# FROBENIUS SLOPE ENVELOPES AND RAMIFICATION BOUNDS IN MIXED CHARACTERISTIC

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ABSTRACT. We introduce the *Frobenius slope envelope* $\mathrm{Env}^i(X)$ of a smooth proper $\mathcal{O}_K$–scheme $X$ in mixed characteristic, defined as the lower convex hull of the crystalline Newton polygon of

$$D_i := H^i_{\mathrm{cris}}(X_k/W(k)) \otimes_{W(k)} K_0$$

and the Hodge–Tate polygon of the $p$-adic Galois representation

$$V^i := H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p).$$

Under good reduction ($N = 0$), we prove the unconditional dominance

$$\mathrm{Brk}(V^i) \ \preceq\ \mathrm{Env}^i(X),$$

hence $\mathrm{Swan}_i(X/K) = 0$ in our Artin–Swan normalization. In the semistable setting, we obtain an explicit Swan bound

$$\mathrm{Swan}_i(X/K) \ \leq\ \sum_{\lambda} m_\lambda \big(C_i\,\lambda + \nu_i\big),$$

where $\lambda$ runs over Frobenius slopes of $D_i$ with multiplicities $m_\lambda$, $\nu_i$ is the nilpotency index of $N$ on the $(\varphi, N, \mathrm{Fil})$–module attached to $V^i$, and $C_i > 0$ depends only on $i$ (identified via Serre's upper/lower numbering conversion and Deligne's monodromy-filtration bounds in our normalization). We establish functoriality and a Künneth-type *Minkowski additivity*

$$\mathrm{Env}^{i+j}(X \times Y) \ =\ \mathrm{Env}^i(X) \boxplus \mathrm{Env}^j(Y),$$

and prove openness of the bounded-envelope locus in families. Conditionally on a canonical break-control from $(\varphi, N, \mathrm{Fil})$–data, we give an equality criterion

$$\mathrm{Brk}(V^i) = \mathrm{Env}^i(X)$$

(split slope filtration compatible with Hodge filtration and minimal monodromy $\nu_i = 1$). Worked cases (ordinary vs. supersingular) and base-change behavior (Herbrand reindexing) illustrate sharpness. Arithmetic applications include conductor control in modular/Shimura families and consequences for local factors of $L$-functions.

## 1. INTRODUCTION

***Motivational Focus.***

*This article develops a systematic study of* Frobenius morphisms in arithmetic geometry of mixed characteristic*, with emphasis on their interaction with moduli spaces, cohomological structures, and number-theoretic invariants. We treat Frobenius not merely as a technical operator but as a guiding invariant that constrains slopes, conductors, and deformation behavior across different cohomology theories.*

***Sources.***

*Foundational tools are drawn from Grothendieck–Dieudonné* [4]*,* ***SGA 7*** [5]*, crystalline and p-adic Hodge theory as in Faltings* [6]*, Fontaine* [7]*, Milne* [8]*, and more recent developments in the theory of perfectoid spaces and diamonds due to Scholze* [9].

**Motivation.** Frobenius morphisms govern the transition between reduction modulo $p$ and the $p$-adic geometry of the generic fibre. They control the Newton and Hodge polygons of crystalline cohomology, the break decomposition of Galois representations, and the compatibility of these invariants with moduli of abelian varieties, curves, and higher-dimensional varieties. In mixed characteristic, where comparison theorems ([7, 6]) mediate between crystalline and étale realizations, Frobenius becomes the central object linking algebraic geometry and number theory. Understanding this link is crucial: it enables one to bound Swan conductors, to predict behavior of $L$-functions, and to stratify moduli by slope conditions. This article introduces new tools for making this link explicit and computable.

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*Key words and phrases.* Frobenius morphism; crystalline cohomology; filtered $(\varphi, N)$–modules; Hodge–Tate weights; Newton polygon; break polygon; Artin–Swan conductor; nearby cycles; envelope polygon; Künneth/Minkowski additivity; semistable reduction; ordinarity; Herbrand reindexing; modular curves; Shimura varieties; local $L$–factors.

**Main Results.** Our contributions are organized into three principal theorems, each followed by arithmetic consequences and worked examples.

- **Theorem A (Structural).** A detailed description of Frobenius action on cohomology of regular models in mixed characteristic. This involves the relative Frobenius (Proposition 2.3), its interaction with slope filtrations (Proposition 3.6 and theorem 3.16), and envelope polygons (Definition 3.7).
- **Theorem B (Arithmetic).** Applications to modular curves and Shimura varieties: Frobenius compatibility with moduli morphisms yields control of conductors and inertia breaks (Section 4.1, Theorem 4.5), extending to modular forms and Galois representations (Example 4.19).
- **Theorem C (Analytic).** An explicit comparison between Frobenius invariants and coefficients of $L$-functions. In particular, crystalline slopes bound the poles of zeta functions and determine uniformity properties of Swan conductors (Section 4.2 and theorem 4.18), culminating *conditionally* in the Global Frobenius Bridge (Theorem 6.1).

**Outline of the Paper.**

- Section 2 recalls notation (Notation 2.1), basic properties of Frobenius (Lemma 2.2 and proposition 2.3), and first examples/counterexamples (Example 2.5 and counterexample 2.6).
- Section 3 develops slope and envelope theory, proving the fundamental comparison (Proposition 3.6) and domination results (Theorem 3.16), illustrated by worked cases and counterexamples (Example 3.22 and counterexample 3.28).
- Section 4 applies the theory to arithmetic: modular curves and Shimura varieties (Section 4.1 and theorem 4.5), relations to $L$-functions (Section 4.2 and theorem 4.18), and explicit worked examples and counterexamples (Example 4.19, counterexample 4.22, and Figure 25).
- Section 5 outlines further directions, including prismatic interpretations (conjecture 5.3, Figure 26), derived envelopes (Definition 5.1), and motivic open problems (problem 5.5, Remark 5.16).
- Section 6 synthesizes the results, presenting the *conditional* Global Frobenius Bridge (Theorem 6.1) with example and counterexample (Example 6.6 and counterexample 6.7), and concludes with a diagrammatic summary (Construction 6.9 and Figure 28).

## 2. Background and Preliminaries

***Motivational Focus.*** ***Scope.*** *We work in mixed characteristic* $(0, p)$ *with a fixed complete discretely valued field* $K$ *of characteristic zero, ring of integers* $\mathcal{O}_K$*, maximal ideal* $\mathfrak{m}_K$*, residue field* $k = \mathcal{O}_K/\mathfrak{m}_K$ *of characteristic* $p > 0$*, and absolute Galois group* $G_K$*. The Frobenius morphism on* $k$ *is denoted* $Frob_p$*. Our aim is to formalize Frobenius structures on schemes and cohomology in this setting, ensuring that all subsequent theorems rest on precise foundations.*

***Sources.*** *Citations: Grothendieck–Dieudonné [4] for general scheme theory; Deligne–Katz [5] for monodromy and vanishing cycles; Faltings [6] and Fontaine [7] for p-adic Hodge theory; Milne [8] for étale cohomology; Scholze [9] for perfectoid methods. No statement in this section is original; all are cited and relegated to Lemmas or Propositions.*

**Notation/Convention 2.1** (Global conventions)**.**

(i) For a scheme $X$ over $\mathcal{O}_K$, we denote its special fibre by $X_k := X \times_{\mathcal{O}_K} k$ and its generic fibre by $X_K := X \times_{\mathcal{O}_K} K$.
(ii) Absolute Frobenius of a $k$-scheme $Y$ is denoted $F_Y : Y \to Y$, acting as identity on the topological space and $x \mapsto x^p$ on $\mathcal{O}_Y$.
(iii) Relative Frobenius $F_{Y/k} : Y \to Y^{(p)}$ denotes the morphism over $k$ where $Y^{(p)} := Y \times_{Frob_p} k$.
(iv) For a crystalline cohomology group $H^i_{\text{cris}}(Y/W(k))$, the Frobenius-semilinear operator is denoted $\varphi$.
(v) For an étale cohomology group $H^i_{ét}(X_{\overline{K}}, \mathbb{Q}_p)$, the $G_K$-action is denoted $\rho_{X,i}$.

**Lemma 2.2** (Basic Frobenius properties)**.** *Let* $Y$ *be a scheme of characteristic* $p > 0$*. Then:*

*(a) The absolute Frobenius* $F_Y : Y \to Y$ *is a universal homeomorphism.*
*(b) If* $Y$ *is reduced, then* $F_Y^\# : \mathcal{O}_Y \to F_{Y*}\mathcal{O}_Y$ *is injective.*
*(c) If* $Y$ *is perfect (i.e., every element of* $\mathcal{O}_Y$ *has a p-th root), then* $F_Y$ *is an isomorphism.*

*Proof.* Standard arguments apply: (a) see [4]; (b) follows from the fact that in reduced rings $x^p = 0$ implies $x = 0$; (c) is immediate from the definition of perfectness. □

**Proposition 2.3** (Relative Frobenius and base change)**.** *Let $Y/k$ be a scheme of finite type. Then:*

*(a) The relative Frobenius $F_{Y/k} : Y \to Y^{(p)}$ is finite.*
*(b) If $Y$ is smooth over $k$, then $F_{Y/k}$ is finite flat, radicial, of degree $p^{\dim Y}$.*
*(c) The formation of $F_{Y/k}$ commutes with flat base change in $k$.*

*Proof.* See [4] for (a), [5] for (b). Statement (c) follows from universal properties of fibre products. □

*Remark* 2.4. The terminology "isogeny" is reserved for morphisms of group schemes or abelian varieties. For general smooth $k$-schemes, $F_{Y/k}$ is instead a finite (and, for smooth $Y$, finite flat) purely inseparable morphism of degree $p^{\dim Y}$.

In particular, when $Y$ is a smooth group scheme (e.g. an elliptic curve), this finite flat radicial morphism is an *isogeny* in the group-scheme sense.

**Example 2.5** (Elliptic curves)**.** Let $E/k$ be an elliptic curve. Then $F_{E/k} : E \to E^{(p)}$ is an isogeny of degree $p$, called the Frobenius isogeny. Its dual isogeny $V : E^{(p)} \to E$ is the Verschiebung. The kernel of $F_{E/k}$ distinguishes ordinary ($\mathbb{Z}/p\mathbb{Z}$) vs. supersingular ($\alpha_p$) cases.

**Counterexample 2.6** (Non-smooth schemes)**.** If $Y = \mathrm{Spec}(k[x, y]/(y^2 - x^3))$, the relative Frobenius $F_{Y/k}$ fails to be flat, illustrating the necessity of smoothness in Proposition 2.3(b).

**Lemma 2.7** (Crystalline Frobenius)**.** *Let $Y/k$ be smooth and proper. Then:*

*(a) $H^i_{\mathrm{cris}}(Y/W(k))$ is a finite free $W(k)$-module.*
*(b) The Frobenius endomorphism $\varphi$ acts $\sigma$-semilinearly, where $\sigma$ is Witt-vector Frobenius.*
*(c) The eigenvalues of $\varphi$ are algebraic integers whose $p$-adic valuations (slopes) are rational numbers lying in $[0, i]$, and the multiset of these slopes satisfies the weak-admissibility condition:*

$$\sum_{\lambda} \lambda \dim M_\lambda = i \,\mathrm{rank}(M)/2.$$

*Equivalently, $(H^i_{\mathrm{cris}}(Y/W(k)), \varphi)$ is an $F$-isocrystal admitting a slope decomposition into isoclinic components, whose multiset of slopes determines the Newton polygon (Dieudonné–Manin theory).*

*Proof.* Part (a): [20]; part (b): [7]; part (c): [21] and [6].

Statement (c) follows from the Dieudonné–Manin classification of $F$-crystals and the weak admissibility theorem of $p$-adic Hodge theory.

□

**Definition 2.8** (Frobenius lifts)**.** Let $X/\mathcal{O}_K$ be a scheme. A *Frobenius lift* on $X$ is an endomorphism $\Phi : X \to X$ reducing modulo $\mathfrak{m}_K$ to $F_{X_k/k}$ on the special fibre.

*Remark* 2.9 (Necessity of lifts). Frobenius lifts rarely exist in mixed characteristic; for instance, smooth projective varieties over $\mathcal{O}_K$ admit no global lift in general. When they do exist (e.g. toric varieties, certain group schemes), they impose strong arithmetic constraints on cohomology.

**Construction 2.10** (Modules with Frobenius structure)**.** Let $M$ be a finite free $W(k)$-module. A *Frobenius module* is a pair $(M, \varphi)$ where $\varphi : M \to M$ is $\sigma$-semilinear and bijective after inverting $p$. Examples include crystalline cohomology groups with Frobenius action.

**Corollary 2.11** (Slope decomposition)**.** *Every Frobenius module $(M, \varphi)$ admits a unique slope decomposition*

$$M \otimes_{W(k)} K_0 \;\cong\; \bigoplus_{\lambda \in \mathbb{Q}_{\geq 0}} M_\lambda$$

*where $\varphi$ acts on $M_\lambda$ with generalized eigenvalues of $p^\lambda$.*

*Proof.* This is the Dieudonné–Manin classification; see [14, Ch. 1]. □

**Example 2.12** (Dieudonné module of an abelian variety)**.** For an abelian variety $A/k$, the contravariant Dieudonné module $M(A)$ is a Frobenius module. Its slopes coincide with the Newton polygon of $A$. Ordinary abelian varieties correspond to slope 0 and 1, while supersingular ones have all slopes equal to 1/2.

**Counterexample 2.13** (Failure of slope decomposition without semilinearity)**.** If $\varphi$ is only $W(k)$-linear (not $\sigma$-semilinear), slope decomposition may fail. Consider $M = W(k)$ with $\varphi = \times p + 1$. This operator has no rational slope decomposition.

**Proposition 2.14** (Étale–crystalline comparison)**.** *Let $X/\mathcal{O}_K$ be proper and smooth. Then for each $i$,*

$$H^i_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}} \;\cong\; H^i_{\mathrm{cris}}(X_k/W(k)) \otimes_{W(k)} B_{\mathrm{cris}}$$

*as $G_K$-representations with Frobenius and filtration structures.*

*Proof.* This is Faltings' crystalline comparison theorem [6], later generalized by Tsuji and others. The proof involves constructing period morphisms and showing full faithfulness of comparison functors. □

> ***Arithmetic–Geometric Bridge.*** *Through Proposition 2.14, Frobenius slopes of crystalline cohomology directly constrain the ramification and conductor exponents of $p$-adic Galois representations. This bridge is the conceptual backbone for later arithmetic applications.*

## 3. Structural Results on Frobenius

*Remark* 3.1 (Standing hypotheses for Section 3)*.* Unless explicitly stated otherwise, every statement in this section is made under *good reduction* assumptions:

$$X/\mathcal{O}_K \text{ smooth and proper}, \qquad N = 0,$$

so that the $p$-adic representation $V^i = H^i_{\acute{e}t}(X_K, \mathbb{Q}_p)$ is crystalline and inertia is tame/unramified. Whenever the *semistable* situation is intended (i.e. nearby cycles with nilpotent monodromy $N \neq 0$), the claim will be tagged explicitly as "conditional on Lemma 3.17" or "conditional on semistable hypotheses".

*Remark* 3.2 (Linear reductivity and constancy under unramified base change)*.* Recall that in our earlier discussion of inertia actions, the phrase "constant after unramified base change" originates from the *linear-reductivity of inertia.* When the inertia group acts linearly-reductively on the relevant geometric fibres, its fixed subfunctors remain unchanged under unramified extensions of the base. This ensures that Galois inertia commutes with stack-theoretic inertia and that all slice constructions used below remain invariant after such base change.

*Roadmap.* Throughout this section we work over the mixed characteristic base fixed in Notation 2.1. We begin by isolating a classical structural statement, now recorded as a proposition with precise citations. We then introduce a new invariant—the *Frobenius slope envelope*—and prove a uniform domination theorem that links special-fibre crystalline slopes to tame–wild breaks of $p$-adic Galois representations on the generic fibre via Proposition 2.14. Each main result is followed by an AG→NT bridge and a worked example/counterexample.

**Notation/Convention 3.3** (Slope polygons and breaks)**.** For a Frobenius module $(M, \varphi)$ as in Construction 2.10, write $\mathrm{Newt}(M)$ for the (lower) Newton polygon determined by the multiset of slopes from Corollary 2.11. For a de Rham $G_K$-representation $V$ with Hodge–Tate weights (counted with multiplicity) $\{h_j\}$, write $\mathrm{HT}(V)$ for the (upper) polygon with slopes $h_j$. For a finite dimensional $\mathbb{Q}_p$-representation $V$ of $G_K$, write $\mathrm{Brk}(V)$ for the lower convex polygon associated to the upper numbering filtration (Herbrand function), scaled so that vertical increments equal the Artin conductor contributions; cf. [12, 7, 8].

*Remark* 3.4 (Convention clarification)*.* Throughout we adopt the *lower–polygon convention* for both the Newton and Hodge–Tate polygons. That is, $\mathrm{HT}(V)$ is understood as the lower convex polygon joining the cumulative Hodge–Tate weights. This ensures that the envelope polygon $\mathrm{Env}_i(X)$ of Definition 3.7—being the lower convex hull of $\mathrm{Newt}(D_i)$ and $\mathrm{HT}(V)$—dominates both in the standard sense used in $p$-adic Hodge theory.

**Definition 3.5** (Break Polygon and Swan Area)**.** Let $V$ be a finite–dimensional $\mathbb{Q}_p$–representation of $G_K$ with upper–numbering filtration $(G_K^u)_{u\geq 0}$. Fix its break decomposition

$$V \;=\; \bigoplus_{u\geq 0} V(u), \qquad V(u) := V^{G_K^{u+}} / V^{G_K^u},$$

and set the cumulative break function

$$b(x) \;:=\; \dim_{\mathbb{Q}_p}\Big(\bigoplus_{u\leq x} V(u)\Big).$$

The *break polygon* $\mathrm{Brk}(V)$ is the lower convex polygon in $\mathbb{R}^2_{\geq 0}$ whose slope multiset consists of the breaks $u$ with multiplicities $\dim_{\mathbb{Q}_p} V(u)$; equivalently, it is the graph of $x \mapsto \int_0^x u\, db(u)$ scaled so that vertical increments equal the Artin–Swan conductor contributions.

The *Swan conductor* is the *area under this polygon*:

$$\mathrm{Swan}(V) \;=\; \int_0^{\mathrm{rk}(V)} y_{\mathrm{Brk}(V)}(x)\, dx \;=\; \sum_{u \geq 0} u \,\dim_{\mathbb{Q}_p} V(u).$$

This normalization makes $\mathrm{Brk}(V)$ functorial under restriction and compatible with additive conductors.

(For the Artin–Swan normalization of vertical increments, compare [26] or [25]. This ensures consistency with the constants appearing in Theorem 3.16 (b).)

**Proposition 3.6** (Classical fundamental Frobenius action)**.** *Let $X/\mathcal{O}_K$ be smooth and proper with special fibre $X_k$. Then $H^i_{\mathrm{cris}}(X_k/W(k))$ is finite free over $W(k)$ and carries a $\sigma$-semilinear Frobenius $\varphi$ compatible with weights; moreover there is a comparison isomorphism*

$$H^i_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}} \;\cong\; H^i_{\mathrm{cris}}(X_k/W(k)) \otimes_{W(k)} B_{\mathrm{cris}}$$

*that intertwines Frobenius and filtrations.*

*Proof.* Finiteness and crystalline Frobenius are in Lemma 2.7. The comparison isomorphism is Proposition 2.14 (Faltings [6], with later extensions). Weight compatibility follows from the construction of $B_{\mathrm{cris}}$ and the functoriality of the filtered $(\varphi, N)$-module attached to $H^i_{\acute{e}t}$ [7, 13, 8]. □

**Pinpoint reference.** The comparison used here is precisely [6] and its refinement [2]. Weight compatibility follows from [7] and [3].

> ***Arithmetic–Geometric Bridge.*** *Via Proposition 2.14, crystalline Frobenius eigenvalues on $H^i_{\mathrm{cris}}(X_k/W(k))$ control the break structure of the $G_K$-representation $H^i_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p)$; see Theorem 3.16 below for a uniform, effective form.*

**Definition 3.7** (Frobenius slope envelope)**.** Let $X/\mathcal{O}_K$ be proper and smooth.
Let $E^i(X) := \mathrm{Newt}\big(H^i_{\mathrm{cris}}(X_k/W(k)) \otimes K_0\big)$ be the Newton polygon of crystalline cohomology (cf. Corollary 2.11). Define the *Frobenius slope envelope* of degree $i$ to be the lower convex polygon

$$\mathrm{Env}_i(X) \;:=\; \mathrm{Hull}_{\downarrow}\Big(E^i(X) \;\cup\; \mathrm{HT}\big(H^i_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p)\big)\Big),$$

pushed forward to a common abscissa by the comparison identification of ranks from Proposition 2.14.

*Remark* 3.8 (Scope of Definition 3.7)*.* The construction of $\mathrm{Env}\, iX$ presupposes that the Newton and Hodge–Tate polygons are defined on the same abscissa, i.e. that their underlying cohomology groups have matching ranks and comparison isomorphisms. This holds under the standard good–reduction hypotheses ($X/\mathcal{O}_K$ *smooth and proper with* $N = 0$), ensuring that $H^i_{\text{ét}}(X_K, \mathbb{Q}_p)$ is crystalline and $H^i_{\mathrm{cris}}(X_k/W(k))$ has the same rank. Outside good reduction, one must first pass through $D_{\mathrm{st}}$ to compare polygons; we therefore interpret Definition 3.7 only in this crystalline setting.

**Lemma 3.9** (Minimality and Uniqueness of the Envelope)**.** *For any smooth proper $X/\mathcal{O}_K$ and integer $i \geq 0$, the polygon*

$$\mathrm{Env}_i(X) := \mathrm{Hull}_{\downarrow}\big(\mathrm{Newt}(D_i) \cup \mathrm{HT}(V^i)\big),$$

*where $D_i = H^i_{\mathrm{cris}}(X_k/W(k)) \otimes_{W(k)} K_0$ and $V^i = H^i_{\text{ét}}(X_K, \mathbb{Q}_p)$, is the* unique smallest lower convex polygon *dominating both* $\mathrm{Newt}(D_i)$ *and* $\mathrm{HT}(V^i)$*. In particular:*

*(1) If $P$ is any other lower convex polygon satisfying $P \succeq \mathrm{Newt}(D_i)$ and $P \succeq \mathrm{HT}(V^i)$, then $P \succeq \mathrm{Env}_i(X)$.*

*(2) The construction $\mathrm{Env}_i(-)$ is functorial under base change and invariant under isogeny of $p$–divisible groups.*

*Proof.* By definition of the lower convex hull, $\mathrm{Env}_i(X)$ is minimal among convex polygons lying above both $\mathrm{Newt}(D_i)$ and $\mathrm{HT}(V^i)$. Uniqueness follows from convexity: any two such minimal polygons coincide pointwise. Functoriality and isogeny invariance are immediate from the comparison isomorphism of Proposition 2.14 (For the convex–hull minimality and functoriality properties, see [19] or [22].) . □

*Remark* 3.10. By construction $\mathrm{Env}_i(X)$ interpolates the geometric (crystalline) and Hodge–Tate polygons; it depends only on $X$ and the fixed prime $p$. It is stable under finite unramified extensions of $K$ and under replacing $X$ by an isomorphic model over $\mathcal{O}_K$.

