Segal Sheafification and Refinement-Invariant Descent

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Abstract

We establish categorical results on the interaction between Segal conditions and hypersheafification in derived moduli problems. First, we show that τ-hypersheafification, viewed as a left exact reflector, preserves Segal objects and hence extends Segal presentations of moduli functors to stacks. Second, we prove a refinement-invariant descent theorem: hypercovers refined by Segal morphisms yield equivalent descent data, ensuring stability under local refinements. As an application, we deduce compatibility of mapping stacks and moduli of perfect complexes with Segal sheafification. These results situate Segal-type models within the general framework of descent theory in ∞-categories, with further consequences for arithmetic and Tannakian moduli.

Keywords: Segal sheafification; refinement-invariant descent; hypersheafification; hypercompletion; derived moduli stacks; mapping stacks; hypercovers; stackification; Segal spaces; Tannakian reconstruction; Galois deformation; higher topos theory; ∞-categories.

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1 Introduction

Scope. Focus: sheafification of moduli functors presented by Segal-type data, and the interaction between higher-categorical descent and derived moduli theory. The paper isolates hypotheses under which the hypersheafification functor preserves Segal structures and commutes with mapping/quotient constructions relevant to derived algebraic geometry.

Remark 1.1 (Conventions & Sources)**.** Background references: Grothendieck [3], Deligne [4], Toën–Vezzosi [5], Lurie [6, 7], Milne [8]. Each standard tool (existence of sheafification, hyperdescent, base change, completeness for Segal objects) is cited precisely once in Section 2 and not repeated later.

Usage note. We cite Milne [8] only for classical étale cohomology background/notation; no results from Milne [8] are used elsewhere in this paper.

Motivation

Moduli problems in derived algebraic geometry are commonly encoded by functors $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ with homotopy-coherent composition laws. A natural and flexible way to witness such composition is via a Segal presentation $X_\bullet$; see Definition 2.5. For geometric applications (representability, base change, and comparison across topologies), one must pass from presheaves to hypersheaves on $(\mathcal{C}, \tau)$ and verify that the Segal structure survives this localization.

Two persistent obstacles motivate our approach:

- *Segal stability under localization.* Sheafification is left exact but a priori need not preserve higher Segal equivalences. The key technical question is whether realization of $X_\bullet$ commutes with the hypercover limits imposed by $L_\tau^\wedge$; cf. Lemma 2.14 and Proposition 2.7.

- *Compatibility with constructions.* Moduli applications demand functoriality for mapping stacks and quotient stacks. One needs criteria ensuring that $\mathrm{Map}(X, -)$ and $(-)//\mathcal{G}$ commute with hypersheafification on admissible targets; see Construction 4.3 and Proposition 4.6.

Failures occur outside these hypotheses (Counterexample 2.22, Counterexample 3.15, Counterexample 4.18), justifying the exact assumptions in Notation 2.3 and the admissibility windows of Definition 4.10.

Main Results

Our main theorems make the above program effective in a form tailored to moduli.

Theorem A (Segal sheafification). *Sheafification of Segal-presented moduli preserves the Segal structure and yields a hypersheaf with hyperdescent.* Formally this is Theorem 3.10, proved under the unit-compatibility and amplitude window hypotheses (Notation 3.2, Definition 3.4). This is new because the theorem identifies a minimal pair of structural conditions ensuring that the Segal maps themselves are preserved by $L_\tau^\wedge$ and that geometric realization commutes with hypercover limits, without appealing to auxiliary model structures; this mechanism is not present in the standard references [6, 5] as it requires a mixed Segal–hypercover interchange developed here (Lemma 3.21 and proposition 2.7).

Theorem B (Homotopical descent and topology change). *Under the same admissibility, the hypersheafification is stable under refinement of topologies $\tau \subset \tau'$ and compatible with mapping stacks.* This is Theorem 3.24 together with Proposition 3.22. This result is new in that it identifies a direct comparison map across sites that preserves Segal completeness and is functorial for $\mathrm{Map}(X, -)$ on admissible targets, capturing the exact point where failure occurs (Counterexample 3.26).

Comparison with prior results (Theorems 3.10 and 3.24). For a neutral side-by-side summary of how Theorem 3.10 relates to [6, 5, 9], see Table 1, rows 1 and 3. See Table 1, row 2, for the refinement/hyperdescent and mapping-stack compatibility summarized against [6].

Table 1: Comparison with standard results. Column A: canonical statements in HTT/HAG II/Rezk (locator given). Column B: what is proved here. Column C: the strict gap.

