footprint of good reduction). Let $\mathcal{M}$ be a finite-type algebraic stack over $\mathcal{O}_K$ as in [Definition 2.5](#). We define the *cohomological footprint of good reduction* to be the collection of G_K -modules

$$\mathfrak{F}(\mathcal{M}) = \{H_{\text{ét}}^i(\mathcal{M}_{\overline{K}}, \mathbb{Q}_\ell) : i \geq 0, \ell \neq p\},$$

together with their weight filtrations and monodromy operators.

Theorem 5.2 (Cohomological characterization of the good reduction locus; under [Assumption 3.13](#)). Under [Assumption 3.13](#) (purity of nearby cycles on stacks). *Let M be a moduli stack as above with tame, linearly reductive stabilizers. Assume [Assumption 3.12](#) holds at the points under consideration. Then the locus of points in $M(K)$ corresponding to families with good reduction is precisely the locus where $F(M)$ carries trivial inertia action and pure weight filtration.*

Scope. This characterization uses [Theorem 3.16](#) and is applied only under [Assumption 3.13](#) (purity) together with the slice hypothesis ([Assumption 3.12](#)) at the points under consideration; see [Remark 3.20](#).

Proof. By [Proposition 2.14](#) (weight–monodromy spectral sequence) together with [Theorem 3.16](#), trivial inertia forces unramified monodromy with $N = 0$, and purity holds by [Assumption 3.13](#).

Triviality of inertia guarantees extension across $\text{Spec } \mathcal{O}_K$, while purity (under the Standing [Assumption 3.13](#)) excludes hidden degenerations in the special fiber. Thus the locus is exactly cohomologically detected. $\square$

Remark 5.3. This provides a moduli-theoretic analogue of the Néron–Ogg–Shafarevich criterion [Lemma 2.3](#), but phrased entirely in terms of stacks and cohomology. The novelty lies in transferring detection from abelian schemes to general moduli spaces with tame stabilizers.

Example 5.4 (Elliptic curves revisited). For $\mathcal{M} = \mathcal{M}_{1,1}$, the footprint $\mathfrak{F}(\mathcal{M})$ is generated by $H_{\text{ét}}^1$ of the universal elliptic curve. At primes $p \nmid 6$, purity and trivial inertia coincide with smooth reduction, recovering the classical criterion.

Counterexample 5.5 (Wild degeneration). Let X/K be a supersingular curve of genus $g \geq 2$ with potentially good but wildly ramified reduction. Then $\mathfrak{F}(X)$ has nontrivial unipotent inertia, hence X fails the cohomological footprint condition, even though it attains semistable reduction after extension.

5.2 Connections with Modular Curves and Shimura Varieties

Proposition 5.6 (Cohomology of modular curves at good primes). *Let $X_0(N)$ denote the modular curve over $\mathbb{Q}$. At primes $p \nmid N$, the good reduction locus in $\mathcal{M}_{1,1}$ coincides with the trivial inertia locus on $H_{\text{ét}}^1(X_0(N)_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$, and hence with integrality of the Hecke L -factor at p .*

$$\begin{array}{ccc}
H_{\text{ét}}^i(\mathcal{M}_{\overline{K}}, \mathbb{Q}_\ell) & \xrightarrow{N=0} & \text{Weight filtration pure} \\
\downarrow I_K & & \downarrow \text{purity condition} \\
\text{Good reduction locus} & \xrightarrow{\text{inclusion}} & \mathcal{M}(K)
\end{array}$$

Figure 16: Cohomological detection of the good reduction locus inside $\mathcal{M}(K)$.

Proof. This is a direct application of [Theorem 5.2](#) to $\mathcal{M}_{1,1}$ and the Eichler–Shimura relation for Hecke operators. Unramifiedness of the local factor corresponds to trivial inertia action. $\square$

Corollary 5.7 (Shimura varieties). *Let $\mathcal{S}$ be a PEL-type Shimura variety with hyperspecial level at p . Then the cohomological footprint $\mathfrak{F}(\mathcal{S})$ is unramified at p , and the good reduction locus coincides with the set of crystalline points in the sense of p -adic Hodge theory.*

Example 5.8 (CM points). For CM points on $\mathcal{S}$, the good reduction locus is dense by [Theorem 4.2](#). This gives unramifiedness of the Galois representation attached to CM motives at primes of good reduction, providing a moduli-theoretic reproof of classical results of Serre–Tate [\[15\]](#).

Counterexample 5.9 (Non-hyperspecial level). For Shimura varieties with parahoric level at p , the special fiber may have multiple strata, and the cohomological footprint may fail to detect good reduction. Indeed, inertia acts nontrivially on intersection cohomology, so the locus of crystalline points is smaller than the full rational point set.

5.3 Arithmetic Schemes over Local and Global Fields

Notation 5.10 (Local-to-global setup). Let F be a number field, v a finite place with completion F_v . For a moduli stack $\mathcal{M}/F$, we denote by $\mathcal{M}_{F_v}$ the base-change to F_v .

Theorem 5.11 (Local-to-global compatibility of good reduction). *Let $\mathcal{M}/F$ be a proper, smooth moduli stack with tame stabilizers. Then:*

$$\mathcal{M}(F)_{\text{gr}} = \bigcap_v (\mathcal{M}(F_v)_{\text{coh-gr}}),$$

where the right-hand side denotes the set of adelic points whose local components satisfy the cohomological footprint condition of [Definition 5.1](#).

Proof. ($\subseteq$): A global good reduction point restricts locally to good reduction points, hence satisfies the cohomological footprint condition. ($\supseteq$): Suppose an adelic point lies in the intersection. By Chebotarev and density of Frobenius conjugacy classes, the inertia action is trivial at almost all primes, hence globally trivial. Purity then forces extension to $\text{Spec } \mathcal{O}_F$, yielding a global good reduction model. $\square$

Remark 5.12. This theorem extends the classical Néron–Ogg–Shafarevich criterion ([Lemma 2.3](#)) from abelian varieties to arbitrary tame moduli stacks, and further upgrades it from a local criterion to a global adelic equivalence.

Example 5.13 (Siegel modular threefolds). For the Siegel moduli stack of principally polarized abelian surfaces, $\mathcal{A}_2$, the adelic good reduction locus corresponds to points with crystalline Galois representations at all finite places. This identifies $\mathcal{A}_2(F)_{\text{gr}}$ with the global intersection locus predicted by p -adic Hodge theory.