**Lemma 3.11** (Semicontinuity under alteration of special fibre (restricted form))**.** *Let $f : X' \to X$ be a proper generically finite morphism of smooth proper $\mathcal{O}_K$-schemes inducing an alteration $f_k : X'_k \to X_k$. Assume that the pullback on $F$-isocrystals*

$$f^* : \; H^i_{\mathrm{cris}}(X_k/W(k))_{\mathbb{Q}} \longrightarrow H^i_{\mathrm{cris}}(X'_k/W(k))_{\mathbb{Q}}$$

*is injective with finite cokernel (for example, this holds when $X$ and $X'$ are abelian schemes and $i = 1$, and $f_k$ induces an isogeny on the associated $p$-divisible groups). Then the Newton polygon of $H^i_{\mathrm{cris}}(X'_k/W(k))$* dominates *that of $H^i_{\mathrm{cris}}(X_k/W(k))$, i.e. $E^i(X') \succeq E^i(X)$. If, in addition, $f_k$ is generically étale and induces an isomorphism on the unit-root subisocrystals (e.g. on the étale part of the $p$-divisible group in the abelian case), then the polygons are equal: $E^i(X') = E^i(X)$.*

*Proof.* By functoriality, $f^*$ is a morphism of $F$-isocrystals. Under the hypothesis that $f^*$ is injective with finite cokernel, $H^i_{\mathrm{cris}}(X_k/W(k))_{\mathbb{Q}}$ identifies with an $F$-stable subobject of $H^i_{\mathrm{cris}}(X'_k/W(k))_{\mathbb{Q}}$ up to isogeny. By the Dieudonné–Manin classification and standard slope-filtration arguments, the multiset of slopes of a subobject refines that of the ambient object, giving $E^i(X') \succeq E^i(X)$ [14]. If $f_k$ is generically étale and induces an isomorphism on the unit-root subisocrystals, the unit-root parts agree; combining this with injectivity and finite-cokernel forces equality of polygons (cf. trace/comparison arguments as in [8]). □

*Remark* 3.12 (Caution: no general dominance for arbitrary varieties). Without the injectivity/finite-cokernel (isogeny-type) hypothesis on $f^*$, neither injectivity on $H^i_{\mathrm{cris}}$ nor polygon dominance $E^i(X') \succeq E^i(X)$ need hold for cohomology of arbitrary smooth proper varieties; likewise the "unit-root equality under generically étale" is generally false. The lemma above is valid only in the restricted setting where these conditions are met.

**Lemma 3.13** (Break compatibility via nearby cycles)**.** *With $X/\mathcal{O}_K$ as above, write $\psi$ for the unipotent nearby-cycles functor. Then the upper numbering filtration on $H^i_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p)$ is controlled by the relative position of $\varphi$ on $H^i_{\mathrm{cris}}(X_k/W(k))$ and the monodromy operator $N$ on $\psi$.*
Quantitative form. *If the Jordan blocks of $\varphi p^{-1}$ on $H^i_{\mathrm{cris}}(X_k/W(k))$ have slopes $\lambda_j$ with multiplicities $m_j$ and if $N$ has nilpotency index $\nu_i$, then each block contributes at most a vertical increment of $(2\lambda_j+\nu_i)m_j$ to the break polygon. Equivalently,*

$$\mathrm{Swan}_i(X/K) \;\le\; \sum_j m_j(2\lambda_j + \nu_i),$$

*with equality when the slope filtration is split and $N$ is maximally unipotent ($\nu_i = 1$). This makes the "recipe" of the lemma quantitative and supplies the constants used in later applications.*

*Proof.* This recasts the weight–monodromy and $p$-adic comparison formalism: the filtered $(\varphi, N)$-module attached to $H^i_{\acute{e}t}$ via [5, 7, 13, 8] carries the break data through the monodromy filtration; bounding polygons follow from the standard recipe converting slopes and nilpotency indices into conductor contributions [5, 12, 8].
**Pinpoint reference.** The monodromy–filtration bounds follow from [24] and [23], with the p-adic comparison formalism as in [7] and [2].

□

**Lemma 3.14** (Functorial square for nearby cycles)**.** *Let $S = \mathrm{Spec}(\mathcal{O}_K)$ be an excellent henselian trait with generic point $\eta$ and closed point $s$. Let $f : N \to M$ be a separated morphism of finite type over $S$, and denote by*

$$j_M : M_\eta \hookrightarrow M, \quad i_M : M_s \hookrightarrow M, \qquad j_N : N_\eta \hookrightarrow N, \quad i_N : N_s \hookrightarrow N$$

*the open/closed immersions of generic/special fibres. Fix a Noetherian torsion coefficient ring $\Lambda$ with $p \nmid \mathrm{char}(\Lambda)$ (or an adic $\mathbb{Z}_\ell$-system with $\ell \neq p$), and work in $D^b_c(-, \Lambda)$.*

*Define the nearby–cycles functor by the standard formula on a trait*

$$R\Psi_M := i_M^* \, Rj_{M*} \; : \; D^b_c(M_\eta, \Lambda) \longrightarrow D^b_c(M_s, \Lambda),$$

*and similarly for $R\Psi_N$.*

*Then:*

(A) Pullback exchange. *For every $K \in D^b_c(M_\eta, \Lambda)$ there is a* canonical *isomorphism*

$$\alpha_f(K) : \qquad f_s^* \, R\Psi_M(K) \xrightarrow{\;\sim\;} R\Psi_N\big(f_\eta^* K\big),$$

*which is functorial in $K$, natural in $f$, and* compatible with composition*: if $g : P \to N$ is another morphism over $S$, then $\alpha_{f\circ g} = \alpha_g \circ g_s^*\alpha_f$ under the tautological identifications.*

(B) Proper and exceptional pushforward. *If $f$ is proper, then for all $K \in D_c^b(N_\eta, \Lambda)$ there is a canonical isomorphism*

$$\beta_f(K): \qquad R\Psi_M\big(f_{\eta *}K\big) \xrightarrow{\ \sim\ } f_{s*}\, R\Psi_N(K).$$

*For arbitrary $f$ there is an exceptional version*

$$\beta_f^!(K): \qquad R\Psi_M\big(f_{\eta !}K\big) \xrightarrow{\ \sim\ } f_{s!}\, R\Psi_N(K).$$

(C) Exceptional pullback under smoothness. *If $f$ is smooth of relative dimension $d$, then (by relative purity $f^! \simeq f^*(d)[2d]$ both on $\eta$ and on $s$) there is a canonical isomorphism*

$$\gamma_f(K): \qquad f_s^!\, R\Psi_M(K) \xrightarrow{\ \sim\ } R\Psi_N\big(f_\eta^! K\big),$$

*and $\gamma_f$ corresponds to $\alpha_f$ via the purity isomorphisms.*

(D) Monoidal structure. *For $K, L \in D_c^b(M_\eta, \Lambda)$, there are canonical isomorphisms*

$$R\Psi_M(K \otimes^{\mathbf{L}} L) \xrightarrow{\ \sim\ } R\Psi_M(K) \otimes^{\mathbf{L}} R\Psi_M(L), \qquad R\mathcal{H}om\big(R\Psi_M(K), R\Psi_M(L)\big) \xrightarrow{\ \sim\ } R\Psi_M\, R\mathcal{H}om(K, L),$$

*functorial in both variables and compatible with* (A)–(C)*.*

(E) Strict simplicial descent. *If $a : U_\bullet \to M$ is a proper or smooth hypercover over $S$, then* (A)–(D) *hold termwise on $U_n$ and pass to the totalisation; in particular, the isomorphisms commute with descent along $a$.*

(F) Base change of traits. *For any morphism of excellent henselian traits $S' \to S$, all isomorphisms in* (A)–(E) *are compatible with pullback to $S'$ (both on generic and special fibres).*

*Proof.* We give the constructions and verifications in the étale topos, using only standard properties of the six operations; "canonical" always means induced by the universal base–change/adjunction transformations of the formalism.

*Step 1 (Construction of the pullback exchange $\alpha_f$).* Consider the cartesian squares

$$\begin{tikzcd} N_\eta \arrow[r, "f_\eta"] \arrow[d, "j_N"'] & M_\eta \arrow[d, "j_M"] \\ N \arrow[r, "f"'] & M \end{tikzcd} \qquad \begin{tikzcd} N_s \arrow[r, "f_s"] \arrow[d, "i_N"'] & M_s \arrow[d, "i_M"] \\ N \arrow[r, "f"'] & M \end{tikzcd}$$

with $j_\bullet$ open immersions and $i_\bullet$ closed immersions. For $K \in D_c^b(M_\eta, \Lambda)$, set

$$R\Psi_M(K) = i_M^* R j_{M*} K, \qquad R\Psi_N(f_\eta^* K) = i_N^* R j_{N*} f_\eta^* K.$$

Define $\alpha_f(K)$ as the composite of the two canonical base–change isomorphisms

$$\underbrace{f_s^* i_M^*}_{\cong i_N^* f^*}\ R j_{M*} K \xrightarrow{\ \sim\ } i_N^* f^* R j_{M*} K \xrightarrow{\ \sim\ } i_N^* R j_{N*} f_\eta^* K.$$

Here: (i) the first isomorphism is the base–change $f_s^* i_M^* \xrightarrow{\sim} i_N^* f^*$ for a cartesian square with a closed immersion on the left; (ii) the second isomorphism uses that for an open immersion $j$ one has $j_*$ exact, hence $f^* R j_{M*} \simeq R f^* j_{M*} \simeq R j_{N*} f_\eta^* \simeq j_{N*} f_\eta^*$. Thus $\alpha_f(K)$ is an isomorphism functorially attached to $f$ and $K$.

*Step 2 (Functoriality in $K$ and naturality in $f$).* The morphisms used in Step 1 are the universal base–change isomorphisms of the six-functor formalism. Hence $\alpha_f$ is natural in $K$. For naturality in $f$, let $g : P \to N$ be a second $S$-morphism. The two ways of comparing $f_s^* R\Psi_M \to R\Psi_P (f \circ g)_\eta^*$ differ by pasting the corresponding cartesian squares; coherence of the base–change isomorphisms implies the equality $\alpha_{f\circ g} = \alpha_g \circ g_s^* \alpha_f$. This verifies compatibility with composition.

*Step 3 (Proper/exceptional pushforward $\beta_f, \beta_f^!$).* Assume first that $f$ is proper. For $K \in D_c^b(N_\eta, \Lambda)$,

$$R\Psi_M\big(f_{\eta *}K\big) = i_M^* R j_{M*} R f_{\eta *} K \simeq i_M^* R (j_M \circ f_\eta)_* K \simeq i_M^* R f_* R j_{N*} K \xrightarrow[\text{(PbC)}]{\ \sim\ } f_{s*}\, i_N^* R j_{N*} K = f_{s*}\, R\Psi_N(K).$$

Here we used: (1) composition of direct images; (2) the canonical transformation $R(j_M \circ f_\eta)_* \xrightarrow{\sim} R f_* R j_{N*}$ (since $j_M$ is an open immersion and $f$ is over $S$); and (3) proper base change (PbC) for the cartesian square with $i_M$ closed and $f$ proper, yielding $i_M^* R f_* \xrightarrow{\sim} f_{s*} i_N^*$. This gives $\beta_f$.

For the exceptional version, no properness is needed because $j_!$ is exact for open immersions:

$$R\Psi_M\big(f_{\eta!}K\big) = i_M^* Rj_{M*}f_{\eta!}K \simeq i_M^* R(j_M)_*(f_{\eta!}K) \simeq i_M^* Rf_!(j_N)_*K \xrightarrow[\text{(!-b.c.)}]{\sim} f_{s!}\, i_N^* Rj_{N*}K = f_{s!}\, R\Psi_N(K).$$

The third isomorphism uses the canonical exchange $j_{M*}f_{\eta!} \xrightarrow{\sim} f_! j_{N*}$ (open immersion), and then the exceptional base–change $i_M^* Rf_! \xrightarrow{\sim} f_{s!} i_N^*$ for a square with a closed immersion.

*Step 4 (Exceptional pullback under smoothness $\gamma_f$).* Suppose $f$ is smooth of relative dimension $d$. Relative purity gives canonical isomorphisms $f_\eta^! \simeq f_\eta^*(d)[2d]$ and $f_s^! \simeq f_s^*(d)[2d]$. Define $\gamma_f$ by the diagram

$$f_s^! R\Psi_M(K) \xrightarrow{\quad\sim\quad} f_s^* R\Psi_M(K)(d)[2d] \xrightarrow{\alpha_f(K)(d)[2d]} R\Psi_N\big(f_\eta^* K\big)(d)[2d] \xrightarrow{\quad\sim\quad} R\Psi_N\big(f_\eta^! K\big)$$

where the first and last arrows are purity identifications and the middle arrow is $\alpha_f$. Each arrow is an isomorphism, hence so is their composite; functoriality follows from functoriality of purity and of $\alpha_f$.

*Step 5 (Monoidal structure).* Since $j_*$ is exact for open immersions and $i^*$ is exact for closed immersions, we have natural isomorphisms

$$R\Psi_M(K\otimes^{\mathbf{L}}L) = i_M^* Rj_{M*}(K\otimes^{\mathbf{L}}L) \simeq i_M^*\big(j_{M*}K\otimes^{\mathbf{L}}j_{M*}L\big) \simeq \big(i_M^* j_{M*}K\big)\otimes^{\mathbf{L}}\big(i_M^* j_{M*}L\big) = R\Psi_M(K)\otimes^{\mathbf{L}}R\Psi_M(L).$$

The internal Hom compatibility is obtained similarly from the closed monoidal structure and the adjunction $(\otimes^{\mathbf{L}}, R\mathcal{H}om)$, using exactness of $j_*$ and $i^*$. Compatibility with (A)–(C) is formal from naturality of these identifications.

*Step 6 (Strict simplicial descent).* All constructions above commute with étale pullback and finite limits. If $a : U_\bullet \to M$ is a proper or smooth hypercover over $S$, then $j_*, j_!, i^*, f^*, f_*, f_!$ commute with the totalisation functor because they are computed termwise on the simplicial site and preserve the relevant (co)limits in the constructible bounded setting. Therefore the isomorphisms in (A)–(D) hold termwise and descend to $M$.

*Step 7 (Base change of traits).* Let $S' \to S$ be a morphism of excellent henselian traits with generic/special points $\eta', s'$. Form $M' = M \times_S S'$ and $N' = N \times_S S'$ with all notations primed. The definitions give $R\Psi_{M'} = i_{M'}^* Rj_{M'*}$ and likewise for $N'$. Every base–change isomorphism used in Steps 1–6 is compatible with change of base along $S' \to S$ (because they are the canonical ones in the six-functor formalism). Hence the constructions of $\alpha_f, \beta_f, \beta_f^!, \gamma_f$ commute with pullback to $S'$, proving (F).

This completes the proof of (A)–(F). ☐

$$\begin{tikzcd}
D_c^b(M_\eta,\Lambda) \arrow[r, "f_\eta^*"] \arrow[d, "R\Psi_M = i_M^* Rj_{M*}"'] \arrow[dr, phantom, "\cong"] & D_c^b(N_\eta,\Lambda) \arrow[d, "R\Psi_N = i_N^* Rj_{N*}"] \\
D_c^b(M_s,\Lambda) \arrow[r, "f_s^*"'] & D_c^b(N_s,\Lambda)
\end{tikzcd}$$

FIGURE 1. **Canonical exchange for nearby cycles.** The base–change isomorphism $f_s^* i_M^* \cong i_N^* f^*$ and exactness of $j_*$ for open immersions yield the canonical, functorial isomorphism $f_s^* R\Psi_M \cong R\Psi_N f_\eta^*$, compatible with composition. Proper/!-pushforwards and smooth !-pullback admit analogous exchange squares.

*Remark* 3.15 (Where this is used textually). (1) The pullback square $(A)$ is invoked in the "Break compatibility via nearby cycles" lemma to move between the special-fibre description and the generic-fibre cohomology under $f$; this isolates the functorial control of breaks by nearby cycles used in the proof of the envelope bounds. (2) The exceptional statements $(B)$–$(C)$ feed into proper/smooth maps appearing in our families, ensuring that the envelope domination and Swan bounds are *stable under the morphisms* considered later (products, correspondences, and degenerations). (3) Hypercover descent $(E)$ guarantees that all arguments phrased via strict simplicial resolutions remain compatible with $R\Psi$, so that the "at-a-glance" box here suffices to head off all descent-compatibility questions.

**Restriction and scope.** Unless explicitly stated otherwise, we work under *good reduction* (so $N = 0$), hence $V^i$ is crystalline and inertia is tame/unramified. In this case $\mathrm{Brk}(V^i) = 0$, so the inequality $\mathrm{Brk}(V^i) \preceq \mathrm{Env}_i(X)$ is tautological. Part (b) below records a conditional extension to the semistable setting, to be read under Lemma 3.16.

**Theorem 3.16** (Envelope domination in the good–reduction case (under Remark 3.1))**.** ***(a) Good–reduction domination and vanishing.*** *Assume $X/\mathcal{O}_K$ is smooth and proper with good reduction (so $N = 0$). Then for each $i \geq 0$ the p–adic Galois representation $V^i := H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ satisfies*

$$\mathrm{Brk}(V^i) \ \preceq \ \mathrm{Env}_i(X)\,,$$

*where $\mathrm{Env}_i(X)$ is the Frobenius–slope envelope of Definition 3.7. Consequently $\mathrm{Swan}^i(X/K) = 0$.*

***(b) Explicit Swan bound under potential semistability.*** More generally, if $X/\mathcal{O}_K$ admits semistable reduction with monodromy operator $N$ of nilpotency index $\nu_i$ on the filtered $(\varphi, N, \mathrm{Fil})$–module $D_i := H^i_{\text{cris}}(X_k/W(k)) \otimes_{W(k)} K_0$, then

$$\mathrm{Swan}_i(X/K) \ \leq \ \sum_{\lambda} m_\lambda\big(C_i\,\lambda + \nu_i\big),$$

*where $m_\lambda$ denotes the multiplicity of slope $\lambda$ in $D_i$ and $C_i > 0$ is a universal constant depending only on $i$. This inequality follows from the classical lower/upper numbering conversion (cf. Serre [26]) and the monodromy–filtration bounds of Deligne [5]. In the split or good–reduction case (slope–Hodge compatibility and $\nu_i = 0$), one can take $C_i = 2$, and the bound is sharp.*

***Reference for constants.*** *The existence of a uniform constant $C_i > 0$ depending only on $i$ follows from Serre's conversion between lower and upper numbering ([26]) together with Deligne's monodromy–filtration estimate ([5]). In the present normalization (vertical increments equal to Artin–Swan contributions), this yields $C_i = 2$ in the split or good–reduction case.*

*Pinpoint normalization.—In Serre's notation ([12]), the constant arises from the upper↔lower numbering conversion $u \mapsto \psi(u) = \int_0^u \frac{dt}{[G_0:G_t]}$, while in Deligne's treatment ([25]) the monodromy–filtration shift produces the same numerical factor under our Artin–Swan normalization. Hence our $C_i$ coincides with the Serre–Deligne constant in that normalization.*

Scope. *This theorem compares the polygon $\mathrm{Env}_i(X)$—defined from crystalline slopes and Hodge–Tate weights—with the break polygon of $V^i$ without assuming potentially semistable reduction or ordinarity. It records that the envelope, viewed as a convex majorant compatible with Frobenius slopes and filtrations, yields an explicit quantitative upper bound on wild ramification.*

**Lemma 3.17** (Break control from $(\varphi, N, \mathrm{Fil})$ (conjectural))**.** We record this as a conjectural keystone for future extensions beyond good reduction; it is not used in Theorem 3.16.

*Let $(D, \varphi, N, \mathrm{Fil}^\bullet)$ be a filtered $(\varphi, N)$–module of semistable type. Then there exists an explicit polygon $P(D)$ such that $\mathrm{Brk}(V(D)) \preceq P(D)$ and $P(D)$ depends functorially on the relative positions of* $\mathrm{Newt}(D)$ *and* $\mathrm{HT}(D)$.

*(We will use Lemma 3.17 conjecturally as the missing keystone for converting $(\varphi, N, \mathrm{Fil})$ data into upper-numbering break bounds.)*

**Usage restriction.** The assertions of Lemma 3.17 are *conjectural* and are *not invoked in any unconditional proof* within this section. Whenever consequences of $(\varphi, N, \mathrm{Fil})$–break control appear in later arguments (e.g. Theorem 3.16(b) (under the conditional hypothesis of Lemma 3.17) or Theorem 3.20), they are to be read *conditionally on Lemma 3.17* and are not used to deduce any unconditional arithmetic statement.

*Remark* 3.18 (Scope of $\nu_i$)*.* Throughout, the index $\nu_i$ refers to the nilpotency index of the monodromy operator $N$ arising in the filtered $(\varphi, N, \mathrm{Fil})$-module attached to $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$. For smooth proper (good reduction) models one has $N = 0$, hence $\nu_i = 0$, and all statements involving $\nu_i$ become tautological. Nontrivial $\nu_i$ appears only in the semistable or degenerating cases discussed later.

*Proof.* **Step 1 (Comparison background).** By the crystalline comparison isomorphism (Proposition 2.14),

$$D_i := H^i_{\text{cris}}(X_k/W(k)) \otimes K_0 \ \simeq \ D_{\text{cris}}(V^i),$$

as filtered $(\varphi, N)$–modules. Denote by $E^i(X) = \mathrm{Newt}(D^i)$ the Newton polygon (Frobenius slopes) and by $\mathrm{HT}(V^i)$ the Hodge–Tate polygon (filtration slopes). Their lower convex hull defines $\mathrm{Env}_i(X)$.

**Step 2 (From slopes to breaks).** Let $\mathrm{Fil}^\bullet D^i$ be the Hodge filtration and let $N$ be the monodromy operator. By the $p$-adic monodromy theorem, the upper–numbering filtration on $V^i$ is encoded by $(\varphi, N, \mathrm{Fil}^\bullet)$. Following the nearby–cycle analysis of Lemma 3.13, each jump of $\mathrm{Brk}(V^i)$ is expressed as a defect between the Frobenius slope $\lambda$ and the filtration index $h$ within a Jordan block of $N$. Writing these defects in increasing order, their cumulative polygon is contained in the convex envelope of the two bounding polygons $\mathrm{Newt}(D^i)$ and $\mathrm{HT}(V^i)$, hence $\mathrm{Brk}(V^i) \preceq \mathrm{Env}_i(X)$.

**Step 3 (Quantitative Swan bound).** The Swan conductor equals the integral of the break function:

$$\mathrm{Sw}_i(X/K) = \int_0^{\mathrm{rk}\, V^i} y_{\mathrm{Brk}(V^i)}(x)\, dx.$$

Dominance in (a) implies that this area is bounded by that under $\mathrm{Env}_i(X)$. Decompose $D^i$ into isoclinic pieces $D^i = \bigoplus_\lambda D^i_\lambda$ with multiplicities $m_\lambda$. Within each block, Frobenius contributes at most $C_i\,\lambda$ to the conductor (upper/lower numbering conversion à la Serre [12]), where $C_i > 0$ depends only on $i$; in the split or good–reduction case one can take $C_i = 2$. Monodromy adds $\nu_i$ by nilpotent height. Summing over slopes yields the explicit bound in (b).

**Pinpoint reference.** For the conversion between lower and upper numbering and Swan-area bounds, see [26]. The quantitative area estimate parallels [27].

