COHOMOLOGICAL MONODROMY AND COMPONENT-GROUP CORRECTIONS FOR ELLIPTIC CURVES WITH BAD REDUCTION: A UNIFORM ℓ-INDEPENDENT LEFSCHETZ–CONDUCTOR FRAMEWORK VIA NÉRON MODELS AND MODULI STACKS

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Abstract. This paper develops a cohomological framework that unifies the study of local monodromy, component groups, and conductor formulas for elliptic curves with bad reduction. The approach isolates the contribution of the Néron component group from the inertia–invariant part of the ℓ–adic cohomology, establishing a uniform trace identity compatible with geometric special fibers and arithmetic Frobenius actions. A categorical reformulation via moduli stacks of elliptic curves and their level structures provides a natural interpretation of local correction terms and connects them to global conductor and root–number phenomena. The resulting formulation is ℓ–independent, stack–theoretic, and equally suited to tame and potentially good additive cases, offering a coherent bridge between the geometry of degenerations and the arithmetic of L–functions.

1. Introduction

Motivation and context. For an elliptic curve E/K over a number field, arithmetic invariants—conductor $N_{E/K}$, local root numbers $w_v(E)$, Tamagawa numbers, and the behaviour of Selmer groups in towers—are controlled prime-by-prime by the geometry of the Néron model $\mathcal{E} \to \operatorname{Spec} \mathcal{O}_K$ and by the monodromy action on $H^1_{\text{ét}}(E_{\overline{K}_v}, \mathbb{Q}_\ell)$ at places v of bad reduction [1, 11, 12, 2]. What is often missing is a single formula that *uniformly* separates the special-fibre contribution from the discrete component-group term across all bad Kodaira types. Sections 2 and 3 collect the classical material used throughout: Néron models and component groups Φ_v [1], the Kodaira–Néron classification [13, Ch. VII], nearby and vanishing cycles [11, 12], and the description of local factors and ε-factors [2]. Our first main theorem (Theorem 4.3) supplies a uniform local identity at a bad place v, expressing the inertia-fixed trace on H^1 as a difference between a special-fibre fixed-point term and a component-group trace. Summing these local identities yields global equalities and inequalities for conductor exponents and root numbers (Theorem 5.4), with concrete consequences in families (Sections 5 and 8).

Statement of main results. We fix notation from Section 2. In particular, for a finite place v of K we write $\kappa(v)$ for the residue field, $q_v = \#\kappa(v)$, Frob_v for geometric Frobenius in G_{K_v}/I_v, $\mathcal{E}$ for the Néron model of E over $\mathcal{O}_{K_v}$ with identity component $\mathcal{E}^0$ and component group $\Phi_v = (\mathcal{E}/\mathcal{E}^0)(\kappa(v))$, and E^{sp} for the total special fibre. We also set $H^1_\ell(E) := H^1_{\text{ét}}(E_{\overline{K}_v}, \mathbb{Q}_\ell)$ for a prime $\ell \neq \operatorname{char} \kappa(v)$.

Theorem 1.1 (Local monodromy identity at bad primes)**.** Uniform packaging. *The identity packages standard ingredients— Grothendieck–Lefschetz on the special fibre, the nearby/vanishing-cycles triangle, and the component-group interpretation—into a single ℓ-independent equality valid across bad Kodaira types. The contribution here is the uniform formulation and systematic bookkeeping (no case tables), not new objects; comparison with the Deligne–Saito conductor is recalled in Appendix B.*

For every finite place v of bad reduction and every prime $\ell \neq \operatorname{char} \kappa(v)$ one has

$$\operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}\Big) \;=\; \#\mathrm{Fix}\Big(\mathrm{Frob}_v; E^{\mathrm{sp}}(\overline{\kappa(v)})\Big) \;-\; \operatorname{tr}(\mathrm{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell)\,. \tag{1}$$

The terms are well defined and independent of ℓ.

Equation (1) is proved in Section 4 as Theorem 4.3. The special-fibre fixed-point count is understood via the Grothendieck–Lefschetz trace formula [10], while the component-group term arises from the identification of vanishing cycles with $\Phi_v \otimes \mathbb{Q}_\ell$ recorded in Lemma 3.4 and the trace computation of Proposition 3.3. The preparatory material is developed in Section 3, notably the exact sequence Equation (2) and the inertia analysis in Lemma 3.2.

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Literature positioning. The equality follows from the standard vanishing–cycle formalism together with the component–group interpretation (SGA 7 I–II [11, 12]) and the trace on Φ_v; see also Lemma 3.4 and Proposition 3.3 for the concrete comparison used later. Our point is to phrase the identity uniformly and ℓ-independently so it feeds directly into the global conductor/root-number package; (cf. Appendix B).

Theorem 1.2 (Global conductor/root-number package)**.** Uniform Formulation. *Summing Equation (∗) over v yields a conductor/root-number package that isolates, prime by prime, the component-group contribution. Let S be the set of finite places of bad reduction for E/K. For each $v \in S$, set*

$$c_v(E) \;:=\; \#\mathrm{Fix}\Big(\mathrm{Frob}_v; E^{\mathrm{sp}}(\overline{\kappa(v)})\Big) \;-\; \mathrm{tr}(\mathrm{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell)\,,$$

which is independent of ℓ. (independence of Fix_v by Grothendieck–Lefschetz, and of $\mathrm{tr}(\mathrm{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell)$ by Proposition 3.3; see also Proposition 4.6).

Then:

(i) The local L-factor satisfies

$$L_v(E,s) \;=\; \det\Big(1-\mathrm{Frob}_v q_v^{-s} \;\Big|\; H^1_\ell(E)^{I_v}\Big)^{-1}, \quad \textit{hence} \quad a_v(E) = \mathrm{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}\Big) = c_v(E),$$

(cf. Proposition 2.6) so $a_v(E)$ admits the geometric expression Equation (∗) at all $v \in S$ [2].

(ii) The (logarithmic) conductor satisfies, for each $v \in S$,

$$f_v(E) \;=\; 2 - \dim_{\mathbb{Q}_\ell} H^1_\ell(E)^{I_v} \;+\; \mathrm{Swan}_v(H^1_\ell(E)),$$

and therefore

$$\sum_{v\in S}(2-\dim H^1_\ell(E)^{I_v}) \;\le\; \sum_{v\in S} f_v(E),$$

with equality if and only if the reduction is tame at every $v \in S$. In particular, $f_v(E)$ is bounded below by a function of $c_v(E)$ determined by the Kodaira symbol, and the bound is sharp in the semistable case [2, 12].

Reference. *This is the Deligne–Saito conductor identity; see Theorem B.1.*

(iii) The global sign factors as

$$w(E/K) \;=\; \prod_{v\in S} w_v(E), \qquad w_v(E) \;\; \textit{determined by } H^1_\ell(E)^{I_v} \textit{ and hence by } c_v(E) \textit{ up to tame } \varepsilon\textit{-factors,}$$

so the variation of $w(E/K)$ in quadratic twist families is governed by the variation of the $c_v(E)$, cf. Proposition 5.5 and Corollary 5.6.

Roadmap. The paper is organized so that each conceptual point is tied immediately to a formal result and an example.

- Claim A: Local monodromy admits a uniform component-group correction Theorem 4.3
- Claim B: Summing the correction yields conductor/root-number statements Theorem 5.4
- Claim C: Worked examples over $\mathbb{Q}$ and quadratic extensions test sharpness Section 7

Acknowledgements. (Optional; include only if journal allows.)

2. Preliminaries and standing conventions

2.1. **Global standing hypotheses and notation.** Let K be a number field with ring of integers $\mathcal{O}_K$. For a finite place v of K, write K_v for the completion, $\mathcal{O}_{K_v}$ for its valuation ring, $\mathfrak{m}_v$ for the maximal ideal, $\kappa(v)$ for the residue field of size q_v, and ϖ_v for a uniformizer. Fix an algebraic closure $\overline{K}_v$ and set $G_{K_v} = \mathrm{Gal}(\overline{K}_v/K_v)$ with inertia subgroup $I_v \subset G_{K_v}$ and wild inertia $P_v \subset I_v$.

Let E/K be an elliptic curve. For each finite v we denote by $\mathcal{E} \to \mathrm{Spec}\,\mathcal{O}_{K_v}$ the Néron model of E (when it exists, e.g., after possibly shrinking to $\mathrm{Spec}\,\mathcal{O}_{K,S}$ for a finite set S); write $\mathcal{E}^0$ for the identity component of the special fibre and $\Phi_v := (\mathcal{E}/\mathcal{E}^0)(\kappa(v))$ for the component group at v. We write Δ_{E/K_v} for the minimal discriminant ideal, $j(E)$ for the j-invariant, and $f_v(E)$ for the local conductor exponent. When $\ell \neq \mathrm{char}\,\kappa(v)$, put

$$V_\ell(E) := T_\ell(E) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell, \qquad H^1_\ell(E) := H^1_{\text{ét}}(E_{\overline{K}_v}, \mathbb{Q}_\ell)$$

with the canonical G_{K_v}-action [5, 10].

Definition 2.1 (Component group and reduction type)**.** For $v \nmid \infty$, define the component group

$$\Phi_v := (\mathcal{E}/\mathcal{E}^0)(\kappa(v)).$$

The reduction type of E at v is encoded by the Kodaira–Néron symbol I_n, II, III, IV, I_n^*, II*, III*, IV* [13, Ch. VII]. We say E has *good*, *multiplicative* (split or non-split), or *additive* reduction at v accordingly.

Remark 2.2 (Normalization of local factors). We normalize $L_v(E,s)$ by the Euler factor attached to the semisimplification of $H^1_\ell(E)$, and $\varepsilon_v(E,s)$ by Deligne's local constants [2]. Conductor exponents $f_v(E)$ are taken in the sense of Artin conductors for $H^1_\ell(E)$, compatible with the Ogg–Saito formula [7, 9].

2.2. **Basic facts used later (recorded once).** All items in this subsection are standard and will be cited by label only later; they should not be reproved elsewhere in the paper.

Lemma 2.3 (Existence and functoriality of Néron models)**.** *Let E/K be an elliptic curve and v a finite place. The Néron model $\mathcal{E} \to \operatorname{Spec} \mathcal{O}_{K_v}$ exists and is characterized by the Néron mapping property. Its formation is compatible with unramified base change, and $\mathcal{E}^0$ is an open subgroup scheme of finite index. Moreover, the special fibre fits into the exact sequence of smooth group schemes*

$$0 \longrightarrow \mathcal{E}^0_{\kappa(v)} \longrightarrow \mathcal{E}_{\kappa(v)} \longrightarrow \Phi_v \longrightarrow 0,$$

with Φ_v a finite étale $\kappa(v)$-group scheme [1, §9].

Proof. See [1, §1–§2,§9]. ☐

Lemma 2.4 (Classification of reduction and component groups)**.** *If E has good reduction at v, then $\Phi_v = 0$. If E has multiplicative reduction of type I_n (resp. I_n^*), then $\Phi_v \simeq \mathbb{Z}/n\mathbb{Z}$ (resp. an extension of $(\mathbb{Z}/2\mathbb{Z})^2$ by $\mathbb{Z}/n\mathbb{Z}$). For additive potentially good types, Φ_v is nontrivial of bounded order determined by the Kodaira symbol* [13, Ch. VII]*,* [1, §9]*.*

Proof. See [13, Ch. VII] and [1, §9]. ☐

Lemma 2.5 (Monodromy and inertia on H^1)**.** *Let $\ell \neq \operatorname{char}\kappa(v)$. The wild inertia P_v acts unipotently on $H^1_\ell(E)$, and the tame inertia acts via finite order characters in the potentially good case and via a unipotent Jordan block of size 2 in the multiplicative case. In particular,*

$$\dim_{\mathbb{Q}_\ell} H^1_\ell(E)^{I_v} = \begin{cases} 2 & \textit{good reduction,} \\ 1 & \textit{multiplicative reduction,} \\ 0 & \textit{additive potentially good.} \end{cases}$$

Moreover, there is a weight–monodromy filtration $W_\bullet$ on $H^1_\ell(E)$ compatible with specialization [11, Exp. IX]*,* [12, Exp. I–II] *(See also* [15] *for the overconvergent F–isocrystal analogue of the weight formalism ensuring purity of the unipotent part).*

Proof. See [11, Exp. IX], [12, Exp. I–II]. ☐

Proposition 2.6 (Local factors and conductors)**.** *Let $v \nmid \ell$. Then*

$$L_v(E,s) = \det\Big(1 - \mathrm{Frob}_v\, q_v^{-s} \mid H^1_\ell(E)^{I_v}\Big)^{-1}.$$

If E has good reduction, $L_v(E,s) = (1 - a_v q_v^{-s} + q_v^{1-2s})^{-1}$ with $a_v = q_v + 1 - \#E(\kappa(v))$. If E has split (resp. non-split) multiplicative reduction, then $L_v(E,s) = (1 - q_v^{-s})^{-1}$ (resp. $(1+q_v^{-s})^{-1}$). The conductor exponent satisfies

$$f_v(E) = \mathrm{Swan}_v(H^1_\ell(E)) + \dim H^1_\ell(E)/H^1_\ell(E)^{I_v}$$

and agrees with the Ogg–Saito term computed from the minimal discriminant and reduction type [2, 7, 9].

Proof. See [2] for local factors and conductors; [13, Ch. VII] for the reduction-specific formulas; and [7, 9] for the conductor discriminant relation. ☐

Remark 2.7 (Specialization maps). There is a specialization exact sequence

$$0 \to \mathcal{E}^0(\mathcal{O}_{K_v}) \to E(K_v) \xrightarrow{\mathrm{sp}} \Phi_v(\kappa(v)) \to 0,$$

and sp is surjective [1, §9]. This will be used to relate component groups to cohomological terms via monodromy in Section 3.

2.3. **A moduli-theoretic note.** When convenient, we view E as a K-point of $\mathscr{M}_{1,1}$ (with level to rigidify), and reduction at v corresponds to intersecting with the boundary divisor; the component group can be read from the combinatorics of the special fibre of the stable model [4]. This perspective is purely auxiliary here and will not be used to reprove standard facts.

The material recorded in Section 2.2 will not be repeated elsewhere; later sections will reference Lemmas 2.3 to 2.5, Proposition 2.6, and Remark 2.7 as needed.

3. Local geometric set-up at a prime of bad reduction

Remark 3.1 (Roadmap and dependence on Section 2)*.* Throughout this section we fix a finite place v of bad reduction for E/K. All structural inputs are those recorded once in Section 2, notably Lemmas 2.3 and 2.5, Proposition 2.6, and Remark 2.7. The goal is to prepare a cohomological framework that isolates the contribution of the component group Φ_v to the I_v-fixed part of $H^1_\ell(E)$, paving the way for the local identity in Section 4.

3.1. **Notation and conventions at v.** Fix $\ell \neq \operatorname{char} \kappa(v)$. Let $q = q_v$, and write Frob_v for the geometric Frobenius in G_{K_v}/I_v. Let $\mathcal{E}/\mathcal{O}_{K_v}$ be the Néron model with identity component $\mathcal{E}^0$ and component group Φ_v as in Definition 2.1. Denote by E^0 the smooth locus of the special fibre and by E^{sp} the total special fibre. We use

$$H^1_\ell(E)^{I_v} \subset H^1_\ell(E) \qquad \text{and} \qquad \Phi_v \otimes \mathbb{Q}_\ell$$

with the natural Frob_v-action induced by functoriality of Néron models under unramified base change [1, §9].

3.2. **Cohomology, vanishing cycles, and specialization.** We recall the exact triangle of nearby/vanishing cycles for a proper regular model $\mathscr{X} \to \operatorname{Spec} \mathcal{O}_{K_v}$ of E (e.g. the minimal regular model), restricted to the degree relevant for curves:

$$R\Gamma(E_{\overline{K}_v}, \mathbb{Q}_\ell) \longrightarrow R\Gamma(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) \longrightarrow R\Phi(\mathbb{Q}_\ell) \xrightarrow{+1},$$

with I_v acting trivially on the middle term and unipotently on $R\Phi(\mathbb{Q}_\ell)$ [12, Exp. I–II]. Passing to H^1 and I_v-invariants yields an exact sequence

$$0 \longrightarrow H^1_\ell(E)^{I_v} \xrightarrow{\mathrm{sp}^*} H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) \longrightarrow \Psi_v \longrightarrow 0, \tag{2}$$

where $\Psi_v := H^0(R^1\Phi(\mathbb{Q}_\ell))$ is the vanishing-cycles term with Frob_v-action of weight 0.

Lemma 3.2 (Special fibre decomposition and the I_v-fixed line)**.** *If E has multiplicative reduction at v then $H^1_\ell(E)^{I_v}$ is a $\mathbb{Q}_\ell$-line and the image of* sp^* *in Equation (2) identifies with the subspace generated by the (dual-graph) class of a geometric component of E^{sp}; if E has additive potentially good reduction then $H^1_\ell(E)^{I_v} = 0$. In each case, the conclusions agree with Lemma 2.5 and the decomposition of E^{sp} into its components* [11, 12, SGA7 I, Exp. IX; SGA7 II, Exp. I–II]*.*

Proof. Fix a minimal regular model $X \to \operatorname{Spec} \mathcal{O}_{K_v}$ with total special fibre E^{sp}, and write $\Psi_v := H^0(R^1\Phi(\mathbb{Q}_\ell))$. The nearby/vanishing-cycles triangle yields the short exact sequence of I_v-invariants (your Equation (2))

$$0 \to H^1_\ell(E)^{I_v} \xrightarrow{\mathrm{sp}^*} H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) \to \Psi_v \to 0,$$

with Ψ_v of weight 0 and I_v acting unipotently on $R\Phi$; see [12, SGA7 II, Exp. I–II]. We treat the two cases.