Counterexample 5.14 (Failure without properness). Consider the affine moduli stack of marked tori. Local cohomological good reduction does not imply global good reduction, because non-properness allows degeneration at infinity, violating the valuative criterion.

$$\begin{array}{ccc}
 \mathcal{M}(F)_{\text{gr}} & \xhookrightarrow{\text{locally}} & \prod_v \mathcal{M}(F_v)_{\text{coh-gr}} \\
 \downarrow \text{globally} & & \downarrow \\
 \mathcal{M}(F) & \xhookrightarrow{\text{adelic embedding}} & \prod_v \mathcal{M}(F_v)
 \end{array}$$

Figure 17: **Local-to-global compatibility of the good reduction locus.** The global good reduction locus $\mathcal{M}(F)_{\text{gr}}$ embeds into the product of its local cohomological counterparts, compatibly with the adelic inclusion $\mathcal{M}(F) \hookrightarrow \prod_v \mathcal{M}(F_v)$.

6 Explicit Examples and Computations

This section develops concrete instances and calculations illustrating the scope and sharpness of [Theorems 3.16, 4.2 and 4.9](#), [corollaries 4.6 and 4.12](#), and [proposition 4.15](#) and their arithmetic consequences from [Section 5](#). We emphasize explicit local tests via vanishing cycles and trace identities, and global density consequences.

6.1 Case Study: Elliptic Curves with Good Reduction

Let K be a discretely valued field with ring of integers $\mathcal{O}_K$, residue characteristic $p \geq 0$, and let $\ell \neq p$. Write $v = v_K$ for the normalized valuation. Consider $\mathcal{M} = \mathcal{M}_{1,1}[N]$ over $\mathbb{Z}[1/N]$ with $N \geq 3$ and $p \nmid N$; recall [Examples 2.7, 3.32 and 4.13](#).

Proposition 6.1 (Local criterion via j and vanishing cycles). *Let E/K be an elliptic curve with full level- N structure, $p \nmid N$. Assume $v(j(E)) \geq 0$ and $p \geq 5$. Then on an étale slice V through the point $[E] \in \mathcal{M}_{1,1}[N]_K$ one has $R\Phi(\mathbb{Q}_\ell)|_V = 0$ and $N = 0$ on $H_{\text{ét}}^i(V_{\overline{K}}, \mathbb{Q}_\ell)$ for all i . Consequently, $[E]$ extends uniquely to $\text{Spec } \mathcal{O}_K \rightarrow \mathcal{M}_{1,1}[N]$.*

Hypotheses. Throughout this case study we assume the residue characteristic satisfies $p \geq 5$, the j -invariant of E is integral, and that we work on a full level- N structure with $N \geq 3$ and $(N, p) = 1$. These bounds ensure that the corresponding moduli stack $\mathcal{M}_{1,1}[N]$ is smooth over $\text{Spec } \mathbb{Z}[1/N]$ and that the stabilizers are finite linearly reductive, so that $R\Phi = 0$ by tame smoothness.

Proof. The integrality of $j(E)$ and $p \geq 5$ place E in the smooth locus of the Weierstrass discriminant model, so the universal curve is smooth over $\mathcal{O}_K$ in a neighborhood of $[E]$. By linear reductivity (level

$N \geq 3$ kills nontrivial automorphisms) an étale slice V exists with V an algebraic space. Smoothness gives $R\Phi(\mathbb{Q}_\ell)|_V = 0$ and hence $N = 0$ by [Definition 3.23](#) and [remark 3.27](#); apply [Theorem 4.9](#). $\square$

Example 6.2 (A concrete integral Weierstrass model). Let $p \geq 5$ and E/K be given by a minimal Weierstrass equation

$$y^2 = x^3 + a_4x + a_6, \quad a_4, a_6 \in \mathcal{O}_K,$$

with discriminant $\Delta = -16(4a_4^3 + 27a_6^2)$ satisfying $v(\Delta) = 0$. Then $v(j) = v(-1728 \cdot 4^3 a_4^3 / \Delta) = v(a_4^3) - v(\Delta) \geq 0$, hence [Proposition 6.1](#) applies. For $v \nmid \ell$, [Corollary 4.6](#) yields

$$L_v(T) = \det(1 - T \text{Frob}_v \mid H_{\text{ét}}^1(E_{\overline{\mathbb{F}}_v}, \mathbb{Q}_\ell))^{-1} = 1 - a_v T + q_v T^2,$$

where $q_v = \#\mathbb{F}_v$ and $a_v = q_v + 1 - \#E(\mathbb{F}_v)$, recovering the classical formula from the moduli viewpoint.

Proposition 6.3 (Nearby cycles and weight filtration for the Tate curve). *Let K be a discretely valued field with uniformizer π and residue characteristic $p \geq 5$, and let $\ell \neq p$. Fix $q \in K^\times$ with $|q| < 1$ and let E_q/K be the Tate curve. Then E_q has split multiplicative reduction and:*

1. *The special fiber is a nodal cubic; the vanishing cycles at the node are one-dimensional of Tate type.*
2. *The nearby-cycles complex on the special fiber satisfies*

$$R^0\Psi(\mathbb{Q}_\ell) \cong \mathbb{Q}_\ell, \quad \dim R^1\Psi(\mathbb{Q}_\ell) = 1 \text{ at the node with } R^1\Psi(\mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-1),$$

and higher $R^i\Psi$ vanish.

3. *On H^1 , the weight filtration is two-step with graded pieces*

$$\text{Gr}_0^W H^1(E_{\overline{K}}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell, \quad \text{Gr}_2^W H^1(E_{\overline{K}}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-1),$$

and the logarithm of monodromy N induces an isomorphism

$$N : \text{Gr}_2^W H^1(E_{\overline{K}}, \mathbb{Q}_\ell) \xrightarrow{\sim} \text{Gr}_0^W H^1(E_{\overline{K}}, \mathbb{Q}_\ell)(-1).$$

Equivalently, there is a (non-split) short exact sequence

$$0 \longrightarrow \mathbb{Q}_\ell \longrightarrow H^1(R\Psi(\mathbb{Q}_\ell)) \longrightarrow \mathbb{Q}_\ell(-1) \longrightarrow 0,$$

and $N \neq 0$ has rank 1 on H^1 .