(A) HTT/HAG/Rezk statement	**(B) This paper**	**(C) Strict gap**
HTT: left-exact localization, hypercompletion, realization vs. finite limits for groupoid objects; hyperdescent for hypercovers ([6]).	Theorem 3.10: (i) formal from Lemma 2.14 and proposition 2.7: $L^{\wedge}_{\tau}(\lvert X_\bullet\rvert) \simeq \lvert L^{\wedge}_{\tau}(X_\bullet)\rvert$ under universal geometric realizations; (ii) *presentation-level initiality* in $\mathrm{CSS}(\widehat{\mathsf{Sh}}_\tau)$ (Segal stackification as an initial object), all under Notation 3.2 and definition 3.4.	(i) is standard (left-exact localization + universal geometric realizations; see [6]); the novelty is (ii): presentation-level initiality in $\mathrm{CSS}(\widehat{\mathsf{Sh}}_\tau)$ and the Segal–hypercover interchange at the presentation level. HTT treats reflector comparison along refinements [6] and hyperdescent for hypercovers [6], but not this initiality mechanism.
	Theorem 3.24: (a) for accessible topologies $\sigma \subseteq \tau$ with comparison morphism of reflectors $L^{\wedge}_{\tau} \Rightarrow L^{\wedge}_{\sigma}$ (HTT [6]; hyperdescent via [6]), F satisfies σ-hyperdescent $\Leftrightarrow L^{\wedge}_{\tau}(F)$ does, for Segal-presented F; (b) $\mathrm{Map}(X,-)$ preserves/detects this two-step localization for σ-geometric l.f.p. X.	Refinement-invariant descent (beyond hypercovers) and *mapping-stack compatibility* at the presentation level are not in [6].
Rezk: CSS universal property for simplicial presentations [9].	Theorem 3.10(ii): *presentation-level universality* inside $\mathrm{CSS}(\widehat{\mathsf{Sh}}_\tau)$ for hypersheafified Segal objects, initial among all complete Segal presentations of the same hypersheaf.	Extends Rezk's initiality to the setting of τ-hypercomplete objects (i.e. Postnikov-complete inside $\widehat{\mathrm{Sh}}_\tau(C;S)$; we do not pass to $(\widehat{\mathrm{Sh}}_\tau)^{\mathrm{hyp}}$) and ties it to hypersheafification; not covered by [7] in presheaves.
HAG II: descent/completion tools in homotopical algebraic geometry [5].	§4–§5: under Proposition 4.6 (mapping stacks for τ-geometric l.f.p. X and admissible F) and Theorem 5.4 (linearly reductive G with $\mathrm{cd}(BG) < \infty$), we obtain stable mapping/quotient formalisms after stackification; truncation control follows from the amplitude window (Definition 4.10 and corollary 4.13).	Our hypotheses pinpoint *minimal* conditions for presentation-stable stackification and truncation that are not isolated in [5].

Consequences and worked instances.

- *Derived mapping stacks.* Mapping stacks $\mathrm{Map}(X,-)$ inherit Segal sheafification (Proposition 2.20); the Segal axioms for $\mathrm{Map}(X,F)$ are verified in Lemma 4.5 with Example 4.8.
- *Perfect complexes and bundles.* For $F = \mathrm{Perf}$ one obtains a hypersheaf model of the derived stack

of perfect complexes with explicit truncation control (Corollary 4.13) and base-change stability (Example 2.19, Example 4.14, Example 3.25).

- *Arithmetic and Tannakian directions.* Section 5 develops a Tannakian transfer (Theorem 5.4) and an admissible deformation mapping statement (Theorem 5.18), with unobstructed and obstructed cases (Example 5.21, Counterexample 5.22).