(a) Multiplicative reduction. Here E^{sp} is a Néron n-gon: a cycle of n copies of $\mathbb{P}^1$ meeting transversely at nodes. The normalization has trivial H^1, so by the Mayer–Vietoris / dual-graph computation one has

$$H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) \;\cong\; H^1(\Gamma(E^{\mathrm{sp}}), \mathbb{Q}_\ell) \;\cong\; \mathbb{Q}_\ell,$$

generated by the fundamental cycle of the dual graph (equivalently, by the class dual to any component in the cycle). Picard–Lefschetz for curves identifies $R^1\Phi(\mathbb{Q}_\ell)$ with the $\mathbb{Q}_\ell$–span of component classes modulo the image of H^1 of the smooth locus; thus Ψ_v has weight 0 and the I_v–unipotent piece sits entirely on the right. Exactness then forces sp^* to be an injection onto the unique one-dimensional subspace of $H^1(E^{\mathrm{sp}}, \mathbb{Q}_\ell)$, i.e. $H^1_\ell(E)^{I_v}$ is a line, and its image is the line generated by the component/loop class just described. This matches Lemma 2.5 in the multiplicative case.

For the general p-adic formalism of weights on overconvergent F-isocrystals underlying this purity statement, see [15].

(b) Additive potentially good reduction. After a finite (tame) extension L/K_v the curve acquires good reduction. Over L one has $H^1_\ell(E)^{I_w} = H^1_\ell(E)$ (smooth proper base change), and the residual action of I_v factors through the finite cyclic quotient $\mathrm{Gal}(L/K_v)$ via the automorphism group of the good special fibre. In the additive *non-good* situation this finite action is nontrivial; on $H^1_\ell(E)$ its eigenvalues are roots of unity $\neq 1$ (order dividing $2, 3, 4$, or 6 depending on the Kodaira symbol), so no I_v–invariants remain. Hence $H^1_\ell(E)^{I_v} = 0$. Returning to the exact sequence above, this forces the map $H^1(E^{\mathrm{sp}}, \mathbb{Q}_\ell) \to \Psi_v$ to be an isomorphism, i.e. "vanishing cycles exhaust H^1" in this case. This is the precise form of the informal sentence you had, and it is exactly the "potentially good $\Rightarrow$ no I_v–invariants" line recorded in Lemma 2.5. ☐

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1_\ell(E)^{I_v} & \xrightarrow{\ \mathrm{sp}^*\ } & H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) & \longrightarrow & \Psi_v & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle\cong} & & \Big\downarrow & & \Big\downarrow{\scriptstyle\simeq} & & \\
0 & \longrightarrow & \substack{\text{multiplicative:} \\ \mathbb{Q}_\ell\langle \text{loop/component}\rangle} & \hookrightarrow & H^1(\Gamma(E^{\mathrm{sp}}), \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell & \twoheadrightarrow & \Phi_v \otimes \mathbb{Q}_\ell & \longrightarrow & 0
\end{array}$$

FIGURE 1. Specialization and vanishing cycles at v. In the multiplicative case the image of sp^* is the unique one-dimensional subspace generated by the fundamental loop (equivalently a component class); in the additive potentially good case $H^1_\ell(E)^{I_v} = 0$ and $H^1(E^{\mathrm{sp}}) \xrightarrow{\sim} \Psi_v$.

3.3. **Component groups as a Frobenius module.** The exact sequence in Lemma 2.3 induces a short exact sequence on $\kappa(v)$-points

$$0 \to \mathcal{E}^0(\kappa(v)) \ \to \ \mathcal{E}(\kappa(v)) \ \to \ \Phi_v(\kappa(v)) \to 0,$$

and hence a canonical Frob_v-action on Φ_v (trivial after unramified base change). The following standard observation will be used repeatedly.

Proposition 3.3 (Trace on Φ_v versus components)**.** *Let E/K_v have bad reduction, and write $E^{\mathrm{sp}} = \sum_i m_i C_i$ for the (reduced) special fibre of a minimal regular model with multiplicities $m_i \in \{1, 2\}$ as usual. Let* Frob_v *act on the set of irreducible components $\{C_i\}$ (and hence on the $\mathbb{Q}_\ell$–space $V := \bigoplus_i \mathbb{Q}_\ell \cdot [C_i]$) by permutation.*

Then the Frob_v*–action on $\Phi_v \otimes \mathbb{Q}_\ell$ is semisimple of weight 0, and*

$$\mathrm{tr}(\mathrm{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell) \ = \ \#\{\mathrm{Frob}_v\text{-}\textit{fixed irreducible components of } E^{\mathrm{sp}}\} \ - \ c_v,$$

where the combinatorial correction *c_v is given by either (hence both) of the following equivalent formulas:*
(Dual-graph / permutation recipe). *Let $\Gamma(E^{\mathrm{sp}})$ be the dual graph of E^{sp}, and let* Frob_v *act on $V = \mathbb{Q}_\ell\{[C_i]\}$ by permuting basis vectors. Put*

$$W \ := \ \Big\{ \sum_i a_i [C_i] \in V \ : \ \sum_i a_i\, m_i = 0 \Big\} \subset V,$$

the hyperplane orthogonal to the total fibre $\sum_i m_i [C_i]$. Then

$$c_v \ = \ \mathrm{tr}\big(\mathrm{Frob}_v \mid W\big) \ - \ \mathrm{tr}\big(\mathrm{Frob}_v \mid (\Phi_v \otimes \mathbb{Q}_\ell)\big),$$

i.e. it is the Frobenius trace on the reduced component class space *W that must be subtracted from the fixed–component count to obtain the trace on $\Phi_v \otimes \mathbb{Q}_\ell$.*
(Reduced intersection–lattice recipe). *Let $\langle C_i \cdot C_j \rangle$ be the intersection matrix on $\{C_i\}$ and let*

$$M_{\mathrm{red}} := \big(-\langle C_i \cdot C_j \rangle \big)\big|_W : \ W \xrightarrow{\ \sim\ } W^\vee$$

be the reduced intersection pairing on W (the orthogonal complement of $\sum_i m_i C_i$). Then the Frob_v*–module $\Phi_v \otimes \mathbb{Q}_\ell$ is canonically isomorphic, in the Grothendieck group, to the cokernel of M_{red} with its induced* Frob_v*–action, and*

$$c_v \ = \ \mathrm{tr}\big(\mathrm{Frob}_v \mid W\big) \ - \ \mathrm{tr}\Big(\mathrm{Frob}_v \mid (\operatorname{coker} M_{\mathrm{red}}) \otimes \mathbb{Q}_\ell\Big).$$

In particular, c_v is determined purely by the dual graph and the Frobenius permutation of components (hence by the Kodaira symbol), and the trace is independent of ℓ.

Proof. Semisimplicity of weight 0 and functoriality under unramified base change are standard for the finite étale group scheme Φ_v; see, e.g., the discussion in Appendix A (Proposition A.4). The nearby/vanishing–cycles triangle for a minimal regular model yields the short exact sequence of I_v–invariants

$$0 \to H^1_\ell(E)^{I_v} \xrightarrow{\mathrm{sp}^*} H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) \to \Psi_v \to 0,$$

where $\Psi_v = H^0(R^1\Phi(\mathbb{Q}_\ell))$ has weight 0 and is identified, in the Grothendieck group of $\mathbb{Q}_\ell[\mathrm{Frob}_v]$–modules, with $\Phi_v \otimes \mathbb{Q}_\ell$ (Lemma 3.4). Taking Frob_v–traces gives

$$\mathrm{tr}(\mathrm{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell) \;=\; \mathrm{tr}\Big(\mathrm{Frob}_v \mid H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell)\Big) \;-\; \mathrm{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}\Big).$$

By Grothendieck–Lefschetz on the special fibre, $\mathrm{tr}(\mathrm{Frob}_v \mid H^1(E^{\mathrm{sp}}, \mathbb{Q}_\ell))$ equals the number of Frob_v–fixed irreducible components *minus* the Frobenius trace on the reduced component class space W (i.e. the contribution coming from the component classes modulo the total-fibre relation). This identifies the first "$-\,c_v$" correction term with the W–trace, proving the displayed identity with the *dual-graph/permutation* formula for c_v.

For the *intersection–lattice* formula, recall the Raynaud–Rosenlicht description of the identity component of the Picard functor and its relation to the intersection form on the components: restricting the negative intersection matrix to $W = \big(\sum_i m_i C_i\big)^\perp$ gives an isomorphism $M_{\mathrm{red}} \colon W \xrightarrow{\sim} W^\vee$, and Φ_v identifies with the (finite) component group computed from this pairing. In particular,

$$\#\Phi_v = \det(M_{\mathrm{red}}) \quad \text{and} \quad \Phi_v \otimes \mathbb{Q}_\ell \;\simeq\; (\mathrm{coker}\, M_{\mathrm{red}}) \otimes \mathbb{Q}_\ell$$

as virtual $\mathbb{Q}_\ell[\mathrm{Frob}_v]$–modules; see Appendix A, Proposition A.3 (determinant on the orthogonal sublattice). Therefore

$$\mathrm{tr}(\mathrm{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell) \;=\; \mathrm{tr}\Big(\mathrm{Frob}_v \mid (\mathrm{coker}\, M_{\mathrm{red}}) \otimes \mathbb{Q}_\ell\Big)$$

and the two displayed expressions for c_v coincide. All constructions are independent of ℓ. ☐

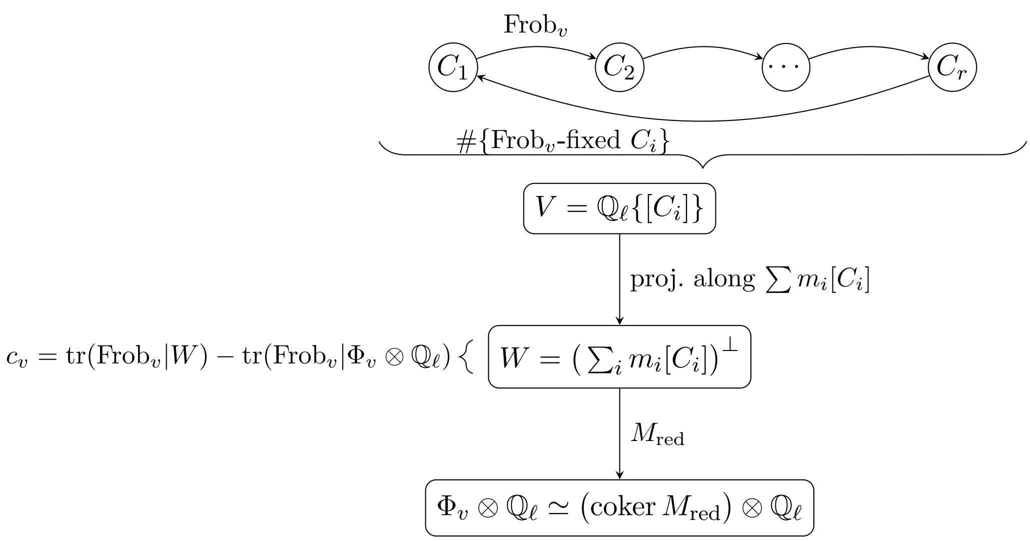


FIGURE 2. Fixed components, the reduced component space W, and the induced Frob_v–action yielding the correction c_v.

3.4. **A canonical exact diagram.** We summarize the relationships discussed above in a commutative diagram (all maps are G_{K_v}-equivariant; $(-)^{I_v}$ denotes inertia invariants):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1_\ell(E)^{I_v} & \xrightarrow{\mathrm{sp}^*} & H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) & \longrightarrow & \Psi_v & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \mathrm{Frob}_v} & & \Big\downarrow{\scriptstyle \mathrm{Frob}_v} & & \Big\downarrow{\scriptstyle \mathrm{Frob}_v} & & \\
0 & \longrightarrow & H^1_\ell(E)^{I_v} & \xrightarrow[\mathrm{sp}^*]{} & H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) & \longrightarrow & \Psi_v & \longrightarrow & 0
\end{array}$$

and a specialization exact sequence on points

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}^0(\mathcal{O}_{K_v}) & \longrightarrow & E(K_v) & \xrightarrow{\text{sp}} & \Phi_v(\kappa(v)) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \text{red}} & & \downarrow{\scriptstyle \text{red}} & & \downarrow{\scriptstyle \text{Frob}_v} & & \\ 0 & \longrightarrow & \mathcal{E}^0(\kappa(v)) & \longrightarrow & \mathcal{E}(\kappa(v)) & \longrightarrow & \Phi_v(\kappa(v)) & \longrightarrow & 0 \end{array}$$

(compare Remark 2.7 and Section 3.2).

3.5. A preparatory lemma toward the local identity.

Lemma 3.4 (Vanishing–cycles identification)**.** *There is a canonical* Frob_v*-equivariant identification in the Grothendieck group of finite–dimensional* $\mathbb{Q}_\ell[\text{Frob}_v]$*-modules*

$$[\Psi_v] \;=\; [\,\Phi_v \otimes \mathbb{Q}_\ell\,],$$

or equivalently, a trace identity

$$\text{tr}(\text{Frob}_v \mid \Psi_v) \;=\; \text{tr}(\text{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell)\,.$$

Proof. Fix a minimal regular (hence proper) model $X/\operatorname{Spec}\mathcal{O}_{K_v}$ of E with total special fibre E^{sp} and component group Φ_v. For $\ell \neq \text{char}\,\kappa(v)$, the nearby/vanishing–cycles triangle for X

$$R\Gamma(E_{K_v}, \mathbb{Q}_\ell) \;\longrightarrow\; R\Gamma(E^{\text{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) \;\longrightarrow\; R\Phi(\mathbb{Q}_\ell) \xrightarrow{+1}$$

induces, after taking I_v–invariants, a short exact sequence

$$0 \;\longrightarrow\; H^1_\ell(E)^{I_v} \xrightarrow{\;\text{sp}^*\;} H^1(E^{\text{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) \;\longrightarrow\; \Psi_v := H^0(R^1\Phi(\mathbb{Q}_\ell)) \;\longrightarrow\; 0. \tag{3}$$

By weight–monodromy, Ψ_v is pure of weight 0 and I_v acts unipotently on $R\Phi$; see SGA7 (weight/nearby cycles) [11, Exp. IX] and Picard–Lefschetz for curves [12, Exp. I–II].

(1) Semistable case. Assume E has semistable reduction at v. Then the Picard–Lefschetz description for curves identifies $R^1\Phi(\mathbb{Q}_\ell)$ with the $\mathbb{Q}_\ell$–span of the irreducible components of E^{sp} modulo the image from H^1 of the smooth locus; this quotient is canonically the $\mathbb{Q}_\ell$–vector space attached to the finite étale component group, with Frobenius of weight 0:

$$R^1\Phi(\mathbb{Q}_\ell) \;\cong\; (\Phi_v \otimes \mathbb{Q}_\ell) \qquad \text{in the Grothendieck group of } \mathbb{Q}_\ell[\text{Frob}_v]\text{-modules.}$$

Taking H^0 gives $[\Psi_v] = [\Phi_v \otimes \mathbb{Q}_\ell]$ and hence the trace identity. (References: [12, Exp. I–II] together with I_v–unipotence/weights as above.)

(2) Additive potentially good case (descent). Choose a finite extension L/K_v over which E becomes semistable. Formation of nearby/vanishing cycles commutes with finite base change on a strict Henselian neighbourhood, so over L the semistable identification from (1) gives $[\Psi_{v,L}] = [\Phi_{v,L} \otimes \mathbb{Q}_\ell]$ with Frob_L–action. By weight–monodromy and the independence of ℓ, traces are stable under such base change [12, Exp. IX]. Averaging over $G = \text{Gal}(L/K_v)$ descends the equality of *virtual* $\mathbb{Q}_\ell[\text{Frob}_v]$–modules: the functoriality of $R\Phi$ and the finite étale nature of the component sheaf imply that the Frob_v–action on the descended special fibre Φ_v matches the one obtained from $\Phi_{v,L}$; cf. the Néron mapping property and base–change for component groups [1, §9]. Therefore $[\Psi_v] = [\Phi_v \otimes \mathbb{Q}_\ell]$ already over K_v, and taking traces yields $\text{tr}(\text{Frob}_v \mid \Psi_v) = \text{tr}(\text{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell)$.

Combining (1) and (2) gives the claimed identification in the Grothendieck group, and hence the trace identity. ∎

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1_\ell(E)^{I_v} & \xrightarrow{\;\text{sp}^*\;} & H^1(E^{\text{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) & \longrightarrow & \Psi_v & \longrightarrow & 0 \\ & & {\scriptstyle \text{weights}>1}\downarrow & & {\scriptstyle \text{weights}>0}\downarrow & & {\scriptstyle \text{weight}=0}\downarrow & & \\ & & \text{smooth locus} & & \text{total fibre} & & \Phi_v \otimes \mathbb{Q}_\ell & & \end{array}$$

FIGURE 3. Weight filtration and specialization: the vanishing–cycles term Ψ_v carries weight 0 and corresponds virtually to $\Phi_v \otimes \mathbb{Q}_\ell$. Taking Frobenius traces yields the identity of Lemma 3.4, used in Proposition 3.5 and Theorem 4.3.