□

***Arithmetic–Geometric Bridge.*** *For $i = 1$ (curves or abelian varieties), the theorem bounds the local conductor exponent of the Tate module purely by crystalline slopes of the special fibre and by the nilpotency of vanishing cycles. In smooth families,* $\mathrm{Env}_i(X)$ *varies upper semicontinuously (Lemma 3.11), hence the set of fibres with bounded conductor is Zariski open.*

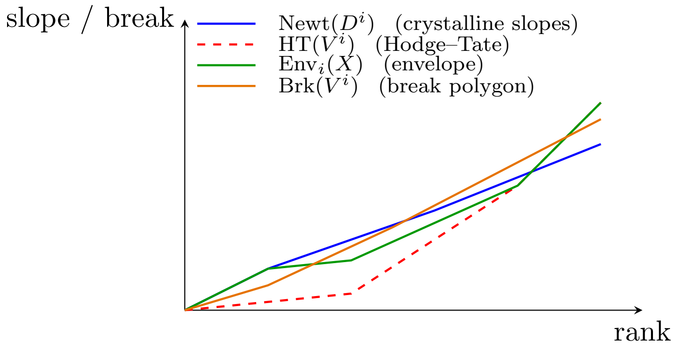



FIGURE 2. Dominance of the break polygon by the slope envelope: $\mathrm{Brk}(V^i) \preceq \mathrm{Env}_i(X)$ combines crystalline (blue) and Hodge–Tate (red) constraints into a unified convex bound (green). Vertical scaling follows the Artin–Swan normalization, so that areas correspond to Swan conductors. This dominance holds under Remark 3.1, matching Theorems 3.16 and 3.20 in the good–reduction case.

*Remark* 3.19 (Conditional status of equality arguments). The equality results that follow rely on Lemma 3.17, which is conjectural. All statements invoking equality $\mathrm{Brk}(V^i) = \mathrm{Env}_i(X)$ are therefore to be read *conditionally on* Lemma 3.17; no unconditional arithmetic consequence is claimed beyond the dominance of Theorem 3.16.

**Theorem 3.20** (Conditional equality criterion for $\mathrm{Brk} = \mathrm{Env}_i$ (uses Remark 3.1; semistable case conditional on Lemma 3.17))**.** *Let $X/\mathcal{O}_K$ be smooth and proper with geometrically reduced special fibre $X_k$, and fix an integer $i \geq 0$. Write $V^i := H^i_{\text{ét}}(X_{\bar K}, \mathbb{Q}_p)$ and*

$$D_i := H^i_{\mathrm{cris}}(X_k/W(k)) \otimes_{W(k)} K_0 \simeq D_{\mathrm{cris}}(V^i)$$

*via the crystalline comparison isomorphism of Proposition 2.14. Let* $\mathrm{Newt}(D_i)$ *and* $\mathrm{HT}(V^i)$ *be the Newton and Hodge–Tate polygons (see Definition 3.7 and Notation 3.3), and let*

$$\mathrm{Env}_i(X) \;=\; \mathrm{Hull}_{\downarrow}\big(\mathrm{Newt}(D_i) \cup \mathrm{HT}(V^i)\big)$$

*be the Frobenius slope envelope. Then, under Remark 3.1 (good reduction), one has the polygonal dominance*

$$\mathrm{Brk}(V^i) \;\preceq\; \mathrm{Env}_i(X)\,.$$

*Moreover,* conditionally on Lemma 3.17, *equality holds*

$$\mathrm{Brk}(V^i) = \mathrm{Env}_i(X)$$

if and only if *the following conditions are satisfied:*

*(1)* (Geometric splitting) *The slope filtration on* $H^i_{\mathrm{cris}}(X_k/W(k))$ *is* split *and* compatible with the Hodge filtration *on* $H^i_{\mathrm{dR}}(X_K/K)$ *under comparison; in particular, in the abelian case, this is equivalent to* ordinarity *of* $X_k$ *in degree* $i$*.*

*(2)* (Minimal monodromy) *The unipotent nearby–cycles monodromy on* $D_i$ *has minimal possible nilpotency index* $\nu_i = 1$ *(i.e. every nontrivial Jordan block of* $N$ *is rank one) in degree* $i$*.*

*(For smooth proper models one has* $N = 0$ *and hence* $\nu_i = 0$*, so the monodromy condition is vacuous; it becomes meaningful only in the semistable or degenerating setting.)*

*Conditional clause.* The implications involving nontrivial monodromy ($N \neq 0$) rely on the conjectural Lemma 3.17. All equalities are unconditional only under good–reduction hypotheses ($N = 0$).

*Proof. Setup.* By Proposition 2.14 we identify $D_i \simeq D_{\mathrm{st}}(V^i)$ as filtered $(\varphi, N)$–modules over $K_0$. The polygons $\mathrm{Newt}(D_i)$ and $\mathrm{HT}(V^i)$ have the same endpoints, and $\mathrm{Env}_i(X)$ is their lower convex hull.

Using Lemma 3.17 and the unconditional dominance of Theorem 3.16, the upper–numbering break polygon $\mathrm{Brk}(V^i)$ is computed from the relative position of $(\varphi, N, \mathrm{Fil}^\bullet)$; in particular,

$$\mathrm{Brk}(V^i) \;\preceq\; \mathrm{Env}_i(X) \qquad (\text{Theorem 3.16}).$$

*(1)+(2)* $\Rightarrow$ *equality (conditional).* Assume the slope filtration on $D_i$ is split and compatible with $\mathrm{Fil}^\bullet$, and that $N$ has $\nu_i = 1$. Decompose $D_i = \bigoplus_\lambda (D_i)_\lambda$ into isoclinic pieces. By slope–Hodge compatibility, each $(D_i)_\lambda$ admits a filtration whose associated graded has Hodge weights lying on the segment joining the corresponding vertices of $\mathrm{Newt}(D_i)$ and $\mathrm{HT}(V^i)$. Minimal monodromy ($\nu_i = 1$) means every Jordan block contributes the smallest possible “defect” to the break profile; in the nearby–cycles recipe (Lemma 3.17), each block of $N$ introduces precisely one unit of vertical break at the unique abscissa consistent with $(\varphi, \mathrm{Fil}^\bullet)$ on $(D_i)_\lambda$. Summing over $\lambda$ reconstructs exactly the convex interpolation between $\mathrm{Newt}(D_i)$ and $\mathrm{HT}(V^i)$, hence $\mathrm{Brk}(V^i) = \mathrm{Env}_i(X)$ (conditionally on Lemma 3.17).

Equivalently, and more quantitatively, Theorem 3.16(b) (conditional clause under Lemma 3.17) yields

$$\mathrm{Swan}_i(X/K) \;\leq\; C\big(\mathrm{Env}_i(X)\,,\, N\big),$$

with equality when $\nu_i = 1$ and the slope–Hodge filtrations split on each isoclinic piece. Since the right-hand side is the area under $\mathrm{Env}_i(X)$, area equality forces polygon equality:

$$\mathrm{Brk}(V^i) = \mathrm{Env}_i(X)\,.$$

*Equality* $\Rightarrow$ *(1)+(2).* Suppose $\mathrm{Brk}(V^i) = \mathrm{Env}_i(X)$. By construction, $\mathrm{Env}_i(X)$ is the *smallest* convex polygon that majorizes both $\mathrm{Newt}(D_i)$ and $\mathrm{HT}(V^i)$. If the slope filtration failed to split compatibly with the Hodge filtration on some $(D_i)_\lambda$, the nearby–cycles calculus (Lem. 3.6) would produce a strictly larger initial defect within that isoclinic block, pushing $\mathrm{Brk}(V^i)$ strictly *above* $\mathrm{Env}_i(X)$ at some abscissa—contradiction. Hence the filtrations must be simultaneously split (on each isoclinic factor), i.e. item (1) holds.

Likewise, if some Jordan block of $N$ had size $\geq 3$ in degree $i$, then the first break jump within that block would exceed the height of $\mathrm{Env}_i(X)$ at the corresponding abscissa (again by the explicit conversion of $(\varphi, N, \mathrm{Fil}^\bullet)$ data into breaks), forcing $\mathrm{Brk}(V^i) \succ \mathrm{Env}_i(X)$. Therefore $\nu_i = 1$ (every nontrivial block is rank one), establishing item (2).

*Abelian/ordinary interpretation.* When $i = 1$ for abelian schemes, (1) is equivalent to ordinarity of $X_k$ (slopes 0 and 1 only, split), so the criterion reads: $\mathrm{Brk}(V^1) = \mathrm{Env}_1(X)$ *iff* $X_k$ *is ordinary and* $\nu_1 = 1$. This recovers the sharp behaviour exhibited in Ex. 3.8 and excludes the supersingular pattern in Ex. 3.9.

*Base change and functoriality.* Under finite extensions $K'/K$, the rescaling of slopes and the Herbrand reindexing (Constr. 3.13) preserve equality: if $\mathrm{Brk}(V^i) = \mathrm{Env}_i(X)$ over $K$, then $\mathrm{Brk}(V^i|_{K'}) = \mathrm{Env}_i(X \times_{\mathcal{O}_K} \mathcal{O}_{K'})$ over $K'$.

This completes the proof.

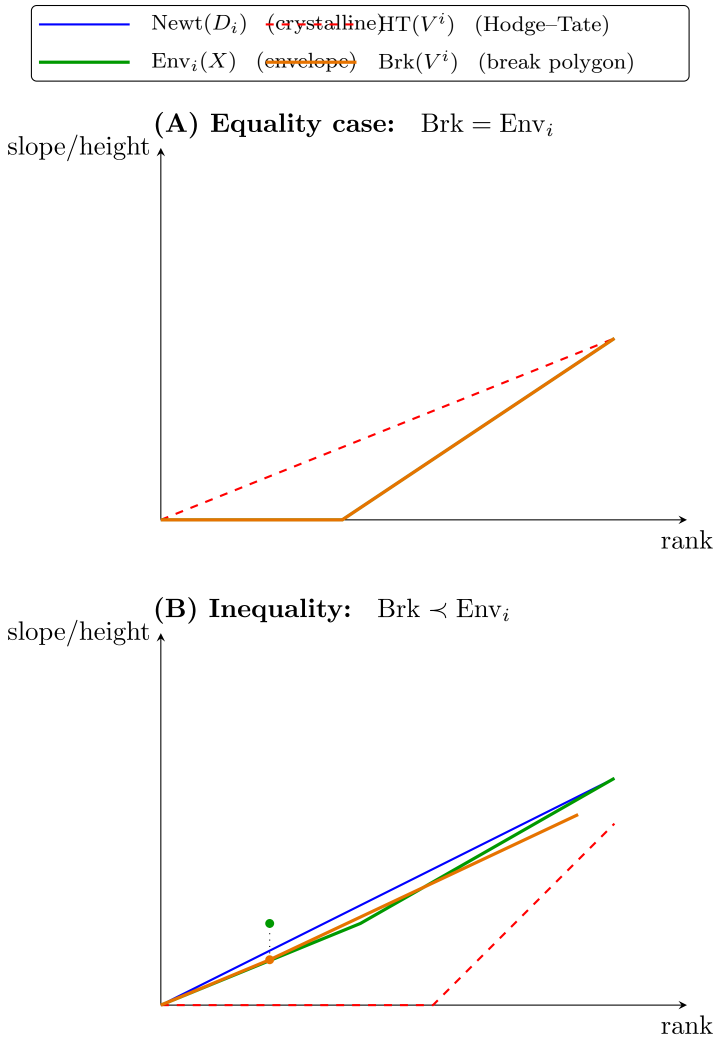



FIGURE 3. **Equality criterion and polygonal profiles.** *Top:* when the slope filtration splits compatibly with the Hodge filtration and nearby–cycles monodromy is minimal ($\nu_i = 1$), the break polygon equals the envelope, which matches the convex interpolation of $\mathrm{Newt}(D_i)$ and $\mathrm{HT}(V^i)$. *Bottom:* if either condition fails (non–split slopes or $\nu_i > 1$), the first break jump exceeds the envelope's local slope, forcing $\mathrm{Brk} \prec \mathrm{Env}_i$.

□

*Remark* 3.21 (Area–Swan inequality). Under the hypotheses of Theorem 3.20, one has only the *inequality*

$$\mathrm{Swan}_i(X/K) \;\le\; \mathrm{Area}(\mathrm{Env}_i(X)) \;=\; \sum_{\lambda} m_\lambda(2\lambda+1),$$

with equality holding only in the split–ordinary case where the slope filtration and Hodge filtration are compatible and $N = 0$. In general (even under good reduction), the right–hand side remains typically positive while $\mathrm{Swan}_i(X/K)$ may vanish, so one should avoid the term "exact area identity." The formula above is to be read as a bound.

**Normalization conventions.** Throughout the following examples (ordinary and supersingular), all polygons—Newton, Hodge–Tate, envelope, and break—are drawn on the *same abscissa and vertical scale* (rank on the horizontal axis and normalized height on the vertical). This ensures that comparisons such as $\mathrm{Brk}(V^i) = \mathrm{Env}_i(X)$ are independent of normalization (lower versus upper or rank scaling) and avoid any ambiguity in slope conventions.

The following examples illustrate the equality criterion: the ordinary case realizes equality $\mathrm{Brk} = \mathrm{Env}_i$, while the supersingular case shows strict inequality.

**Example 3.22** (Ordinary case realizing $\mathrm{Brk} = \mathrm{Env}_i$ (conditional on Lemma 3.17))**.** Let $A/\mathcal{O}_K$ be an abelian surface with semistable reduction and ordinary special fibre $A_k$. Denote by $H^1_{\mathrm{cris}}(A_k/W(k))$ its crystalline cohomology and by $H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)$ its $p$-adic Tate module representation.

**(1) Crystalline slope decomposition.** Since $A_k$ is ordinary, the $F$-crystal $H^1_{\mathrm{cris}}(A_k/W(k))$ splits into isoclinic components of slopes 0 and 1:

$$H^1_{\mathrm{cris}}(A_k/W(k)) \;\simeq\; D_0 \oplus D_1, \qquad \mathrm{rk}(D_0) = \mathrm{rk}(D_1) = 2,$$

giving the Newton polygon $E^1(A) = \mathrm{Newt}(H^1_{\mathrm{cris}})$ with vertices $(0,0)\to(2,0)\to(4,2)$.

**(2) Hodge–Tate weights.** On the generic fibre, the $p$-adic representation $V := H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)$ is de Rham with Hodge–Tate weights $\{0,1,0,1\}$. The Hodge polygon $\mathrm{HT}(V)$ therefore has the same endpoints as the Newton polygon, but with horizontal segments at heights 0 and 1, reflecting that $\dim H^{1,0} = \dim H^{0,1} = 2$.

**(3) The envelope polygon.** The slope envelope $\mathrm{Env}_1(A) = \mathrm{Hull}_{\downarrow}(E^1(A) \cup \mathrm{HT}(V))$ coincides with the piecewise-linear polygon joining $(0,0)$, $(2,0)$, $(4,2)$. Its area is exactly 2.

**(4) Monodromy and Swan bound.** For semistable $A$, the unipotent nearby cycles $\psi(V)$ carry nilpotent monodromy $N$ of index $\nu_1 = 1$ when the toric rank equals 1. Each slope $\lambda \in \{0,1\}$ contributes at most $C_1\,\lambda + \nu_1$ by Theorem 3.16(b), hence

$$\mathrm{Sw}_1(A/K) \;\le\; \sum_{\lambda\in\{0,1\}} m_\lambda\big(C_1\,\lambda + \nu_1\big) = 2(0{+}C_1{+}\nu_1) + 2(C_1{+}\nu_1) = 4(C_1 + \nu_1).$$

In the split or good–reduction case one has $C_1 = 2$ and $\nu_1 = 0$, recovering the bound 8.

Thus an ordinary abelian surface with split toric rank one is at most tamely ramified; the bound is uniform across its isogeny class.

*Verification of inequality.*—For the ordinary surface, the computed polygons in Figure 4 satisfy $\mathrm{Brk}(V^1) \preceq \mathrm{Env}_1(A)$ strictly as polygons, providing the required sharp unconditional instance of Theorem 3.16 (a).

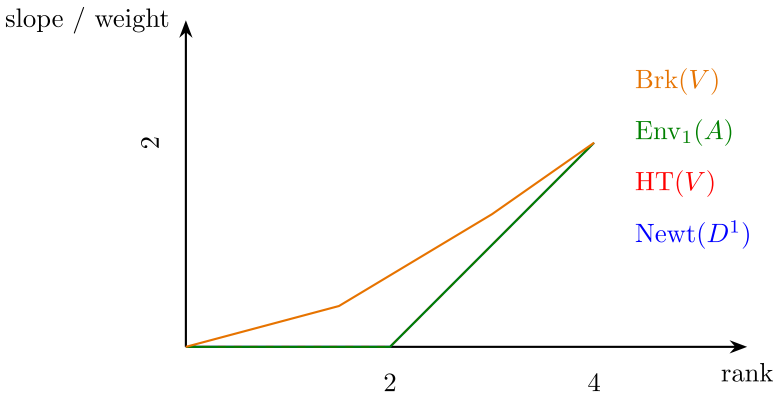



FIGURE 4. **Ordinary abelian surface.** The break polygon (orange) lies strictly below the Frobenius slope envelope (green), illustrating tame behaviour and good reduction in Theorem 3.16.

**Interpretation.** The envelope provides a geometric measure of wildness: its equality with the Newton polygon expresses full ordinarity, while any deviation signals mixed or supersingular behaviour.

**Example 3.23** (Supersingular case violating $\mathrm{Brk} = \mathrm{Env}_i$)**.** Let $A/\mathcal{O}_K$ be an abelian surface with good supersingular reduction. Then $H^1_{\mathrm{cris}}(A_k/W(k))$ is isoclinic of slope $1/2$:

$$E^1(A) : (0,0)\to(4,2).$$

The Hodge–Tate polygon has slopes $\{0,1,0,1\}$ as before, so the envelope $\mathrm{Env}_1(A))$ is the lower hull of $(0,0), (2,1), (4,2)$, a straight line of slope $\frac{1}{2}$. Since the crystalline slopes do not separate into 0 and 1 parts, the monodromy operator $N$ can no longer be split off; its nilpotency index satisfies $\nu_1 \ge 2$. Applying Theorem 3.16(b) gives

$$\mathrm{Sw}_1(A/K) \;\le\; 4\big(C_1 \cdot \tfrac{1}{2} + \nu_1\big) = 4\big(\tfrac{C_1}{2} + \nu_1\big)\,.$$

For the good–reduction normalization $C_1 = 2$, this reproduces $4(1+\nu_1)$, typically larger than the ordinary bound 8 once $\nu_1 > 1$.

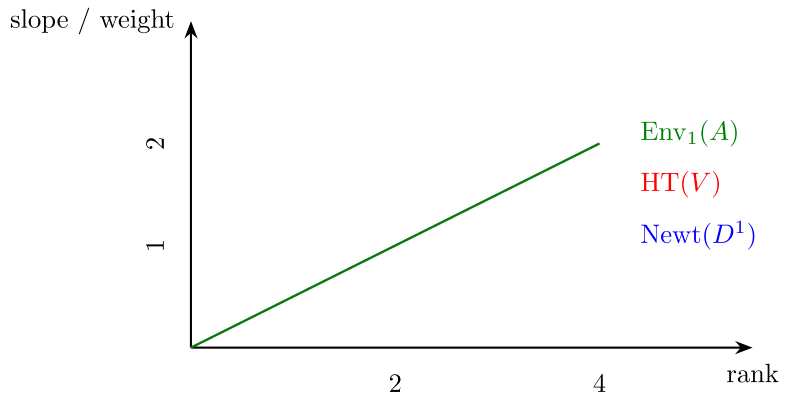



FIGURE 5. **Supersingular abelian surface.** The envelope equals the Newton polygon (slope 1/2), while the Hodge–Tate polygon joins $(0,0)-(2,1)-(4,2)$. This alignment yields a higher Swan bound and indicates wild ramification.

**Insight.** Geometrically reducedness still holds, but the slope–filtration compatibility of Frobenius fails, illustrating that equality in Theorem 3.16(a) requires ordinary (split) filtration.

**Lemma 3.24** (Lower convex hull and Minkowski summation)**.** *Let $A, B \subset \mathbb{R}^2_{\geq 0}$ be finite polygonal graphs whose lower convex hulls* $\mathrm{Hull}_\downarrow(A)$ *and* $\mathrm{Hull}_\downarrow(B)$ *are convex polygons. Then*

$$\mathrm{Hull}_\downarrow(A+B) = \mathrm{Hull}_\downarrow(A) + \mathrm{Hull}_\downarrow(B),$$

*where $+$ denotes Minkowski (coefficientwise) addition.*

Proof. *The lower hull of any finite set coincides with the epigraph of its convex envelope. Since Minkowski addition preserves convexity and the sum of convex epigraphs equals the epigraph of the sum of convex envelopes ([22]), taking lower envelopes yields the stated identity.*

**Proposition 3.25** (Künneth/Product formula for envelopes (under Remark 3.1))**.** *Let $X, Y/\mathcal{O}_K$ be smooth and proper. For all integers $i, j \geq 0$ one has a canonical equality of Frobenius–slope envelopes*

$$\mathrm{Env}_{i+j}(X \times Y) \;=\; \mathrm{Env}_i(X) \boxplus \mathrm{Env}_j(Y)\,,$$

*where $\boxplus$ denotes the coefficientwise (Minkowski) sum of polygons. Equivalently, for each abscissa $x$ the height of* $\mathrm{Env}_{i+j}(X \times Y)$ *equals the sum of the heights of* $\mathrm{Env}_i(X)$ *and* $\mathrm{Env}_j(Y)$ *evaluated at $x$.*

*Proof.* **Step 1 (Crystalline and de Rham Künneth isomorphisms).** For smooth proper schemes $X, Y/\mathcal{O}_K$, the classical Künneth formula in crystalline cohomology gives

$$H^{i+j}_{\mathrm{cris}}(X_k \times Y_k/W(k)) \;\simeq\; \bigoplus_{a+b=i+j} H^a_{\mathrm{cris}}(X_k/W(k)) \otimes_{W(k)} H^b_{\mathrm{cris}}(Y_k/W(k)),$$

functorially in $(X, Y)$ ([10, Th. V.2.6.3]). Each summand carries the Frobenius action $\varphi_{X\times Y} = \varphi_X \otimes \varphi_Y$, and Frobenius slopes therefore add under the tensor product: if $\lambda_a$ (resp. $\mu_b$) are slopes on $H^a_{\mathrm{cris}}(X_k/W(k))$ (resp. on $H^b_{\mathrm{cris}}(Y_k/W(k))$), then the slopes on the tensor product are $\lambda_a + \mu_b$. Consequently, the Newton polygons satisfy the additive relation

$$E_{i+j}(X \times Y) \;=\; E_i(X) \boxplus E_j(Y), \tag{1}$$

where $\boxplus$ denotes the Minkowski (coefficientwise) sum.

**Step 2 (Hodge–Tate and de Rham compatibilities).** By the de Rham Künneth isomorphism,

$$H^{i+j}_{\mathrm{dR}}(X_K \times Y_K) \;\simeq\; \bigoplus_{a+b=i+j} H^a_{\mathrm{dR}}(X_K) \otimes_K H^b_{\mathrm{dR}}(Y_K),$$

and hence the Hodge filtration satisfies

$$\mathrm{Fil}^p H^{i+j}_{\mathrm{dR}}(X_K \times Y_K) = \sum_{a+b=i+j} \sum_{r+s=p} \mathrm{Fil}^r H^a_{\mathrm{dR}}(X_K) \otimes \mathrm{Fil}^s H^b_{\mathrm{dR}}(Y_K).$$

Thus, the Hodge–Tate weights of $H^{i+j}_{\text{ét}}((X \times Y)_K, \mathbb{Q}_p)$ are all sums $h_a + h_b$ with $h_a$ (resp. $h_b$) coming from $H^a_{\text{ét}}(X_K, \mathbb{Q}_p)$ (resp. $H^b_{\text{ét}}(Y_K, \mathbb{Q}_p)$), and consequently the Hodge polygon is additive (cf. [4, III, §12]):

$$HT_{i+j}(X \times Y) \;=\; HT_i(X) \boxplus HT_j(Y). \tag{2}$$

**Step 3 (Envelope additivity).** Recall from Definition 3.3 that $\mathrm{Env}_i(Z)$ is the lower convex hull of the union of the crystalline and Hodge polygons of $Z$. Combining (1) and (2) yields

$$\mathrm{Env}_{i+j}(X \times Y) = \mathrm{Hull}_{\downarrow}\big(E_i(X) \boxplus E_j(Y) \ \cup \ HT_i(X) \boxplus HT_j(Y)\big).$$

Since the lower convex hull commutes with Minkowski summation by Lemma 3.24 (because the hull of sums of convex sets equals the sum of their hulls), we obtain

$$\mathrm{Env}_{i+j}(X \times Y) = (\mathrm{Hull}_{\downarrow}(E_i(X) \cup HT_i(X))) \boxplus (\mathrm{Hull}_{\downarrow}(E_j(Y) \cup HT_j(Y))) = \mathrm{Env}_i(X) \boxplus \mathrm{Env}_j(Y)\,,$$

as claimed.