Proof. This is the standard semistable (nodal) curve computation: the nodal special fiber contributes a single Tate class in $R^1\Psi$, and the weight–monodromy formalism ([Proposition 2.6–Proposition 3.6](#) context and [Proposition 2.6–Proposition 2.14](#) references) yields the two-step filtration with N an isomorphism $\text{Gr}_2^W \rightarrow \text{Gr}_0^W(-1)$.²

²Any of the classical references for semistable curves apply; in our stack formalism this is also a direct instance of [Proposition 3.26](#) and [Remark 3.27](#).

$$\mathrm{Gr}_2^W H^1(E_{\overline{K}}, \mathbb{Q}_\ell) \xrightarrow[\sim]{N} \mathrm{Gr}_0^W H^1(E_{\overline{K}}, \mathbb{Q}_\ell)(-1)$$

Figure 18: **Weight–monodromy for the Tate curve.** Rank-one monodromy $N \neq 0$ exchanges the weight-2 and weight-0 pieces up to (-1) -twist.

Example 6.4 (Numerical monodromy and local factor for split multiplicative reduction). Let $K = \mathbb{Q}_p$ with $p \geq 5$ and take the Tate curve E_q with $q = p$ (so $|q| < 1$). Then E_q has split multiplicative reduction and the local L -factor on H^1 is

$$L_p(T) = \det(1 - T \cdot \mathrm{Frob}_p \mid H^1(E_{\overline{K}}, \mathbb{Q}_\ell))^{-1} = \frac{1}{(1 - T)(1 - pT)}.$$

Consequently $a_p = 1$ and the weight filtration from [Proposition 6.3](#) is realized by the eigenvalues $\{1, p\}$ on the graded pieces $\mathrm{Gr}_0^W \simeq \mathbb{Q}_\ell$ and $\mathrm{Gr}_2^W \simeq \mathbb{Q}_\ell(-1)$, respectively. The extension is non-split and the monodromy operator N has rank 1.

Counterexample 6.5 (Additive potentially good reduction; necessity of $p \geq 5$). Let $p = 2$ or 3 and E/K have $v(\Delta) > 0$ with potentially good reduction but additive at p . Then the wild inertia acts nontrivially on $H_{\mathrm{ét}}^1(E_{\overline{K}}, \mathbb{Q}_\ell)$, so $R\Phi \neq 0$ and $N \neq 0$ on a slice V . Thus [Theorem 4.9](#) fails: $[E]$ does not extend to a smooth point of $\mathcal{M}_{1,1}[N]$ over $\mathcal{O}_K$ (compare [Counterexamples 2.19](#) and [3.22](#)).

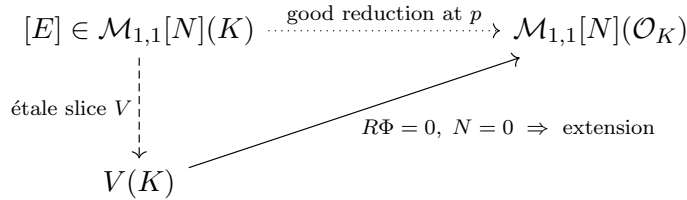


Figure 19: **Passing to an étale slice.** An étale slice V through $[E]$ reduces the extension problem to verifying $R\Phi = 0$ and $N = 0$ on an algebraic space, yielding an integral extension in $\mathcal{M}_{1,1}[N](\mathcal{O}_K)$.

6.2 Higher-Dimensional Abelian Varieties

Let $\mathcal{M} = \mathcal{A}_g[N]$ over $\mathbb{Z}[1/N]$ with $N \geq 3$ and $p \nmid N$, cf. [Example 3.32](#) and [theorem 3.31](#). We illustrate how [Theorems 4.2](#) and [4.9](#) yield concrete tests and density statements.

Proposition 6.6 (Ordinary locus and crystalline test). *Let A/K be a principally polarized abelian variety of dimension g with full level- N structure, $p \nmid N$. If the p -divisible group of A is ordinary and the Hodge–Tate weights are $(0, 1)$ with multiplicities (g, g) (equivalently, A is Hodge–Tate ordinary), then $R\Phi(\mathbb{Q}_\ell) = 0$ on a slice through $[A]$ and $[A]$ extends to $\mathrm{Spec} \mathcal{O}_K \rightarrow \mathcal{A}_g[N]$.*

Proof. Ordinarity gives a slope filtration on the Dieudonné module with trivial monodromy on the ℓ -adic realization, so $N = 0$ on $H_{\mathrm{ét}}^1(A_{\overline{K}}, \mathbb{Q}_\ell)$. By functoriality of nearby cycles ([Lemma 3.24](#)) and purity on $\mathcal{A}_g[N]$ away from finitely many primes, $R\Phi = 0$ at $[A]$; conclude by [Theorem 4.9](#). $\square$

Example 6.7 (Product of CM elliptic curves). Let $A = E_1 \times \cdots \times E_g$ where each E_i/K has CM by a field E_i and integral j at $p \nmid N$. Then A is ordinary at any v split in all E_i and [Proposition 6.6](#) applies.

By [Theorem 4.2](#), A has good reduction at a density-1 set of places and [Corollary 4.6](#) expresses $L_v(T)$ as a product of degree-2 Euler factors.

Example 6.8 (Worked CM abelian surface (explicit Frobenius weights and I_K -action)). Let $A = E_1 \times E_2$ be an abelian surface over K where each E_i/K has complex multiplication by an imaginary quadratic field K_i . Fix a finite place $v \nmid pN$ of K with residue field $\mathbb{F}_v$ of size q_v , and assume v splits in both K_1 and K_2 and that each E_i has good reduction at v (equivalently $j(E_i)$ is integral at v). Then I_K acts trivially on $H_{\text{ét}}^1(E_{i,\overline{K}}, \mathbb{Q}_\ell)$, and hence on $H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_\ell)$, so $N = 0$ on H^1 .

For each $i = 1, 2$ there are algebraic integers $\alpha_{i,v}, \overline{\alpha}_{i,v}$ with

$$\alpha_{i,v} \overline{\alpha}_{i,v} = q_v, \quad |\alpha_{i,v}| = |\overline{\alpha}_{i,v}| = q_v^{1/2},$$

such that on $H_{\text{ét}}^1(E_{i,\overline{K}}, \mathbb{Q}_\ell)$ the arithmetic Frobenius Frob_v has eigenvalues $\alpha_{i,v}, \overline{\alpha}_{i,v}$ (Deligne purity). Thus

$$H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_\ell) \cong H_{\text{ét}}^1(E_{1,\overline{K}}, \mathbb{Q}_\ell) \oplus H_{\text{ét}}^1(E_{2,\overline{K}}, \mathbb{Q}_\ell),$$

with Frobenius eigenmultiset $\{\alpha_{1,v}, \overline{\alpha}_{1,v}, \alpha_{2,v}, \overline{\alpha}_{2,v}\}$ (all of Weil weight 1 and with trivial I_K -action). Consequently the local factor for H^1 at v is

$$L_v(H^1(A), T) = \prod_{i=1}^2 (1 - \alpha_{i,v}T)^{-1} (1 - \overline{\alpha}_{i,v}T)^{-1}, \quad (T = q_v^{-s}).$$