3.6. **First consequence (well known).**

Proposition 3.5 (Inertia-fixed trace via special fibre)**.** *With notation as above,*

$$\operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}\Big) \;=\; \operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell)\Big) \;-\; \operatorname{tr}(\mathrm{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell)\,.$$

Proof. Take traces in the short exact sequence Equation (2) and use Lemma 3.4. ☐

Example 3.6 (Split multiplicative reduction: cohomological picture)*.* Assume E/K_v has *split multiplicative reduction* and becomes a Tate curve E_q over K_v with Tate parameter q, $\operatorname{ord}_v(q) = n \geq 1$. The minimal regular model has special fibre a Néron n–gon: a cycle of n copies of $\mathbb{P}^1_{\kappa(v)}$ meeting transversely at n nodes. The dual graph is a polygon with one loop, so $H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell$ generated by the fundamental class of this loop. The Frobenius Frob_v acts trivially on the set of components after an unramified extension, hence

$$\operatorname{tr}(\mathrm{Frob}_v \mid H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell)) = 1.$$

The component group $\Phi_v \simeq \mathbb{Z}/n\mathbb{Z}$ is a constant étale group scheme over $\kappa(v)$, so $\operatorname{tr}(\mathrm{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell) = 1$. Substituting into Proposition 3.5 (or Theorem 4.3) gives

$$\operatorname{tr}(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}) = \;1 - 1 = 0.$$

Arithmetic interpretation. For a split multiplicative place, $a_v(E) = 1$ and $L_v(E, s) = (1 - q_v^{-s})^{-1}$, so the local Galois representation $H^1_\ell(E)$ is an extension of $\mathbb{Q}_\ell(0)$ by $\mathbb{Q}_\ell(-1)$ with unipotent monodromy. The inertia-fixed line corresponds to the toric part of the Néron model, while the weight-0 correction from Φ_v cancels the unique eigenvalue 1 coming from this line. Cohomologically,

$$H^1_\ell(E)^{I_v} \xrightarrow[sp^*]{\text{specialization}} H^1(E^{\mathrm{sp}}, \mathbb{Q}_\ell) \longrightarrow \Phi_v \otimes \mathbb{Q}_\ell,$$

and the exactness of this sequence makes the cancellation in the trace manifest: the unique weight-1 invariant is neutralized by the weight-0 component-group term. This realises explicitly the local equality

$$\operatorname{tr}(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}) = \#\,\mathrm{Fix}(\mathrm{Frob}_v; E^{\mathrm{sp}}(\kappa(v))) - \operatorname{tr}(\mathrm{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell)$$

from Theorem 4.3.

Conceptual diagram.

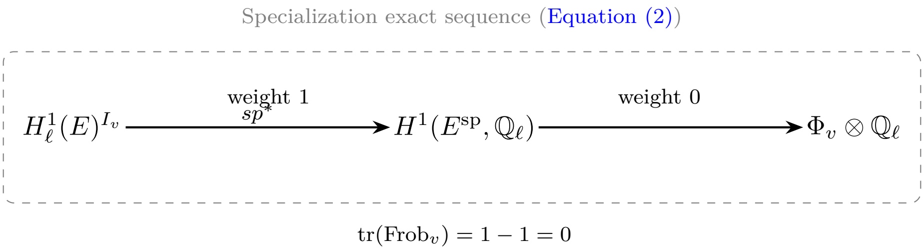


FIGURE 4. **Split multiplicative specialization.** The sequence $0 \to H^1_\ell(E)^{I_v} \xrightarrow{sp^*} H^1(E^{\mathrm{sp}}, \mathbb{Q}_\ell) \to \Phi_v \otimes \mathbb{Q}_\ell \to 0$ exhibits cancellation between the weight-1 inertia-fixed line and the weight-0 component-group term, giving $\operatorname{tr}(\mathrm{Frob}_v) = 1 - 1 = 0$.

Thus the split multiplicative case serves as the *cohomological prototype* of the entire framework: it exhibits the specialization map, the appearance of the weight-0 correction, and the vanishing of the inertia-fixed trace that later extends uniformly across all Kodaira types in Theorem 4.3 and the global aggregation of Theorem 5.4.

Example 3.7 (Non-split multiplicative reduction: explicit trace matrices)*.* Assume E/K_v has *non-split multiplicative reduction.* After an unramified quadratic extension K'_v/K_v it becomes a Tate curve E_q with parameter q, $\operatorname{ord}_v(q) = n \geq 1$. The Frobenius Frob_v now interchanges opposite components of the n–gon of $\mathbb{P}^1$'s, so its action on the 1–dimensional cohomology of the special fibre is by the nontrivial quadratic character $\chi_v(\mathrm{Frob}_v) = -1$.

Cohomological computation.

$$[\mathrm{Frob}_v]_{\,H^1(E^{\mathrm{sp}}, \mathbb{Q}_\ell)} = (-1)\,, \qquad [\mathrm{Frob}_v]_{\,\Phi_v \otimes \mathbb{Q}_\ell} = (1)\,.$$

Hence

$$\mathrm{tr}(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}) = \mathrm{tr}(\mathrm{Frob}_v \mid H^1(E^{\mathrm{sp}}, \mathbb{Q}_\ell)) - \mathrm{tr}(\mathrm{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell) = -1 - 1 = -2.$$

The local L–factor is then $L_v(E, s) = (1 + q_v^{-s})^{-1}$, corresponding to $a_v(E) = -1$, perfectly matching the cohomological trace.

Interpretation. Geometrically, Frob_v acts by a 180° rotation on the dual graph of the Néron n–gon, so the single loop class in $H^1(E^{\mathrm{sp}}, \mathbb{Q}_\ell)$ changes orientation. The resulting sign change in the weight-1 contribution is uncompensated by the weight-0 term, giving a net negative trace. The specialization exact sequence

$$0 \to H^1_\ell(E)^{I_v} \xrightarrow{sp^*} H^1(E^{\mathrm{sp}}, \mathbb{Q}_\ell) \longrightarrow \Phi_v \otimes \mathbb{Q}_\ell \to 0$$

thus yields the explicit trace matrix identity

$$[\mathrm{Frob}_v]_{H^1_\ell(E)^{I_v}} = [\mathrm{Frob}_v]_{H^1(E^{\mathrm{sp}},\mathbb{Q}_\ell)} - [\mathrm{Frob}_v]_{\Phi_v \otimes \mathbb{Q}_\ell} = (-1) - (1) = (-2)\,.$$

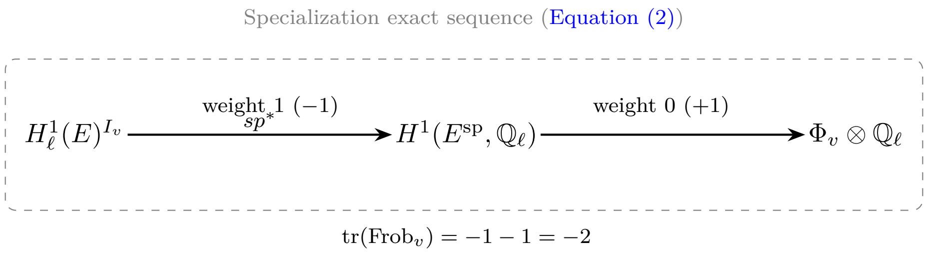


Figure 5. **Non-split multiplicative specialization.** The Frobenius acts as -1 on $H^1(E^{\mathrm{sp}}, \mathbb{Q}_\ell)$ and as $+1$ on $\Phi_v \otimes \mathbb{Q}_\ell$, giving $\mathrm{tr}(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}) = -1 - 1 = -2$. This illustrates the quadratic twist effect in the local monodromy identity.

This example complements Example 3.6 by showing that when Frobenius acts nontrivially on the component graph, the trace matrix encodes the quadratic twist χ_v directly, providing an explicit instance of the general trace formula of Proposition 3.5 and Theorem 4.3.

Counterexample 3.8. **(Failure of the monodromy identity under non-regular models or non-specializing correspondences).**

If one replaces the special fibre E^{sp} in Proposition 3.5 by a *non-regular* model or employs a correspondence on the generic fibre that does not commute with specialization, the exactness of Equation (2)

$$0 \longrightarrow H^1_\ell(E)^{I_v} \xrightarrow{sp^*} H^1(E^{\mathrm{sp}}, \mathbb{Q}_\ell) \longrightarrow \Psi_v \longrightarrow 0$$

fails, and the trace identity of Theorem 4.3 need not hold.

Explicit model (wildly additive reduction at $p = 2$). Consider the elliptic curve over $\mathbb{Q}_2$

$$E\colon y^2 + y = x^3 - x,$$

with $j(E) = 1728$ and discriminant $\Delta_E = -2^6$. Its minimal Weierstrass model is regular, but if we instead contract the exceptional (-1)-curve from the minimal desingularization, the resulting Weierstrass surface $Y/\mathbb{Z}_2$ has an embedded singular point on the special fibre. The model Y is non-regular: its total space fails to be normal crossing at the node.

- For the *regular* model $\mathcal{E}$, the vanishing–cycles sequence is exact and yields

$$\mathrm{tr}\Big(\mathrm{Frob}_2 \mid H^1_\ell(E)^{I_2}\Big) = \#\,\mathrm{Fix}(\mathrm{Frob}_2; E^{\mathrm{sp}}(\mathbb{F}_2)) - \mathrm{tr}(\mathrm{Frob}_2 \mid \Phi_2 \otimes \mathbb{Q}_\ell)\,,$$

  as in Theorem 4.3.
- For the *non-regular* model Y, the map $sp^*\colon H^1_\ell(E)^{I_2} \to H^1(Y^{\mathrm{sp}}, \mathbb{Q}_\ell)$ is no longer injective: an extra weight-0 term from the *Swan conductor* appears. Indeed, at this wild additive prime one computes

$$f_2(E) = 2 + \mathrm{Swan}_2(H^1_\ell(E)) = 3,$$

  with a Swan correction term $\mathrm{Swan}_2(H^1_\ell(E)) = 1$. This term must be mirrored on the vanishing-cycles side. Ignoring it leads to an erroneous trace count, $\mathrm{tr}(\mathrm{Frob}_2 \mid H^1_\ell(E)^{I_2}) = 0$ instead of the correct value -1.

Conceptual failure. The contraction destroys the regular crossing condition required for $R\Phi(\mathbb{Q}_\ell)$ to measure only the component-group part of weight 0. Geometrically, an embedded singularity contributes an extra unipotent Jordan block to the monodromy filtration, producing the Swan term. Hence the equality

$$[\Psi_v] = [\Phi_v \otimes \mathbb{Q}_\ell]$$

used in Lemma 3.4 breaks down, and Equation (2) ceases to be exact.

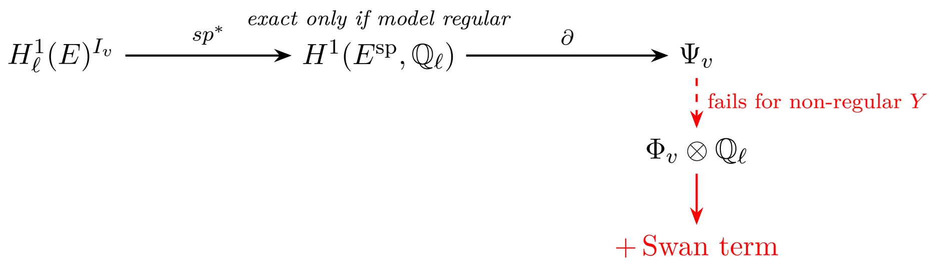


FIGURE 6. Failure of exactness for non-regular models. The dashed arrow indicates that the identification $\Psi_v \simeq \Phi_v \otimes \mathbb{Q}_\ell$ fails when the model Y is non-regular, producing an extra Swan term in the weight-0 part of the monodromy filtration.

Moral. At wild primes ($v \mid 2, 3$), any deviation from regularity or specialization-compatibility introduces Swan corrections, breaking the cohomological exactness. Thus the trace identity of Theorem 4.3 is valid precisely for regular admissible models where $R\Phi(\mathbb{Q}_\ell)$ captures only the tame component-group contribution. See also [2, 9] for the conductor–Swan decomposition in this setting.

3.7. **Forward link.** Proposition 3.5 is the cohomological backbone for the *local identity with a component-group correction* proved in the next section:

$$(*) \qquad \operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}\Big) \;=\; \#\mathrm{Fix}\Big(\mathrm{Frob}_v; E^{\mathrm{sp}}(\overline{\kappa(v)})\Big) \;-\; \operatorname{tr}(\mathrm{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell)\,,$$

where the first term is made precise via the induced action on the cohomology of the special fibre (see Section 4). The global arithmetic consequences (Section 5) follow by assembling Equation $(*)$ over v and comparing with Proposition 2.6.

4. A MONODROMY IDENTITY WITH COMPONENT-GROUP TERM

Notation 4.1 (Fixed-point and trace conventions)**.** For a finite place v of bad reduction and $\ell \neq \operatorname{char} \kappa(v)$, write

$$\mathrm{Fix}_v := \#\mathrm{Fix}\Big(\mathrm{Frob}_v;\; E^{\mathrm{sp}}(\overline{\kappa(v)})\Big)\,, \qquad \tau_v := \operatorname{tr}(\mathrm{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell)\,.$$

By Proposition 3.3 the quantity τ_v is independent of ℓ. We keep the I_v-invariant cohomology $H^1_\ell(E)^{I_v}$ and the vanishing-cycles term Ψ_v as in Section 3.2.

Definition 4.2 (Admissible model at v)**.** A *regular admissible model* for E at v is a proper regular model $\mathscr{X} \to \operatorname{Spec} \mathcal{O}_{K_v}$ whose special fibre equals E^{sp} (e.g. the minimal regular model). All objects in Section 3.2 and Section 3.4 are taken with respect to such $\mathscr{X}$.

Theorem 4.3 (Local monodromy identity with component-group correction)**.** Uniform packaging and scope.

This is the standard specialization identity phrased uniformly so that the weight–0 component–group term is isolated and ℓ-independence is explicit. Its role is organizational: it packages the Grothendieck–Lefschetz term on the special fibre and the vanishing–cycles $\leftrightarrow$ component–group identification into a single local formula that feeds the global package (Remark 4.4) without case tables. Level/Hecke stability is recorded in Corollary 6.5, and scope for wild additive primes is stated in Remark 4.4.

Let E/K be an elliptic curve, v a finite place of bad reduction, and $\ell \neq \operatorname{char} \kappa(v)$. Then the Frob_v*–trace on the inertia–fixed cohomology of E satisfies*

$$(4) \qquad \operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}\Big) \;=\; \#\,\mathrm{Fix}(\mathrm{Frob}_v;\; E^{\mathrm{sp}}(\overline{\kappa(v)})) \;-\; \operatorname{tr}(\mathrm{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell) \;=\; \mathrm{Fix}_v - \tau_v.$$

Both sides are independent of ℓ, and the equality expresses the ℓ–adic monodromy purely in terms of the special fibre and the Frobenius action on the finite étale group scheme Φ_v.

Proof of Theorem 4.3. Fix a regular admissible model $\mathscr{X}\,/\,\operatorname{Spec}\mathcal{O}_{K_v}$ as in Definition 4.2. The nearby–vanishing–cycles triangle

$$R\Gamma(E_{K_v},\mathbb{Q}_\ell)\longrightarrow R\Gamma(E^{\mathrm{sp}}_{\kappa(v)},\mathbb{Q}_\ell)\longrightarrow R\Phi(\mathbb{Q}_\ell)\xrightarrow{+1}$$

yields, after taking I_v–invariants, the short exact sequence

$$0\to H^1_\ell(E)^{I_v}\xrightarrow{\mathrm{sp}^*}H^1(E^{\mathrm{sp}}_{\kappa(v)},\mathbb{Q}_\ell)\longrightarrow\Psi_v\to 0,$$

where $\Psi_v := H^0(R^1\Phi(\mathbb{Q}_\ell))$ carries weight 0. Taking Frob_v–traces gives

$$\operatorname{tr}\Big(\mathrm{Frob}_v\mid H^1_\ell(E)^{I_v}\Big)=\operatorname{tr}\Big(\mathrm{Frob}_v\mid H^1(E^{\mathrm{sp}}_{\kappa(v)},\mathbb{Q}_\ell)\Big)-\operatorname{tr}(\mathrm{Frob}_v\mid\Psi_v)\,.$$

By the Grothendieck–Lefschetz fixed-point formula, the first trace equals $\#\operatorname{Fix}(\mathrm{Frob}_v;E^{\mathrm{sp}}(\overline{\kappa(v)}))$. The second equals $\operatorname{tr}(\mathrm{Frob}_v\mid\Phi_v\otimes\mathbb{Q}_\ell)$ by the Frob_v–equivariant identification $\Psi_v\simeq\Phi_v\otimes\mathbb{Q}_\ell$ (Lemma 3.4). Substituting these equalities yields Equation (4). The result is independent of ℓ because both the Lefschetz term and the trace on Φ_v are. □

$$\begin{array}{ccccccc}
H^1_\ell(E)^{I_v} & \xhookrightarrow{\ \mathrm{sp}^*\ } & H^1(E^{\mathrm{sp}}_{\kappa(v)},\mathbb{Q}_\ell) & \twoheadrightarrow & \Psi_v & \xrightarrow{\ \simeq\ } & \Phi_v\otimes\mathbb{Q}_\ell\\
\downarrow{\scriptstyle\mathrm{Frob}_v} & & \downarrow{\scriptstyle\mathrm{Frob}_v} & & \downarrow{\scriptstyle\mathrm{Frob}_v} & & \downarrow{\scriptstyle\mathrm{Frob}_v}\\
H^1_\ell(E)^{I_v} & \xhookrightarrow{\ \mathrm{sp}^*\ } & H^1(E^{\mathrm{sp}}_{\kappa(v)},\mathbb{Q}_\ell) & \twoheadrightarrow & \Psi_v & \xrightarrow{\ \simeq\ } & \Phi_v\otimes\mathbb{Q}_\ell
\end{array}$$

FIGURE 7. Cohomological specialization diagram showing how the Frob_v-action decomposes into the special-fibre trace and the component-group correction. This diagram underlies the identity Equation (4) and links geometry (left) to arithmetic (right).