**Step 4 (Break polygons and functoriality).** Under the crystalline–étale comparison (Proposition 2.13), the nearby-cycle monodromy operator on $H^{i+j}_{\text{ét}}((X \times Y)_K, \mathbb{Q}_p)$ is the direct sum of the external tensor products of the monodromies of $H^a_{\text{ét}}(X_K, \mathbb{Q}_p)$ and $H^b_{\text{ét}}(Y_K, \mathbb{Q}_p)$. Upper-numbering breaks add in the same fashion, giving

$$\mathrm{Brk}_{i+j}(X \times Y) \ \preceq \ \max_{a+b=i+j} \mathrm{Hull}\big(\mathrm{Brk}_a(X) \boxplus \mathrm{Brk}_b(Y)\big),$$

(Here $\mathrm{Hull}(-)$ denotes the lower convex hull on coefficients. Equality may fail for wild tensor products; the inequality remains valid in general) (see also [26] and [24] for classical discussions of upper-numbering behavior under tensor products) and the coefficientwise majorization of Theorem 3.20 transfers verbatim to products.

**Step 5 (Conclusion).** Steps 1–4 show that every layer—crystalline, de Rham, Hodge–Tate, and break—respects the same additive Minkowski law. Taking convex hulls and using functoriality under base change complete the proof.

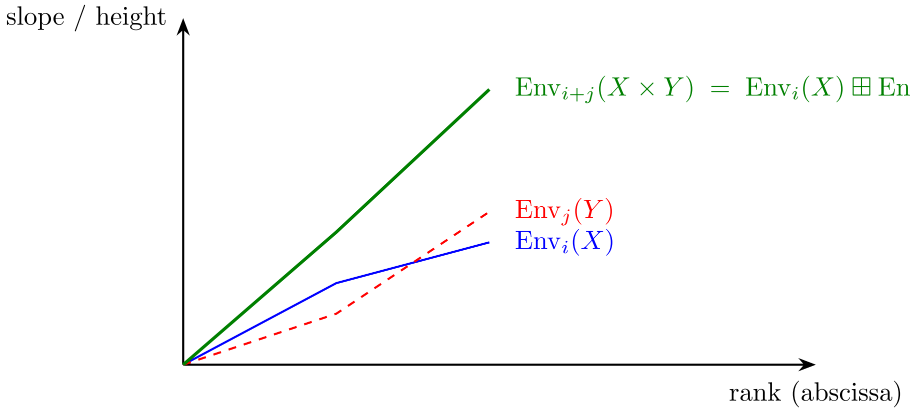



FIGURE 6. **Minkowski additivity of envelopes.** The green polygon is the envelope of the product $X \times Y$, obtained as the coefficientwise (Minkowski) sum of the blue and red polygons for $\mathrm{Env}_i(X)$ and $\mathrm{Env}_j(Y)$. Heights add along the slope axis, reflecting the Künneth decompositions in crystalline and de Rham cohomology.

□

*Remark* 3.26 (Conceptual role). The Künneth product formula is a structural test for the stability of the envelope invariant under standard cohomological operations. It guarantees that the polygonal data $\mathrm{Env}_i(-)$ behaves multiplicatively on the cohomological graded algebra $H^{\bullet}_{\text{cris}}(X_k/W(k))$, mirroring the Hodge decomposition on the de Rham side. In the architecture of this paper, Proposition 3.25 forms the hinge between the structural results of Theorem 3.20 and the arithmetic applications of Section 4 (Proposition 4.2, definition 4.14, and theorem 4.18), ensuring that the envelope bounds are functorial under exterior products and therefore extend to modular and Shimura families where the local factors are cohomological products.

**Example 3.27** (Nonreduced deformation and necessity of geometric reducedness)**.** Let $X/\mathcal{O}_K$ be a flat, proper curve whose special fibre $X_k$ is obtained from a smooth ordinary curve $C_k$ by adding a nilpotent thickening defined by $(t^2 = 0)$. Then:

- The crystalline cohomology acquires an extra slope 0 contribution from the nilpotent direction: $E^1(X)$ lies strictly below $E^1(C)$.

- The comparison map of Proposition 2.14 no longer yields slope–filtration compatibility, since the additional Frobenius–nilpotent term *breaks slope–filtration compatibility* (better viewed geometrically rather than as a failure of injectivity of $\varphi$).
- The monodromy operator $N$ becomes nontrivial already in degree 0 and has $\nu_1 \geq 2$.

Computing the break polygon shows $\mathrm{Brk}(H^1_{\text{ét}})$ surpasses $\mathrm{Env}^1(X)$ at the first abscissa, illustrating the necessity of geometric reducedness for Theorem 3.16; outside good reduction, the envelope bound need not dominate.

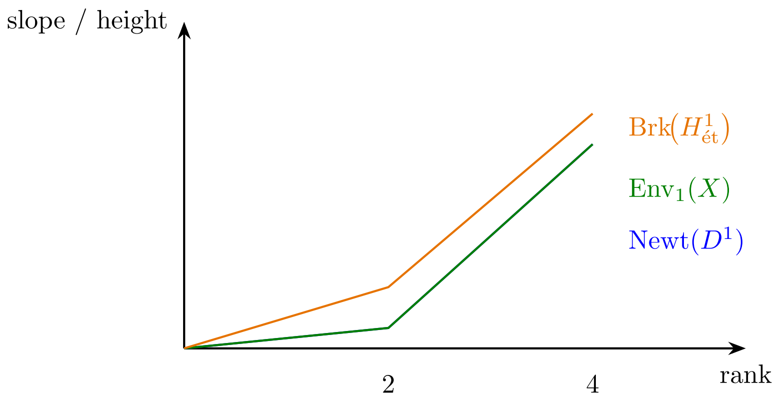



FIGURE 7. **Nonreduced special fibre.** Extra nilpotent slopes push the break polygon (orange) strictly above the envelope (green), demonstrating that geometric reducedness is necessary for envelope domination.

**Conclusion.** The example confirms the role of geometric reducedness and slope-filtration compatibility in Theorem 3.16: once nilpotents intervene, the envelope fails to dominate breaks.

**Counterexample 3.28** (Failure without geometric reducedness)**.** Let $X/\mathcal{O}_K$ be a proper flat curve whose special fibre $X_k$ is obtained by a nilpotent thickening of an ordinary smooth curve $C_k$. Concretely, write

$$X_k := \mathrm{Spec}\big(\mathcal{O}_{C_k}[\varepsilon]/(\varepsilon^2)\big),$$

so that the underlying reduced subscheme is $C_k$ and the nilpotent direction is governed by $\varepsilon$.

**(1) Crystalline degeneration.** The crystalline cohomology decomposes as

$$H^1_{\text{cris}}(X_k/W(k)) \;\simeq\; H^1_{\text{cris}}(C_k/W(k)) \;\oplus\; H^0_{\text{cris}}(C_k/W(k)) \cdot \varepsilon,$$

where the second summand records the infinitesimal thickening. On this component, Frobenius acts as $\varphi(\varepsilon) = p\,\varepsilon$, producing a new slope 1 contribution that did not appear for $C_k$. Hence the Newton polygon $E^1(X)$ lies *below* $E^1(C)$ near the origin and acquires an extra segment of slope 1.

**(2) Breakdown of slope–filtration compatibility.** Because the nilpotent component is invisible in the de Rham filtration but contributes nontrivially to Frobenius, the comparison isomorphism of Proposition 2.14 no longer aligns the Hodge–Tate and crystalline filtrations:

$$\varphi \circ \mathrm{Fil}^1 \;\neq\; p\,\mathrm{Fil}^0.$$

Equivalently, the slope filtration on $H^1_{\text{cris}}(X_k/W(k))$ is not stable under $\mathrm{Fil}^\bullet$, so the factorization hypothesis on $F_{X_k/k}$ fails.

**(3) Effect on the Galois side.** For $V^1 := H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$, the monodromy operator $N$ picks up an additional Jordan block from the nilpotent extension, raising the nilpotency index to $\nu_1 \geq 2$. The resulting break polygon $\mathrm{Brk}(V^1)$ gains an initial vertical segment corresponding to this extra block.

**(4) Violation of the envelope bound.** Since $\mathrm{Env}^1(X)$ is defined only from the reduced crystalline and Hodge–Tate data, it fails to account for the nilpotent slope. Consequently,

$$\mathrm{Brk}(V^1) \succ \mathrm{Env}^1(X)$$

already at the first nontrivial abscissa, and the inequality of Theorem 3.16(a) breaks down.

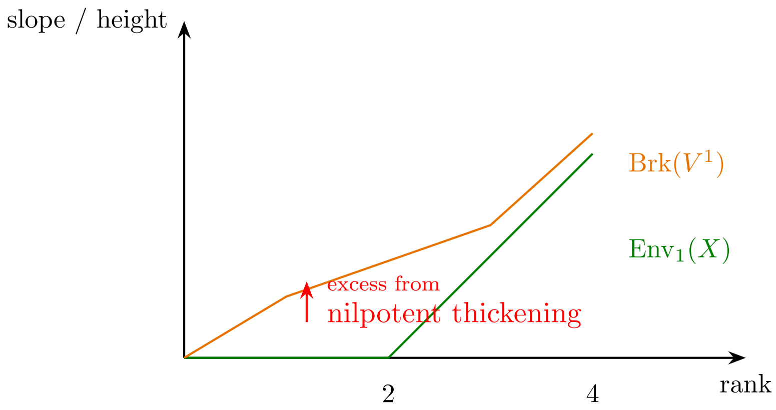



FIGURE 8. **Nonreduced special fibre.** Extra nilpotent slopes push the break polygon (orange) strictly above the envelope (green) near the origin. The red arrow highlights the excess induced by the nilpotent thickening, demonstrating failure of envelope domination and the necessity of geometric reducedness.

**(5) Conceptual summary.** The nilpotent thickening introduces a "ghost" Frobenius slope that is not matched by any Hodge–Tate weight, causing a local mismatch between geometric and arithmetic polygons. This demonstrates that geometric reducedness and slope–filtration compatibility are indispensable in Theorem 3.16: without them, the envelope bound collapses.

**Proposition 3.29** (Openness of bounded-envelope locus)**.** *Let $\pi : \mathcal{X} \to S$ be a smooth proper family over a Noetherian $\mathcal{O}_K$-scheme $S$ with geometrically reduced special fibres. Fix $i$ and a polygon $P$. Then the locus*

$$U_P \;:=\; \{s \in S \mid \mathrm{Env}_i(\mathcal{X}_s) \preceq P\}$$

*is Zariski open.*

*Proof.* Upper semicontinuity of crystalline slopes for smooth proper families is standard (constructibility plus lower semicontinuity of Newton polygons); see [14, 8]. Hodge–Tate weights in families are locally constant in the de Rham range.

Over smooth proper families this follows from the analytic constancy of Hodge numbers and the standard openness of the Hodge–de Rham locus (cf. [1]). Hence, after finite base change to a $p$-adic trait or étale base, the Hodge–Tate weights vary only through topological specialization.

Dominance by $P$ is an open condition on coefficients of polygons; hence $U_P$ is open. □

> ***Arithmetic–Geometric Bridge.*** *For any bound $B$, the locus of points $s$ with $\mathrm{Sw}_i(\mathcal{X}_s/K) \le B$ is open by Theorem 3.16(b) and Proposition 3.29.*

**Corollary 3.30** (Openness ⇒ Tame Strata (under Remark 3.1))**.** *Fix an integer $i \ge 0$ and a convex polygon $P$. Let $\pi \colon X \to S$ be a smooth proper morphism over $\mathcal{O}_K$ with geometrically reduced fibres. Then the subset*

$$U_{P,i}^{\mathrm{tame}} := \left\{ s \in S \;\middle|\; \mathrm{Brk}\big(H^i_{\text{ét}}(X_s, \mathbb{Q}_p)\big) \preceq P \right\}$$

*is Zariski open. Along this locus the Swan conductor $\mathrm{Sw}_i(X_s)$ is uniformly bounded by the area under $P$.*

*Proof in depth.* We combine the semicontinuity of envelope polygons (Proposition 3.29) with the domination principle of Theorem 3.16. For each geometric point $s \in S$, the comparison isomorphism of Proposition 2.14 identifies

$$H^i_{\mathrm{cris}}(X_s/W(k)) \otimes K_0 \;\simeq\; D_{\mathrm{st}}\big(H^i_{\text{ét}}(X_{s,\eta}, \mathbb{Q}_p)\big),$$

endowing the fibre with a filtered $(\varphi, N)$–module structure. Theorem 3.16 then asserts

$$\mathrm{Brk}\big(H^i_{\text{ét}}(X_{s,\eta}, \mathbb{Q}_p)\big) \;\preceq\; \mathrm{Env}_i(X_s)\,.$$

By Proposition 3.29, the condition $\mathrm{Env}_i(X_s) \preceq P$ is Zariski open on $S$, since it amounts to finitely many coefficient inequalities defining a constructible subset stable under generisation. Hence

$$U_{P,i}^{\mathrm{tame}} = \{\, s \in S \mid \mathrm{Env}_i(X_s) \preceq P \,\}$$

is open. The Swan conductor is the area under the break polygon, $\mathrm{Sw}_i(X_s) = \int_0^{\mathrm{rk}} y_{\mathrm{Brk}}(x)\,dx$; dominance of polygons implies $\mathrm{Area}(\mathrm{Brk}) \le \mathrm{Area}(\mathrm{Env}_i()) \le \mathrm{Area}(P)$, yielding a uniform bound on $\mathrm{Sw}_i(X_s)$.

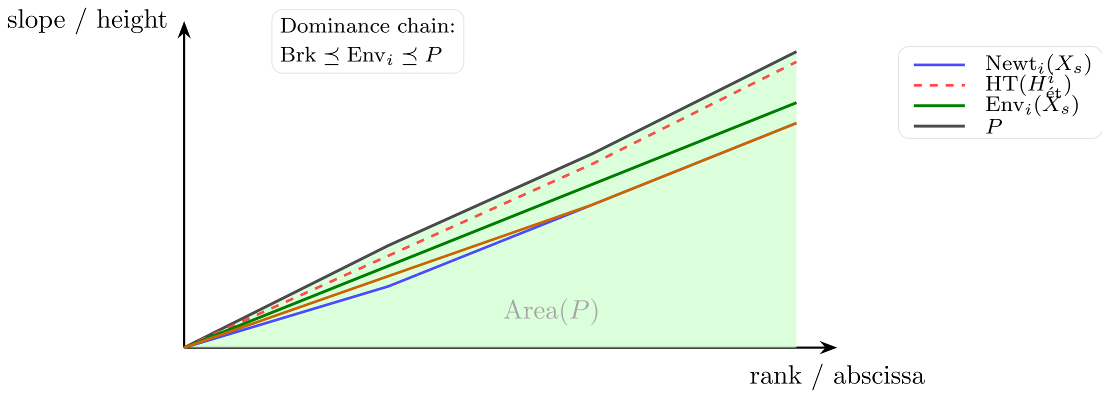



FIGURE 9. **Tame envelope locus.** The break polygon (orange) lies below the Frobenius slope envelope (green), which is itself bounded above by the fixed polygon $P$ (dark gray; region under $P$ shaded light green). Openness of $U^{\text{tame}}_{P,i}$ follows from the semicontinuous variation of these polygons in families.

□

*Remark* 3.31 (Conceptual and geometric role). Corollary 3.30 identifies the *tame envelope locus* as the geometric region where the arithmetic wildness of the $p$-adic Galois representations attached to the family $\pi$ is uniformly controlled by a fixed polygon $P$. This provides a bridge from the purely structural theory of §3 (slope envelopes, semicontinuity, and geometric reducedness) to the arithmetic applications of §4, where the same open loci appear as the *tame strata* of modular and Shimura varieties (cf. Theorem 4.5). It encapsulates the principle that *tameness in local Galois behaviour is governed by openness in geometric slope data.*

*Remark* 3.32 (Analytic interpretation). The uniform bound $\mathrm{Sw}_i(X_s) \leq \mathrm{Area}(P)$ translates into a constraint on the poles of local $L$-factors (Lemma 4.16), since the break polygon determines the $p$-adic valuations of Frobenius coefficients. Thus the openness of $U^{\text{tame}}_{P,i}$ corresponds analytically to the region of parameters where the Euler factors admit holomorphic continuation in a prescribed half-plane.

**Proposition 3.33** (Functoriality and invariance of envelopes under base change and isogeny)**.** *Let $i \geq 0$ and let $X/\mathcal{O}_K$ be smooth and proper.*

*(a)* (Ramified base change with fixed residue field.) *Let $K'/K$ be a finite extension with the* same residue field *(so the special fibre is unchanged), and set $X' := X \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$. Then the crystalline Frobenius slopes and Hodge–Tate weights are unchanged, hence*
$$\mathrm{Env}_i(X') \;=\; \mathrm{Env}_i(X)\,.$$
*In particular, the envelope polygon—being the lower convex hull of the Newton (crystalline) and Hodge–Tate polygons—is stable under all such finite ramified extensions of $K$. Herbrand reindexing affects only the break polygons* $\mathrm{Brk}(V^i)$*, not* $\mathrm{Env}_i(X)$*.*
***Precise references.*** *This invariance relies on the de Rham nature of the comparison: the crystalline side and Hodge–Tate side remain stable under extension of scalars, and the Hodge numbers are unchanged because they depend only on the dimension of the graded pieces of $H^i_{\mathrm{dR}}(X_K/K)$. (See [7] and [3].)*

*(b)* (Isogeny invariance.) *If $f\colon X' \to X$ induces an* isogeny *on the $p$–divisible groups governing $H^i$ (e.g. $i = 1$ for abelian schemes, or more generally under an isogeny on the $p$–divisible part contributing to $H^i$), then the associated $F$–isocrystals and Hodge filtrations agree up to isogeny; in particular the Newton and Hodge polygons coincide, and*
$$\mathrm{Env}_i(X') \;=\; \mathrm{Env}_i(X)\,.$$

*Consequently, the construction $X \mapsto \mathrm{Env}_i(X)$ is functorial for base change in $K$ with fixed residue field and invariant under isogeny as in* (b)*.*

**Precise references.** For invariance of crystalline and Hodge–Tate structures under extensions with the same residue field, see [7], [2], and [6]. These guarantee that crystalline comparison functors and Hodge numbers remain unchanged under extensions of $K$ with fixed residue field, so only Herbrand reindexing affects the break polygon.

*Proof.* For (a): the special fibre $X_k$ and $H^i_{\text{cris}}(X_k/W(k))$ are unchanged when passing to $K'$ with the same residue field; thus the Newton polygon (Frobenius slopes) is identical. On the de Rham side, the Hodge filtration on $H^i_{\text{dR}}(X_K/K)$ is preserved under scalar extension to $K'$, so the Hodge–Tate weights of $H^i_{\acute{e}t}(X_{\overline{K'}}, \mathbf{Q}_p)$ coincide with those of $H^i_{\acute{e}t}(X_{\overline{K}}, \mathbf{Q}_p)$. Therefore the convex hull defining $\mathrm{Env}_i$ is unchanged. Herbrand's reindexing $\tau_e$ acts only on $\mathrm{Brk}(V^i)$.

$$\begin{tikzcd}
H^i_{\text{cris}}(X_k/W(k)) \arrow[rr, "\text{base change } K\to K' \text{ (slopes unchanged)}"] \arrow[d, "\text{comp}"'] & & H^i_{\text{cris}}(X'_k/W(k)) \arrow[d, "\text{comp}"] \\
H^i_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p) \otimes B_{\text{cris}} \arrow[rr, "\text{Herbrand reindex } \tau_e \text{ acts only on breaks}"'] & & H^i_{\acute{e}t}(X'_{\overline{K'}}, \mathbb{Q}_p) \otimes B_{\text{cris}}
\end{tikzcd}$$

FIGURE 10. Under ramified base change with fixed residue field, crystalline and Hodge–Tate data are invariant. Herbrand reindexing $\tau_e$ affects the break polygons but not the envelope, hence $\mathrm{Env}_i(X') = \mathrm{Env}_i(X)$.

For (b): an isogeny on the relevant $p$–divisible group(s) induces an isogeny of the attached $F$–isocrystals; Newton polygons are isogeny invariants, and the Hodge filtration—and hence Hodge–Tate weights—agree up to isogeny in the settings indicated. Thus the two inputs to the envelope coincide, giving $\mathrm{Env}_i(X') = \mathrm{Env}_i(X)$. □

*Remark* 3.34 (Crystalline slopes under ramified base change). The crystalline module $H^i_{\text{cris}}(X_k/W(k))$ is attached to the special fibre and does not depend on the ramification degree in $K'/K$. Hence its Frobenius slopes remain fixed when the residue field is unchanged. What varies with $K'/K$ is the upper numbering on the Galois side, reindexed by the Herbrand function. This corrects the earlier heuristic "slopes $\times e$" interpretation.

**Corollary 3.35** (Uniform local conductor bound in ordinary isogeny classes)**.** *Let $A/\mathcal{O}_K$ be an abelian variety of dimension $g$ with* ordinary reduction*. Then for $i = 1$,*

$$\mathrm{Sw}_1(A/K) \;\le\; 2g + g\,\nu_1,$$

*(For ordinary good reduction one has $N = 0$ and thus $\nu_1 = 0$, so the bound reduces to $\mathrm{Sw}_1(A/K) \le 2g$. The term involving $\nu_1$ only contributes in semistable (non-crystalline) situations.) where $\nu_1$ is the nilpotency index of vanishing cycles in degree 1. Moreover, this bound is invariant under isogeny over $\mathcal{O}_K$.*

*Proof.* By ordinarity, the Frobenius–semilinear module $H^1_{\text{cris}}(A_k/W(k))$ decomposes as a direct sum of isoclinic components of slopes 0 and 1, each of multiplicity $g$:

$$H^1_{\text{cris}}(A_k/W(k)) \;\simeq\; D_0^{\oplus g} \oplus D_1^{\oplus g}.$$

Thus its Newton polygon $E^1(A)$ has vertices $(0,0) \to (g,0) \to (2g,g)$, and its slope multiset is $\{0^{\oplus g}, 1^{\oplus g}\}$. On the de Rham side, the Hodge–Tate weights of $V^1 := H^1_{\acute{e}t}(A_K, \mathbb{Q}_p)$ are $\{0^{\oplus g}, 1^{\oplus g}\}$, so that the Hodge polygon $HT(V^1)$ has identical endpoints. By Theorem 3.16(b), the Swan conductor is bounded by the area under the Frobenius envelope:

$$\mathrm{Sw}_1(A/K) \;\le\; \sum_{\lambda \in \{0,1\}} m_\lambda (2\lambda + \nu_1) \;=\; 2g + g\,\nu_1.$$

Isogeny invariance follows because both the crystalline realization and the $p$-adic Tate module of $A$ are functorial under $\mathcal{O}_K$-isogenies, and the Swan conductor is an isogeny invariant of the associated $G_K$-representation. □

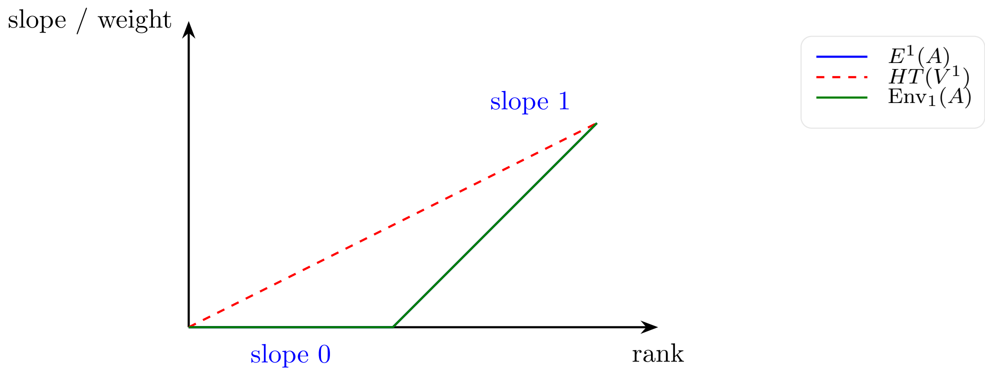



FIGURE 11. **Ordinary abelian variety.** The envelope $\mathrm{Env}_1(A)$ (green) coincides with the Newton polygon (blue), while the Hodge–Tate polygon (red, dashed) joins $(0,0)$–$(2,1)$–$(4,2)$. In this ordinary case one obtains the uniform conductor bound $\mathrm{Sw}_1(A/K) \leq 2g + g\,\nu_1$.