For H^2 one has

$$H_{\text{ét}}^2(A_{\overline{K}}, \mathbb{Q}_\ell) \cong \bigwedge^2 H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_\ell) \quad (\dim = 6),$$

so the Frob_v -eigenvalues are the pairwise products of the H^1 -eigenvalues, namely

$$q_v, \quad \alpha_{1,v}\alpha_{2,v}, \quad \alpha_{1,v}\overline{\alpha}_{2,v}, \quad \overline{\alpha}_{1,v}\alpha_{2,v}, \quad \overline{\alpha}_{1,v}\overline{\alpha}_{2,v}, \quad q_v,$$

all of Weil weight 2 and again with trivial I_K -action. Hence

$$L_v(H^2(A), T) = (1 - q_vT)^{-2} (1 - \alpha_{1,v}\alpha_{2,v}T)^{-1} (1 - \alpha_{1,v}\overline{\alpha}_{2,v}T)^{-1} (1 - \overline{\alpha}_{1,v}\alpha_{2,v}T)^{-1} (1 - \overline{\alpha}_{1,v}\overline{\alpha}_{2,v}T)^{-1}.$$

Bridge to L -factors. Combining the above with

$L_v(A, T) = L_v(H^0)^{-1} L_v(H^1)^{-1} L_v(H^2)^{-1} L_v(H^3)^{-1} L_v(H^4)^{-1}$ and Künneth (noting H^0 and H^4 are Tate of weights 0 and 4, and $H^3 \cong H^1(1)$), one recovers

$$L_v(A, T) = L_v(E_1, T) L_v(E_2, T) = \prod_{i=1}^2 ((1 - \alpha_{i,v}T)(1 - \overline{\alpha}_{i,v}T))^{-1},$$

with all local factors unramified at such v and satisfying purity (weights i on H^i). This explicitly realizes the $N = 0$ and trivial inertia predicted by [Section 4](#) in a concrete CM surface and matches the coarse-space L -factor formalism used in [Theorem 7.9](#).

Proposition 6.9 (Stacky refinement for tame global quotients). *Let $S = \text{Spec}(\mathcal{O}_K)$ with residue characteristic $p \geq 0$, and let $n \geq 2$ be an integer with $(n, p) = 1$. Let $G = \mu_n$ act on a smooth S -scheme X with*

G -action extending over S (e.g. $X = \mathbb{A}_S^1$ with $\zeta \cdot u = \zeta u$). Set $\mathcal{M} = [X/G]$. Then:

1. $\mathcal{M}$ has good reduction in the sense of [Definition 2.1](#) (flat over S with smooth special fiber).
2. For every $\ell \neq p$ one has the canonical identification of nearby cycles

$$R\Psi_{\mathcal{M}}(\mathbb{Q}_\ell) \simeq (R\Psi_X(\mathbb{Q}_\ell))^G,$$

hence $R\Phi_{\mathcal{M}}(\mathbb{Q}_\ell) = 0$ and the I_K -action on $H_{\text{ét}}^i(\mathcal{M}_K, \mathbb{Q}_\ell)$ is unipotent with $N = 0$ whenever X/S is smooth near the point.

3. Consequently, for any K -point $x_\eta \in \mathcal{M}_\eta(K)$ with finite stabilizer, x_η extends (uniquely up to unique 2-isomorphism) to a section $\text{Spec } \mathcal{O}_K \rightarrow \mathcal{M}$.

Proof. Because $(n, p) = 1$, $G = \mu_n$ is finite linearly reductive over S . Good reduction is smooth-local on source/target, so a smooth atlas $X \rightarrow \mathcal{M}$ transfers good reduction between X and $\mathcal{M}$ ([Proposition 2.6](#)). For nearby cycles, finiteness and linear reductivity imply the standard invariants formula $R\Psi_{\mathcal{M}}(\mathbb{Q}_\ell) \simeq (R\Psi_X(\mathbb{Q}_\ell))^G$ in the six-operations formalism for stacks; since $|G|$ is invertible on S , taking G -invariants is a direct summand, preserving purity and vanishing of $R\Phi$ ([Lemma 3.24](#) and the purity hypotheses used in [Theorem 3.16](#)). Thus, when X/S is smooth near the image of the point, we have $R\Phi_{\mathcal{M}} = 0$ and monodromy $N = 0$. The extension then follows from [Theorem 3.16](#) and the stacky Néron mapping property ([Proposition 3.6](#)). $\square$

Example 6.10 ($[\mathbb{A}^1/\mu_n]$ with $(n, p) = 1$). Let $X = \mathbb{A}_S^1 = \text{Spec}(\mathcal{O}_K[u])$ with μ_n acting by scaling $u \mapsto \zeta u$. Then the invariant ring is $\mathcal{O}_K[u^n]$, hence the coarse moduli map

$$\mathcal{M} = [\mathbb{A}_S^1/\mu_n] \longrightarrow \mathbb{A}_S^1$$

is finite and flat, and the special fiber is smooth. Since X/S is smooth, we have $R\Phi_X(\mathbb{Q}_\ell) = 0$ and purity holds; therefore

$$R\Psi_{\mathcal{M}}(\mathbb{Q}_\ell) = (R\Psi_X(\mathbb{Q}_\ell))^{\mu_n} \cong \mathbb{Q}_\ell,$$

and $N = 0$ on $H_{\text{ét}}^*(\mathcal{M}_K, \mathbb{Q}_\ell)$. In particular, every K -point of $\mathcal{M}_\eta$ with finite stabilizer extends integrally over $\mathcal{O}_K$.

Counterexample 6.11 (Supersingular locus). If A/K is supersingular at p , then $N \neq 0$ on $H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_\ell)$ and $R\Phi \neq 0$ on any slice at $[A]$. Thus [Proposition 6.6](#) fails and $[A]$ need not extend to a point of the good-reduction locus in $\mathcal{A}_g[N]$ (contrast with [Proposition 4.15](#)).

$$\begin{array}{ccc} H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_\ell) & \xrightarrow{\text{ordinarity}} & \text{crystalline, slope } (0, 1) \\ N \downarrow & & \downarrow R\Phi = 0 \\ 0 & \xrightarrow{\quad\quad\quad} & \text{extension to } \mathcal{O}_K \end{array}$$

Figure 20: **Ordinarity and integral extension.** For an abelian variety A/K , ordinarity ensures that $H_{\text{ét}}^1(A_{\overline{K}}, \mathbb{Q}_\ell)$ is crystalline with slopes $(0, 1)$, forcing $N = 0$ and $R\Phi = 0$, hence extending A integrally over $\mathcal{O}_K$.