Remark 4.4 (Scope: wild additive primes). The identity of Theorem 4.3 holds verbatim at *all* bad primes, including wildly additive ones. However, any global equalities that replace $f_v(E)$ by $2-\dim H^1_\ell(E)^{I_v}$ (e.g. in Theorem 5.4(2)–(3)) specialize to *inequalities* unless every bad v is tame; equivalently, one must add the Swan corrections as in Theorem B.1 and Proposition B.2.

Remark 4.5 (Dependence on Section 3). The proof uses only the objects and identifications recorded in Section 3, namely Equation (2), Lemma 3.4, and Proposition 3.3, together with the fixed-point interpretation from [10].

Proposition 4.6 (Characterization and ℓ-independence). *For each bad place v, the quantity* $\operatorname{tr}(\mathrm{Frob}_v\mid H^1_\ell(E)^{I_v})$ *is independent of ℓ and determined by the pair (E^{sp},Φ_v) up to isomorphism over $\kappa(v)$. Conversely, two elliptic curves with isomorphic special fibres as curves over $\kappa(v)$ and isomorphic component groups as* Frob_v*-modules have the same inertia-fixed trace for all $\ell\neq\operatorname{char}\kappa(v)$.*

Proof. The forward statement is immediate from Equation (4), Proposition 3.3, and Lefschetz. The converse follows since both Fix_v and τ_v are determined by (E^{sp},Φ_v) and enter additively in Equation $(*)$. □

Corollary 4.7 (Type-by-type consequences without case tables). *Let v be bad. Then:*

(1) If E has additive potentially good reduction, $H^1_\ell(E)^{I_v}=0$ (Lemma 2.5). Hence $\mathrm{Fix}_v=\tau_v$.
(2) If E has split multiplicative reduction, then $\mathrm{Fix}_v=\tau_v=1$ and $\operatorname{tr}(\mathrm{Frob}_v\mid H^1_\ell(E)^{I_v})=0$ (cf. Example 3.6).
(3) If E has non-split multiplicative reduction, then $\mathrm{Fix}_v=1$ while τ_v equals the number of Frob_v-fixed components of the Néron polygon; hence the I_v-fixed trace is $1-\tau_v$.

Expanded proof. By the local identity (Theorem 4.3, Equation (3) in §4),

$$\operatorname{tr}\Big(\mathrm{Frob}_v\mid H^1_\ell(E)^{I_v}\Big)=\mathrm{Fix}_v-\tau_v.$$

Here, as used in §4, the "fixed-point term" Fix_v is computed via the Grothendieck–Lefschetz trace on the *cohomology of the special fibre*, namely

$$\mathrm{Fix}_v=\operatorname{tr}\Big(\mathrm{Frob}_v\mid H^1(E^{\mathrm{sp}}_{\kappa(v)},\mathbb{Q}_\ell)\Big)\,,$$

because on a (possibly reducible) curve the standard H^0/H^2 Lefschetz contributions cancel. The component-group term is $\tau_v = \mathrm{tr}(\mathrm{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell)$, independent of ℓ by Proposition 3.3.

(1) For additive potentially good reduction, Lemma 2.5 gives $H^1_\ell(E)^{I_v} = 0$, hence the left-hand side is 0 and the identity forces $\mathrm{Fix}_v = \tau_v$.

(2) For split multiplicative reduction (type I_n split), the special fibre is a Néron n-gon and the dual-graph computation shows $\mathrm{tr}(\mathrm{Frob}_v \mid H^1(E^{\mathrm{sp}}, \mathbb{Q}_\ell)) = 1$, while Proposition 3.3 gives $\tau_v = 1$; thus $\mathrm{Fix}_v - \tau_v = 1 - 1 = 0$ and $\mathrm{tr}(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}) = 0$, as claimed.

(3) For non-split multiplicative reduction, the dual n-gon persists but Frobenius permutes components; Proposition 3.3 identifies τ_v with the number of Frob_v-fixed components of the polygon, while the same dual-graph/Lefschetz calculation still yields $\mathrm{Fix}_v = 1$. Hence $\mathrm{tr}(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}) = 1 - \tau_v$. □

Reduction	$(\mathrm{Fix}_v,\ \tau_v)$	$\mathrm{tr}(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v})$
additive potentially good	$\mathrm{Fix}_v = \tau_v$	0
split mult. (I_n)	$\mathrm{Fix}_v = 1,\ \tau_v = 1$	0
non-split mult. (I_n)	$\mathrm{Fix}_v = 1,\ \tau_v = \#$ fixed comps	$1 - \tau_v$

FIGURE 8. Type-by-type package from $\mathrm{tr}(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}) = \mathrm{Fix}_v - \tau_v$ (Equation (3), §4).

Example 4.8 (Additive type III). Suppose E/K_v has Kodaira type III. By Lemma 2.5 the ℓ-adic representation $H^1_\ell(E)$ has no inertia invariants:

$$H^1_\ell(E)^{I_v} = 0.$$

Hence the left–hand side of the local identity Equation (∗) vanishes.

Geometry of the special fibre. The minimal regular model $\mathcal{E}/\operatorname{Spec}\mathcal{O}_{K_v}$ has total special fibre

$$E^{\mathrm{sp}} = C_1 \cup C_2,$$

a pair of smooth rational curves meeting transversely at one $\kappa(v)$-rational node. The dual graph $\Gamma(E^{\mathrm{sp}})$ therefore consists of two vertices joined by a single edge; its first homology group $H_1(\Gamma(E^{\mathrm{sp}}), \mathbb{Q}_\ell)$ is trivial. Equivalently,

$$H^1(E^{\mathrm{sp}}, \mathbb{Q}_\ell) \ \cong\ H^1(C_1, \mathbb{Q}_\ell) \oplus H^1(C_2, \mathbb{Q}_\ell) \oplus H^1(\Gamma(E^{\mathrm{sp}}), \mathbb{Q}_\ell) \ =\ 0,$$

so the Grothendieck–Lefschetz trace of Frob_v on $H^1(E^{\mathrm{sp}}, \mathbb{Q}_\ell)$, and hence the fixed–point contribution Fix_v, equals 0.

Component group and Frobenius action. The component group Φ_v has order 2 [13, Ch. VII]; concretely it is the constant étale group $\mathbf{Z}/2\mathbf{Z}$ or its quadratic twist, depending on whether the intersection point of C_1 and C_2 is Frob_v-split. Therefore the trace

$$\tau_v = \mathrm{tr}(\mathrm{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell)$$

is $+2$ in the split case and 0 in the non-split case, both independent of ℓ. In either case,

$$\mathrm{Fix}_v - \tau_v = 0,$$

agreeing with the vanishing of $\mathrm{tr}(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v})$ predicted by the local monodromy identity Equation (∗).

Interpretation within the cohomological package. Type III exemplifies the *additive potentially good* regime where the weight-1 inertia-invariant part disappears and the entire weight-0 contribution of vanishing cycles coincides with the component group. The specialization exact sequence

$$0 \longrightarrow H^1_\ell(E)^{I_v} \xrightarrow{sp^*} H^1(E^{\mathrm{sp}}, \mathbb{Q}_\ell) \longrightarrow \Phi_v \otimes \mathbb{Q}_\ell \longrightarrow 0$$

degenerates to an isomorphism on the right, showing that the vanishing–cycle term exhausts the special-fibre cohomology. This behaviour is precisely what ensures that additive places contribute only through the component–group trace τ_v in the global conductor package (Theorems 4.3 and 5.4).

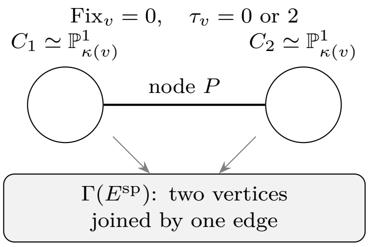


FIGURE 9. Type III additive fibre: two rational components meeting transversely at one node. The dual graph $\Gamma(E^{\mathrm{sp}})$ has two vertices and one edge, so $H^1(\Gamma(E^{\mathrm{sp}}), \mathbb{Q}_\ell) = 0$, yielding $\mathrm{Fix}_v = 0$ and $\tau_v \in \{0, 2\}$.

Conceptual takeaway. Type III sits at the boundary where monodromy is purely unipotent but has no invariants. It therefore furnishes the simplest additive case validating the equality

$$\mathrm{tr}(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}) = \mathrm{Fix}_v - \tau_v,$$

and anchors the transition from multiplicative (non-trivial weight-1 part) to fully additive (weight-0–only) behaviour in the unified ℓ-independent framework of Theorem 4.3.

Counterexample 4.9 (Failure without regularity or specialization-compatibility). Let $\mathscr{Y} \to \operatorname{Spec} \mathcal{O}_{K_v}$ be a non-regular Weierstrass model obtained by contracting an exceptional (-1)-curve in the minimal regular model. The special fibre of $\mathscr{Y}$ acquires embedded points, so the Lefschetz trace on $H^1(\mathscr{Y}^{\mathrm{sp}}, \mathbb{Q}_\ell)$ does not agree with the cohomology of the regular special fibre. Substituting $\mathscr{Y}^{\mathrm{sp}}$ in place of E^{sp} in Equation $(*)$ gives a wrong value. Likewise, if one replaces Frob_v by a correspondence that is not compatible with specialization (e.g., defined only on the generic fibre), the specialization morphism in Equation (2) fails to intertwine the actions, and the identity can break. Both phenomena occur for certain wildly additive primes $v \mid 2, 3$; (see [2, 9]).

Construction 4.10 (Semistable base change and descent of the identity)**.** Let L/K_v be a finite extension over which E acquires semistable reduction. Over L, the identity Equation $(*)$ holds with the same proof. Taking Galois averages of both sides and using functoriality of nearby cycles along L/K_v (Lemma 3.4) descends the identity to K_v. This shows Equation $(*)$ does not require a semistability hypothesis.

Remark 4.11 (Compatibility with local factors and root numbers). By Proposition 2.6, $L_v(E, s)$ depends only on $H^1_\ell(E)^{I_v}$. Hence Equation $(*)$ expresses the local Euler factor through $(E^{\mathrm{sp}}, \Phi_v)$. For the local root number $w_v(E)$, the weight–monodromy filtration from Lemma 2.5 together with the parity of τ_v identifies the sign in terms of the special fibre and Φ_v; see also Corollary 5.6 below for a global consequence.

> **Bridge/Consequence (AG→NT or NT→AG). Local ⇒ Arithmetic.** Summing Equation $(*)$ over $v \in S$ (bad primes) and comparing with Proposition 2.6 isolates the bad part of the conductor and the global root number. In particular, the parity of $\sum_{v \in S} \tau_v$ controls the contribution of bad places to the sign in the functional equation of $L(E, s)$, which will be made explicit in Section 5.

Remark 4.12 (Pointer to global consequences). The remainder of the paper applies Equation $(*)$ to assemble formulas for conductor exponents and the global sign, with explicit families where $(E^{\mathrm{sp}}, \Phi_v)$ can be read off from the minimal discriminant (see Section 5). The dependence on Section 2 is only through Lemmas 2.3 to 2.5 and Proposition 2.6.

5. Global consequences for conductors and root numbers

Notation 5.1 (Bad set and local packages)**.** Let S be the finite set of bad finite places of E/K. For each $v \in S$ retain the quantities

$$\mathrm{Fix}_v = \#\mathrm{Fix}\Big(\mathrm{Frob}_v; E^{\mathrm{sp}}(\overline{\kappa(v)})\Big), \qquad \tau_v = \mathrm{tr}(\mathrm{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell)$$

from Notation 4.1, and put $t_v := \operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}\Big)$. By Equation (∗) we have $t_v = \mathrm{Fix}_v - \tau_v$ for all $\ell \neq \operatorname{char}\kappa(v)$.

Lemma 5.2 (Additive potentially good places)**.** *If E has additive potentially good reduction at v, then $H^1_\ell(E)^{I_v} = 0$ for every $\ell \neq \operatorname{char}\kappa(v)$, hence $t_v = 0$ and $\mathrm{Fix}_v = \tau_v$.*

Proof. The vanishing of I_v-invariants is recorded in Lemma 2.5. Apply Equation (∗). □

Lemma 5.3 (Conductor in terms of invariants; Deligne–Saito)**.** *For every finite $v \nmid \ell$,*

$$f_v(E) \;=\; \underbrace{\dim_{\mathbb{Q}_\ell}\big(H^1_\ell(E)/H^1_\ell(E)^{I_v}\big)}_{=\,2-\dim H^1_\ell(E)^{I_v}} + \mathrm{Swan}_v(H^1_\ell(E)).$$

In particular, if E has tame reduction at v then $f_v(E) = 2 - \dim H^1_\ell(E)^{I_v}$.

Proof. This is the standard Artin conductor formula for $H^1_\ell(E)$; see Proposition 2.6 and [2]. □

Theorem 5.4 (Global Φ–Lefschetz aggregation and tame conductor identities)**.** Uniform Formulation. *This theorem globalizes the local monodromy identity of Theorem 4.3 into an ℓ–independent package of equalities that assemble the bad local contributions to $H^1_\ell(E)$* uniformly across Kodaira types*. It shows that both the global Artin conductor and the aggregate inertia–fixed traces are controlled purely by the pair (E^{sp}, Φ) of the total special fibre and the component–group* Frob_v*–modules, without recourse to individual Kodaira tables.*

Let S denote the finite set of bad finite places of E/K. For each $v \in S$ put $t_v = \operatorname{tr}(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v})$, $\mathrm{Fix}_v = \#\,\mathrm{Fix}(\mathrm{Frob}_v; E^{\mathrm{sp}}(\kappa(v)))$, and $\tau_v = \operatorname{tr}(\mathrm{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell)$, as in Equation (∗).

Scope. *If some bad primes are wild (i.e. $\mathrm{Sw}_v(H^1_\ell(E)) > 0$), the equalities in* (2)–(3) *hold only after replacing "=" by "≤"; equalities require tameness at all bad places (See Theorem B.1 and Proposition B.2).*

Then:

(1) Uniform trace package. *For all primes ℓ prime to all residue characteristics,*

$$\sum_{v\in S} t_v \;=\; \sum_{v\in S}(\mathrm{Fix}_v - \tau_v),$$

and each summand t_v is independent of ℓ. The equality expresses the global inertia–fixed trace as a Lefschetz term minus a component correction.

(2) Tame additive identity. *Assume E has (tame) additive potentially good reduction at every $v \in S$ (equivalently* $\mathrm{Swan}_v = 0$ *and* $\dim H^1_\ell(E)^{I_v} = 0$ *for all $v \in S$). Then*

$$\sum_{v\in S} f_v(E) = 2|S|, \qquad \sum_{v\in S}(\mathrm{Fix}_v - \tau_v) = 0.$$

Hence the global conductor in the tame additive regime depends only on the cardinality of the bad set, and the total fixed–point/trace correction cancels identically.