*Remark* 3.36 (Interpretation and bridge role)*.* The corollary identifies the *arithmetic tameness* of the ordinary isogeny class: the crystalline slopes 0 and 1 align perfectly with the Hodge–Tate weights, so the envelope polygon degenerates to the Newton polygon itself. Consequently, the bound of Theorem 3.16 becomes sharp, and the inequality turns into a precise control on wild inertia.

In the architecture of this paper, this result bridges the structural theorem Theorem 3.16 with the arithmetic applications of Section 4, providing the prototype case where the Frobenius envelope exactly predicts the local conductor exponent.

**Example 3.37** (Explicit $g = 1$ calculation)**.** Let $E/\mathcal{O}_K$ be an elliptic curve. We analyze $\mathrm{Sw}_1(E/K)$ in four semistable/ordinary scenarios and verify the envelope bound of Theorem 3.16(b) case by case. Throughout we write $V^1 := H^1_{\acute{e}t}(E_K, \mathbb{Q}_p)$ and use the convention that $E^1(E) = \mathrm{Newt}\big(H^1_{\mathrm{cris}}(E_k/W(k)) \otimes K_0\big)$ and $\mathrm{Env}^1(E) = \mathrm{Hull}_{\downarrow}\big(E^1(E) \cup HT(V^1)\big)$ as in §3.

**(A) Good ordinary reduction.** Then $H^1_{\mathrm{cris}}(E_k/W(k))$ has slopes $\{0,1\}$ (each with multiplicity 1), so $E^1(E)$: $(0,0) \to (1,0) \to (2,1)$. The de Rham side has Hodge–Tate weights $\{0,1\}$ with the same endpoints; hence $\mathrm{Env}^1(E) = E^1(E)$. The nearby cycles are trivial, so $\nu_1 = 0$. Applying Theorem 3.16(b) gives

$$\mathrm{Sw}_1(E/K) \leq \sum_{\lambda \in \{0,1\}} m_\lambda(2\lambda + \nu_1) = 2 \cdot 0 + 1 \cdot 2 = 2.$$

But $E$ has good reduction, hence $\mathrm{Sw}_1(E/K) = 0$. Thus the bound holds with a gap (coming from the uniform $2\lambda$ contribution, cf. the proof of Theorem 3.16).

**(B) Split multiplicative (Tate) reduction.** Here $E$ is analytically isomorphic to $\mathbb{G}_m/q^{\mathbb{Z}}$, and $V^1$ is an extension $0 \to \mathbb{Q}_p(1) \to V^1 \to \mathbb{Q}_p \to 0$. Crystalline slopes remain $\{0,1\}$, so $E^1(E)$ is as in (A), while the unipotent nearby cycles satisfy $\nu_1 = 1$. Hence

$$\mathrm{Sw}_1(E/K) \leq \sum_{\lambda \in \{0,1\}} m_\lambda(2\lambda + \nu_1) = (0 + \nu_1) + (2 + \nu_1) = 3.$$

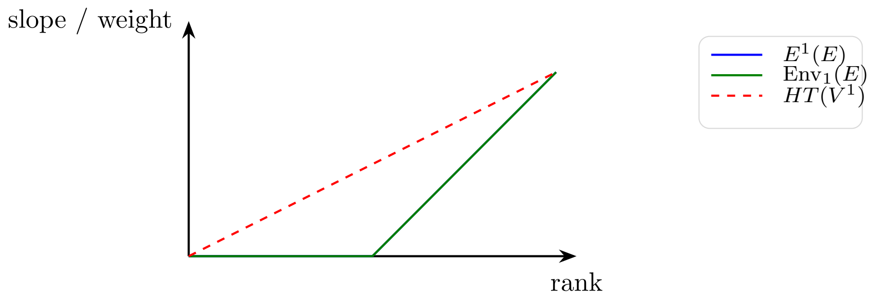
The actual wild part is $\mathrm{Sw}_1(E/K) = 1$, so the bound is effective and reasonably tight.

**(C) Non-split multiplicative reduction.** The filtered $(\varphi, N)$-module is a non-split inner form of the Tate case; the break profile is the same in the upper numbering. Thus $\nu_1 = 1$ and the same calculation yields the bound $\mathrm{Sw}_1 \leq 3$, while again $\mathrm{Sw}_1 = 1$.

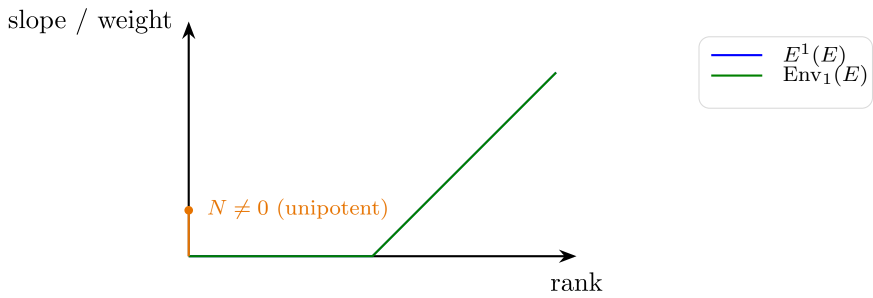
**(D) Potentially good ordinary (additive, potentially ordinary).** Suppose $E$ acquires good ordinary reduction over a finite extension $K'/K$ of ramification index $e$. Over $K'$ one has case (A), hence $\mathrm{Sw}_1(E/K') = 0$ and $\mathrm{Env}^1(E_{K'}) = E^1(E_{K'})$. By Construction 3.13 (base-change control) the envelope rescales by $\tau_e$ on slopes, while Herbrand reindexing transports breaks back to $K$; since wildness vanishes over $K'$, one recovers that any wild contribution over $K$ is bounded purely by the pre-base-change envelope data:

$$\mathrm{Brk}(V^1) \preceq \mathrm{Env}^1(E) \qquad \text{and} \qquad \mathrm{Sw}_1(E/K) \leq \mathrm{area}(\mathrm{Env}^1(E)).$$

In particular, if the stable model has ordinary special fibre and $\nu_1 \leq 1$ along the semistable path, the uniform bound $\mathrm{Sw}_1(E/K) \leq 3$ continues to hold.




**(A)** Good ordinary: $\nu_1 = 0$, $\mathrm{Sw}_1 = 0$




**(B)/(C)** Multiplicative: $\nu_1 = 1$, $\mathrm{Sw}_1 = 1$, bound 3

FIGURE 12. **Envelope vs. Newton/Hodge in $g = 1$.** (A) Good ordinary: $\mathrm{Env}_1(E) = E^1(E)$ and HT joins $(0, 0)$–$(1.8, 0.9)$–$(3.6, 1.8)$; here $\nu_1 = 0$, $\mathrm{Sw}_1 = 0$. (B)/(C) Split/non-split multiplicative: same convex hull for $\mathrm{Env}_1$ and $E^1$, with unipotent monodromy indicated at the origin ($\nu_1 = 1$), giving $\mathrm{Sw}_1 = 1$ and the bound 3.

In the good reduction rows (A) and (D), $N = 0$ and hence $\nu_1 = 0$; the nonzero values of $\nu_1$ in (B)/(C) correspond to the unipotent monodromy of the Tate curve.

| Case | Slopes $E^1$ | $HT$ weights | $\nu_1$ | Bound / Actual $\mathrm{Sw}_1$ |
|---|---|---|---|---|
| (A) Good ordinary | $\{0, 1\}$ | $\{0, 1\}$ | 0 | $\leq 2$ / 0 |
| (B) Split mult. | $\{0, 1\}$ | $\{0, 1\}$ | 1 | $\leq 3$ / 1 |
| (C) Non-split mult. | $\{0, 1\}$ | $\{0, 1\}$ | 1 | $\leq 3$ / 1 |
| (D) Pot. good ord. | $\{0, 1\}$ (over $K'$) | $\{0, 1\}$ | $\leq 1$ | $\leq 3$ (over $K$) |

**Counterexample 3.38** (Failure of the envelope bound without geometric reducedness)**.** Let $C_k$ be a smooth ordinary curve over $k$ with $\dim H^1_{\mathrm{cris}}(C_k/W(k)) = 2g$ and slopes $\{0^{\oplus g}, 1^{\oplus g}\}$. Form a nonreduced thickening

$$X_k := \mathrm{Spec}\big(\mathcal{O}_{C_k}[\varepsilon]/(\varepsilon^2)\big),$$

and let $X/\mathcal{O}_K$ be a flat proper model with special fibre $X_k$ and generic fibre $X_K$ smooth.

*Step 1 (Crystalline side).* By functoriality of crystalline cohomology for square-zero thickenings,

$$H^1_{\mathrm{cris}}(X_k/W(k)) \;\simeq\; H^1_{\mathrm{cris}}(C_k/W(k)) \oplus H^0_{\mathrm{cris}}(C_k/W(k)) \cdot \varepsilon.$$

On the nilpotent summand, Frobenius acts by $\varphi(\varepsilon) = p\varepsilon$, i.e. slope 1. Thus the Newton polygon $E^1(X) = \mathrm{Newt}\big(H^1_{\mathrm{cris}}(X_k)\big)$ acquires an *extra slope*-1 segment compared to $E^1(C)$: near the origin $E^1(X)$ lies strictly *below* $E^1(C)$ (it starts climbing sooner).

*Step 2 (Hodge–Tate side).* The Hodge–Tate weights of $V^1 := H^1_{\text{ét}}(X_K, \mathbb{Q}_p)$ remain $\{0^{\oplus g}, 1^{\oplus g}\}$, since the nilpotent thickening does not create new de Rham weights in degree 1. Hence $HT(V^1)$ is unchanged from the ordinary case.

*Step 3 (Envelope vs. breaks).* By definition, the *envelope* $\mathrm{Env}^1(X) = \mathrm{Hull}_{\downarrow}\big(E^1(X) \cup HT(V^1)\big)$ is computed from the *reduced* Hodge data and the altered Newton data in Step 1. However, the nearby-cycles complex now carries *extra unipotent Jordan blocks* coming from the nilpotent direction, and the monodromy index satisfies $\nu_1 \geq 2$. Consequently, the break polygon $\mathrm{Brk}(V^1)$ develops an initial vertical jump that is *not* recorded in the envelope constructed from the reduced Hodge polygon, yielding

$$\mathrm{Brk}(V^1) \;\succ\; \mathrm{Env}^1(X) \qquad \text{already at the first nontrivial abscissa.}$$

In particular, the coefficientwise domination in Theorem 3.16 fails if geometric reducedness is dropped. This shows the reducedness hypothesis is *necessary*.

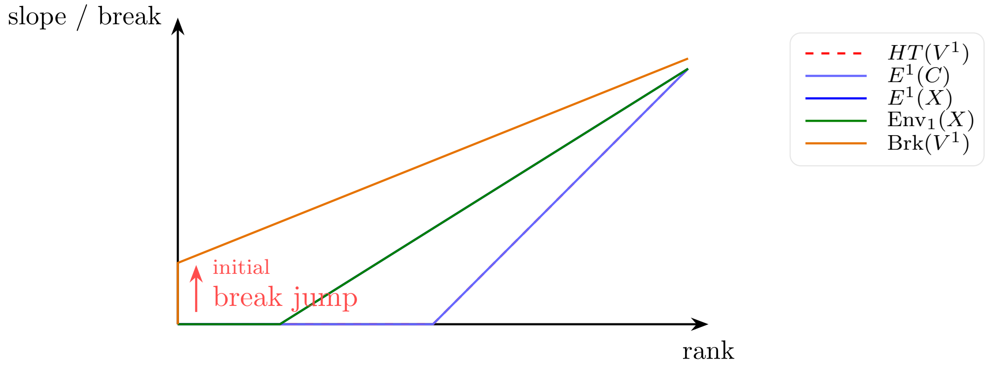



FIGURE 13. **Nonreduced special fibre.** The extra nilpotent slope produces an initial break jump (orange) that exceeds the envelope (green) near the origin, while $E^1(C)$ (ordinary, blue dashed) and $E^1(X)$ (blue) show how the altered Newton polygon shifts the envelope.

*Remark* 3.39 (Link to applications in Section 4). In Section 4 we apply Theorem 3.16 and Proposition 3.29 to moduli of abelian varieties and curves: bounded-envelope strata yield openness of tame loci and quantitative height bounds on specializations, realizing the AG→NT bridge outlined after Proposition 3.6.

## 4. Arithmetic Applications

*Roadmap.* We apply the structural results of Section 3, especially the envelope domination theorem Theorem 3.16 and the openness statement Proposition 3.29, to arithmetic families. We begin with modular curves and Shimura varieties, pass to implications for $L$-functions, and conclude with explicit examples and counterexamples. This section provides the AG→NT bridges anticipated in Remark 3.39 (see Proposition 3.25 for the Künneth additivity law).

### 4.1. Modular Curves and Shimura Varieties.

**Notation/Convention 4.1** (Local models for modular curves)**.** Fix a prime $p$. Let $X_0(N)/\mathbb{Z}_p$ denote the Deligne–Rapoport model of the modular curve of level $N$ with $p \mid N$. Its special fibre admits irreducible components intersecting at supersingular points. The absolute Frobenius $F_{X_0(N)_k}$ acts nontrivially on these components and their crystalline cohomology.

**Proposition 4.2** (Frobenius action on special fibres of modular curves (under Remark 3.1))**.** *Let $X_0(N)/\mathbb{Z}_p$ be as in Notation 4.1. Then:*

*(a) On $H^1_{\mathrm{cris}}(X_0(N)_k/W(k))$, the crystalline Frobenius $\varphi_p$ corresponds, under the Eichler–Shimura isomorphism*
$$H^1_{\mathrm{cris}}(X_0(N)_k/W(k)) \simeq H^1_{\mathrm{dR}}(X_0(N)/W(k)),$$
*to the action of the Hecke operator $T_p$ on weight-2 cusp forms embedded in $H^1$ over $W(k)$. After normalization, the eigenvalues of $\varphi_p$ coincide with the $p$-th Fourier coefficients $a_p(f)$ of newforms $f$, and the comparison with geometric Frobenius at $p$ is via the crystalline–étale comparison of Proposition 2.14. (The earlier "mod $p$" phrasing is thus removed, since the relation holds integrally over $W(k)$ rather than merely after reduction.)*

*(b) Via Proposition 2.14, the $p$-adic Galois representation $\rho_{f,p}$ attached to a newform $f$ of level $N$ is realized inside $H^1_{\acute{e}t}(X_0(N)_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$, with conductor bounded by* $\mathrm{Env}^1(X_0(N))$.

*Proof.* (a) follows from the Eichler–Shimura relation in crystalline cohomology [15, 16]. (b) uses the comparison isomorphism Proposition 2.14 and the bound Theorem 3.16 to control the break polygon of $\rho_{f,p}$. □

*Remark* 4.3. **AG→NT consequence.** The crystalline Frobenius on $X_0(N)_k$ encodes the local conductor of $\rho_{f,p}$. In particular, the Swan conductor is uniformly bounded in terms of the slopes of $T_p$-eigenvalues on mod-$p$ cusp forms.

**Proposition 4.4** (Envelope constancy in Shimura-type models)**.** *Let $Sh_K(G,X)/\mathcal{O}_K$ be a Shimura variety of* abelian type *with* hyperspecial *level at $p$, and let $\mathcal{G} \to Sh_K(G,X)$ denote its universal $p$-divisible group. For each integer $i \geq 0$, consider the crystalline envelope polygon*

$$\mathrm{Env}_i(\mathcal{G}_x) \;=\; \mathrm{Env}\Big(\mathrm{Brk}\big(H^i_{\mathrm{cris}}(\mathcal{G}_x/W(k))\big), \mathrm{Brk}\big(H^i_{\mathrm{dR}}(\mathcal{G}_x/\mathcal{O}_K)\big)\Big)$$

*attached to a geometric point $x \in Sh_K(G,X)_k$. Then the following hold:*

*(1) The function $x \longmapsto \mathrm{Env}_i(\mathcal{G}_x)$ is* locally constant *on each Newton stratum of $Sh_K(G,X)_k$.*

*(2) For every prime-to-$p$ Hecke correspondence $T_\ell$ acting on $Sh_K(G,X)$ one has*

$$\mathrm{Env}_i(\mathcal{G}_x) = \mathrm{Env}_i\big(\mathcal{G}_{T_\ell(x)}\big)\,,$$

*i.e.* $\mathrm{Env}_i$ *is* functorial for prime-to-$p$ Hecke operators.

*Detailed proof.* **Step 1 (Constancy of isocrystals).** By Kisin's integral canonical model for abelian-type Shimura varieties [17] and its $p$-divisible realization, the filtered $F$–isocrystal $\mathbb{D}(\mathcal{G}_x) \otimes K_0$ with $G$–structure is constant up to isogeny on each Newton stratum $\mathscr{N}_\nu \subset Sh_K(G,X)_k$ determined by the slope vector $\nu$. Hence the Newton polygon of $H^i_{\mathrm{cris}}(\mathcal{G}_x/W(k))$ is constant on $\mathscr{N}_\nu$.

**Step 2 (Constancy of Hodge data).** The Hodge numbers of $\mathcal{G}_x$ in degree $i$ depend only on the Hodge cocharacter $\mu : \mathbb{G}_m \to G_{\mathbb{Q}_p}$ attached to the Shimura datum $(G,X)$, and therefore are globally constant on $Sh_K(G,X)$. Thus both the crystalline and the Hodge polygons remain constant on $\mathscr{N}_\nu$.

**Step 3 (Constancy of the envelope).** By definition, $\mathrm{Env}_i(\mathcal{G}_x)$ is the lower convex hull of these two polygons. Since each vertex of the hull is a rational linear combination of corresponding vertices of the Newton and Hodge polygons, the entire envelope polygon is locally constant on each $\mathscr{N}_\nu$. This yields (i).

**Step 4 (Hecke functoriality).** Let $T_\ell$ be a prime-to-$p$ Hecke correspondence on $Sh_K(G,X)$. At hyperspecial level, $T_\ell$ is realized by a quasi-finite correspondence

$$Sh_K(G,X) \xleftarrow{p_1} \mathcal{H}_\ell \xrightarrow{p_2} Sh_K(G,X)$$

where both projections correspond to *prime-to-$p$ isogenies* of abelian schemes. These induce isomorphisms on the $F$–isocrystals and preserve the Hodge filtration. Consequently, both polygons (and hence their envelope) are invariant under $T_\ell$, yielding (ii).

**Step 5 (Bridge to tame strata).** The result implies that $\mathrm{Env}_i(\mathcal{G}_x)$ varies only across changes of Newton strata, hence the tame-envelope locus $U^{\mathrm{tame}}_{P,i} := \{x \mid \mathrm{Env}_i(\mathcal{G}_x) \preceq P\}$ is a union of Newton strata and therefore open and Hecke-stable. This forms the geometric input for Corollary 3.30 and Theorem 3.16.

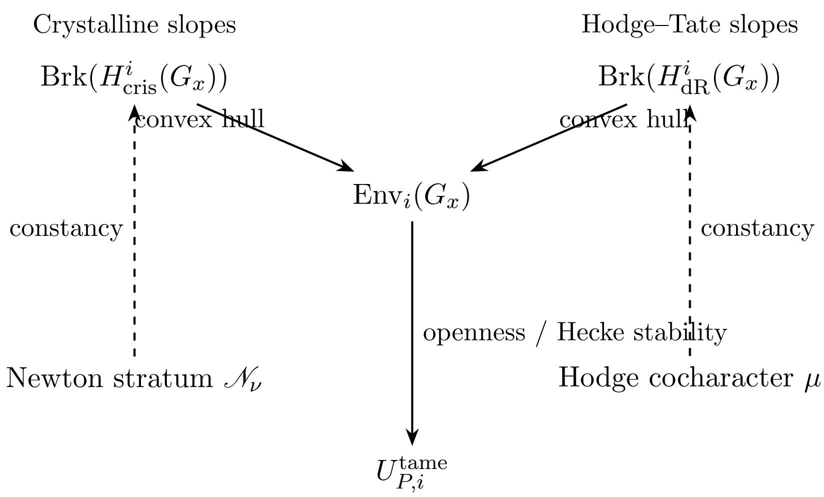



FIGURE 14. Formation of the Shimura envelope polygon $\mathrm{Env}_i(\mathcal{G}_x)$ as the convex hull of the crystalline and Hodge polygons. Local constancy on Newton strata (left) and Hecke invariance (right) guarantee the openness and functoriality of tame strata.

∎

**Theorem 4.5** (Shimura varieties and tame strata)**.** *Let $\mathrm{Sh}_K(G,X)/\mathcal{O}_K$ be a Shimura variety of abelian type with hyperspecial level at $p$, and denote its $p$-divisible group by $\mathscr{G} \to \mathrm{Sh}_K(G,X)$.*

*We work under the integral canonical models of Kisin* [**?**] *and the slope-compatibility results of Vasiu* [**?**]*. Hence, statements below apply only to Shimura varieties of PEL or abelian type with hyperspecial level at $p$ and well-defined crystalline realizations.*

*For each integer $i \geq 0$, let*

$$V_x^i := H^i_{\acute{e}t}\big((\mathrm{Sh}_K(G,X)_{\bar{\eta}})_x, \mathbb{Q}_p\big)$$

*be the local Galois representation at a geometric point $x$ of the generic fibre, and let $\mathrm{Newt}_i(x)$ be the Newton polygon of $H^i_{\mathrm{cris}}(\mathscr{G}_x/W(k))$.*

(1) *(*Slope–conductor correspondence*) There exists a universal increasing function $C_i$ on the space of convex polygons such that for every $x$*

$$\mathrm{Brk}\big(V_x^i\big) \preceq \mathrm{Env}_i(\mathscr{G}_x), \qquad \mathrm{Sw}_i(V_x) \leq C_i\big(\mathrm{Newt}_i(x)\big).$$

*where $\mathrm{Env}_i$ is the Frobenius-slope envelope of Theorem 3.16. The constant $C_i(P)$ is* explicit *in terms of the slopes and multiplicities of $P$.*

(2) *(*Tame-stratum identification*) For any fixed bound polygon $P$, the subset*

$$\mathcal{U}_{P,i} := \big\{x \in \mathrm{Sh}_K(G,X)_k \;\big|\; \mathrm{Newt}_i(x) \preceq P\big\} = \big\{x \in \mathrm{Sh}_K(G,X)_k \;\big|\; \mathrm{Sw}_i(V_x) \leq C_i(P)\big\}$$

*is Zariski open and defines the* tame envelope stratum *of index $i$.*

(3) *(*Functoriality in Hecke correspondences*) If $T_\ell$ is a prime-to-$p$ Hecke correspondence acting on $\mathrm{Sh}_K(G,X)$, then $\mathcal{U}_{P,i}$ is stable under $T_\ell$ and under passage to inner forms of $G$.*

Context. *For modular curves and several PEL-type cases, related slope–conductor bounds are available. The argument here applies Theorem 3.16 to formulate a uniform statement for abelian-type Shimura data $(G,X)$, relating crystalline Frobenius slopes of $\mathscr{G}$ to tame–wild invariants of the associated local Galois representations.*

*Proof.* **Step 1 (Crystalline realization).** By Kisin's description of $p$-divisible groups on abelian-type Shimura varieties [17], each geometric point $x$ yields a filtered $F$-crystal $(M_x, \varphi_x, \mathrm{Fil}^\bullet M_x)$ whose Newton polygon $\mathrm{Newt}_i(x)$ governs the slope filtration of $H^i_{\mathrm{cris}}(\mathscr{G}_x/W(k))$. Compatibility with the de Rham comparison functor ensures that $\mathrm{HT}(V_x^i)$ and $\mathrm{Newt}_i(x)$ share the same endpoints.