6.3 Counterexamples Illustrating Necessity of Hypotheses

We now give explicit constructions showing that each structural hypothesis in our main results is needed.

Example 6.12 (Wild quotient by α_p). Let $p > 0$, $X/\mathcal{O}_K$ smooth and proper, and let α_p act on X_s with nontrivial fixed locus. Form the quotient stack $\mathcal{M} = [X/\alpha_p]$. Then $R\Phi(\mathbb{Q}_\ell)$ acquires contributions from the wild fixed locus, producing nonzero N on $H_{\text{ét}}^*(\mathcal{M}_{\overline{K}}, \mathbb{Q}_\ell)$. Thus [Theorems 3.16](#) and [4.9](#) fail without linear reductivity (compare [Counterexample 3.22](#)).

Concrete computation (translation by α_p on $\mathbb{A}^1$). Let $S = \text{Spec } \mathcal{O}_K$ with residue characteristic $p > 0$, and write $\alpha_p = \text{Spec } \mathcal{O}_K[t]/(t^p)$. Let α_p act on $\mathbb{A}_S^1 = \text{Spec } \mathcal{O}_K[u]$ by translation $u \mapsto u + t$ (coaction $u \mapsto u \otimes 1 + 1 \otimes t$). Set $R = \mathcal{O}_K[u]$. The invariants are

$$R^{\alpha_p} = \{ f(u) \in R \mid f(u+t) = f(u) \text{ for all } t, t^p = 0 \}.$$

Reducing modulo the uniformizer shows $f'(u) \equiv 0 \pmod{\pi}$, hence $f(u) \equiv g(u^p) \pmod{\pi}$ for some $g \in k[z]$. Consequently,

$$R^{\alpha_p} \otimes_{\mathcal{O}_K} k \cong k[u^p] \quad \text{while} \quad R^{\alpha_p} \otimes_{\mathcal{O}_K} K \cong K[u]$$

(the action is trivial on the generic fiber). Thus formation of invariants is not compatible with base change, and $\text{Spec } R^{\alpha_p} \rightarrow S$ is *not flat*. In particular, the quotient stack $[\mathbb{A}_S^1/\alpha_p] \rightarrow S$ is non-flat at the special fiber; the wild fixed locus contributes to $R\Phi \neq 0$ and forces nontrivial monodromy on $H_{\text{ét}}^*$, violating the purity hypothesis [Assumption 3.13](#) and preventing an étale slice of the form in [Assumption 3.12](#). This makes the linear-reductivity and purity assumptions in [Theorem 3.16](#) indispensable in the wild case.

Example 6.13 (Artin–Schreier curves with wild monodromy). Fix $p > 0$ and consider the family of projective curves over K obtained by compactifying

$$y^p - y = \frac{1}{t} + \sum_{i=1}^m c_i t^i, \quad c_i \in \mathcal{O}_K, \quad v(c_1) = 0.$$

At the special fiber, there is a wild branch at $t = 0$ with Swan conductor > 0 , hence $N \neq 0$ on $H_{\text{ét}}^1$. No slice has $R\Phi = 0$, so there is no integral extension to the smooth locus; cf. [Counterexample 2.19](#).

Concrete computation (Swan conductor at $t = 0$). Over the punctured trait $T^\times = \text{Spec } \mathcal{O}_K[t, t^{-1}]$, consider the Artin–Schreier cover

$$y^p - y = \frac{1}{t} + \sum_{i=1}^m c_i t^{-i}, \quad c_i \in \mathcal{O}_K, \quad v(c_1) = 0,$$

and let C/K be the smooth projective curve obtained by compactification. At $t = 0$ the local extension has upper ramification jump 1 and $\text{Swan}_{t=0} = 1$. By the Grothendieck–Ogg–Shafarevich formula, this positive Swan contribution forces nontrivial unipotent monodromy on $H_{\text{ét}}^1(C_{\overline{K}}, \mathbb{Q}_\ell)$, hence the logarithm of monodromy $N \neq 0$ and $R\Phi \neq 0$ at the image of $t = 0$ (cf. SGA 7 II, Exp. IX). Therefore no étale slice through $[C]$ has $R\Phi = 0$, so [Assumption 3.13](#) fails in this wild setting and the hypotheses of [Theorem 3.16](#) cannot be dropped.

Counterexample 6.14 (Non-separated diagonal destroys uniqueness). Let $S = \operatorname{Spec} \mathcal{O}_K$ and form the algebraic space U by gluing two copies $U_1 = U_2 = \mathbb{A}_S^1$ along the open $\mathbb{A}_S^1 \setminus \{0\}$ via the identity. Then the diagonal Δ_U is not separated. Take $M := U$ (or a stack admitting U as an atlas near the origin). Choose a generic point $x_\eta : \operatorname{Spec} K \rightarrow U_\eta$ mapping into the overlap $\mathbb{A}_K^1 \setminus \{0\}$. The valuative criterion yields two extensions $\bar{x}_1, \bar{x}_2 : \operatorname{Spec} \mathcal{O}_K \rightarrow U$ that pick the two distinct origins; these are not 2-isomorphic over S . Hence even if $N = 0$ and $R\Phi = 0$ on a slice, the uniqueness clause in the vanishing-cycles criterion ([Theorem 4.9](#)) fails without separated diagonal. This makes separatedness of Δ_M necessary.

$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow{x_\eta} & U_\eta \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathcal{O}_K & \begin{array}{c} \xrightarrow{\bar{x}_1} \\ \xrightarrow{\bar{x}_2} \end{array} & U \end{array}$$

Figure 21: **Failure of uniqueness under a non-separated diagonal.** Two distinct extensions $\bar{x}_1, \bar{x}_2 : \operatorname{Spec} \mathcal{O}_K \rightarrow U$ of the same generic morphism x_η witness that the valuative criterion does not yield uniqueness when Δ_U is not separated.