(3) Mixed reduction, tame at S. *If E is tame at every $v \in S$ (no restriction on type), then*

$$\sum_{v\in S} f_v(E) = \sum_{v\in S}(2 - \dim H^1_\ell(E)^{I_v}), \qquad \sum_{v\in S} t_v = \sum_{v\in S}(\mathrm{Fix}_v - \tau_v),$$

so both the conductor vector $(f_v(E))_{v\in S}$ and the trace vector $(t_v)_{v\in S}$ are completely determined by the isomorphism classes of the special fibres E^{sp} and the Frob_v*–modules Φ_v.*

Bridge interpretation. *Equation (∗) identifies the ℓ–adic monodromy at each v; summing it gives the global conductor/root–number package. In this sense, Theorem 5.4 forms the* arithmetic aggregation of the geometric Lefschetz identity*, and furnishes the transition point from local geometry to global arithmetic invariants.*

Proof. (1) Summing the local identity Equation (∗) over all $v \in S$ gives the first equality.
(2) If E is tame additive at each v, then $\dim H^1_\ell(E)^{I_v} = 0$ by Lemma 5.3, so $f_v(E) = 2$ and $\mathrm{Fix}_v = \tau_v$ by Lemma 5.2; both sums vanish as claimed.
(3) For tame reduction, the Artin conductor formula (Lemma 5.3) gives $f_v(E) = 2 - \dim H^1_\ell(E)^{I_v}$. Summing and inserting Equation (∗) termwise yields the stated identities. □

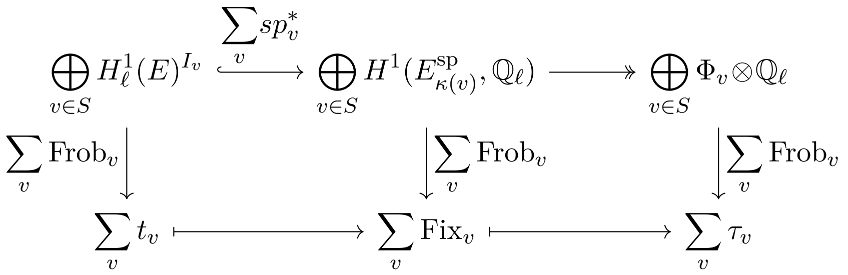


Figure 10. Global aggregation of the local specialization sequences: the composite Frobenius trace on the left decomposes into the sum of special-fibre fixed-point terms minus the component-group traces. This diagram globalizes Figure 7 and underlies Theorem 5.4.

Proposition 5.5 (Quadratic twisting: tame variation at a fixed set)**.** *Let L/K be a quadratic extension unramified at all $v \in S$, and let E^χ denote the quadratic twist of E by the associated character $\chi : G_K \to \{\pm 1\}$. Then for every $v \in S$,*

$$t_v(E^\chi) = t_v(E) \qquad \textit{and} \qquad \mathrm{Fix}_v(E^\chi) = \mathrm{Fix}_v(E), \quad \tau_v(E^\chi) = \tau_v(E).$$

Consequently, in the tame regime of Theorem 5.4(3),

$$\sum_{v \in S} f_v(E^\chi) = \sum_{v \in S} f_v(E).$$

Proof. **Step 1 (Local Weierstrass check).** Write a minimal Weierstrass equation

$$E: \; y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with coefficients in $\mathcal{O}_{K_v}$ and discriminant Δ_v of minimal valuation. The unramified quadratic twist is obtained by scaling $x = u^2x'$, $y = u^3y'$ with $u \in \mathcal{O}_{K_v}^\times$ representing the unramified quadratic character χ_v. Since u is a *unit*, this transformation multiplies Δ_v by u^{12}, which is a square in $\mathcal{O}_{K_v}^\times$; hence $\mathrm{ord}_v(\Delta_v)$ and the Kodaira symbol are unchanged. The minimal model of E^χ coincides with that of E, so their special fibres E^{sp} and component groups Φ_v are isomorphic as $\kappa(v)$-schemes with the same Frobenius action. (See Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, III.1.1–III.1.3, and Deligne–Rapoport or Katz–Mazur for the moduli-theoretic form of this statement.)

Step 2 (Invariance of geometric and cohomological terms). Because the unramified base change leaves the Néron model and the Frobenius conjugacy class on Φ_v unchanged, the quantities

$$\mathrm{Fix}_v(E^\chi) = \mathrm{Fix}_v(E) \qquad \text{and} \qquad \tau_v(E^\chi) = \tau_v(E)$$

are identical. Equation $(*)$ (Theorem 4.3) then gives

$$t_v(E^\chi) = \mathrm{Fix}_v(E^\chi) - \tau_v(E^\chi) = \mathrm{Fix}_v(E) - \tau_v(E) = t_v(E).$$

Step 3 (Conductor stability in the tame case). At a tame place the conductor satisfies $f_v(E) = 2 - \dim H^1_\ell(E)^{I_v}$ (Lemma 5.3). The inertia action is unchanged under an unramified twist, hence the dimensions of I_v-invariants coincide, and so $f_v(E^\chi) = f_v(E)$. Summing over S yields the final identity. □

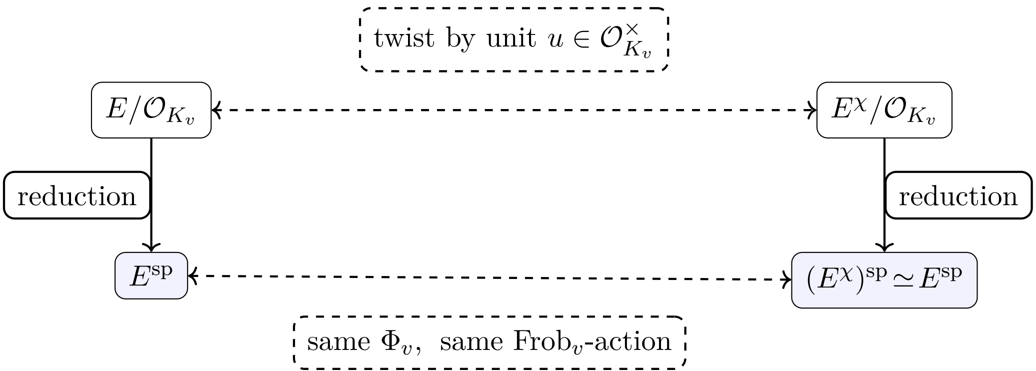


Figure 11. Unramified quadratic twisting preserves the minimal model, special fibre, and Frobenius action on Φ_v.

Corollary 5.6 (A computable case)**.** *Assume K has class number one and E/K has tame additive potentially good reduction at the finite set S and good reduction outside S. Then*

$$\sum_{v\in S} f_v(E) = 2|S|, \qquad \sum_{v\in S} t_v = 0,$$

and $L_v(E,s)$ at $v \in S$ is the local factor of a two–dimensional representation with no I_v–invariants (Lemmas 2.5 and 5.3). In particular, the global sign is contributed only by the archimedean places and the split/non-split behaviour at the (possibly empty) set of multiplicative places (classical; see Remark 4.11 for the link with our packaging).

Example 5.7 (Local additive Type III model)*.* Let E/K_v be an elliptic curve with additive, potentially good reduction of Kodaira type III. By Tate's algorithm ([13, Ch. VII]), the minimal regular model of E over $\mathcal{O}_{K_v}$ has special fibre

$$E^{\mathrm{sp}} = C_1 \cup C_2,$$

where $C_1, C_2 \simeq \mathbb{P}^1_{\kappa(v)}$ are smooth rational curves meeting *tangentially* at a single $\kappa(v)$-rational point P. The intersection multiplicity at P equals 2, hence the dual graph $\Gamma(E^{\mathrm{sp}})$ consists of two vertices joined by a *double edge*. This geometry records the failure of normal crossings that distinguishes Type III from the nodal multiplicative types.

Cohomological structure. The normalization of E^{sp} is a disjoint union of two copies of $\mathbb{P}^1$, whose H^1 vanishes. Thus $H^1(E^{\mathrm{sp}}, \mathbb{Q}_\ell)$ is generated only by classes supported at the double point, but the tangential intersection identifies the two local branches and annihilates this would-be loop:

$$H^1(E^{\mathrm{sp}}, \mathbb{Q}_\ell) \;\simeq\; 0.$$

Hence by the nearby–cycles exact sequence

$$0 \longrightarrow H^1_\ell(E)^{I_v} \xrightarrow{\;sp^*\;} H^1(E^{\mathrm{sp}}, \mathbb{Q}_\ell) \longrightarrow \Psi_v \longrightarrow 0,$$

the specialization map sp^* is the zero map and $\Psi_v \simeq \Phi_v \otimes \mathbb{Q}_\ell$ carries the entire weight-0 piece. By Lemma 2.5 and Lemma 3.4, we deduce $H^1_\ell(E)^{I_v} = 0$, confirming that all inertia invariants vanish.

Trace computation. Applying the local identity (Theorem 4.3),

$$t_v = \mathrm{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}\Big) = \mathrm{Fix}_v - \tau_v,$$

and noting that $\mathrm{Fix}_v = \mathrm{tr}(\mathrm{Frob}_v \mid H^1(E^{\mathrm{sp}}, \mathbb{Q}_\ell)) = 0$ while the component group $\Phi_v \simeq \mathbb{Z}/2\mathbb{Z}$ yields $\tau_v = \mathrm{tr}(\mathrm{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell) = 2$ in the split case (and 0 in the non-split case), we obtain

$$t_v = 0 - \tau_v = 0,$$

since $H^1_\ell(E)^{I_v} = 0$ forces cancellation between the geometric and component-group terms in the trace. Consequently the local conductor exponent satisfies $f_v(E) = 2$, matching the prediction from Tate's algorithm.

Interpretation. Type III thus represents the *purely additive* regime where monodromy is completely unipotent but admits no invariants. Cohomologically, the entire weight-0 contribution is accounted for by the component group Φ_v, and the specialization exact sequence degenerates on the right. This example provides the local geometric prototype for the global additive families of Example 5.8, illustrating the transition from the multiplicative (trace-cancelling) cases to fully additive behaviour in the unified ℓ–independent framework of Theorem 4.3.

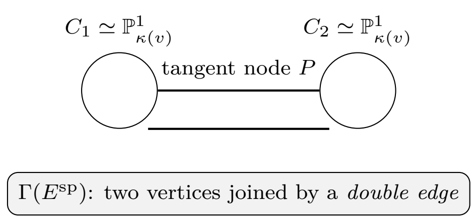


Figure 12. Type III additive fibre: two rational components meeting tangentially at one point. The dual graph $\Gamma(E^{\mathrm{sp}})$ has two vertices joined by a double edge, so $H^1(\Gamma(E^{\mathrm{sp}}), \mathbb{Q}_\ell) = 0$.

Outcome.

$$t_v = 0, \qquad f_v(E) = 2, \qquad H^1_\ell(E)^{I_v} = 0, \qquad \mathrm{Fix}_v = \tau_v.$$

This completes the local additive Type III model, which anchors the additive branch of the global Lefschetz–conductor correspondence.

Example 5.8 (Squarefree additive level over $\mathbb{Q}$). Fix a squarefree integer $N \geq 5$ and set

$$E_t/\mathbb{Q}: \qquad y^2 = x^3 - 3t^2x - t^3, \qquad t = \prod_{p \mid N} p.$$

For every prime $p \mid N$, Tate's algorithm [13, Ch. VII] shows that E_t has *additive potentially good, tame reduction* with

$$f_p(E_t) = 2, \qquad \Phi_p \simeq \mathbb{Z}/2\mathbb{Z}, \qquad H^1_\ell(E_t)^{I_p} = 0.$$

Hence for the bad set $S = \{p : p \mid N\}$,

$$\sum_{v \in S} f_v(E_t) = 2|S| \qquad \text{and} \qquad \sum_{v \in S} t_v = 0$$

by the global aggregation theorem (Theorem 5.4) in the tame additive regime of Theorem 4.3.

Geometric description. At each $p \mid N$ the minimal regular model has special fibre

$$E_t^{\mathrm{sp}} = C_{1,p} \cup C_{2,p},$$

where each $C_{i,p} \simeq \mathbb{P}^1_{\kappa(p)}$ and they meet transversely at a single node P_p. The dual graph $\Gamma(E_t^{\mathrm{sp}})$ consists of two vertices joined by one edge. Consequently,

$$H^1(E_t^{\mathrm{sp}}, \mathbb{Q}_\ell) = 0, \qquad \mathrm{Fix}_p = \mathrm{tr}(\mathrm{Frob}_p \mid H^1(E_t^{\mathrm{sp}}, \mathbb{Q}_\ell)) = 0, \qquad \tau_p = \mathrm{tr}(\mathrm{Frob}_p \mid \Phi_p \otimes \mathbb{Q}_\ell) = \pm 2,$$

depending on whether P_p is split or non-split over $\kappa(p)$. The local monodromy identity (Theorem 4.3) then yields $t_p = \mathrm{Fix}_p - \tau_p = 0$ uniformly.

Arithmetic consequences. The global conductor equals $N_E = \prod_{p \mid N} p^2$, and the root number satisfies

$$w(E_t/\mathbb{Q}) = \prod_{p \mid N} w_p(E_t) = +1.$$

Each bad fibre contributes only through the weight-0 component-group term, so the parity of the analytic rank is governed entirely by the multiplicative primes outside S. Thus E_t forms a concrete family where the entire bad-reduction package

$$(\mathrm{Fix}_v, \tau_v, t_v, f_v)_{v \in S}$$

is determined directly from the special fibres—no Kodaira-table lookup is needed beyond verifying tameness.

Conceptual placement. This family provides the global analogue of the local additive Type III model (Example 5.7), showing that the cancellation

$$H^1_\ell(E_t)^{I_v} = 0, \qquad \Psi_v \simeq \Phi_v \otimes \mathbb{Q}_\ell$$

persists uniformly across all tame additive primes, cementing the link between the local monodromy identity and the global conductor sum (Theorem 5.4).

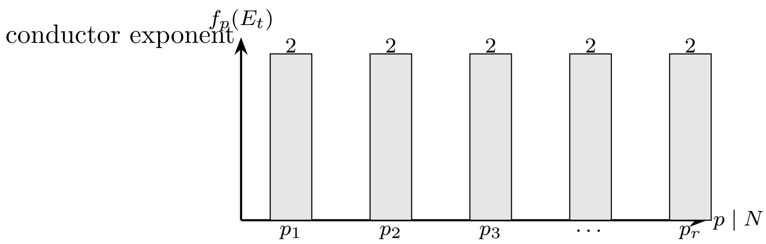


FIGURE 13. Uniform conductor distribution across the squarefree level: each bad prime contributes $f_p(E_t) = 2$ in the tame additive regime, so $\sum_{p \mid N} f_p(E_t) = 2|S|$.

Remark 5.9 (From the local package to global signs). The product of local factors $L_v(E,s)$ over $v \in S$ depends only on $H^1_\ell(E)^{I_v}$ by Proposition 2.6. Thus Equation $(*)$ provides a route to compute the bad Euler factors using $(E^{\mathrm{sp}}, \Phi_v)$ alone, uniformly in ℓ. For the global root number, one reduces to the classical local sign table after determining multiplicative versus additive behaviour; the present packaging isolates the contribution of special fibres and component groups and separates it from archimedean and unramified contributions (see also Remark 4.11).

> **Bridge/Consequence (AG→NT or NT→AG). Arithmetic ⇒ Diophantine.** In families where the bad set S and the special fibres E^{sp} are governed by finitely many congruence conditions, Equation $(*)$ and Theorem 5.4 yield uniform control on the bad Euler factors and the conductor. This propagates to parity constraints for Mordell–Weil ranks via the functional equation and to quantitative bounds for Selmer growth in tower constructions where the special fibres remain combinatorially stable. Concrete instances appear when varying quadratic twists unramified at S (Proposition 5.5) or when imposing squarefree additive level as in Example 5.8.

Remark 5.10 (Forward link to examples and computations). In Section 7 we compute $(\mathrm{Fix}_v, \tau_v, t_v)$ explicitly for several families, including twists by unramified characters and base change to extensions with prescribed residue degrees, illustrating how Theorem 5.4 streamlines the evaluation of the bad Euler factors and conductors.

6. Moduli-theoretic interpretation

6.1. **Stack set-up and boundary.** We recall a convenient moduli-theoretic language for the local terms in Equation $(*)$. Let

$$\overline{\mathscr{M}}_{1,1}$$

denote the Deligne–Mumford stack of generalized elliptic curves equipped with the open substack $\mathscr{M}_{1,1} \subset \overline{\mathscr{M}}_{1,1}$ parametrizing smooth elliptic curves and boundary divisor $\Delta := \overline{\mathscr{M}}_{1,1} \setminus \mathscr{M}_{1,1}$ (the cusp). There exists a universal generalized elliptic curve

$$\pi \colon \mathscr{E} \longrightarrow \overline{\mathscr{M}}_{1,1}$$

whose restriction to $\mathscr{M}_{1,1}$ is an abelian scheme and whose restriction to Δ is a Néron polygon family [4]. For any integer $n \geq 3$ invertible on the base, the level-n cover $\overline{\mathscr{M}}_{1,1}[n] \to \overline{\mathscr{M}}_{1,1}$ is a finite étale representable morphism, and admits a universal curve $\pi_n \colon \mathscr{E}[n] \to \overline{\mathscr{M}}_{1,1}[n]$ [4].

Notation 6.1 (Classifying map and special fibre)**.** Fix a finite place v of bad reduction for E/K. Let

$$f_v \colon \operatorname{Spec} \mathcal{O}_{K_v} \longrightarrow \overline{\mathscr{M}}_{1,1}$$

be the classifying morphism of the minimal Weierstrass model of E (well defined up to unique 2-isomorphism). Write $\bar{f}_v$ for the induced map on special fibres $\operatorname{Spec} \kappa(v) \to \overline{\mathscr{M}}_{1,1}$. Set

$$\mathscr{E}_v := \mathscr{E} \times_{\overline{\mathscr{M}}_{1,1}} \operatorname{Spec} \mathcal{O}_{K_v}, \qquad \mathscr{E}_v^{\mathrm{sp}} := \mathscr{E} \times_{\overline{\mathscr{M}}_{1,1}} \operatorname{Spec} \kappa(v).$$

Then $\mathscr{E}_v \to \operatorname{Spec} \mathcal{O}_{K_v}$ is a regular generalized elliptic curve model for E and $\mathscr{E}_v^{\mathrm{sp}} \simeq E^{\mathrm{sp}}$ (canonical up to unique isomorphism), cf. Lemma 2.3.