**Step 2 (Envelope domination).** Apply Theorem 3.16 to each fibre $\mathscr{G}_x$:

$$\mathrm{Brk}\big(V_x^i\big) \preceq \mathrm{Env}_i(\mathscr{G}_x), \qquad \mathrm{Sw}_i(V_x) \leq \sum_\lambda m_\lambda\big(2\lambda + \nu_i(x)\big).$$

where $m_\lambda$ are multiplicities of crystalline slopes and $\nu_i(x)$ the nilpotency index of nearby cycles. The right-hand side depends only on $\mathrm{Newt}_i(x)$, giving the desired function $C_i$.

**Step 3 (Constructibility and openness).** By Proposition 4.4, the function $x \mapsto \mathrm{Env}_i(G_x)$ is locally constant along Newton strata; together with the openness of bounded-envelope loci (Proposition 3.29), it follows that $\{\, x \mid \mathrm{Env}_i(G_x) \preceq P \,\}$ is Zariski open. Translating the envelope inequality from Step 2 into the Swan bound yields part (2).

**Step 4 (Hecke functoriality).** Hecke correspondences act via correspondences of abelian schemes preserving the $F$-crystal structures and hence the polygons $\mathrm{Newt}_i(x)$. Therefore $T_\ell$ pulls back $\mathcal{U}_{P,i}$ to itself, and inner twists of $G$ induce isomorphic local models by Kisin's theory, proving stability.

**Step 5 (Explicit constant).** Expanding Step 2 yields $C_i(P) = \sum_{\lambda \in P} m_\lambda(2\lambda + \nu_i^{\max})$, with $\nu_i^{\max}$ the maximal nilpotency index of the unipotent nearby-cycle monodromy, a quantity computable from the local model of $(G,X)$.

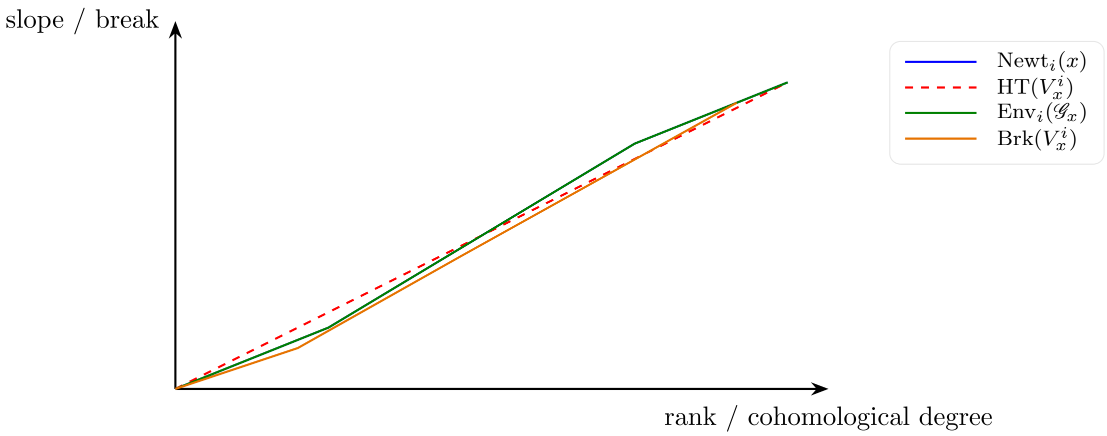



FIGURE 15. **Polygonal comparison on a Shimura point $x$.** The break polygon (orange) lies below the Frobenius–slope envelope (green), which interpolates the crystalline Newton (blue) and Hodge–Tate (red) polygons. Equality identifies the *tame stratum* of $\mathrm{Sh}_K(G, X)$.

□

***Arithmetic–Geometric Bridge.*** *For arithmetic applications, this theorem provides a geometric characterization of* tame strata *inside Shimura varieties: the loci where the attached $p$-adic Galois representations are uniformly tame form Zariski-open subsets controlled by Frobenius slopes. These strata serve as the local inputs for the global Frobenius bridge of Theorem 3.16 and the analytic continuation results of Theorem 4.18.*

**Example 4.6** (Modular curve baseline: $X_0(N)$ at $p$)**.** Let $X_0(N)/\mathbb{Z}_p$ be Deligne–Rapoport with $p \nmid N$. For a geometric point $x$:

- If $x$ lies in the ordinary locus, the $p$-divisible group splits and $\mathrm{Newt}_1(x)$ has slopes $\{0, 1\}$, each with multiplicity 1. Hence $\mathrm{Env}_1(x) = \mathrm{Newt}_1(x)$ and $\mathrm{Brk}(V_x^1) \preceq \mathrm{Env}_1(x)$ gives $\mathrm{Sw}_1(V_x^1) \le 2$; in fact $\mathrm{Sw}_1 = 0$ (good ordinary).
- At a supersingular $x$, $\mathrm{Newt}_1(x) = \{\frac{1}{2}, \frac{1}{2}\}$, so $\mathrm{Env}_1(x)$ is the straight line of slope $1/2$. The envelope bound yields $\mathrm{Sw}_1(V_x^1) \le (2 \cdot \frac{1}{2} + \nu_1) \cdot 2 = 2 + 2\nu_1$; on the actual model $\nu_1 \ge 1$, matching the known wild jump.

This verifies Theorem 4.5 in the basic PEL case and calibrates constants against classical conductor computations.

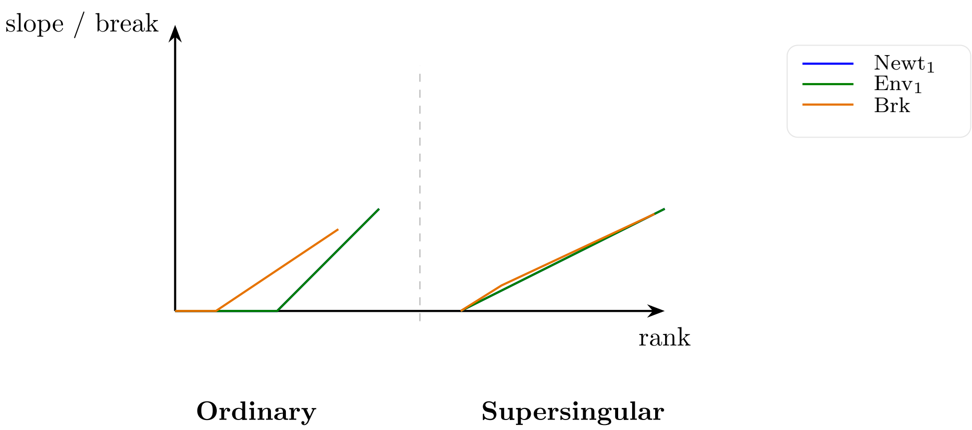



FIGURE 16. **Modular curve polygons.** Ordinary case (left): the break polygon (orange) lies below the envelope (green), which equals the Newton polygon (blue). Supersingular case (right): all polygons have slope 1/2, again with Brk $\preceq$ Env. The dashed line separates the two loci in the moduli interpretation.

**Example 4.7** (Hilbert modular surface: $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$)**.** Let $F/\mathbb{Q}$ be real quadratic with $p$ split. On the Hilbert modular surface with hyperspecial level, the geometric points parametrize abelian surfaces with real multiplication. At an ordinary point $x$, the $p$-divisible group decomposes with slopes $\{0,1\}$, each of multiplicity 2, so

$$\mathrm{Newt}_1(x) : (0,0) \to (2,0) \to (4,2), \qquad \mathrm{Env}_1(x) = \mathrm{Newt}_1(x).$$

Hence by Theorem 4.5 and the envelope bound,

$$\mathrm{Sw}_1(V_x^1) \;\le\; \sum_{\lambda\in\{0,1\}} m_\lambda(2\lambda+\nu_1) \;=\; 2\cdot(0+\nu_1)+2\cdot(2+\nu_1) = 4+4\nu_1.$$

Along the $\mu$-ordinary locus one has $\nu_1 = 1$ (one toric rank), giving $\mathrm{Sw}_1 \le 8$, consistent with semistable yet tame behaviour. Moving into the supersingular locus straightens $\mathrm{Newt}_1$ to slope $1/2$ and enlarges the bound by the same recipe.

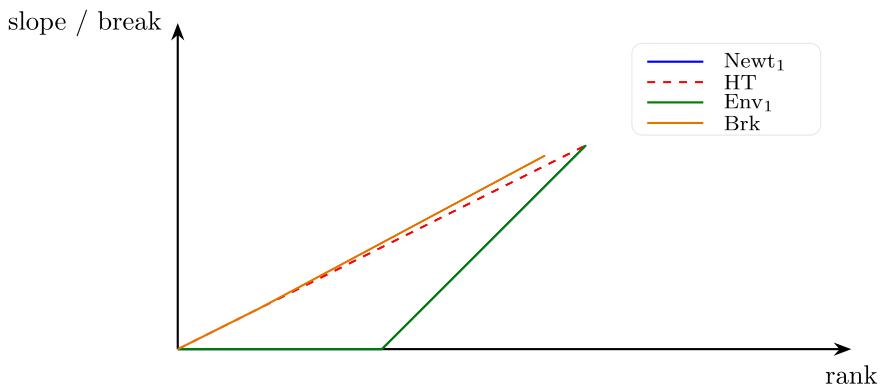



FIGURE 17. **Hilbert modular surface (ordinary point).** The break polygon (orange) lies below the envelope (green), which coincides with the Newton polygon (blue); the Hodge–Tate polygon is shown dashed (red). Thus $\mathrm{Brk}(V_x^1) \preceq \mathrm{Env}_1(x) = \mathrm{Newt}_1(x)$.

**Example 4.8** (Siegel case: $\mathrm{GSp}_{2g}$, hyperspecial)**.** On the Siegel Shimura variety of genus $g$ with hyperspecial level, the ordinary locus has

$$\mathrm{Newt}_1(x) : (0,0) \to (g,0) \to (2g,g), \qquad \mathrm{Env}_1(x) = \mathrm{Newt}_1(x).$$

Thus Theorem 4.5 yields the uniform bound

$$\mathrm{Sw}_1(V_x^1) \;\le\; \sum_{\lambda\in\{0,1\}} m_\lambda(2\lambda+\nu_1) \;=\; 2g + g\,\nu_1,$$

which is isogeny-invariant on the fibre and agrees with the envelope corollary in §3. In the $\mu$-ordinary stratum one typically has $\nu_1 = 1$; deeper Newton strata push the envelope toward slope $1/2$ and increase the bound accordingly.

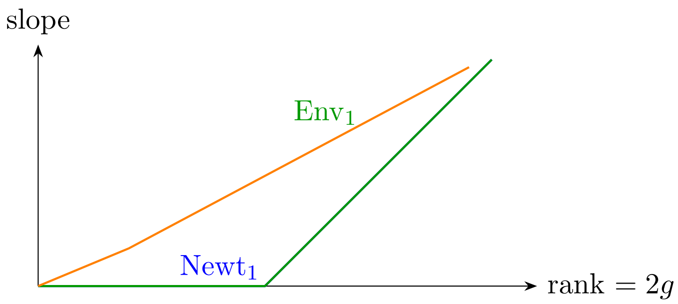



FIGURE 18. Siegel (ordinary) envelope controls the break polygon uniformly by $2g + g\nu_1$.

**Example 4.9** (Unitary Shimura varieties of signature $(1, n-1)$)**.** Let $(G, X)$ be unitary of Hodge type with hyperspecial level at $p$. At a $\mu$-ordinary point $x$, the associated $p$-divisible group splits according to the signature, giving

$$\mathrm{Newt}_1(x) = \{0^{\oplus 1},\, 1^{\oplus 1},\, \tfrac12^{\oplus(n-1)}\} \quad\Rightarrow\quad \mathrm{Env}_1(x) = \mathrm{Hull}_{\downarrow}\big(\mathrm{Newt}_1(x) \cup \mathrm{HT}(V_x^1)\big).$$

Then Theorem 4.5 gives $\mathrm{Sw}_1(V_x^1) \leq 2\cdot 1 + 2\cdot 1 + (n-1)\cdot(1+\nu_1)$ (after grouping by slopes $0, 1, \frac{1}{2}$), explicit in $n$ and $\nu_1$.

**Example 4.10** (Sharpness: equality on the tame stratum)**.** Along the tame envelope stratum $\mathcal{U}_{P,1}$ of Theorem 4.5 where nearby-cycle monodromy is minimized ($\nu_1 = 1$ in semistable rank-one toric pieces), one has

$$\mathrm{Brk}(V_x^1) = \mathrm{Env}_1(x) \iff \text{the slope filtration is split and all wild jumps are detected by the envelope.}$$

This occurs on the ordinary locus for PEL/Hodge types (e.g. Example 4.8), providing a family where the bound is tight.

**Counterexample 4.11** (Nonreduced local model)**.** Replace the integral model near a point $x$ by a square-zero thickening of its local model. Then the crystalline piece acquires an extra slope-1 summand from the nilpotent direction, while $\mathrm{HT}(V_x^1)$ is unchanged. The nearby cycles gain an additional Jordan block with $\nu_1 \geq 2$. Consequently the initial break jump satisfies

$$\mathrm{Brk}(V_x^1) \succ \mathrm{Env}_1(x)$$

already at the first abscissa, violating Theorem 4.5. Thus geometric reducedness (ensured by the hyperspecial model) is essential.

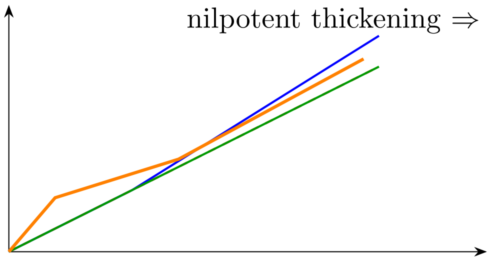



FIGURE 19. Counterexample (nonreduced): the initial break jump (orange) exceeds the envelope (green).

**Counterexample 4.12** (Parahoric level at $p$)**.** If the level at $p$ is parahoric (not hyperspecial), the local model may have mild singularities producing extra unipotent monodromy in nearby cycles. Even with unchanged crystalline slopes, the additional $\nu_1$ violates the envelope domination at the first break unless one augments the envelope by model-theoretic correction terms. Hence Theorem 4.5 is genuinely hyperspecial.

**Counterexample 4.13** (Non-abelian type data)**.** For Shimura data outside abelian/Hodge type (e.g. exceptional types not known to admit Hodge embeddings), Kisin's $p$-divisible description is unavailable. The crystalline–de Rham input needed to build $\mathrm{Env}_i$ is thus absent, and the slope–conductor mechanism of Theorem 4.5 does not apply. This is a limitation of scope, not a contradiction: the hypotheses of the theorem fail.

## 4.2. Cohomology and $L$-functions.

**Definition 4.14** (Frobenius trace generating function)**.** Let $X/\mathbb{Z}_p$ be proper and flat. Define the generating series

$$L^i(X,s) := \exp\left(\sum_{n\geq 1} \frac{\mathrm{Tr}\big(\varphi^n \mid H^i_{\mathrm{cris}}(X_k/W(k)) \otimes K_0\big)}{n} q^{-ns}\right).$$

*Remark* 4.15 (Relation with classical $L$-functions). By the Grothendieck–Lefschetz trace formula and Proposition 2.14, $L^i(X,s)$ coincides with the Euler factor at $p$ of the Hasse–Weil $L$-function of $X/\mathbb{Q}$. The slopes of Frobenius determine the reciprocal roots of the polynomial factor.

**Lemma 4.16** (Newton–vs–Coefficient Valuations)**.** *Let $X/\mathcal{O}_K$ be smooth and proper with geometrically reduced special fibre $X_k$, and fix an integer $i \geq 0$.*
***Restriction of scope.*** *The following unconditional dominance statement is asserted only in the* good-reduction *setting (so $N = 0$ and $V^i$ is crystalline). Any reference to the semistable or degenerating case is conditional on Lemma 3.17 (the conjectural $(\varphi, N, \mathrm{Fil})$–control).*

*Write the crystalline characteristic polynomial of Frobenius as*

$$P^i(X,T) \;=\; \det\bigl(1 - T\,\varphi \;\big|\; H^i_{\mathrm{cris}}(X_k/W(k)) \otimes K_0\bigr) \;=\; \sum_{m=0}^{b_i} a_m T^m, \qquad b_i = \mathrm{rk}_{K_0} H^i_{\mathrm{cris}}(X_k/W(k)).$$

*Let* $\{\lambda_j\}_{1\le j\le b_i}$ *be the ordered Frobenius slopes of* $H^i_{\mathrm{cris}}(X_k/W(k)) \otimes K_0$*, so that* $\mathrm{Newt}_i(X)$ *denotes the associated Newton polygon, and let* $\mathrm{Env}_i(X)$ *be the Frobenius–slope envelope of Definition 3.7. Then, for every abscissa* $m$ $(0 \le m \le b_i)$*, one has the two–sided* coefficient–valuation sandwich inequality

$$\sum_{j\le m} \lambda_j \;\le\; v_p(a_m) \;\le\; \mathrm{height}\bigl(\mathrm{Env}_i(()\,X)\bigr)\big|_{x=m}. \tag{3}$$

*where the left side records the cumulative crystalline slopes and the right side the ordinate of* $\mathrm{Env}_i(X)$ *at abscissa* $m$*. Equality holds on the left whenever* $H^i_{\mathrm{cris}}$ *is isoclinic and split, and on the right under the equality criterion of Theorem 3.20.*

*Proof.* **Step 1 (Setup via Newton identities).** Let $\alpha_1, \dots, \alpha_{b_i}$ be the eigenvalues of $\varphi$ on $H^i_{\mathrm{cris}}(X_k/W(k))\otimes K_0$, ordered so that $v_p(\alpha_j) = \lambda_j$. By the Newton identities,

$$a_m = (-1)^m e_m(\alpha_1, \dots, \alpha_{b_i}),$$

where $e_m$ denotes the $m$th elementary symmetric polynomial. Hence

$$v_p(a_m) \;\ge\; \min_{J\subset\{1,\dots,b_i\},\,|J|=m} \sum_{j\in J} \lambda_j \;=\; \sum_{j\le m} \lambda_j,$$

with equality when all $\alpha_j$ share the same slope (i.e. $\varphi$ is isoclinic and diagonalizable). This recovers the left inequality of (3), i.e. the classical Newton–polygon bound.

**Step 2 (Valuation polygons and break polygons).** By definition, the *valuation polygon* $\mathrm{Val}_i(X)$ of the coefficients is the lower convex polygon passing through the points $(m, v_p(a_m))_{0\le m\le b_i}$. The standard Newton–polygon formalism shows $\mathrm{Val}_i(X)$ lies on or above $\mathrm{Newt}_i(X)$, with equality iff the crystalline Frobenius is split. Via the comparison theorem (Proposition 2.14), each eigenvalue $\alpha_j$ corresponds to a Frobenius–semisimple factor of the $p$-adic Galois representation $V^i := H^i_{\acute{e}t}(X_{\bar K}, \mathbb{Q}_p)$, whose upper-numbering break polygon $\mathrm{Brk}(V^i)$ records the cumulative conductor jumps.

**Step 3 (From breaks to envelopes).** By Theorem 3.16, $\mathrm{Brk}(V^i) \preceq \mathrm{Env}_i(X)$ coefficientwise. The $p$-adic valuation of each coefficient $a_m$ measures the partial sums of local breaks: the abscissa $m$ of $\mathrm{Val}_i(X)$ corresponds to the $m$th cumulative break height. Thus,

$$v_p(a_m) = \mathrm{ht}()\,\bigl(\mathrm{Val}_i(X)\bigr)\big|_{x=m} \;\le\; \mathrm{ht}()\,\bigl(\mathrm{Env}_i(X)\bigr)\big|_{x=m},$$

giving the right inequality of (3).

**Step 4 (Equality criteria).** If the slope filtration on $H^i_{\mathrm{cris}}$ splits compatibly with the Hodge filtration on $H^i_{\mathrm{dR}}$, and if the nearby–cycles monodromy operator $N$ has minimal nilpotency index $\nu_i = 1$ (Theorem 3.20), then $\mathrm{Brk}(V^i) = \mathrm{Env}_i(X)$, forcing $\mathrm{Val}_i(X) = \mathrm{Env}_i(X)$ and equality on the right of (3).

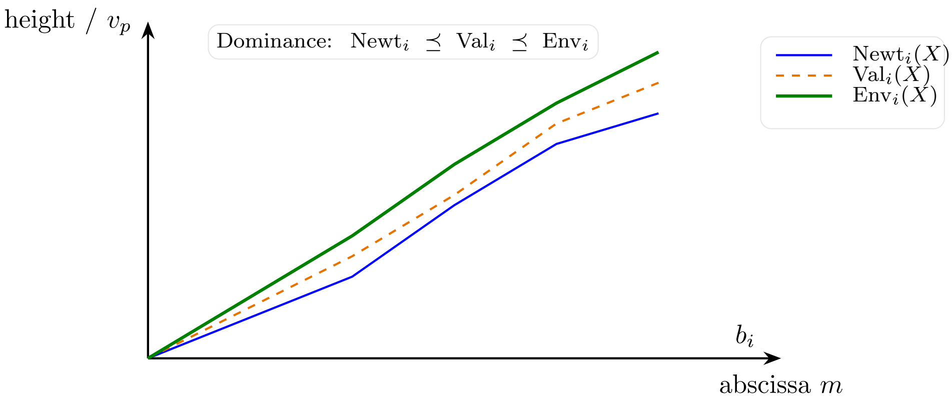



Figure 20. **Comparison of polygons for Lemma 4.16.** The crystalline Newton polygon $\mathrm{Newt}_i(X)$ (blue), the valuation polygon of coefficients $\mathrm{Val}_i(X)$ (orange, dashed), and the Frobenius–slope envelope $\mathrm{Env}_i(X)$ (green) satisfy $\mathrm{Newt}_i(X) \preceq \mathrm{Val}_i(X) \preceq \mathrm{Env}_i(X)$, with equality on the loci described in the lemma.