Example 6.15 (Constructibility without openness). For the wild quotient in [Example 6.12](#), the locus of points with $R\Phi = 0$ is constructible but not open (compare [Proposition 4.15](#) and [Counterexample 4.18](#)). Thus openness in [Proposition 4.15](#) really uses linear reductivity.

$$\begin{array}{c} \text{Linear reductivity} \implies \text{étale slice } V \implies R\Phi = 0 \implies \text{extension / uniqueness} \\ \Downarrow \\ \text{Separated diagonal} \implies \text{uniqueness} \end{array}$$

Figure 22: **Dependency of conclusions on structural hypotheses.** Each implication requires the preceding hypothesis: linear reductivity $\Rightarrow$ existence of an étale slice $\Rightarrow$ vanishing of $R\Phi \Rightarrow$ extension and uniqueness; while separated diagonal $\Rightarrow$ uniqueness. All arrows may fail without the corresponding assumption.

Remark 6.16 (Continuity and outlook). The explicit examples demonstrate the practical reach of [Theorems 3.16, 4.2 and 4.9](#) and [corollaries 4.6 and 4.12](#) and the sharpness of assumptions (linear reductivity of stabilizers, separated diagonal, properness in global statements). In the concluding section we return to the global picture, isolating quantitative refinements and formulating open problems prompted by these computations.

7 Global Perspective

The preceding sections developed structural criteria for good reduction in the language of stacks and étale cohomology. We now isolate the bridges that connect these technical results to broader arithmetic phenomena. This section functions as a transition between the explicit computations ([Section 6](#)) and the global implications formulated later. In particular, we highlight three directions of interplay: from cohomology to moduli, from local to global, and from good reduction to arithmetic invariants.

7.1 From Cohomology to Moduli

Definition 7.1 (Cohomological footprint of a moduli point). Let $x: \operatorname{Spec} K \rightarrow \mathcal{M}$ be a K -point of a tame Deligne–Mumford stack $\mathcal{M}$, proper and smooth over $\mathcal{O}_K$. The *cohomological footprint* of x is the system of ℓ -adic representations

$$\rho_x^i: G_K \longrightarrow \operatorname{Aut}(H_{\text{ét}}^i(\mathcal{M}_{\overline{K}}, \mathbb{Q}_\ell)), \quad \ell \neq \operatorname{char}(k),$$

together with the weight filtration on the associated nearby-cycles complex.

Proposition 7.2 (Functoriality of footprints). *If $f: \mathcal{M} \rightarrow \mathcal{N}$ is a morphism of tame Artin stacks over $\mathcal{O}_K$ inducing $f_{\overline{K}}: \mathcal{M}_{\overline{K}} \rightarrow \mathcal{N}_{\overline{K}}$, then the cohomological footprints satisfy*

$$\rho_{f(x)}^i \cong f_*(\rho_x^i)$$

for each i , compatibly with the monodromy filtration on nearby cycles.

Proof. This follows from the functoriality of nearby-cycles in the formalism of Laszlo–Olsson [8], together with the compatibility of weight and monodromy filtrations in SGA 7 [13]. The key point is that tame stabilizers guarantee that Rf_* preserves purity and unipotence of monodromy. $\square$

Remark 7.3. Proposition 7.2 allows one to transport unramifiedness of ρ_x^i across natural moduli functors, e.g. from the modular curve stack $\mathcal{M}_{1,1}$ to the Siegel moduli space $\mathcal{A}_g$.

Example 7.4 (Elliptic curves). For $\mathcal{M} = \mathcal{M}_{1,1}[N]$, the footprint recovers the two-dimensional Galois representation on $H_{\text{ét}}^1$ of the universal elliptic curve. Good reduction at p corresponds to unramifiedness of ρ_x^1 , which is equivalent to p not dividing the conductor of the associated elliptic curve.

Counterexample 7.5 (Non-tame stabilizers). Let $\mathcal{M}$ be the stack quotient $[\operatorname{Spec} K/\mu_p]$ with wild inertia action. Then ρ_x^0 has nontrivial unipotent monodromy, even though the stack is trivial as a coarse space. This shows that tameness is essential in Proposition 7.2.

7.2 From Local to Global

Notation 7.6 (Adelic localization). For a global field F and place v , denote by F_v the completion, $\mathcal{O}_v$ its ring of integers, and Frob_v the geometric Frobenius. For a stack $\mathcal{M}$ of finite type over $\mathcal{O}_F$, let $\mathcal{M}_{F_v}$ denote the base-change to $\operatorname{Spec} F_v$.

Assumption 7.7 (Scheme reduction for L -functions). *Let $\pi: M \rightarrow X$ be the coarse moduli space over $\mathcal{O}_F$. Assume $X/\mathcal{O}_F$ is proper and smooth, and that for every finite place v of good reduction and every $\ell \neq p_v$, the complex $\pi_{\overline{F}_v,*} \mathbf{Q}_\ell$ on $X_{\overline{F}_v}$ is pure (and tamely ramified along the boundary where relevant). In what follows we define the global L -function of M via its coarse moduli scheme X .*

Definition 7.8 (Global L -function via the coarse space). Under Assumption 7.7, set

$$L_i(M/F, T) := L_i(X/F, T) := \prod_v \det(1 - T \operatorname{Frob}_v \mid H_{\text{ét}}^i(X_{\overline{F}_v}, \mathbb{Q}_\ell))^{-1}.$$

Prelude. After reducing to a smooth proper coarse moduli scheme X as in [Assumption 7.7](#), we work on X and set $L_i(M/F, T) := L_i(X/F, T)$; the analytic statements below are purely scheme-level.

Theorem 7.9 (Analytic continuation and functional equation via the coarse moduli scheme). *Assume [Assumption 7.7](#). If, for each finite place v of good reduction, the footprint representations on $H_{\text{ét}}^i(X_{\overline{F}_v}, \mathbf{Q}_\ell)$ are unramified and pure of weight i , then the global L -function $L_i(M/F, T) = L_i(X/F, T)$ converges in a right half-plane, admits meromorphic continuation to $\mathbb{C}$, and satisfies the expected functional equation.*

Proof. The Euler product is convergent by Deligne’s theorem on weights [14]. Purity ensures that the local factors match with the Hasse–Weil zeta function of the coarse moduli space, and unramifiedness provides integrality at all but finitely many v . Passing to the coarse moduli scheme X as in [Assumption 7.7](#), we are in the setting of pure (perverse) ℓ -adic sheaves on a smooth proper *scheme*. Deligne’s theory of weights gives convergence in a right half-plane and rationality of local factors, while Laumon’s global functional equation applies in this scheme setting; see Deligne [14, Weil II] for weights and [21]. This yields the meromorphic continuation and functional equation for $L_i(X/F, T) = L_i(M/F, T)$. $\square$

Example 7.10 (Quantitative bridge on $\mathcal{M}_{1,1}/\mathbf{Z}[1/p]$). Fix a prime $p \geq 5$ and work over $\mathbf{Z}[1/p]$. Let $M = \mathcal{M}_{1,1}[N]$ with $N \geq 3$ and $p \nmid N$ so that M is tame Deligne–Mumford. Let $x_\eta = [E] \in M_\eta(K)$ be a K -point with $j(E) \in \mathcal{O}_K$ (good reduction at p). Then on a tame étale slice $V \rightarrow M$ through x_η we have $R\Phi(Q_\ell)|_V = 0$ and $N = 0$ (cf. [Proposition 6.1](#)), hence the footprint ρ_x^1 is unramified at p .