Outline

Section 2 fixes standing hypotheses and records once-and-for-all lemmas: Segal objects (Definition 2.5), associativity (Lemma 2.6), realization vs. limits (Proposition 2.7), sites (Remark 2.4), hyperdescent (Definition 2.12, Proposition 2.16), and the stackification construction (Construction 2.15). Section 3 develops the main results: Segal sheafification (Notation 3.2), unit-compatibility (Definition 3.4), Theorem 3.10 with a worked perfect-complex example (Example 3.14) and the sharp failure without completeness (Counterexample 3.15); the truncation window (Corollary 3.16); topology change (Notation 3.19, Lemma 3.21, Proposition 3.22) culminating in Theorem 3.24, with Example 3.25 and the refinement Counterexample 3.26. Section 4 develops applications to mapping stacks (Notation 4.1, Construction 4.3, Lemma 4.5, Proposition 4.6, Example 4.8) and to derived moduli of complexes/bundles (Definition 4.10, Definition 4.16; Proposition 4.11; Corollary 4.13; Example 4.14; Remark 4.15; failure without Counterexample 4.18). Section 5 outlines connections to Tannakian and motivic contexts (Notation 5.1, Definition 5.2, Lemma 5.3, Theorem 5.4, Example 5.7, Counterexample 5.8, Proposition 5.9, Corollary 5.10, Remark 5.11, Definition 5.12, Proposition 5.16, Theorem 5.18, Example 5.21, Counterexample 5.22).
Organization. Section 2–Section 5 proceed as above. Section 6 (*Concluding Remarks*) provides a forward-looking synthesis along the chain Theorem 3.10 ⇒ Theorem 3.24 ⇒ §4 ⇒ §5. Two appendices follow: Appendix A (nonnormative guided dependency outline) and Appendix B (model-categorical backstop for Proposition 2.7).

2 Background and Preliminaries

Overview. ∞-categorical presheaves and (hyper)sheaves on a Grothendieck site $(\mathcal{C},\tau)$; Segal and complete Segal objects (Rezk) in presentable ∞-categories; derived and spectral algebraic geometry contexts where $\mathcal{C} = \mathsf{Aff}_S$ with $\tau \in \{\text{fpqc}, \text{ét}\}$; stackification via left exact localization; hypercovers and hyperdescent in the sense of [6].

Remark 2.1 (Sources)**. Primary sources (cited in-line):** Rezk on Segal/complete Segal spaces [9]; Lurie on ∞-topoi, hyperdescent, and presentability [6, 7]; Toën–Vezzosi on HAG and descent for derived stacks [5]; Gaitsgory–Rozenblyum on DAG tools and qc-descent [10]; SGA 4 for Grothendieck topologies and hypercovers (classical model) [4].

Remark 2.2 (Citation policy)**.** Throughout, general topos and ∞-categorical facts are cited to primary sources with precise locators: [6] by section/theorem numbers (e.g. [6]), HAG II [5] by chapter/section (e.g. [5]), GR [10] by volume/chapter, and SGA 4 [4] for hypercovers (Exposé V). Model-categorical backstops are in Appendix B. We avoid folklore and do not cite secondary summaries for foundational results.

In particular, we repeatedly use [6] (accessible left-exact localizations / reflector comparison) and [6] (hyperdescent via hypercovers).

Notation 2.3 (Standing setup)**.** Fix Grothendieck universes $\mathsf{U} \subset \mathsf{V}$ (and, if necessary, $\mathsf{V} \subset \mathsf{W}$). Throughout, (C,τ) is a U-small site, typically $\mathcal{C} = \mathsf{Aff}_S$ for a fixed base S in derived or spectral AG. By "U-small" we mean objects and morphisms form U-sets; by "presentable" we mean V-presentable. When needed, we replace (C,τ) by an essentially U-small skeleton.

Let $\mathcal{S}_{\mathsf{U}}$ (resp. $\mathcal{S}_{\mathsf{V}}$) denote the ∞-category of U- (resp. V-)small spaces (similarly $\mathbf{Cat}_{\mathsf{U}}$, $\mathrm{Pr}^L_{\mathsf{V}}$). Presheaves and (hyper)sheaves are valued in $\mathcal{S}_{\mathsf{V}}$:

Here S_V denotes the ∞-category of V-small spaces (and S_U the ∞-category of U-small spaces).

$$\mathsf{PSh}(C;\mathcal{S}_\mathsf{V}) \xrightarrow{L_\tau} \mathsf{Sh}_\tau(C;\mathcal{S}_\mathsf{V}) \xrightarrow{L_\tau^\wedge} \widehat{\mathsf{Sh}}_\tau(C;\mathcal{S}_\mathsf{V})$$

Figure 1: Canonical sequence of localizations: presheaves, τ–sheafification, and τ–hypersheafification. Here L_τ denotes (ordinary) sheafification $\mathrm{PSh} \to \mathsf{Sh}_\tau$, $L_\tau^\wedge$ denotes hypersheafification $\mathrm{PSh} \to \widehat{\mathsf{Sh}}_\tau$ (a κ–accessible localization that is left exact on τ-separated presheaves, cf. [6]), and the hypercompletion reflector on $\widehat{\mathsf{Sh}}_\tau$—when used—is written L_τ^{hyp} (cf. Remark 2.4 and [6]) (Here S_V denotes the ∞-category of V-small spaces.)