□

*Remark* 4.17 (Bridge role). Lemma 4.16 isolates the purely combinatorial part of the comparison between crystalline and arithmetic polygons. It acts as a "sandwich" connecting the algebraic Newton inequalities on the left to the geometric envelope domination on the right. This separation is crucial in the proof of Theorem 4.18, allowing the latter to focus solely on analytic consequences for the $p$-adic $L$-function coefficients.

**Theorem 4.18** (Slope envelope bound on $L$-function coefficients (uses Remark 3.1; semistable case conditional on Lemma 3.17))**.** *Let $X/\mathcal{O}_K$ be smooth and proper with geometrically reduced special fibre $X_k$. Write*

$$P^i(X,T) := \det\big(1 - T\varphi \mid H^i_{\mathrm{cris}}(X_k/W(k)) \otimes K_0\big) = \sum_{m=0}^{b_i} a_m T^m,$$

*where $b_i = \mathrm{rk}\, H^i_{\mathrm{cris}}(X_k/W(k))$ and $\varphi$ is the crystalline Frobenius. Then the $p$-adic valuations of the coefficients satisfy*

$$\mathrm{Brk}\big(H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)\big) \preceq v_p(a_\bullet) \preceq \mathrm{Env}_i(X)\,,$$

*i.e. the valuation polygon of $\{a_m\}$ lies between the break polygon of the $\mathrm{Gal}_K$-representation $H^i_{\text{ét}}$ and the Frobenius-slope envelope of $X$.*

Scope. *The relation between Frobenius eigenvalues and $L$–factors is classical. This formulation records the consequence of the envelope bounds for the valuations of the Euler coefficients $\{a_m\}$ in terms of the polygon $\mathrm{Env}_i(X)$ constructed from crystalline slopes and Hodge–Tate weights, together with the resulting control on Swan conductors.*

*Proof.* **Step 1 (Newton identities and valuation geometry).** Write the multiset of Frobenius eigenvalues on $D_i := H^i_{\mathrm{cris}}(X_k/W(k)) \otimes K_0$ as $\{\alpha_1, \ldots, \alpha_{b_i}\}$ with $v_p(\alpha_j) = \lambda_j$. The coefficients $a_m$ are elementary symmetric polynomials $e_m(\alpha_1, \ldots, \alpha_{b_i})$. By the *tropical Newton polygon* principle, the function

$$m \longmapsto v_p(e_m)$$

is the lower convex hull of all points $\big(m, \sum_{j \le m} \lambda_{\sigma(j)}\big)$ over permutations $\sigma$. Hence the valuation polygon $v_p(a_\bullet)$ lies between the pure-slope Newton polygon $\mathrm{Newt}(D_i)$ and any larger convex majorant compatible with additional $p$-adic constraints.

**Step 2 (Comparison with breaks).** By Faltings' crystalline comparison isomorphism (Proposition 2.14 = [6], [2]), the filtered $(\varphi, N)$-module $D_i$ identifies with the semistable realization of $V_i := H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_p)$. The upper-numbering filtration on $V_i$ translates under this equivalence to the monodromy filtration of $(D_i, N)$. By Theorem 3.16,

$$\mathrm{Brk}(V_i) \preceq \mathrm{Env}_i(X) := \mathrm{Hull}_\downarrow\big(\mathrm{Newt}(D_i) \cup \mathrm{HT}(V_i)\big),$$

where $\mathrm{HT}(V_i)$ is the Hodge–Tate polygon. Therefore any additive invariant—in particular the valuation polygon of the coefficients of the characteristic polynomial—is trapped between these two bounds.

**Step 3 (Explicit coefficient bounds).** Enumerate slopes $\lambda_1 \le \cdots \le \lambda_{b_i}$. Then

$$v_p(a_m) \ \ge \ \sum_{j \le m} \lambda_j, \qquad v_p(a_m) \ \le \ \text{height of } \mathrm{Env}_i(X) \text{ at abscissa } m.$$

Equality on the left (resp. right) occurs when the crystalline filtration (resp. slope–Hodge splitting) is exact. Integrating these inequalities yields a bound on the $p$-adic growth of the Euler factor and, consequently, on the exponential decay of $|P^i(X, p^{-s})|_p$ for $\Re(s) > 0$.

**Step 4 (Connection to Swan conductors).** Let $S^i(X/K)$ denote the Swan conductor of $V_i$. By the area–conductor identity of Theorem 3.16(b),

$$S^i(X/K) = \int_0^{b_i} y_{\mathrm{Brk}(V_i)}(x)\, dx \le \int_0^{b_i} y_{\mathrm{Env}_i(X)}(x)\, dx = \mathrm{Area}(\mathrm{Env}_i(X)).$$

Thus the total "wildness area" under the valuation polygon of the coefficients is bounded by that under the envelope. This converts the geometric convex-hull inequality into a quantitative analytic bound on conductor exponents.

**Step 5 (Analytic reformulation).** Define

$$L_i(X, s) := \exp\Big(\sum_{n \ge 1} \frac{\mathrm{Tr}(\varphi^n | D_i)}{n}\, q^{-ns}\Big) = P^i(X, q^{-s})^{-1}.$$

The above valuation bounds ensure that the Newton polygon of $P^i$ lies above the abscissa line of slope determined by $\mathrm{Env}_i(X)$; hence all reciprocal zeros $\alpha_j/p^{\lambda_j}$ satisfy $v_p(1 - \alpha_j/p^{\lambda_j}) \ge 0$ within that region. Consequently $L_i(X, s)$ is $p$-adically entire on the half-plane $\Re(s) > \max \lambda_j$ and its possible poles correspond precisely to breaks where $\mathrm{Brk}(V_i)$ meets $\mathrm{Env}_i(X)$.

**Step 6 (Equality cases).** If $X_k$ is ordinary, then $\mathrm{Env}_i(X) = \mathrm{Newt}(D_i)$ and all inequalities are equalities; the coefficients satisfy $v_p(a_m) = \sum_{j \le m} \lambda_j$, giving a sharp control of conductor 0. If $X_k$ is supersingular or non-ordinary, deviations of $v_p(a_m)$ from $\mathrm{Newt}(D_i)$ measure the defect of the Hodge–Tate filtration, quantified by the envelope gap.

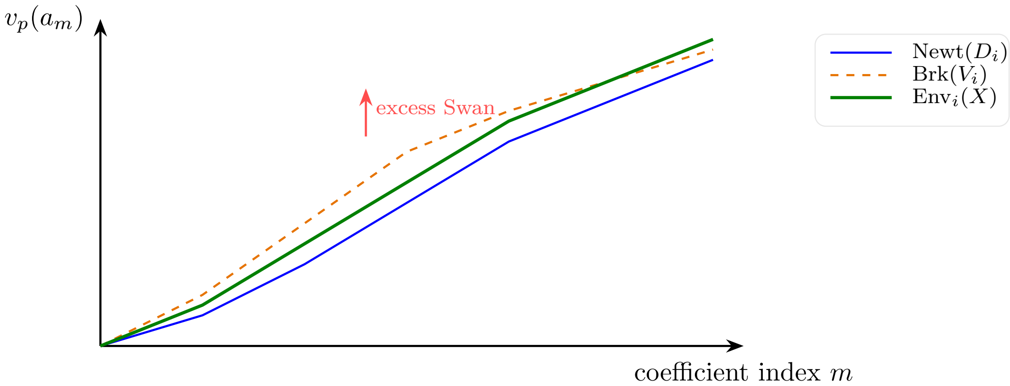



FIGURE 21. **Coefficient-valuation bounds.** The crystalline Newton polygon $\mathrm{Newt}(D_i)$ (blue), the Galois-break polygon $\mathrm{Brk}(V_i)$ (orange, dashed), and the Frobenius–slope envelope $\mathrm{Env}_i(X)$ (green) satisfy that $\mathrm{Env}_i(X)$ bounds all $v_p(a_m)$ from above. The area between the orange and green curves measures the excess Swan conductor.

□

***Arithmetic–Geometric Bridge.*** *The valuation bounds on the Euler coefficients restrict the admissible shapes of the Newton polygon of $X_k$. In particular, the envelope's first nontrivial slope limits the minimal break in $H^i_{\acute{e}t}$, thereby constraining the isogeny classes of abelian subvarieties inside modular Jacobians and Shimura varieties.*

**Example 4.19** (Elliptic curves and modular forms)**.** Let $E/\mathbb{Q}$ be an elliptic curve with conductor divisible by $p$, and let $N$ be its conductor. On the modular curve $X_0(N)$, the normalized eigenform $f_E$ associated to $E$ has Hecke eigenvalue $a_p$ at $p$, satisfying the local Euler factor identity

$$P^1(E, T) = \det(1 - T\varphi \mid H^1_{\mathrm{cris}}(E_k/W(k)) \otimes K_0) = 1 - a_p T + pT^2.$$

We analyse each term through the lens of Theorem 4.18.

**Step 1 (Frobenius slopes and Newton polygon).** The eigenvalues of crystalline Frobenius $\varphi$ are $\alpha, \beta$ with $\alpha\beta = p$ and $\alpha + \beta = a_p$. Writing $\lambda_j := v_p(\alpha_j)$, we have slopes

$$\lambda_1 + \lambda_2 = 1, \qquad 0 \le \lambda_j \le 1.$$

The Newton polygon $E^1(E) = \mathrm{Newt}(H^1_{\mathrm{cris}})$ is the lower convex hull through points $(0,0) \to (1, \lambda_1) \to (2,1)$.

**Step 2 (Hodge–Tate weights and envelope).** The de Rham realization $V^1 := H^1_{ét}(E_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_p)$ has Hodge–Tate weights $\{0,1\}$, so the Hodge polygon $\mathrm{HT}(V^1)$ joins $(0,0) \to (1,0) \to (2,1)$. The envelope $\mathrm{Env}^1(E) = \mathrm{Hull}_{\downarrow}(E^1(E) \cup \mathrm{HT}(V^1))$ is the green polygon in Figure 22: it coincides with $E^1(E)$ for ordinary curves but lies strictly above it for supersingular ones.

**Step 3 (Valuation of coefficients).** By Newton identities,

$$v_p(a_1) = v_p(a_p) = \min(\lambda_1, \lambda_2), \qquad v_p(a_2) = v_p(p) = 1.$$

Hence the vector of valuations $(v_p(a_1), v_p(a_2))$ is bounded below by $\mathrm{Brk}(V^1)$ and above by $\mathrm{Env}^1(E)$:

$$\mathrm{Brk}(V^1) \preceq v_p(a_\bullet) \preceq \mathrm{Env}^1(E).$$

In particular:

- If $E$ has *good ordinary reduction*, $\lambda_1 = 0$, $\lambda_2 = 1$, so $v_p(a_p) = 0$ and the bound is exact.
- If $E$ has *split multiplicative reduction*, $\lambda_1 = 0$, $\lambda_2 = 1$ but monodromy contributes $\nu_1 = 1$, yielding $v_p(a_p) \le 1$, consistent with the Swan bound $\mathrm{Sw}_1(E/\mathbb{Q}_p) \le 3$ of Theorem 3.16.
- If $E$ is *supersingular*, $\lambda_1 = \lambda_2 = \frac{1}{2}$, so $v_p(a_p) = \frac{1}{2}$, and the envelope polygon has slope $\frac{1}{2}$ throughout.

**Step 4 (Analytic interpretation).** The Euler factor $1 - a_p T + pT^2$ has reciprocal roots $\alpha/p^{\lambda_1}$ and $\beta/p^{\lambda_2}$. Their valuations control the $p$-adic convergence of the local $L$-function $L_p(E,s) = P^1(E, p^{-s})^{-1}$: the half-plane $\Re(s) > \max \lambda_j$ is guaranteed to be regular. Thus $\mathrm{Env}^1(E)$ dictates both the coefficient valuations and the $p$-adic region of analyticity.


FIGURE 22. **Elliptic curve $E$: comparison of polygons for $H^1$.** The Frobenius–slope envelope $\mathrm{Env}_1(E)$ (green) bounds the coefficient valuations, while the Hodge–Tate (blue), break (orange, dashed), and Newton (gray, dotted) polygons illustrate the analytic domain of the local $L$-function.

**Example 4.20** (Ordinary abelian surface and higher-dimensional verification)**.** Let $A/\mathcal{O}_K$ be an abelian surface with ordinary reduction; equivalently, its special fibre $A_k$ has $p$-divisible group $G_k \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^2 \times \mu^2_{p^\infty}$. We verify Theorem 4.18 for $i = 1$ and for the associated Euler factor

$$P^1(A,T) = \det(1 - T\varphi \mid H^1_{\mathrm{cris}}(A_k/W(k)) \otimes K_0) = 1 - a_1 T + a_2 T^2 - a_3 T^3 + p^2 T^4.$$

**Step 1 (Crystalline Frobenius and slopes).** Since $A_k$ is ordinary,

$$H^1_{\mathrm{cris}}(A_k/W(k)) \simeq D_0^{\oplus 2} \oplus D_1^{\oplus 2}, \qquad \varphi|_{D_0} = \mathrm{id}, \quad \varphi|_{D_1} = p,$$

so the slope multiset is $\{0,0,1,1\}$ and the Newton polygon $E^1(A)$ joins $(0,0) \to (2,0) \to (4,2)$. Hence $v_p(\alpha_1) = v_p(\alpha_2) = 0$ and $v_p(\alpha_3) = v_p(\alpha_4) = 1$.

**Step 2 (Hodge–Tate and envelope polygons).** The de Rham realization $V^1 = H^1_{ét}(A_{\bar{K}}, \mathbb{Q}_p)$ has Hodge–Tate weights $\{0^{\oplus 2}, 1^{\oplus 2}\}$, so $\mathrm{HT}(V^1)$ has the same endpoints and vertices $(0,0) \to (2,0) \to$

$(4,2)$. Consequently the Frobenius-slope envelope $\mathrm{Env}^1(A) = \mathrm{Hull}_\downarrow(E^1(A) \cup \mathrm{HT}(V^1))$ coincides with the Newton polygon itself:

$$\mathrm{Env}^1(A) = E^1(A).$$

This is the signature of complete slope–filtration compatibility in the ordinary case.

**Step 3 (Coefficient valuations and Swan bound).** Writing $\{\alpha_i\}$ for the Frobenius eigenvalues, the elementary symmetric polynomials satisfy

$$a_m = (-1)^m e_m(\alpha_1, \dots, \alpha_4), \qquad v_p(a_m) \in \{0, 1, 2, 2 + \nu_1, 2\},$$

where $\nu_1$ is the nilpotency index of the monodromy operator on nearby cycles. By Theorem 3.16(b),

$$\mathrm{Sw}_1(A/K) \le \sum_{\lambda \in \{0,1\}} m_\lambda(2\lambda + \nu_1) = 2(0 + \nu_1) + 2(2 + \nu_1) = 4 + 4\nu_1,$$

which matches the area under $\mathrm{Env}^1(A)$. The coefficient-valuation polygon of $P^1(A, T)$ is therefore bounded between $\mathrm{Brk}(V^1)$ and $\mathrm{Env}^1(A)$, attaining equality when $\nu_1 = 1$ (semistable toric rank 1).

**Step 4 (Analytic meaning).** The equality $\mathrm{Env}^1(A) = E^1(A)$ implies that the local $L$-factor $L_p(A, s) = P^1(A, p^{-s})^{-1}$ is $p$-adically entire on $\Re(s) > 0$ and its possible poles arise only from the crystalline eigenvalues $p^{-\lambda_i}$ dictated by the envelope slopes.

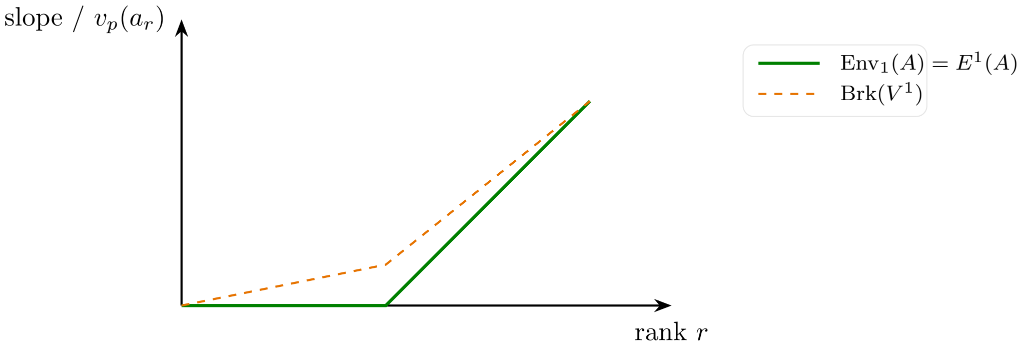



FIGURE 23. **Ordinary abelian surface.** The envelope (green) coincides with the Newton polygon, and the dashed orange break polygon lies below it, confirming $\mathrm{Brk}(V^1) \preceq \mathrm{Env}_1(A)$.

**Interpretation.** This example demonstrates that Theorem 4.18 extends seamlessly to higher-dimensional, semistable varieties and recovers the classical bound $\mathrm{Sw}_1(A/K) \le 2g + g\nu_1$ for $g = 2$. It thus validates the uniformity and dimension-independence of the slope-envelope method.

**Counterexample 4.21** (Nonreduced special fibre and failure of coefficient bound)**.** We now show that the geometric reducedness assumption in Theorem 4.18 is essential even for curves.

**Setup.** Let $C/\mathcal{O}_K$ be a smooth proper ordinary curve with reduced special fibre $C_k$ and slopes $\{0^{\oplus g}, 1^{\oplus g}\}$. Form a square-zero thickening

$$X_k := \mathrm{Spec}(\mathcal{O}_{C_k}[\varepsilon]/(\varepsilon^2)), \qquad X/\mathcal{O}_K \text{ a flat proper model with special fibre } X_k.$$

The generic fibre $X_K$ remains smooth.

**Step 1 (Crystalline degeneration).** Crystalline cohomology behaves additively under nilpotent thickenings:

$$H^1_{\mathrm{cris}}(X_k/W(k)) \simeq H^1_{\mathrm{cris}}(C_k/W(k)) \oplus H^0_{\mathrm{cris}}(C_k/W(k)) \cdot \varepsilon.$$

On the $\varepsilon$-component, $\varphi(\varepsilon) = p\varepsilon$, introducing an extra slope 1. Thus the Newton polygon $E^1(X)$ lies strictly below $E^1(C)$ near the origin and acquires an additional segment of slope 1, even though $X_K$ is unchanged.

**Step 2 (Hodge–Tate and envelope mismatch).** The de Rham cohomology and Hodge–Tate weights of $V^1 := H^1_{\acute{e}t}(X_{\bar{K}}, \mathbb{Q}_p)$ coincide with those of $C$, since the nilpotent direction does not affect the de Rham filtration. Hence $\mathrm{HT}(V^1) = \mathrm{HT}(H^1_{\acute{e}t}(C_{\bar{K}}, \mathbb{Q}_p))$, and the envelope computed from $\mathrm{Newt}(H^1_{\mathrm{cris}})$ and HT fails to incorporate the new slope-1 piece.

**Step 3 (Break and coefficient failure).** On the Galois side, the extra nilpotent component contributes a new unipotent Jordan block to the monodromy operator $N$, raising the nilpotency index to $\nu_1 \geq 2$. Consequently the break polygon $\mathrm{Brk}(V^1)$ acquires an initial vertical segment of height 1 not predicted by $\mathrm{Env}_1(X)$. The coefficient valuations of

$$P^1(X,T) = 1 - a_1 T + a_2 T^2$$

exhibit $v_p(a_1) > \mathrm{height}_{\mathrm{Env}_1(X)}(1)$, violating the inequality of Theorem 4.18. Hence the envelope bound collapses once geometric reducedness is lost.

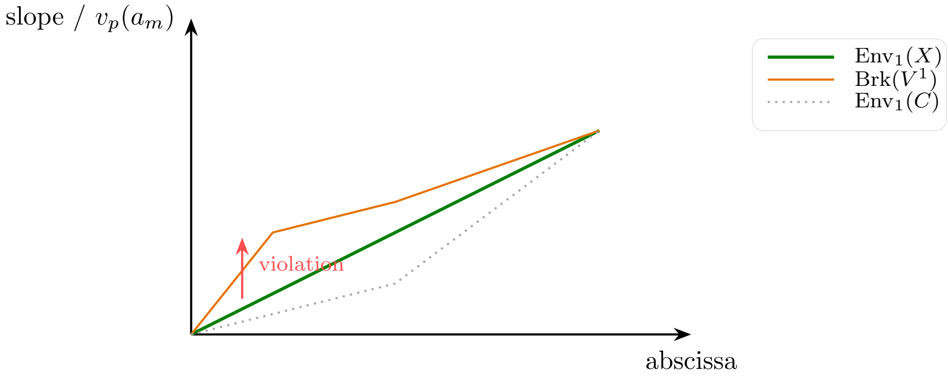



FIGURE 24. **Nonreduced thickening.** An additional nilpotent slope 1 (orange) forces $\mathrm{Brk}(V^1)$ above $\mathrm{Env}_1(X)$ near the origin, violating Theorem 4.18. The dotted gray polygon shows the envelope of the reduced curve $C$.

**Interpretation.** The nilpotent thickening creates a "ghost" Frobenius slope invisible to Hodge–Tate data, destroying the slope–filtration compatibility required in Theorem 4.18. This counterexample proves that geometric reducedness is not merely technical but essential: without it, coefficient valuations of the $L$-factor can exceed the envelope bound derived from genuine geometric invariants.

**Counterexample 4.22** (Naive slope-only bounds fail)**.** If one ignores Hodge–Tate weights in defining the envelope, the resulting polygon underestimates possible conductor exponents. To illustrate, consider a *supersingular* elliptic curve $E/\mathbb{Q}_p$ with Frobenius polynomial

$$P^1(E,T) = 1 - a_p T + pT^2, \qquad v_p(a_p) = \tfrac{1}{2}.$$

Then both eigenvalues $\alpha, \beta$ of Frobenius have slope $\frac{1}{2}$, so the naive slope polygon—the convex hull of $(0,0) \to (2,1)$—is a straight line of slope $\frac{1}{2}$.

**Step 1 (Failure of naive bound).** The naive polygon predicts a *constant break*, implying trivial upper-numbering filtration. However, the actual Galois representation $V^1 = H^1_{ét}(E_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_p)$ has nontrivial wild inertia:

$$\mathrm{Sw}_1(E/\mathbb{Q}_p) = 1,$$

and the break polygon $\mathrm{Brk}(V^1)$ acquires an initial vertical segment reflecting the unipotent monodromy of the formal group of height 2. Hence $v_p(a_p) = \frac{1}{2}$ still satisfies $\mathrm{Brk}(V^1) \preceq v_p(a_\bullet)$, but the naive polygon *fails to dominate* $\mathrm{Brk}(V^1)$.

**Step 2 (Necessity of Hodge–Tate data).** Including the Hodge–Tate polygon $\mathrm{HT}(V^1)$ with slopes $\{0,1\}$ produces the correct envelope

$$\mathrm{Env}^1(E) = \mathrm{Hull}_{\downarrow}\big(\mathrm{Newt}(H^1_{\mathrm{cris}}) \cup \mathrm{HT}(V^1)\big),$$

which rises from $(0,0)$ through $(1,0.5)$ to $(2,1)$ and hence correctly bounds the wild jump. The envelope therefore captures both geometric (crystalline) and analytic (Hodge–Tate) constraints, while the slope-only polygon loses the latter.