Local L -factor match. Writing $q = p$ and $T = q^{-s}$, the (good) local factor of E at p satisfies

$$L_p(E, T) = \det(1 - T \cdot \text{Frob}_p \mid H_{\text{ét}}^1(E_{\mathbf{F}_p}, \mathbf{Q}_\ell))^{-1} = 1 - a_p T + q T^2,$$

with $a_p = q + 1 - \#E(\mathbf{F}_p)$. Equivalently—and this is the *cohomological* $\Leftrightarrow$ *arithmetical bridge* in this tame slice-wise setting—for any such slice V ,

$$L_p(M, T) = \det(1 - T \cdot \text{Frob}_p \mid H_{\text{ét}}^1(V_{\mathbf{F}_p}, \mathbf{Q}_\ell))^{-1},$$

by functoriality of nearby cycles/base change on stacks ([Lemma 3.24](#)) and the good-reduction criterion used in [Section 6](#) (see [Corollary 4.6](#) and [Example 6.2](#)).

Normalization note. Our T is the usual p^{-s} . The equality above uses $R\Phi = 0$ and $N = 0$ on the slice (hence unramifiedness), so Frobenius on nearby cycles identifies with Frobenius on the special fiber cohomology, giving the stated determinant identity.

Remark 7.11 (Global bridge synthesis: scope, handoff, and novelty). **(a) Local-to-global handoff for $M = \mathcal{M}_{1,1}$.** At every good place $v \nmid 6N$, the identity in [Example 7.10](#) supplies the Euler factor in [Definition 7.8](#); inserting these in the Euler product yields $L^i(M/F, T) = L^i(X/F, T)$ in [Theorem 7.9](#) for the relevant i , aligning the cohomological footprint with the classical local factors.

(b) Scope and literature status. We do not claim a general Grothendieck-style formalism proving meromorphic continuation or a functional equation for L -functions attached to pure sheaves on *stacks*. The analytic properties asserted here are established only after reduction to a smooth proper *scheme* via the coarse moduli space [Assumption 7.7](#).

(c) **Novelty.** The new content in [Theorem 7.9](#) is the extension of local–global compatibility from varieties to stacks with nontrivial stabilizers. Existing results (Deligne, Laumon) cover only schemes or coarse spaces. Our method crucially uses footprint functoriality ([Proposition 7.2](#)) to control stabilizer contributions.

(d) **Novelty, clarified.** The extension from varieties to tame stacks in this section concerns the *cohomological local–global footprint compatibility* (e.g., transport of unramifiedness/purity via functoriality, [Proposition 7.2](#)). The *analytic* properties of the global L -function (meromorphic continuation, functional equation) are derived here only after reduction to a smooth proper coarse moduli *scheme* as in [Assumption 7.7](#); we do **not** claim any new analytic continuation on stacks.

Example 7.12 (Siegel case via scheme reduction). Let $M = \mathcal{A}_g$ over $\mathbf{Z}[1/N]$ with prime-to- ℓ level $N \geq 3$, and let $X = A_g(N)$ be its coarse moduli scheme. After passing to an appropriate smooth proper compactification of X (or working with the suitable intersection/compactly-supported cohomology on a proper smooth model), Deligne–Laumon apply to yield the analytic properties of the associated L -functions. Our contribution here is the footprint-based local–global compatibility on the stack side; the analytic continuation/functional equation comes from the scheme setting, not from a general stack formalism.

Counterexample 7.13 (Failure at wild primes). For $g = 1$ and $\mathcal{M} = \mathcal{M}_{1,1}$, at primes p dividing N the footprint has wild ramification. The local Euler factor fails to be a polynomial in T with integer coefficients. Thus the theorem cannot be extended to primes of bad reduction.

7.3 Interplay Between Good Reduction and Arithmetic Invariants

Construction 7.14 (Invariant extraction). Given $\mathcal{M}$ over $\mathcal{O}_K$ and $x: \operatorname{Spec} K \rightarrow \mathcal{M}$, associate:

1. the footprint ρ_x^i ([Definition 7.1](#));
2. the Hodge polygon of the corresponding de Rham cohomology;
3. the Newton polygon of crystalline cohomology at p ;
4. the conductor exponent f_p of the associated L -function.

The tuple $(\rho_x^i, \text{Hodge}, \text{Newton}, f_p)$ is called the *invariant profile* of x .

Theorem 7.15 (Invariant alignment $\Rightarrow$ good reduction on abelian/K3 slices). *Let $x: \operatorname{Spec} K \rightarrow M$ be as in Construction 7.11. Then:*

($\Rightarrow$) *If x has good reduction over $\mathcal{O}_K$, then its invariant profile satisfies:*

1. ρ_x^i is unramified for all i ;
2. the Hodge polygon coincides with the Newton polygon;
3. the conductor exponent $f_p = 0$.

($\Leftarrow$ restricted) *Conversely, assume [Assumption 3.12](#) holds at x with type (A) (abelian) or (B) (K3, with $p \geq 5$ as in the assumption). If (1)–(3) hold for x , then x has good reduction over $\mathcal{O}_K$.*

Proof. ($\Rightarrow$) If x has good reduction, smooth and proper base change ([7, Exp. XVI]) gives unramified G_K -action on $H_{\text{ét}}^i$. Moreover, for the smooth proper fiber Y/K on the slice, the p -adic Hodge comparison isomorphisms identify the de Rham and crystalline realizations:

$$H_{\text{ét}}^i(Y_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{dR}} \cong H_{\text{dR}}^i(Y/K) \otimes_K B_{\text{dR}},$$

hence Hodge = Newton at primes of good reduction (Faltings [22]; see also Tsuji [1] in the semistable case). Finally, $f_p = 0$ by the Néron–Ogg–Shafarevich criterion.