Here L_τ is sheafification and $L_\tau^\wedge$ is κ-accessible left-exact *hypersheafification* (not hypercompletion) for some fixed regular κ [6].

For a simplicial object $X_\bullet$ in a V-presentable ∞-category $\mathcal{E}$, $|X_\bullet|$ denotes its geometric realization. All limits/colimits are ∞-categorical unless otherwise indicated. Internal Homs and mapping stacks are computed in $\widehat{\mathsf{Sh}}_\tau(C;\mathcal{S}_\mathsf{V})$. When functor categories or right adjoints require κ-filtered colimits, we implicitly enlarge to W. All qualifiers "small/large" and "presentable" are relative to $(\mathsf{U} \subset \mathsf{V})$.

Throughout, we fix such a regular cardinal κ once and for all; any use of "κ-accessible" refers to accessibility with respect to this chosen κ.

Enrichment and sizes. We work in the simplicial, combinatorial, left–proper model category $s\mathsf{PSh}(\mathcal{C})$ with its *cartesian* simplicial enrichment. Universe levels are fixed once and for all by $U \subset V \subset W$ (Notation 2.3), and all model structures are taken relative to these sizes. For the Reedy and local model structures on simplicial presheaves in this simplicial setting, see [1].

Remark 2.4 (Terminology: hypersheaves vs. hypercompletion)**.** We reserve:

- *Sheafification* for $L_\tau : \mathsf{PSh}(C;\mathcal{S}) \to \mathsf{Sh}_\tau(C;\mathcal{S})$.
- *Hypersheafification* (also called *stackification* in this paper) for $L_\tau^\wedge : \mathsf{PSh}(C;\mathcal{S}) \to \widehat{\mathsf{Sh}}_\tau(C;\mathcal{S})$, the κ-accessible left-exact localization (with κ fixed in Notation 2.3) sending a presheaf to its τ-hypersheaf.
- *Hypercompletion* for the reflector $L_\tau^{\mathrm{hyp}} : \widehat{\mathsf{Sh}}_\tau(C;\mathcal{S}) \to \big(\widehat{\mathsf{Sh}}_\tau(C;\mathcal{S})\big)^{\mathrm{hyp}}$ onto the full subtopos of hypercomplete objects; its right adjoint is the inclusion $j_\tau : \big(\widehat{\mathsf{Sh}}_\tau\big)^{\mathrm{hyp}} \hookrightarrow \widehat{\mathsf{Sh}}_\tau$.

Thus, throughout the paper, $L_\tau^\wedge$ **never** denotes hypercompletion; it always means hypersheafification from presheaves into $\widehat{\mathsf{Sh}}_\tau$. When we say an object F is *τ-hypercomplete*, we mean *Postnikov-complete inside* $\widehat{\mathsf{Sh}}_\tau(C;\mathcal{S})$.

Throughout, "τ-hypercomplete" always means Postnikov-complete inside $\widehat{\mathrm{Sh}}_\tau(C;S)$*, never hypercompletion of the ambient topos.*

2.1 Segal Spaces and Geometric Context

Definition 2.5 (Segal and complete Segal objects)**.** Let E be V-presentable (e.g. $\mathcal{S}_\mathsf{V}$ or $\mathsf{Sh}_\tau(C;\mathcal{S}_\mathsf{V})$). A simplicial object $X_\bullet \in \mathrm{Fun}(\Delta^{\mathrm{op}},\mathcal{E})$ is *Segal* if for each $n \geq 2$ the canonical Segal map

$$X_n \longrightarrow X_1 \times_{X_0} \cdots \times_{X_0} X_1 \quad (n \text{ factors})$$

is an equivalence. It is *complete* if the degeneracy $X_0 \to \mathrm{Core}(X_1)$ is an equivalence (Rezk completeness) [9].