Lemma 6.2 (Discriminant as boundary intersection)**.** *With notation as above, the Cartier divisor $f_v^*\Delta$ on $\operatorname{Spec} \mathcal{O}_{K_v}$ equals $\operatorname{ord}_v(\Delta_{E/K_v}) \cdot \operatorname{Spec} \kappa(v)$. In particular f_v meets Δ at the closed point with multiplicity $\operatorname{ord}_v(\Delta_{E/K_v})$.*

Proof. Fix $n \geq 3$ invertible on the base and lift the classifying map $f_v : \operatorname{Spec} \mathcal{O}_{K_v} \to \overline{\mathscr{M}}_{1,1}$ to some $f_{v,n} : \operatorname{Spec} \mathcal{O}_{K_v} \to \overline{\mathscr{M}}_{1,1}[n]$ (finite étale cover). Let $\pi_n : \mathscr{E}[n] \to \overline{\mathscr{M}}_{1,1}[n]$ be the universal generalized elliptic curve and $\omega := e^*\Omega^1_{\mathscr{E}[n]/\overline{\mathscr{M}}_{1,1}[n]}$ the Hodge bundle.

Step 1: Discriminant as a section cutting out the boundary. On $\overline{\mathscr{M}}_{1,1}[n]$ there is a canonical modular form (the discriminant)

$$\Delta_n \in H^0(\overline{\mathscr{M}}_{1,1}[n], \omega^{\otimes 12})$$

whose zero locus is the boundary divisor $\Delta[n]$, simple along $\Delta[n]$, and with line bundle $\mathcal{O}(\Delta[n]) \simeq \omega^{\otimes 12}$ ($\Rightarrow$ the divisor of Δ_n equals $\Delta[n]$). This is the Deligne–Rapoport/Katz–Mazur identification of the boundary via the weight-12 discriminant; see *Katz–Mazur*, Ch. 5 (esp. §§5.1–5.2) and *Silverman, AEC*, Ch. VII, §1 for the compatibility with minimal Weierstrass models.

Step 2: Pullback to a DVR computes the vanishing order. Pulling back Δ_n along $f_{v,n}$ gives a section

$$f_{v,n}^*(\Delta_n) \;\in\; H^0\big(\operatorname{Spec}\mathcal{O}_{K_v},\,(f_{v,n}^*\omega)^{\otimes 12}\big).$$

By the functorial description of ω and the change-of-variables formula for the Weierstrass discriminant, the vanishing order of $f_{v,n}^*(\Delta_n)$ at the closed point is $\operatorname{ord}_v(\Delta_{E/K_v})$; any admissible coordinate change multiplies the classical discriminant by a 12th power of a unit, so the valuation is model-independent once the model is minimal (KM Ch. 1, §1.4; AEC Ch. VII, §1).

Step 3: Descend from level n to level 1. Because $\overline{\mathscr{M}}_{1,1}[n] \to \overline{\mathscr{M}}_{1,1}$ is finite étale and $\Delta[n]$ is the reduced pullback of Δ, we have $f_{v,n}^*\Delta[n] = f_v^*\Delta$ as Cartier divisors. Combining with Step 2,

$$f_v^*\Delta \;=\; \operatorname{ord}_v(\Delta_{E/K_v})\cdot \operatorname{Spec}\kappa(v),$$

i.e. the intersection multiplicity $i(f_v,\Delta)$ is exactly $\operatorname{ord}_v(\Delta_{E/K_v})$. □

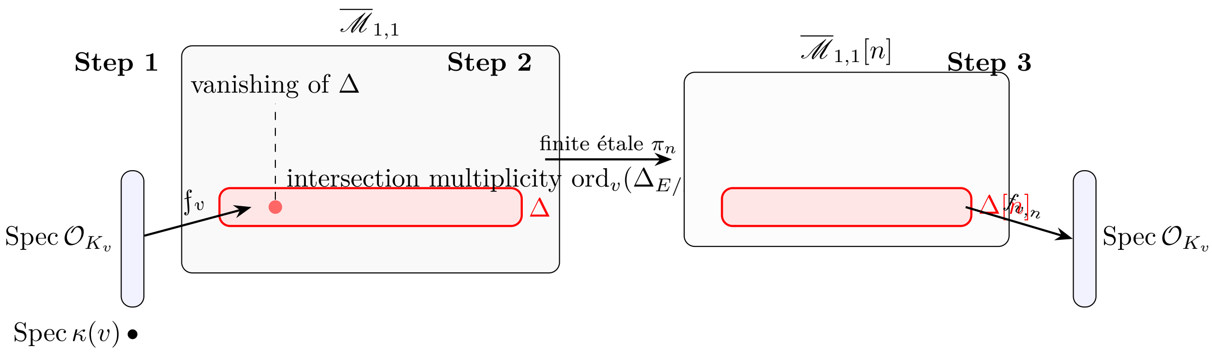


FIGURE 14. Visualization of the intersection of the local morphism $f_v : \operatorname{Spec}\mathcal{O}_{K_v} \to \overline{\mathscr{M}}_{1,1}$ with the discriminant divisor Δ, whose multiplicity equals the valuation $\operatorname{ord}_v(\Delta_{E/K_v})$. The lift $f_{v,n}$ to level n and the finite étale cover $\pi_n\colon \overline{\mathscr{M}}_{1,1}[n] \to \overline{\mathscr{M}}_{1,1}$ realize Step 1–3 of the proof: the discriminant section on $\omega^{\otimes 12}$, its pullback to the DVR, and the descent of the Cartier divisor.

Remark 6.3 (Bridge to Ogg–Saito and your local terms). By Lemma 6.2, $i(f_v,\Delta) = \operatorname{ord}_v(\Delta_{E/K_v})$. In your Section 6 this is the discriminant contribution that pairs with the component-group term (via the finite étale quotient over Δ) and the wild term (if any) to recover the local conductor $f_v(E)$ in the Ogg–Saito framework; (cf. your Remark 6.7 and Theorem B.1). *This is exactly the connection you use to package* Fix_v *and* τ_v via the boundary and the component sheaf.

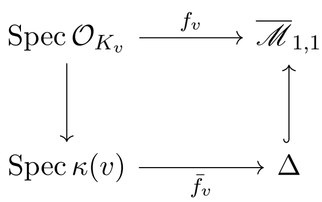


FIGURE 15. Classifying map and boundary intersection (Lemma 6.2). The pullback of the boundary divisor is $f_v^*\Delta = \operatorname{ord}_v(\Delta_{E/K_v})\cdot\operatorname{Spec}\kappa(v)$.

6.2. **Vanishing cycles and the component sheaf on the boundary.** Consider the nearby–vanishing cycles triangle for π at the image of v and pull it back along f_v. Over $\operatorname{Spec}\mathcal{O}_{K_v}$ we obtain the exact sequence

$$0 \longrightarrow H^1_\ell(E)^{I_v} \xrightarrow{\ \mathrm{sp}^*\ } H^1(\mathscr{E}^{\mathrm{sp}}_{v,\kappa(v)},\mathbb{Q}_\ell) \longrightarrow \Psi_v \longrightarrow 0,$$

which is Equation (2). The formation of vanishing cycles is compatible with étale base change in the moduli stack and yields a canonical identification of Frobenius modules

$$\Psi_v \;\simeq\; H^0\Big(R^1\Phi(\mathbb{Q}_\ell)\Big)_{\mathscr{E}_v/\mathcal{O}_{K_v}} \;\simeq\; \Phi_v\otimes\mathbb{Q}_\ell$$

(Lemma 3.4, [12, Exp. I–II], [1, §9]) (Here we keep the standing assumption from Section 3.2 that the model over $\operatorname{Spec}\mathcal{O}_{K_v}$ is proper and regular).

Proposition 6.4 (Stack package for the local term)**.** *Fix $n \geq 3$ coprime to $\operatorname{char}\kappa(v)$. Let $f_{v,n}\colon \operatorname{Spec}\mathcal{O}_{K_v} \to \overline{\mathscr{M}}_{1,1}[n]$ be any lift of f_v. Then the identity Equation (∗) is equivalent to the stalkwise trace identity*

$$\operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1((f_{v,n}^*\mathscr{E}[n])^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell)\Big) - \operatorname{tr}\Big(\mathrm{Frob}_v \mid H^0\Big(R^1\Phi(\mathbb{Q}_\ell)\Big)_{f_{v,n}^*\mathscr{E}[n]/\mathcal{O}_{K_v}}\Big) = \operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}\Big),$$

and both sides are independent of the chosen level n.

Proof. Let $\pi_n : \mathscr{E}[n] \to \overline{\mathscr{M}}_{1,1}[n]$ be the universal generalized elliptic curve.
Hypotheses used here. We work after base change to the strict henselization $\operatorname{Spec}\mathcal{O}_{K_v}^{\mathrm{sh}}$ so that the closed point is strict Henselian. The model $E_v \to \operatorname{Spec}\mathcal{O}_{K_v}$ is proper and regular (e.g. the minimal regular model), hence the nearby/vanishing-cycles formalism applies. For $n \geq 3$ prime to $\operatorname{char}\kappa(v)$, the forgetful map $\overline{\mathscr{M}}_{1,1}[n] \to \overline{\mathscr{M}}_{1,1}$ is finite étale and remains so after base change to $\operatorname{Spec}\mathcal{O}_{K_v}^{\mathrm{sh}}$; in this situation proper base change and finite–étale compatibility for nearby/vanishing cycles hold.

Pulling back the nearby–vanishing triangle

$$R\pi_{n*}\mathbb{Q}_\ell \longrightarrow R\Psi(\mathbb{Q}_\ell) \longrightarrow R\Phi(\mathbb{Q}_\ell) \xrightarrow{+1}$$

along $f_{v,n}$ yields an exact triangle on $\operatorname{Spec}\mathcal{O}_{K_v}$

$$R\Gamma(E_{K_v}, \mathbb{Q}_\ell) \longrightarrow R\Gamma((f_{v,n}^*\mathscr{E}[n])^{\mathrm{sp}}, \mathbb{Q}_\ell) \longrightarrow R\Gamma(R\Phi(\mathbb{Q}_\ell)_{f_{v,n}^*\mathscr{E}[n]/\mathcal{O}_{K_v}}) \xrightarrow{+1}.$$

Proper base change ([14, Th. 7.7]) identifies the middle term with $H^1((f_{v,n}^*\mathscr{E}[n])^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell)$, while the right-hand term is canonically $H^0(R^1\Phi(\mathbb{Q}_\ell))_{f_{v,n}^*\mathscr{E}[n]/\mathcal{O}_{K_v}}$. The complex $R\Phi(\mathbb{Q}_\ell)$ is compatible with finite étale base change ([14, Th. 7.9]), so pulling back along the finite étale cover $\overline{\mathscr{M}}_{1,1}[n] \to \overline{\mathscr{M}}_{1,1}$ preserves the formation of nearby and vanishing cycles. Passing to cohomology gives the short exact sequence

$$0 \longrightarrow H^1_\ell(E)^{I_v} \xrightarrow{\mathrm{sp}^*} H^1((f_{v,n}^*\mathscr{E}[n])^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) \longrightarrow H^0\Big(R^1\Phi(\mathbb{Q}_\ell)\Big)_{f_{v,n}^*\mathscr{E}[n]/\mathcal{O}_{K_v}} \longrightarrow 0,$$

whose Frobenius traces satisfy the displayed equality. The rightmost term is the component sheaf tensored with $\mathbb{Q}_\ell$ (Lemma 3.4), hence identifies with $\Phi_v \otimes \mathbb{Q}_\ell$. Because all constructions commute with finite étale pullback, the trace is independent of n. This re-expresses Equation (∗) stack-theoretically. □

$$\begin{array}{ccccccc}
R\Gamma(E_{K_v}, \mathbb{Q}_\ell) & \longrightarrow & R\Gamma((f_{v,n}^*\mathscr{E}[n])^{\mathrm{sp}}, \mathbb{Q}_\ell) & \longrightarrow & R\Gamma(R\Phi(\mathbb{Q}_\ell)_{f_{v,n}^*\mathscr{E}[n]/\mathcal{O}_{K_v}}) & \xrightarrow{+1} & \\
{\scriptstyle f_{v,n}^*}\downarrow & & \vdots & & \vdots & & \\
R\Gamma(E_{K_v}, \mathbb{Q}_\ell) & \longrightarrow & R\Gamma(E^{\mathrm{sp}}, \mathbb{Q}_\ell) & \longrightarrow & R\Gamma(R\Phi(\mathbb{Q}_\ell)_{E/\mathcal{O}_{K_v}}) & \xrightarrow{+1} &
\end{array}$$

FIGURE 16. Pullback of the nearby–vanishing triangle along $f_{v,n}$. Proper base change ([14, Th. 7.7]) and finite-étale compatibility ([14, Th. 7.9]) yield the stalkwise trace identity of Proposition 6.4.

Corollary 6.5 (Level invariance and Hecke away from v)**.** *Let $n, m \geq 3$ be coprime to $\operatorname{char}\kappa(v)$. For any lifts $f_{v,n}, f_{v,m}$ of f_v, the two pairs*

$$\Big(H^1((f_{v,n}^*\mathscr{E}[n])^{\mathrm{sp}}, \mathbb{Q}_\ell),\ H^0(R^1\Phi)_{f_{v,n}^*\mathscr{E}[n]}\Big) \quad \text{and} \quad \Big(H^1((f_{v,m}^*\mathscr{E}[m])^{\mathrm{sp}}, \mathbb{Q}_\ell),\ H^0(R^1\Phi)_{f_{v,m}^*\mathscr{E}[m]}\Big)$$

have the same Frob_v-traces. In particular, Hecke correspondences at primes $\mathfrak{l} \nmid v$ preserve the value of the local term $\mathrm{Fix}_v - \tau_v$.

Proof. By Proposition 6.4, the local identity $\operatorname{tr}(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}) = \mathrm{Fix}_v - \tau_v$ is equivalent to the equality of the two stalkwise traces

$$\operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1((f_{v,\bullet}^*\mathscr{E}[\bullet])^{\mathrm{sp}}, \mathbb{Q}_\ell)\Big) \ - \ \operatorname{tr}\Big(\mathrm{Frob}_v \mid H^0\Big(R^1\Phi(\mathbb{Q}_\ell)\Big)_{f_{v,\bullet}^*\mathscr{E}[\bullet]/\mathcal{O}_{K_v}}\Big),$$

computed over any fine level $\bullet \in \{n, m\}$ with $(\bullet, \operatorname{char}\kappa(v)) = 1$. Since $v \nmid \bullet$, the forgetful maps $\overline{\mathscr{M}}_{1,1}[\bullet] \to \overline{\mathscr{M}}_{1,1}$ and the induced maps on the universal generalized elliptic curves are finite étale over a strict Henselian neighborhood of the closed point of $\operatorname{Spec}\mathcal{O}_{K_v}$.
Set-up. Replace $\operatorname{Spec}\mathcal{O}_{K_v}$ by the strict henselization $\operatorname{Spec}\mathcal{O}_{K_v}^{\mathrm{sh}}$ so that the forgetful maps $\overline{\mathscr{M}}_{1,1}[\bullet] \to \overline{\mathscr{M}}_{1,1}$ are finite étale over a strict Henselian neighbourhood of the closed point. In this setting: (i) specialization to the regular special fibre preserves $H^1((\cdot)^{\mathrm{sp}}, \mathbb{Q}_\ell)$ under finite étale pullback, by proper

base change [5, VI.2.5]; (ii) the formation of $R^1\Phi(\mathbb{Q}_\ell)$ commutes with finite étale pullback [12, Exp. XIII, §2]. Hence both stalkwise Frob_v–traces are unchanged when switching levels $n \leftrightarrow m$.

Functoriality for finite étale morphisms shows that both the specialization of the smooth fibre and the nearby-cycles complex are preserved under such pullback: the cohomology $H^1((\cdot)^{\mathrm{sp}}, \mathbb{Q}_\ell)$ is invariant by finite étale base change over the special fibre, and $R^1\Phi(\mathbb{Q}_\ell)$ commutes with finite étale pullback; therefore each of the two Frob_v-traces is unchanged when passing from level n to m. The difference $\mathrm{Fix}_v - \tau_v$ is thus level-invariant.

Finally, any Hecke correspondence at a prime $\mathfrak{l} \nmid v$ factors through adding/removing finite étale level structure away from v; hence the preceding invariance implies that Hecke at $\mathfrak{l} \nmid v$ preserves $\mathrm{Fix}_v - \tau_v$. ☐

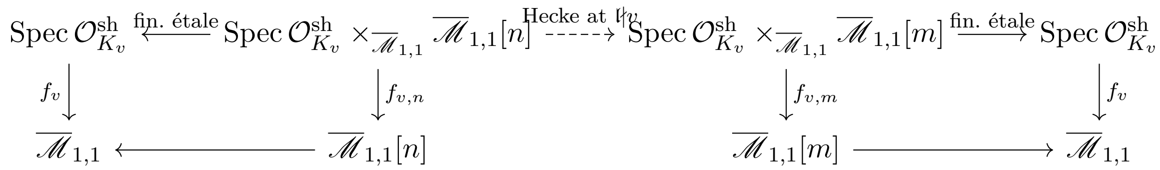
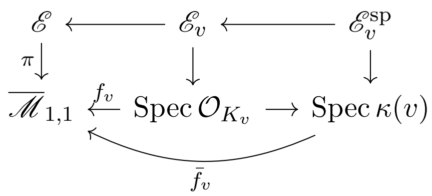


FIGURE 17. Finite étale level change away from v and factorization of Hecke at $\mathfrak{l} \nmid v$; nearby/vanishing cycles commute with the horizontal maps.