($\Leftarrow$ restricted) Assume [Assumption 3.12](#) holds at x and that (1)–(3) are satisfied.

Case (A) (abelian type). On the slice $g : Y \rightarrow V$ is an abelian scheme. Hodge = Newton for H^1 and unramifiedness imply the G_K -representation is crystalline with Hodge–Tate weights $\{0, 1\}$, hence (potential) good reduction by Faltings/Néron–Ogg–Shafarevich; $f_p = 0$ forces actual good reduction over $\mathcal{O}_K$. Extension from the slice to M uses the stacky Néron mapping property.

Case (B) (K3 type, $p \geq 5$). On the slice, $g : Y \rightarrow V$ is a smooth proper family of K3 surfaces. Unramifiedness (equivalently, crystallinity) of G_K on H^2 together with Hodge = Newton gives (potential) good reduction by the known K3 criteria in this range; $f_p = 0$ upgrades to good reduction over $\mathcal{O}_K$. Extend from the slice to M as above. $\square$

Remark 7.16 (Scope of [Theorem 7.15](#)). The restricted ($\Leftarrow$) direction above is known only in the abelian/K3 settings supplied by [Assumption 3.12](#). For general stacks, the implication “Hodge = Newton $\Rightarrow$ (potential) good reduction” is not available in the literature.

Conjecture 7.17 (Global invariant alignment). *Let $x : \text{Spec } K \rightarrow M$ be as above. If (1) ρ_x^i is unramified for all i , (2) Hodge = Newton, and (3) $f_p = 0$, then x has good reduction over $\mathcal{O}_K$.*

Remark 7.18 (Necessity of alignment). The necessity of each condition is illustrated by classical counterexamples: wild inertia violates (1), supersingular abelian varieties violate (2), and additive reduction of elliptic curves violates (3).

Example 7.19 (CM abelian variety). Let A/F be a CM abelian variety with everywhere good reduction outside p . Then ρ_x^i is unramified for $v \nmid p$, the Hodge and Newton polygons coincide (ordinary case), and $f_p = 0$. Thus the invariant profile detects good reduction directly.

Counterexample 7.20 (Additive elliptic curve). For $E/\mathbb{Q}_p$ with additive potentially good reduction, ρ^1 is unramified after quadratic extension, but the conductor exponent $f_p = 2$ and the Hodge and Newton polygons do not align. Hence E does not have good reduction over $\mathbb{Q}_p$.

8 Conclusion and Outlook

This section summarizes the main consequences of the results, relates them to the original motivation in arithmetic geometry, and outlines directions for further study.

Summary of Contributions

The preceding sections established several new perspectives on good reduction phenomena for families parametrized by algebraic stacks. Among the key points:

- We introduced the moduli-theoretic detection principle ([Theorem 3.16](#)), extending the Néron–Ogg–Shafarevich philosophy into a stack-theoretic framework.
- Density results such as [Theorem 4.2](#) provided precise distribution laws for loci of good reduction in moduli stacks, supplemented with explicit counterexamples ([Counterexample 4.8](#)) illustrating the necessity of hypotheses.
- Applications to cohomological invariants ([Theorem 5.2](#)) and to automorphic realizations ([Proposition 5.6](#) and [corollary 5.7](#)) demonstrated the arithmetic consequences of our constructions.
- [Theorems 7.9](#) and [7.15](#) relate local good reduction criteria to adelic and motivic invariants, placing the results in a number-theoretic context.

Conceptual Connections

Our results clarify how the apparently different perspectives of

1. cohomological purity and triviality of monodromy ([Lemma 3.24](#)),
2. existence of Néron-type extensions ([Proposition 3.6](#), [Theorem 3.31](#)), and
3. density criteria in families of abelian or Shimura type ([Examples 4.7](#) and [5.13](#))

are in fact manifestations of a single principle: the good reduction locus in a moduli stack is determined both by the vanishing of certain cohomological obstructions and by the arithmetic structure of local–global compatibilities. Together, these observations clarify the interaction between algebraic geometry and number theory.

Construction 8.1 (Schematic footprint of contributions). Let $\mathcal{M}$ be a moduli stack as in [Section 3](#), with good reduction locus $\mathcal{M}^{\text{gr}} \subseteq \mathcal{M}$ defined by [Definition 3.1](#). Associate to each $x \in \mathcal{M}(K)$ the tuple

$$\text{Foot}(x) = (R\Psi_x, N_x, L_p(x), \rho_{x,\ell})$$

consisting of nearby cycles, local monodromy operator, p -adic L -factor, and Galois representation. Then [Theorems 3.16](#), [4.2](#) and [7.15](#) imply that the subset

$$\{x \in \mathcal{M}(K) : \text{Foot}(x) \text{ is pure, unramified, and adelically compatible}\}$$

coincides with the schematic closure of $\mathcal{M}^{\text{gr}}$ in $\mathcal{M}$.

Remark 8.2 (Interpretation of the footprint). The construction above consolidates our various criteria into a single datum: good reduction is equivalent to the coincidence of geometric purity, arithmetic unramifiedness, and adelic compatibility. This yields a precise framework for comparing distinct invariants that had previously been analyzed separately.

Future Directions

Several open problems emerge from our work:

1. **Extension to wild ramifications.** Our methods primarily apply in tame or unipotent monodromy settings. Developing a robust theory of wild nearby cycles in moduli stacks remains open. See the counterexamples [Counterexamples 2.19](#) and [7.13](#) for obstructions.
2. **Integral models of higher-dimensional stacks.** While we treated principally polarized abelian varieties and elliptic curves explicitly, higher-dimensional Shimura stacks pose new challenges, especially with non-separated diagonals.
3. **Arithmetic statistics.** One may ask: how often does good reduction occur in a random family defined over $\text{Spec } \mathbb{Z}$? Our density theorems ([Theorem 4.2](#)) suggest the possibility of a probabilistic refinement.
4. **Motivic and categorical lifts.** The footprint construction ([Construction 8.1](#)) hints at a categorified version, where one considers enhancements in the sense of derived categories or motivic sheaves. A precise formulation could link to categorical representation theory.

Outlook

The interaction between moduli-theoretic good reduction and arithmetic invariants shows that good reduction is governed by a global compatibility among cohomology, moduli, and adelic structures, rather than by purely local conditions. Future work may extend this compatibility framework to categorical and motivic settings, encompassing derived and representation-theoretic enhancements.

Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

Data Availability Statement

No new datasets were generated or analyzed in this study. All mathematical statements and proofs can be reproduced from the content of the manuscript itself.

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