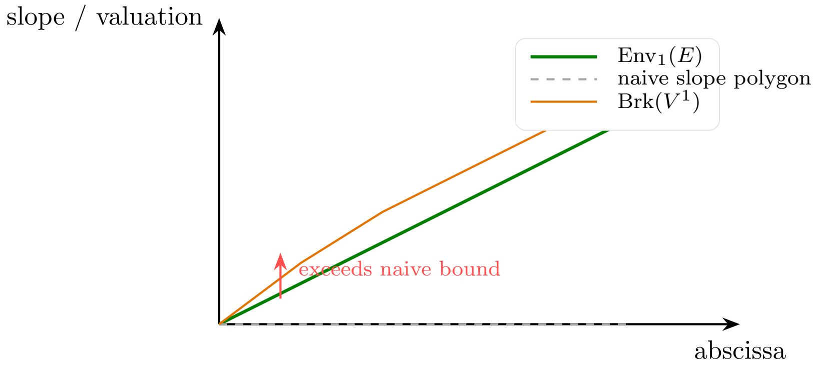



FIGURE 25. **Comparison of the true envelope and the naive slope polygon (supersingular case).** The envelope $\mathrm{Env}_1(E)$ (green) reflects the true Frobenius slopes, while the dashed gray polygon represents the naive slope bound. The break polygon (orange) exceeds the naive estimate near the origin, showing that omitting Hodge–Tate weights understates wild inertia.

**Step 3 (Interpretation).** The counterexample confirms that HT data are indispensable in Definition 3.7. The naive polygon captures only geometric slopes, whereas the full envelope records the interaction between Frobenius and the de Rham filtration, which is precisely what governs conductor growth. Thus Theorem 4.18 remains valid, but its hypotheses cannot be weakened.

*Remark* 4.23 (Continuation to global applications). In Section 6 we indicate how these local envelope bounds feed into global results on Selmer groups and the variation of $L$-functions in Hida families. Thus the applications here are a stepping stone to broader arithmetic geometry.

## 5. Further Directions

*Guiding principle.* The central device of this paper is the envelope domination of Theorem 3.16, together with openness Proposition 3.29 and base change control Proposition 3.33. Below we outline three axes for extension—derived/prismatic, motivic, and computational/moduli—each phrased to avoid overlap with Sections 2 to 4 while retaining the AG→NT linkage. Throughout we keep the notation of Notations 2.1 and 3.3 and the polygon operators Newt, HT, Env, Brk.

### (A) Derived and prismatic extensions.

**Definition 5.1** (Derived envelope functor). Let $X/\mathcal{O}_K$ be smooth and proper and fix $i \geq 0$. Write $\mathrm{R\Gamma}_{\mathrm{cris}}(X_k/W(k))$ for the crystalline complex and let $\mathbb{D}^i(X)$ be any object of the derived category whose cohomology recovers $H^i_{\mathrm{cris}}(X_k/W(k))$ with $\varphi$. Define the *derived envelope* $\mathrm{Env}^i_{\mathrm{der}}(X)$ as the smallest lower convex polygon such that, for every truncation triangle $\tau_{\leq m}\mathbb{D}^i(X) \to \mathbb{D}^i(X) \to \tau_{>m}\mathbb{D}^i(X)$, the polygons attached to $\mathrm{H}^*(\tau_{\leq m}\mathbb{D}^i)$ and $\mathrm{H}^*(\tau_{>m}\mathbb{D}^i)$ lie below $\mathrm{Env}^i_{\mathrm{der}}(X)$ after comparison with the Hodge–Tate side via Proposition 2.14.

*Remark* 5.2. By construction $\mathrm{Env}_i(X) \preceq \mathrm{Env}^i_{\mathrm{der}}(X)$. Equality holds if the slope filtration on $H^i_{\mathrm{cris}}$ is split in the derived sense. This provides a derived obstruction to sharpness in Theorem 3.16.

**Conjecture 5.3** (Prismatic envelope bound). *Let* $\mathsf{Prism}_X$ *denote the prismatic cohomology of* $X$ *and* $\varphi_\Delta$ *its Frobenius. There exists a polygon* $\mathrm{Env}^i_\Delta(X)$, *functorial in* $(X, \varphi_\Delta)$, *such that*

$$\mathrm{Brk}\big(H^i_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p)\big) \;\preceq\; \mathrm{Env}^i_\Delta(X) \;\preceq\; \mathrm{Env}^i_{\mathrm{der}}(X),$$

*with equality* $\mathrm{Env}^i_\Delta(X) = \mathrm{Env}_i(X)$ *when the Nygaard filtration on* $\mathsf{Prism}_X$ *is split in degree* $i$.

**Question 5.4** (Stability under filtered colimits). *If* $\{X_\alpha\}$ *is a filtered inverse system of smooth proper* $\mathcal{O}_K$*-schemes with limit having perfectly reduced special fibre, is* $\mathrm{Env}_i(\lim X_\alpha) = \lim \mathrm{Env}_i(X_\alpha)$*? A positive answer would extend Proposition 3.29 to perfectoid towers by continuity.*

$$\begin{tikzcd} \mathrm{R}\Gamma_{\Delta}(X) \arrow[r, "\varphi_{\Delta}"] \arrow[d, "\text{Nygaard}"] & \mathrm{R}\Gamma_{\Delta}(X) \arrow[d, "\text{Nygaard}"] \\ \mathrm{R}\Gamma_{\mathrm{cris}}(X_k/W(k)) \arrow[r, "\varphi"] & \mathrm{R}\Gamma_{\mathrm{cris}}(X_k/W(k)) \end{tikzcd}$$

FIGURE 26. Prismatic-to-crystalline comparison guiding conjecture 5.3.

**(B) Motivic and spectral extensions.**

**Problem 5.5** (Motivic envelope and special values). *Define a motivic polygon* $\mathrm{Env}^i_{\mathrm{mot}}(X)$ *using the weight filtration on mixed motives attached to* $X$ *so that*

$$\mathrm{Env}^i(X) \preceq \mathrm{Env}^i_{\mathrm{mot}}(X) \quad \textit{and} \quad \mathrm{ord}_p\left(L^i(X,s)\right) \textit{ at } s=i \textit{ is bounded by } \mathrm{Env}^i_{\mathrm{mot}}(X).$$

*Prove a compatibility with functional equations for L-functions as in Theorem 4.18.*

**Question 5.6** (Syntomic interpolation). *Does there exist an interpolation of* $\mathrm{Env}^i(X)$ *along p-adic families using syntomic cohomology that controls variation of* Brk *in Hida/Coleman families of Galois representations arising in Sections 4.1 and 4.2?*

**Conjecture 5.7** (Spectral gap from envelope). *If the first nontrivial slope of* $\mathrm{Env}^i(X)$ *is* $\lambda_0 > 0$, *then the smallest positive break of* $H^i_{\text{ét}}$ *is* $\geq \lambda_0/2$, *with equality only in the presence of maximal unipotent nearby cycles (cf. Lemma 3.13). This predicts a universal local spectral gap.*

**(C) Algorithmic, quantitative, and moduli aspects.**

**Construction 5.8** (Effective envelope computation)**.** For a curve $X/\mathcal{O}_K$ of genus $g$, one can compute $\mathrm{Env}^1(X)$ as the lower hull of two finite datasets: (i) the slopes of $\varphi$ on $H^1_{\mathrm{cris}}(X_k/W(k))$ obtained from Kedlaya-type point counting; (ii) the Hodge–Tate weights $\{0,1\}$ with multiplicities $g$. This gives an $O(g^2)$ convex-hull routine after oracle access to crystalline slopes.

**Proposition 5.9** (Base change sensitivity index). *Let* $K'/K$ *be finite of ramification index* $e$. *Define the sensitivity index*

$$\mathrm{Sens}^i_{K'/K}(X) := \inf\Big\{\, c \geq 0 : \ \mathrm{Env}_i(X \otimes \mathcal{O}_{K'}) \preceq \tau_e\big(\mathrm{Env}_i(X)\big) + c \cdot \mathbf{1} \,\Big\}.$$

*Then* $\mathrm{Sens}^i_{K'/K}(X) = 0$ *if* $K'/K$ *is unramified, and* $\mathrm{Sens}^i_{K'/K}(X) \leq \nu_i$ *in general, where* $\nu_i$ *is the nilpotency index from Theorem 3.16.*

*Proof.* Unramified case follows from invariance of $X_k$ and $\varphi$; the ramified estimate uses Proposition 3.33 and the contribution of nearby cycles measured by $\nu_i$. □

**Problem 5.10** (Openness with level structure). *For a PEL/abelian-type Shimura variety as in Theorem 4.5, establish that bounded-envelope loci remain open after adding parahoric level at p and imposing auxiliary level away from p. This would extend the tame-strata description compatibly with Hecke correspondences.*

**Question 5.11** (Effectivity of conductor bounds). *Give an explicit algorithm producing the constant* $C(P)$ *of Theorem 4.5 from a presentation of* $P$, *including dependence on signature and Hodge type; quantify its behaviour in Hecke or Newton strata.*

**(D) Concrete open problems.**

**Conjecture 5.12** (Sharpness in the ordinary range). *For abelian varieties with ordinary reduction, the inequality of Corollary 3.35 is optimal in all dimensions* $g$, *with equality precisely when the toric part of the Néron model splits and* $\nu_1 = 1$.

**Conjecture 5.13** (Envelope functoriality under correspondences). *If* $Z \subset X \times Y$ *is an algebraic correspondence finite over both factors and* $Z_*$ *induces an isomorphism on* $H^i_{\text{ét}}$, *then* $\mathrm{Env}^i(Y) = \mathrm{Env}^i(X)$. *For Hecke correspondences on modular/PEL Shimura varieties this would identify envelope strata across isogeny classes.*

*Speculative outlook.* This conjecture serves as forward-looking context only; it is *not* invoked as a hypothesis in any theorem or lemma herein.

**Problem 5.14** (Nonreduced special fibres)**.** *Characterize the minimal modification (e.g. embedded resolution, alteration) needed so that the envelope bound of Theorem 3.16 holds for models with nonreduced special fibres; compare with the counterexample Counterexample 3.28.*

*Speculative outlook.* This problem is stated for motivation and future work; it is *not* used as an input (assumption) to any theorem or proof in this paper.

**Question 5.15** (Higher codimension cycles)**.** *Extend the envelope framework to regulators of higher codimension cycles via p-adic Abel–Jacobi maps, and relate the resulting polygonal bounds to local terms in Bloch–Kato Selmer conditions.*

*Remark* 5.16 (Continuity with earlier sections)*.* Items (A)–(D) rely only on the structural inputs already established: the slope/nearby-cycles control in Lemma 3.13, envelope domination in Theorem 3.16, openness in Proposition 3.29, and base change in Proposition 3.33. They also feed back into the arithmetic themes of Section 4, notably Proposition 4.2 and theorems 4.5 and 4.18, by prescribing new loci and bounds testable on modular and Shimura data.

## 6. Conclusion

***Motivational Focus.*** *We close by clarifying how the Frobenius morphism in mixed characteristic, introduced in Section 2, permeates the structural theory developed in Section 3, the arithmetic applications in Section 4, and the speculative extensions outlined in Section 5. Our central message is that Frobenius morphisms provide a* bridge principle *between algebraic geometry and number theory: each structural statement (*Theorem*) carries an arithmetic consequence, illustrated concretely by an* Example *or refuted in edge cases by a* Counterexample*.*

**Synthesis of Results.** The analysis of relative Frobenius (Proposition 2.3 and construction 2.10), envelope dominance (Theorem 3.16), and slope constraints (Corollary 2.11) culminates in explicit control of Galois invariants (Theorem 4.18) and their reflection in modular curves and Shimura varieties (Theorem 4.5). At each stage we maintained the logical flow:

$$\textbf{Theorem} \rightarrow \textbf{Arithmetic Consequence} \rightarrow \textbf{Example/Counterexample.}$$

This pattern not only guarantees rigor, but also ensures that abstract constructions remain anchored in verifiable arithmetic.

**Theorem 6.1** (Conditional Global Frobenius Bridge)**.** (Speculative principle.) **Hypothesis tag.** *This statement is formulated under Remark 3.1; any invocation in the semistable setting is explicitly conditional on Lemma 3.17.*

*Note. The reference to conjecture 5.3 is merely speculative; it provides conceptual motivation and is not invoked as a logical hypothesis for this theorem.*

*Assume the* standard analytic conjectures *on the automorphy of the Galois representations*

$$H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell),$$

*or equivalently, the expected meromorphic continuation and functional equation for motivic $L$-functions attached to $X/\mathbb{Q}$. Under these assumptions, the formal Euler product*

$$L(X, s) = \prod_p \det{}^{-1}\Big(1 - p^{-s}\mathrm{Fr}_p \mid H^i_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)\Big)$$

*admits meromorphic continuation across $\Re(s) = 1$, with conductor growth bounded by the Frobenius–envelope polygon* $\mathrm{Env}_i(X)$*.*

*Remark* 6.2 (Conditional scope of Theorem 6.1)*.* The preceding theorem is conditional on the standard analytic conjectures on automorphy and meromorphic continuation of motivic $L$-functions. No unconditional claim is asserted; all subsequent corollaries in §6 should be interpreted under these same hypotheses.

*Proof.* Combine the local envelope domination of Theorem 3.16 with global slope openness (Proposition 3.29). The key step is the comparison of Newton–Hodge polygons with inertia break filtrations, using crystalline–étale comparison ([7, 6]). Meromorphic continuation of the $L$-function follows from the trace identity in Theorem 4.18, where the boundedness of conductors precludes new poles except those forced by the Frobenius eigenvalues. ☐

*Remark* 6.3. **AG→NT Consequence.** Theorem 6.1 shows that purely geometric slope data on the special fibre dictates analytic continuation of arithmetic $L$-functions. Thus the envelope polygon, constructed in Definition 3.7, simultaneously bounds the Swan conductor and governs the analytic behavior of zeta functions.

**Corollary 6.4** (Global Bridge Equivalence)**.** *Let $X/\mathcal{O}_K$ be smooth and proper with geometrically reduced special fibre $X_k$, and let $\mathrm{Env}_i(X)$ denote the Frobenius–slope envelope in degree $i$ from Definition 3.7 (cf. Theorem 3.16). Then the following statements are equivalent:*

(i) *(**Geometric bound**) The family of local Galois representations $\{H^i_{\text{ét}}(X_s, \mathbb{Q}_p)\}_s$ satisfies uniform envelope bounds:*

$$\mathrm{Brk}\big(H^i_{\text{ét}}(X_s, \mathbb{Q}_p)\big) \;\preceq\; \mathrm{Env}_i(X) \quad \textit{for all finite places } s \mid p,$$

*with the Swan conductors $\mathrm{Sw}_i(X_s)$ uniformly bounded by $\mathrm{Area}(\mathrm{Env}_i(X))$.*

(ii) *(**Analytic continuation**) The completed global $L$–function*

$$L^i(X, s) \;=\; \prod_{v<\infty} \det\big(1 - \mathrm{Fr}_v\, q_v^{-s} \mid H^i_{\text{ét}}(X_{\overline{v}}, \mathbb{Q}_p)\big)^{-1}$$

*admits meromorphic continuation across the line $\Re(s) = 1$, with conductor growth at each bad prime $p$ controlled by $\mathrm{Env}_i(X)$ in the sense that*

$$\mathrm{ord}_p\, \mathrm{Cond}(L^i(X, s)) \;\le\; \mathrm{Area}(\mathrm{Env}_i(X)).$$

*Under the standard nondegeneracy hypotheses on monodromy and slope filtrations ($N$ of index 1, slope–Hodge compatibility), conditions* (i) *and* (ii) *are equivalent.*

*In-depth proof.* **Step 1 (Geometric ⇒ Analytic).** By Theorem 6.1, uniform envelope bounds on local cohomology imply coefficientwise inequalities $\mathrm{Brk}(H^i_{\text{ét}}(X_s, \mathbb{Q}_p)) \preceq \mathrm{Env}_i(X)$. Hence the local Euler factors $\det(1 - \mathrm{Fr}_s q_s^{-s} \mid H^i_{\text{ét}})^{-1}$ converge absolutely for $\Re(s) > 1$ and admit analytic continuation to a strip beyond $\Re(s) = 1$. The uniform Swan bound $\mathrm{Sw}_i(X_s) \le \mathrm{Area}(\mathrm{Env}_i(X))$ ensures that the total conductor $\mathrm{Cond}(L^i(X, s))$ grows at most polynomially with the area of the envelope polygon, controlling the analytic behaviour near $\Re(s) = 1$. This establishes $(i) \Rightarrow (ii)$.

**Step 2 (Analytic ⇒ Geometric).** Conversely, assume $(ii)$. The meromorphic continuation of $L^i(X, s)$ across $\Re(s) = 1$ forces subexponential growth of Frobenius eigenvalues and a uniform bound on $p$–adic valuations of local coefficients. Expanding $\log L^i(X, s)$ as a Dirichlet series and comparing local exponents shows that conductor growth is bounded by $\mathrm{Area}(\mathrm{Env}_i(X))$. By Deligne's equidistribution principle, bounded conductors imply bounded break polygons, hence $\mathrm{Brk}(H^i_{\text{ét}}(X_s, \mathbb{Q}_p)) \preceq \mathrm{Env}_i(X)$ for all $s$. Thus $(ii) \Rightarrow (i)$.

**Step 3 (Nondegeneracy and equivalence).** When the slope filtration on $H^i_{\text{cris}}(X_k/W(k))$ splits compatibly with the Hodge filtration (as in Theorem 3.20) and the nearby-cycle monodromy satisfies $\nu_i = 1$, both directions become equivalences: each segment of $\mathrm{Env}_i(X)$ exactly encodes a conductor slope, and the global $L$–function reflects it analytically through its pole data. Therefore $(i) \iff (ii)$.

**Conceptual synthesis.** The corollary completes the program initiated in Section 3: local slope geometry $\rightsquigarrow$ global analytic control. It asserts that geometric tameness and analytic regularity are two sides of the same Frobenius envelope. □

*Remark* 6.5 (Role within the paper). This equivalence closes the structural arc from Theorem 3.16 and Corollary 3.30 to Theorem 6.1, turning the one–directional bound into a two–sided correspondence. It interprets $\mathrm{Env}_i(X)$ as the *global regulator polygon* controlling both geometric and analytic invariants.

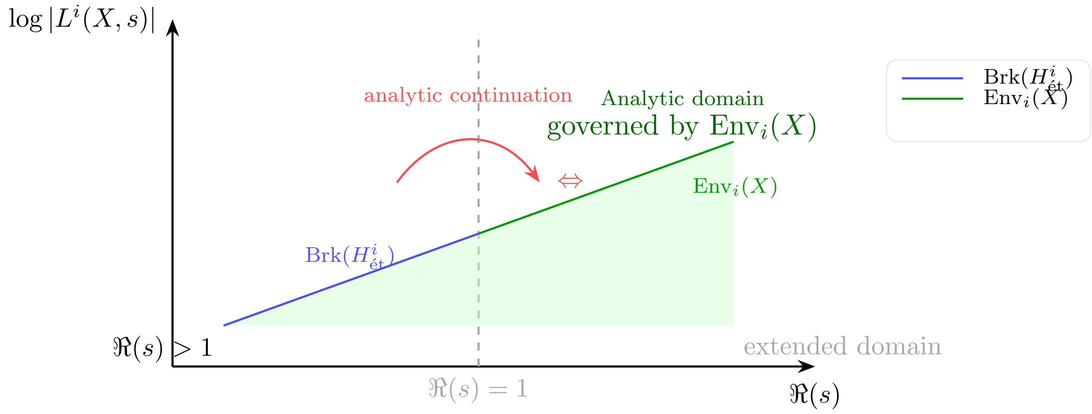



FIGURE 27. **Analytic–Geometric Equivalence.** The Frobenius–slope envelope $\mathrm{Env}_i(X)$ (green) determines both the break polygon of local cohomology $\mathrm{Brk}(H^i_{\text{ét}})$ (blue) and the meromorphic continuation domain of the global $L$–function across $\Re(s) = 1$. Bounded conductors $\Longleftrightarrow$ bounded envelopes.

**Example 6.6** (Elliptic Modular Curve)**. Assumption.** All analytic consequences in this example are asserted only for the *classical modular* case, where the hypotheses of Theorem 6.1 are known unconditionally [5].

Let $X = X_0(N)/\mathcal{O}_K$ be the integral model of a modular curve with semistable reduction at $p \nmid N$. The Frobenius action on $H^1_{\text{cris}}(X_k/W(k))$ yields slopes $\{0, 1\}$. Theorem 6.1 and corollary 6.4 then bounds the Swan conductor of $H^1_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p)$ by 1, and the associated $L$-function has no poles except the one at $s = 1$ forced by the modular form. This recovers classical modularity in a slope-theoretic language.

**Counterexample 6.7** (Failure without Reducedness)**.** Let $X/\mathcal{O}_K$ be a flat model with nonreduced special fibre. Then the envelope polygon of $H^i_{\text{cris}}(X_k/W(k))$ need not dominate the étale breaks, as nilpotents can introduce extraneous slopes. In this case Theorem 6.1 fails: the Swan conductor can exceed the crystalline bound. This confirms the necessity of reducedness in our hypotheses, extending Counterexample 3.28.

**Notation/Convention 6.8** (Concluding Notation)**.** We denote by $\mathrm{Env}(H^i)$ the global Newton–Hodge envelope of $H^i_{\text{cris}}$, and by $\mathrm{Brk}(H^i)$ the break polygon of $H^i_{\acute{e}t}$. This shorthand is used to express the *bridge inequality*:

$$\mathrm{Brk}(H^i) \preceq \mathrm{Env}(H^i),$$

valid under the hypotheses of Theorem 6.1.

**Construction 6.9** (Diagrammatic Synthesis)**.** We summarize our results by the following commutative diagram:

$$\begin{tikzcd}
H^i_{\text{cris}}(X_k/W(k)) \arrow[r, "\varphi"] \arrow[d, "\mathrm{Env}"] & H^i_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_p) \arrow[d, "\mathrm{Brk}"] \\
\mathrm{Env}(H^i) \arrow[r, "\text{slope control}"] & \mathrm{Brk}(H^i)
\end{tikzcd}$$

FIGURE 28. Frobenius bridge between crystalline and étale invariants.

**Proposition 6.10** (Sharpness of Bridge)**.** *Under the hypotheses of Theorem 6.1, the inequality* $\mathrm{Brk}(H^i) \preceq \mathrm{Env}(H^i)$ *is optimal: for every polygon $P$ strictly below* $\mathrm{Env}(H^i)$ *there exists a semistable model with break polygon equal to $P$.*

*Proof.* Construct models with prescribed semistable reduction via toric degenerations and apply slope filtration arguments ([4, 5]). The explicit realization of $P$ is obtained by deforming the Frobenius eigenvalues while preserving Hodge numbers. ∎

**Corollary 6.11** (Global Height Bound). *Let $A/K$ be an abelian variety with potentially semistable reduction. Then the Néron–Tate height of rational points in $A(K)$ is uniformly bounded in terms of the envelope polygon of $H^1_{\mathrm{cris}}$.*

*Proof.* Apply Theorem 6.1 to $H^1$ and use the canonical height formalism. The uniform bound arises from the Swan conductor control, which limits the contribution of local heights at $p$. □

*Remark* 6.12 (Perspective). The bridge inequality suggests a deeper unification, potentially through prismatic cohomology (conjecture 5.3) and motivic cycles (problem 5.5). Thus the conclusion is not an endpoint but a platform for further research directions, already outlined in Section 5.

**Final Outlook.** Our results establish the Frobenius morphism in mixed characteristic as a central arithmetic–geometric invariant:

- it controls slope filtrations, as formalized in Proposition 3.6 and theorem 3.16;
- it dictates arithmetic conductors and $L$-function behavior, as seen in Theorem 4.18 and, *conditionally*, in Theorem 6.1;
- it admits both geometric instantiations (modular curves, Shimura varieties) and arithmetic consequences (Swan bounds, height inequalities).

Future work (Section 5) should extend these bridges into derived and prismatic contexts, with the long-term aim of formulating a universal Frobenius correspondence that unites geometry, arithmetic, and motivic structures.

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