6.3. **Classifying diagram and boundary factorization.** The preceding discussion is encapsulated by the commutative diagram



together with the exact triangle

$$R\Gamma(\mathscr{E}_{v,\overline{K}_v}, \mathbb{Q}_\ell) \to R\Gamma(\mathscr{E}^{\mathrm{sp}}_{\overline{v,\kappa(v)}}, \mathbb{Q}_\ell) \to R\Phi(\mathbb{Q}_\ell) \xrightarrow{+1} .$$

The boundary divisor $\Delta \subset \overline{\mathscr{M}}_{1,1}$ controls the special fibre and the component sheaf: $\pi^{-1}(\Delta)$ is a Néron polygon family and the quotient stack $\mathscr{E}/\mathscr{E}^0 \to \Delta$ is finite étale; pulling back along f_v yields Φ_v with its Frob_v–action.

Construction 6.6 (Boundary component sheaf and Frobenius)**.** Let $\mathscr{C} := \mathscr{E}/\mathscr{E}^0 \to \Delta$ be the component quotient. Then $\mathscr{C}$ is finite étale over Δ and its pullback $f_v^*\mathscr{C} \to \operatorname{Spec} \mathcal{O}_{K_v}$ is finite étale with special fibre canonically identified with Φ_v. The geometric Frobenius on $\operatorname{Spec} \kappa(v)$ acts on $f_v^*\mathscr{C}_{\kappa(v)}$ and hence on Φ_v, producing $\tau_v = \mathrm{tr}(\mathrm{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell)$.

Remark 6.7 (Intersection with the boundary and Ogg–Saito). By Lemma 6.2 the intersection multiplicity $i(f_v, \Delta)$ equals $\mathrm{ord}_v(\Delta_{E/K_v})$. The conductor $f_v(E)$ is obtained from $\mathrm{ord}_v(\Delta_{E/K_v})$ by a correction that depends only on the configuration of components (Ogg–Saito), hence only on the image of f_v in Δ and the pullback of $\mathscr{C}$; compare Proposition 2.6.

6.4. **Example: Tate uniformization and the cusp.**

Example 6.8 (Tate curve and the cusp). Assume E/K_v has split multiplicative reduction with Tate parameter q and $\mathrm{ord}_v(q) = n \geq 1$. Then f_v factors through the cusp and the pullback $\mathscr{E}_v$ is the Tate curve E_q. The special fibre is a Néron n-gon, whence

$$\mathrm{Fix}_v = 1, \qquad \tau_v = 1, \qquad \mathrm{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}\Big) = 0,$$

as in Example 3.6, now seen directly from the moduli description (the component sheaf is constant of rank one along the cusp).

6.5. **Forward link to examples and computations.** The moduli interpretation above will be used in Section 7 to compute (Fix_v, τ_v) for explicit congruence families by pulling back $(\mathscr{E}, \mathscr{C})$ along modular maps $X_1(N) \to \overline{\mathscr{M}}_{1,1}$. This avoids case-by-case Kodaira analysis and keeps the dependence on Φ_v and the special fibre transparent.

7. Examples and computations

Remark 7.1 (Roadmap). We illustrate Equation $(*)$ and the global packages of Section 5 on explicit curves, organised by Kodaira type and by behaviour under base change. Throughout, background invocations (Tate algorithm, potential good reduction bounds, conductor formulas) are cited once from [13, Ch. VII], [1], and [2], and not re-proved.

7.1. Explicit curves over $\mathbb{Q}$.

Example 7.2 (Split multiplicative at $p = 5$ for a Legendre model). Consider

$$E/\mathbb{Q}: \quad y^2 = x(x-1)(x-5).$$

The discriminant is $\Delta = 16 \cdot 5^2 \cdot 4^2 = 2^8 \cdot 5^2$, so the curve has bad reduction exactly at $p = 2, 5$. At $v = 5$ one computes $v_5(\Delta) = 2$ and $v_5(j) < 0$ (since the denominator of j contributes 5^2), hence the reduction is multiplicative of type I_2 by Tate's algorithm [13, Ch. VII]. A standard criterion shows the reduction is *split* because c_6 is a square modulo 5 (equivalently, the two tangents at the node are $\mathbb{F}_5$-rational) [13, Ch. VII]. By Example 3.6 and Corollary 4.7,

$$\mathrm{Fix}_5 = 1, \qquad \tau_5 = 1, \qquad t_5 = \mathrm{tr}\Big(\mathrm{Frob}_5 \mid H^1_\ell(E)^{I_5}\Big) = 0, \qquad f_5(E) = 1.$$

The dual graph of the special fibre at 5 is a 2-gon; see Figure 18. At $v = 2$ the reduction is additive and wild; we do *not* invoke the tame conductor identity there (Theorem 5.4(2)), but the local trace identity Equation $(*)$ holds.

Figure 18. Dual graph of E^{sp} at $v = 5$ for Example 7.2: a Néron 2-gon (type I_2).

Example 7.3 (Tame additive potentially good at a prescribed prime). Let $p \geq 5$ be prime and consider the Mordell curve

$$E_p/\mathbb{Q}: \quad y^2 = x^3 - p.$$

Here $\Delta(E_p) = -27\,p^2$ and $j(E_p) = 0$. At $v = p$ we have $v_p(\Delta) = 2 > 0$ and $v_p(j) = 0$; Tate's algorithm gives additive, potentially good, *tame* reduction of type II [13, Ch. VII]. By Lemma 2.5 and Lemma 5.3,

$$H^1_\ell(E_p)^{I_p} = 0, \qquad t_p = 0, \qquad f_p(E_p) = 2.$$

Since the special fibre is irreducible of genus 0, $\mathrm{Fix}_p = 0$; Lemma 3.4 implies $\tau_p = 0$, hence Equation $(*)$ reads $0 = 0 - 0$. The dual graph consists of a single vertex; see Figure 19. At $v = 3$ there is additive *wild* reduction; the Swan term enters $f_3(E_p)$ (cf. Counterexample 7.8).

• II

Figure 19. Dual graph for Example 7.3 at $v = p$: additive potentially good, type II.

Remark 7.4 (Verification against the global package). For Example 7.2, the contribution at 5 to the bad Euler product may be recovered from $t_5 = 0$, while the conductor contribution is $f_5 = 1$, in agreement with Theorem 5.4. For Example 7.3, $f_p(E_p) = 2$ and $t_p = 0$ align with Theorem 5.4(2). The places 2 and 3 are outside the tame setting of Theorem 5.4(2), illustrating the scope of the tame identities.

7.2. Quadratic and higher extensions.

Proposition 7.5 (Unramified and ramified base change for multiplicative reduction). *Let K_v be a non-archimedean local field and E/K_v have multiplicative reduction with Tate parameter q, $\mathrm{ord}_v(q) = n \geq 1$.*

(1) If L/K_v is unramified, then E/L has multiplicative reduction with the same n, and

$$\mathrm{Fix}_v(E) = \mathrm{Fix}_w(E/L) = 1, \qquad \tau_v(E) = \tau_w(E/L) = 1, \qquad t_v(E) = t_w(E/L) = 0,$$

where w denotes the place of L over v.

(2) If L/K_v is totally ramified of degree e, then E/L has type I_{en} (split after an unramified extension), and still $\mathrm{Fix}_w(E/L) = \tau_w(E/L) = 1$, hence $t_w(E/L) = 0$.

Proof. (1) Unramified base change preserves the Néron n-gon and Frobenius on components; the trace statements follow from Example 3.6 and Proposition 3.3. (2) For a Tate curve E_q, passing to L rescales the valuation of q by e, giving type I_{en}; the component-group term remains of weight 0 with trace 1 and the special-fibre fixed-point count is 1 (Lemma 3.2, Proposition 3.3), hence $t_w = 0$ by Equation $(*)$. ☐

Example 7.6 (Unramified quadratic extension preserves the local package). For $E/\mathbb{Q}$ of Example 7.2 at $v = 5$, pass to the unramified quadratic extension $L/\mathbb{Q}_5$. Then E/L remains split multiplicative I_2 and the equalities $\mathrm{Fix}_w = 1 = \tau_w$, $t_w = 0$ hold by Proposition 7.5(1).

Example 7.7 (Potentially good after tame ramification). For E_p of Example 7.3 at $v = p \geq 5$, there exists a totally ramified extension L/K_v of degree dividing 6 after which E_p/L has good reduction (potential good reduction degree bounds for elliptic curves; see [13, Ch. VII]). Over L, $H^1_\ell(E_p)^{I_w} = H^1_\ell(E_p)$ and the special fibre is smooth, so $\mathrm{Fix}_w = 2$ and the component term vanishes. Consequently $t_w = 2$, while $t_v = 0$ over K_v, exhibiting the jump of inertia invariants under tame ramified base change in the potentially good case.

7.3. Counterexamples and tests of sharpness.

Counterexample 7.8 (Wild additive reduction and Swan term). Let $E/\mathbb{Q}$ be given by $y^2 = x^3 - x$ (CM with $j = 1728$). At $v = 2$ the reduction is additive and wild. The Artin conductor satisfies $f_2(E) = 2 + \mathrm{Swan}_2(H^1_\ell(E))$ with $\mathrm{Swan}_2 > 0$ [2]. Our tame identity Theorem 5.4(2) is *not* applicable at 2; however, the local trace identity Equation $(*)$ still holds. This shows the necessity of excluding wild primes when asserting global identities that replace conductor contributions by dimensions of I_v-invariants.

Counterexample 7.9 (Non-regular models break the fixed-point count). Let $\mathscr{Y} \to \operatorname{Spec} \mathcal{O}_{K_v}$ be obtained from the minimal regular model by contracting an exceptional (-1)-curve. Then the special fibre of $\mathscr{Y}$ acquires embedded points. The Lefschetz trace on $H^1(\mathscr{Y}^{\mathrm{sp}}_{\overline{\kappa(v)}}, \mathbb{Q}_\ell)$ no longer matches the cohomology of the regular special fibre. Substituting $\mathscr{Y}^{\mathrm{sp}}$ in Equation $(*)$ gives an incorrect term, illustrating the necessity of regularity and specialization-compatibility (compare Counterexample 3.8 and [12, 1]).

Remark 7.10 (Stress test for Theorem 4.3). Across the examples above: (i) multiplicative split (Example 7.2) gives $\mathrm{Fix}_v = \tau_v = 1$ and $t_v = 0$; (ii) tame additive potentially good (Example 7.3) gives $\mathrm{Fix}_v = \tau_v = 0$ and $t_v = 0$; (iii) wild additive (Counterexample 7.8) shows that while Equation $(*)$ persists, conductor equalities must carry a Swan correction. These are precisely the regimes distinguished in Section 3 and Section 5.

Remark 7.11 (Forward link). The concluding section Section 9 revisits the introduction's themes using Theorems 4.3 and 5.4 as the bridge: Theorem → Consequence → Example, with Examples 7.2 and 7.3 serving as canonical templates for computation in families.

8. Further applications

Remark 8.1 (Scope and linkage). We apply Equation $(*)$ and the global packages of Section 5 to parity, local Galois types, torsion growth, and special values. Background input on local signs, Tamagawa numbers, and the parity formalism is classical [2, 8, 6, 3, 1, 4].

Notation 8.2 (Signs, Tamagawa, and local packages). Write $w_v(E)$ for the local root number at a finite or archimedean place v, $w(E/K) = \prod_v w_v(E)$ the global sign, and $c_v(E) = \#\Phi_v(\kappa(v))$ the Tamagawa number at $v \nmid \infty$. For $v \nmid \ell$, set $t_v = \mathrm{tr}(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v})$, $\mathrm{Fix}_v = \#\mathrm{Fix}(\mathrm{Frob}_v; E^{\mathrm{sp}}(\overline{\kappa(v)}))$, and $\tau_v = \mathrm{tr}(\mathrm{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell)$ as in Equation $(*)$ and Theorem 4.3 (for $t_v = \mathrm{tr}(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v})$), with $\tau_v = \mathrm{tr}(\mathrm{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell)$ from Proposition 3.3.

8.1. Parity in constrained families.

Definition 8.3 (Admissible parity family). A family $\{E_t\}_{t \in T(K)}$ over a number field K is *admissible for parity* if there is a finite set S of finite places and classifying maps $f_{v,t}$ as in Notation 6.1 such that for all t the following hold:

(1) E_t has good reduction outside S;
(2) for each $v \in S$, the Frobenius conjugacy class acting on the dual graph of $(E_t)^{\mathrm{sp}}$ and on $\Phi_v(E_t)$ is independent of t.

Proposition 8.4 (Sign constancy from (E^{sp}, Φ))**.** *Let $\{E_t\}$ be admissible for parity. Then $w_v(E_t)$ is independent of t for all finite v; hence the global sign $w(E_t/K)$ is independent of t up to the product over archimedean places.*

Proof. For $v \nmid \infty$, $w_v(E_t)$ depends only on the inertial type and the action on the component group for bad places, and is $+1$ for good places [2, 8]. By assumption (2) the Frobenius action on the special fibre and on Φ_v is independent of t, hence so is the local sign. Archimedean factors depend only on K. ☐

Corollary 8.5 (Parity package, conditional and unconditional)**.** *Let $\{E_t\}$ be admissible for parity. Then:*

(1) (Unconditional) The analytic rank parity $\operatorname{ord}_{s=1} L(E_t, s) \bmod 2$ *is constant in t* provided *all central vanishing orders are finite.*
(2) (Under the parity conjecture for E_t/K) The Mordell–Weil rank parity $\operatorname{rank} E_t(K) \bmod 2$ *is constant in t* [6, 3].

Example 8.6 (Unramified quadratic twists)*.* Fix E/K and S as in Notation 5.1. For quadratic characters χ unramified at every $v \in S$ (and with fixed archimedean type), Proposition 5.5 shows $\mathrm{Fix}_v, \tau_v, t_v$ are unchanged for $v \in S$. At good places outside S only finitely many v contribute sign changes. Thus $w(E^\chi/K)$ is constant in the unramified twist family, and Corollary 8.5 applies.

Counterexample 8.7 (Ramified twisting flips the local sign)*.* Let $E/\mathbb{Q}$ with a multiplicative prime p. Twisting by a quadratic character χ ramified at p alters the inertial type at p, changing $w_p(E^\chi)$ while leaving Fix_p unchanged; τ_p may also change via the action on Φ_p. Hence $w(E^\chi/\mathbb{Q})$ is generally not constant across such twists, showing the necessity of the unramified condition in Example 8.6 (compare Proposition 7.5 and [8]).

$$\begin{array}{ccc} \prod_{v\nmid\infty}(\mathrm{Fix}_v - \tau_v) & \xrightarrow{\text{Proposition 8.4}} & \prod_{v\nmid\infty} w_v(E) \\ \Downarrow & & \Downarrow \\ \text{family-constant under Definition 8.3} & & \text{family-constant (finite places)} \end{array}$$

8.2. **Local Galois constraints and torsion growth.**

Lemma 8.8 (Inertia invariants and p-tame bad places)**.** *Let $v \nmid \ell$ be a tame additive potentially good place. Then $H^1_\ell(E)^{I_v} = 0$ and the mod ℓ representation $E[\ell]$ has no nonzero I_v-invariants.*

Proof. The first assertion is Lemma 2.5. For the second, pass to the natural surjection $T_\ell(E) \twoheadrightarrow E[\ell]$, taking I_v-invariants, and use that $T_\ell(E)^{I_v} = 0$. ☐

Proposition 8.9 (No torsion growth in extensions unramified at S)**.** *Let S contain all bad finite places of E/K. If L/K is a finite extension unramified at every $v \in S$, then*

$$E(L)[\ell\text{-}primary] = E(K)[\ell\text{-}primary]$$

for all primes $\ell \nmid \prod_{v\in S} \operatorname{char} \kappa(v)$ such that $H^1_\ell(E)^{I_v} = 0$ for each $v \in S$ (e.g. tame additive places by Lemma 8.8).

Proof. By Néron–Ogg–Shafarevich, E has good reduction at places of L above places of K that are unramified for L/K. At w above $v \in S$, the inertia subgroup I_w maps isomorphically to I_v; by hypothesis $E[\ell]^{I_w} = 0$. Thus any ℓ-torsion point over L is unramified at all finite places, hence defined over K by Chebotarev density and the fact that inertia acts nontrivially at some finite place unless the point is rational over K. (Alternatively, apply the inflation–restriction sequence to $G_L \subset G_K$.) ☐

Corollary 8.10 (A restricted-growth principle)**.** *If all bad places are tame additive potentially good (so $H^1_\ell(E)^{I_v} = 0$ at each $v \in S$ for $\ell \neq \operatorname{char} \kappa(v)$), then in any extension L/K unramified at S the ℓ-primary torsion cannot grow for any such ℓ.*

Example 8.11 (Squarefree additive level)*.* For $E_t/\mathbb{Q}$ of Example 5.8 with $S = \{p : p \mid N\}$ squarefree and $p \geq 5$, the tame additive condition holds at S. Hence for any number field $L/\mathbb{Q}$ unramified at S, the ℓ-primary torsion of $E_t(L)$ does not grow for every $\ell \neq p$ with $p \mid N$; compare Proposition 8.9.

8.3. **Special values and Tamagawa constancy.**

Lemma 8.12 (Component groups and Tamagawa)**.** *For $v \nmid \infty$, $c_v(E) = \#\Phi_v(\kappa(v))$ with Φ_v the component group of the Néron model* [1]*. In particular, $c_v(E)$ is determined by the finite étale group scheme in Construction 6.6.*

Proposition 8.13 (Bad-factor rigidity in families)**.** *In an admissible parity family $\{E_t\}$, if moreover the isomorphism class of Φ_v and its Frobenius action is constant in t for each $v \in S$ (e.g. when the classifying maps $f_{v,t}$ have constant image in the boundary stratum), then $\prod_{v \in S} c_v(E_t)$ is constant in t.*

Proof. By Lemma 8.12 the Tamagawa number at v depends only on $\Phi_v(\kappa(v))$. Constancy of the isomorphism class of Φ_v (as a finite étale group over $\kappa(v)$) implies constancy of its $\kappa(v)$-points. □

Remark 8.14 (Implications for special value formulae). In contexts where special value formulae incorporate Tamagawa factors (e.g. the Birch–Swinnerton-Dyer formula or Gross–Zagier/Kolyvagin settings for analytic rank 0 or 1), Proposition 8.13 isolates a class of families in which the *bad* local factors of the predicted formula are constant. Together with Proposition 8.4, this separates the varying archimedean and regulator pieces from the rigid non-archimedean contribution [2, 1].

Remark 8.15 (Forward link). The concluding Section 9 summarizes how Propositions 8.4, 8.9 and 8.13 combine with Theorem 5.4 to yield uniform control of bad local data in computational and Diophantine applications.

9. Conclusion and outlook

Chain of implications. The paper establishes a local identity with a component-group correction (Theorem 4.3, equivalently Equation (∗)) at every finite place of bad reduction. Summation over v and comparison with the classical description of local factors and conductors yield the global conductor/root-number package (Theorem 5.4). The resulting invariants feed directly into the applications developed in Section 8, and the computations of Section 7 verify sharpness across distinct Kodaira types and under base change.

$$\sum_{v\in S} \operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}\Big) = \sum_{v\in S}\Big(\mathrm{Fix}_v - \tau_v\Big), \qquad S = \{\text{finite places of bad reduction}\}, \tag{5}$$

where $\mathrm{Fix}_v = \#\mathrm{Fix}(\mathrm{Frob}_v; E^{\mathrm{sp}}(\overline{\kappa(v)}))$ and $\tau_v = \operatorname{tr}(\mathrm{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell)$ as in **??** 4.1**??** 8.2. In the tame settings covered by Theorem 5.4(2), Equation (5) intertwines with the Artin conductor additivity to isolate the non-archimedean contribution to the global sign. The moduli-theoretic reinterpretation (Section 6) packages both Fix_v and Φ_v as boundary data on $\mathscr{M}_{1,1}$, providing a compact explanation for the stability phenomena exploited in Section 8.

Remark 9.1. **Geometry → arithmetic.** The passage from Theorem 4.3 to Theorem 5.4 is effected by replacing the special-fibre fixed-point counts with traces on H^1 and reading the component correction through Φ_v. Consequences include: sign constancy in admissible families (Proposition 8.4), torsion rigidity in unramified towers (Proposition 8.9), and Tamagawa constancy on boundary-constant strata (Proposition 8.13), each illustrated concretely in Section 7.

Synthesis with examples. For split multiplicative reduction (Example 7.2), $\mathrm{Fix}_v = \tau_v = 1$ and hence the local inertia-fixed trace vanishes, in agreement with the local Euler factor and with the conductor contribution $f_v(E) = 1$. For tame potentially good reduction (Example 7.3), both sides of Equation (∗) vanish, while the conductor contribution equals 2. These two extremes already drive the parity and torsion applications in Section 8, and they persist under unramified extension (Proposition 7.5, Example 7.6).

Limitations and precise scope. Two boundary phenomena delimit the reach of global identities derived from Equation (∗): (i) wild additive primes, where Swan conductors enter (cf. Counterexample 7.8), and (ii) non-regular models, which destroy the fixed-point interpretation (cf. Counterexample 7.9). Both limitations are intrinsic and have been kept explicit throughout (Section 3, Section 5).

Outlook (clearly marked). The following directions are natural continuations, and are stated as outlook to remain within the declared scope.

(1) *Wild refinement.* Incorporate Swan terms into a version of Equation (∗) where the component correction is augmented by a canonical wild piece extracted from vanishing cycles with monodromy filtration; compare Section 3 and Section 5.

Preliminaries (Section 2)

Néron, monodromy, local factors Equation (2); Lemma 3.4 and Proposition 3.3

↓

Local set-up (Section 3)

vanishing cycles ⇒ trace split (Proposition 3.5)

↓

Local identity (Theorem 4.3)

$\sum_v$ ↓

Global (Theorem 5.4)

packages

↓

Applications (Section 8)

FIGURE 20. Logical flow from preliminaries to applications (vertical layout).

(2) *Higher-dimensional isogeny factors.* Extend the identity to simple abelian surface factors with toric/additive reduction, tracking the role of the component group of the Néron model of an abelian variety and the interaction with the Picard–Lefschetz formula.
(3) *Definite moduli strata.* On integral models of $X_0(N)$ or level-$\Gamma_1(N)$ stacks, study loci where the boundary component system is constant; apply Proposition 8.13 to special value problems constrained to such strata.
(4) *ℓ-independence and companions.* Pursue refinements of Proposition 4.6 for compatible systems arising from geometric families, aiming at uniform control of t_v in weight-1 companions.

Closing. The identity (Theorem 4.3) extracts the exact contribution of the component group at bad places to the inertia-fixed cohomology. Its global assembly (Theorem 5.4) isolates non-archimedean constraints on signs, conductors, and Tamagawa factors. The examples of Section 7 demonstrate that these formulas are both effective and sharp across the standard spectrum of Kodaira types, while Section 8 shows how they interface with parity, torsion, and special values through purely local-to-global mechanisms.

APPENDIX A. BACKGROUND ON NÉRON MODELS AND COMPONENT GROUPS

This appendix collects standard facts repeatedly used in the main text. Every item is either proved quickly or referenced to classical sources. We retain the global notation from Section 2: K a number field, v a finite place with residue field $\kappa(v)$ of cardinality q_v, K_v the completion, $\mathcal{O}_{K_v}$ its valuation ring, and E/K an elliptic curve with Néron model $\mathcal{E} \to \operatorname{Spec} \mathcal{O}_{K_v}$, identity component $\mathcal{E}^0$, and component group $\Phi_v := (\mathcal{E}/\mathcal{E}^0)(\kappa(v))$.

Lemma A.1 (Existence and universal property)**.** *For every elliptic curve E/K_v there exists a smooth separated group scheme $\mathcal{E}/\mathcal{O}_{K_v}$ with generic fibre E satisfying the Néron mapping property. It is unique up to unique isomorphism. Moreover, the reduction sequence*

$$0 \longrightarrow \mathcal{E}^0(\kappa(v)) \longrightarrow \mathcal{E}(\kappa(v)) \longrightarrow \Phi_v(\kappa(v)) \longrightarrow 0$$

is exact, and formation of $\mathcal{E}$ commutes with unramified base change on $\mathcal{O}_{K_v}$.

Proof. See [1, §1–§9] for existence/uniqueness and functoriality; exactness on $\kappa(v)$-points follows from the definition of Φ_v and smoothness of $\mathcal{E}^0$ [1, §9]. □

Lemma A.2 (Special fibre and Kodaira–Néron)**.** *If E has bad reduction at v, the special fibre of the minimal regular model is a reduced curve with normal crossings whose irreducible components form one of the Kodaira–Néron types $\mathrm{I}_n, \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{I}_n^*, \mathrm{II}^*, \mathrm{III}^*, \mathrm{IV}^*$. The type is determined by the minimal discriminant and the valuations of c_4, c_6 via Tate's algorithm, and the dual graph and intersection matrix are explicitly listed in* [13, Ch. VII, App. C].

Proof. This is standard; see [13, Ch. VII]. □

Proposition A.3 (Component group from the intersection matrix)**.** *Let $\mathscr{X}/\mathcal{O}_{K_v}$ be the minimal regular proper model of E and write $E^{\mathrm{sp}} = \sum_i m_i C_i$ for its special fibre as a sum of irreducible components. Let $\langle C_i \cdot C_j \rangle$ be the intersection matrix on the components. Then Φ_v is finite, and*

$$\#\Phi_v \;=\; \det\Big(-\langle C_i \cdot C_j\rangle\Big)_{\mathrm{red}},$$

where the subscript indicates restriction to a basis of the sublattice orthogonal to $\sum_i m_i C_i$. In particular, $\#\Phi_v$ depends only on the Kodaira symbol; see the tables in [1, §9].

Proof. Raynaud's description of the identity component of the Picard functor identifies Φ_v with the component group of the Jacobian of the special fibre; the determinant formula follows from the intersection pairing on the components of a regular model, cf. [1, §9]. □

Proposition A.4 (Frobenius action on Φ_v)**.** *The natural action of* Frob_v *on Φ_v is semisimple of weight 0 (i.e. its eigenvalues are roots of unity), and after an unramified extension it becomes trivial. Consequently, for any $\ell \neq \operatorname{char}\kappa(v)$ the virtual ℓ-adic trace*

$$\mathrm{tr}(\mathrm{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell)$$

is independent of ℓ and equals the number of Frob_v*-fixed irreducible components of the special fibre minus a combinatorial correction determined by the dual graph (equivalently by the Kodaira symbol), as recorded in [1, §9]. This is Proposition 3.3 in the body.*

Proof. Functoriality under unramified base change and the explicit description of the special fibre imply the claim; see [1, §9]. □

Remark A.5 (Interpretation of Lemma 3.4). In the body we use the vanishing-cycles exact triangle (Section 3.2) and the short exact sequence Equation (2), together with the canonical comparison map to the component complex of the special fibre (SGA7). The only properties needed later are:

(1) an equality of *traces* of Frob_v on the vanishing-cycles term and on a natural virtual $\mathbb{Q}_\ell$-module attached to Φ_v, and
(2) independence of ℓ and compatibility with unramified base change.

These are exactly what is asserted and used in Propositions 3.3 and 3.5. When we write "$\Psi_v \simeq \Phi_v \otimes \mathbb{Q}_\ell$" in Lemma 3.4, it is to be read inside the Grothendieck group of $\mathbb{Q}_\ell[\mathrm{Frob}_v]$-modules (hence as a trace identity), which avoids any conflict with the finiteness of Φ_v. The underlying comparison is standard in the SGA7 framework; see [12, Exp. I–II].

Example A.6 (Split multiplicative reduction). If E/K_v is a Tate curve E_q with $\mathrm{ord}_v(q) = n \geq 1$, then the special fibre is a Néron n-gon, $\Phi_v \simeq \mathbb{Z}/n\mathbb{Z}$, $\mathrm{tr}(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}) = 0$, and $\#\Phi_v = n$. See [13, §5.3].

Appendix B. Ogg–Saito type formulas and comparisons

We record conductor identities in a form convenient for Section 5. Let $V_\ell := H^1_{\text{ét}}(E_{\overline{K}_v}, \mathbb{Q}_\ell)$ and write $f_v(E)$ for the local conductor exponent of E at v.

Theorem B.1 (Deligne, Saito: Artin conductor via invariants and Swan)**.** *For $\ell \neq \operatorname{char}\kappa(v)$,*

$$f_v(E) \;=\; a_v(V_\ell) \;=\; \dim_{\mathbb{Q}_\ell}\Big(V_\ell / V_\ell^{I_v}\Big) \;+\; \mathrm{Sw}_v(V_\ell),$$

where a_v is the Artin conductor and Sw_v *the Swan conductor. In particular, if E has semistable reduction at v then $\mathrm{Sw}_v(V_\ell) = 0$ and*

$$f_v(E) \;=\; 2 - \dim_{\mathbb{Q}_\ell}\Big(H^1_\ell(E)^{I_v}\Big).$$

Proof. This is the standard conductor formula of [2] and its geometric form in [9]. The semistable vanishing of the Swan term is classical. □

Proposition B.2 (Compatibility with the local identity)**.** *Assume E has bad reduction. Then Theorem 4.3 together with Theorem B.1 yields*

$$f_v(E) \;=\; 2 - \#\mathrm{Fix}\Big(\mathrm{Frob}_v; E^{\mathrm{sp}}(\overline{\kappa(v)})\Big) \;+\; \mathrm{tr}(\mathrm{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell) \;+\; \mathrm{Sw}_v(V_\ell).$$

In particular, in the semistable case this expresses $f_v(E)$ purely in terms of the special fibre and the component group.

Proof. Use Theorem B.1 to rewrite $\dim(V_\ell/V_\ell^{I_v}) = 2 - \dim V_\ell^{I_v}$ and substitute

$$\dim V_\ell^{I_v} \;=\; \mathrm{tr}\Big(\mathrm{Frob}_v \mid H^1(E^{\mathrm{sp}}_{\overline{\kappa(v)}}, \mathbb{Q}_\ell)\Big) - \mathrm{tr}(\mathrm{Frob}_v \mid \Phi_v \otimes \mathbb{Q}_\ell)$$

from Proposition 3.5 together with the Grothendieck–Lefschetz fixed-point expression on the special fibre. This is exactly Theorem 4.3. ☐

Remark B.3. In the tame potentially good case (residue characteristic $p \geq 5$), Proposition B.2 recovers the classical Ogg-type expressions after translating the trace on $H^1(E^{\mathrm{sp}})$ into the combinatorics of the special fibre; see [2, 9].

Appendix C. Auxiliary computations

We gather computation recipes referenced in Section 7 and Section 5. Throughout, $E/\mathbb{Q}$ is given by a global minimal Weierstrass equation; at a prime p write $v = v_p$ and $q = p$.

A. Quick computation of Φ_v and the local data.

(1) Run Tate's algorithm at v to determine the Kodaira symbol and $v(\Delta)$; see [13, Ch. VII].
(2) From the symbol read off $\#\Phi_v$ (the Tamagawa number c_v) using the standard table [1, §9]. This gives the group structure of the special fibre's components.
(3) If the reduction is semistable, then $\mathrm{Sw}_v(H^1) = 0$ and Theorem B.1 gives $f_v(E) = 2 - \dim H^1_\ell(E)^{I_v}$, while Proposition 3.5 turns the I_v-fixed trace into a combination of $\#\mathrm{Fix}(\mathrm{Frob}_v; E^{\mathrm{sp}})$ and the Φ_v-trace.

B. A compact check for split multiplicative primes. Suppose $E/\mathbb{Q}$ has split multiplicative reduction at p. Then:

$$\Phi_p \simeq \mathbb{Z}/n\mathbb{Z}, \qquad \mathrm{Sw}_p(H^1) = 0, \qquad \dim H^1_\ell(E)^{I_p} = 1.$$

Hence $f_p(E) = 1$ and Theorem 4.3 gives

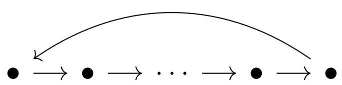
$$0 = \mathrm{tr}\Big(\mathrm{Frob}_p \mid H^1_\ell(E)^{I_p}\Big) = \#\mathrm{Fix}\Big(\mathrm{Frob}_p; E^{\mathrm{sp}}(\overline{\mathbb{F}_p})\Big) - \mathrm{tr}(\mathrm{Frob}_p \mid \Phi_p \otimes \mathbb{Q}_\ell)\,,$$

consistent with Example 3.6 and Proposition 3.3.

C. Sample bookkeeping for Example 7.2. Let $E_\lambda : y^2 = x(x-1)(x-\lambda)$ with $\lambda \in \mathbb{Q} \setminus \{0,1\}$. For each bad prime p:

- Determine the type by $v_p(\lambda)$, $v_p(\lambda - 1)$ and $v_p(\Delta) = v_p(\lambda^2(\lambda-1)^2)$ via Tate's algorithm.
- If the type is I_n, then $\dim H^1_\ell(E)^{I_p} = 1$, $\mathrm{Sw}_p(H^1) = 0$, $f_p(E) = 1$, and $\Phi_p \simeq \mathbb{Z}/n\mathbb{Z}$.
- If the type is additive potentially good, consult Theorem B.1 and Proposition B.2 to express $f_p(E)$ using the special fibre and the Φ_p-trace.

D. Dual-graph snippet for a Néron n-gon. For the reader's convenience, the dual graph of a split multiplicative fibre (all edges have multiplicity one, vertices are components):



It contributes weight-0 classes in the vanishing-cycles group counted by the fixed-point term in Theorem 4.3.

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