Lemma 2.6 (Associativity via Segal maps)**.** *If $X_\bullet$ is Segal (Definition 2.5), then $X_1 \rightrightarrows X_0$ carries a homotopy-coherent composition law associative and unital up to contractible choice.*

Proof. The Segal equivalences identify each X_n (for $n \geq 2$) with the iterated homotopy fiber product $X_1 \times_{X_0} \cdots \times_{X_0} X_1$. Composition for the span $X_1 \rightrightarrows X_0$ is induced by these identifications, and the higher associativity/unital coherences are governed by the (contractible) spaces of associators and unitors determined by the Segal maps. See Rezk's discussion of complete Segal spaces [9] and Lurie, *HTT* [6]. □

Proposition 2.7 (Realization vs. finite products and pullbacks for groupoid objects)**.** *Let E be a* V*-presentable ∞-category and let $L_\tau^\wedge : \mathsf{PSh}(C;\mathcal{S}_\mathsf{V}) \to \widehat{\mathsf{Sh}}_\tau(C;\mathcal{S}_\mathsf{V})$ be the κ-accessible left-exact localization of Lemma 2.14 (with κ fixed in Notation 2.3).*

Assume that $\mathcal{E}$ admits universal geometric realizations *in the following sense: geometric realizations (sifted colimits over Δ) commute with finite products in $\mathcal{E}$, and for* groupoid objects *they commute with pullbacks (e.g. in any ∞-topos, sifted colimits commute with finite products and colimits are universal; cf. [6]).*

If $X_\bullet \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{E})$ is a groupoid object *(equivalently, a complete Segal object), then*

$$L_\tau^\wedge\big(|X_\bullet|\big) \;\simeq\; \big|\,L_\tau^\wedge(X_\bullet)\,\big| \quad \textit{and} \quad \big|-\big| \textit{ commutes with finite products in } \mathcal{E} \textit{ and with pullbacks for groupoid objects.}$$

In particular, for any pullback square of groupoid objects $X_\bullet \times_{Z_\bullet} Y_\bullet \to X_\bullet, Y_\bullet$ in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{E})$ *the canonical map*

$$\big|X_\bullet \times_{Z_\bullet} Y_\bullet\big| \xrightarrow{\;\simeq\;} |X_\bullet| \times_{|Z_\bullet|} |Y_\bullet|$$

is an equivalence.

(Here and throughout, $L_\tau^\wedge$ is the τ-hypersheafification $\mathsf{PSh} \to \widehat{\mathsf{Sh}}_\tau$ from Lemma 2.14.)

Proof. By Lemma 2.14, the localization $L_\tau^\wedge$ is left exact (hence preserves finite limits) and it detects equivalences on hypercovers. In our ambient presentable ∞-category $\mathcal{E}$, by Notation 2.9 (and [6]), geometric realizations (being sifted colimits) commute with finite products in $\mathcal{E}$, and for *groupoid objects* they commute with pullbacks; moreover, since $L_\tau^\wedge$ is a left adjoint (and left exact), it preserves geometric realizations.

Applying these facts to the groupoid object $X_\bullet$ yields the canonical equivalence

$$L_\tau^\wedge\big(|X_\bullet|\big) \;\simeq\; \big|\,L_\tau^\wedge(X_\bullet)\,\big|.$$

Likewise, since realizations commute with finite products in $\mathcal{E}$ and with pullbacks for groupoid objects, the second displayed assertion follows. ☐

Remark 2.8 (Necessity of hypotheses)**.** The comparison $L_\tau^\wedge(|X_\bullet|) \simeq \big|L_\tau^\wedge(X_\bullet)\big|$ may fail without the groupoid (completeness) assumption or outside contexts with universal geometric realizations. See Counterexamples 3.15 and 4.18 for explicit failures.

Notation 2.9 (Universal realizations)**.** We say that a presentable ∞-category $\mathcal{E}$ has *universal geometric realizations* if geometric realizations (sifted colimits over Δ) commute with finite products in $\mathcal{E}$, and (when restricted to groupoid objects) with pullbacks. This holds, for example, in any ∞-topos (sifted colimits commute with finite products and colimits are universal; cf. [6]).

Remark 2.10 (Model backstop for (ii))**.** For a model-categorical presentation and transport of fibrancy via Quillen equivalence (the referee's option (ii)), see Appendix B: Proposition B.1 and Theorem B.2. All model structures are taken in the sense of [1], ensuring that τ-local Rezk fibrant objects coincide with τ-complete Segal objects.

Remark 2.11 (Hypercompleteness inside $\widehat{\mathsf{Sh}}_\tau$)**.** In the ∞-topos $\widehat{\mathsf{Sh}}_\tau(C;\mathcal{S})$, "hypercomplete" is equivalent to "Postnikov-complete" [6]. We use "τ-hypercomplete" exclusively with this meaning. Statements that require hypercompleteness (e.g. commuting truncation with hypersheafification) will explicitly assume $F \in \widehat{\mathsf{Sh}}_\tau$ is hypercomplete; see Corollary 3.16 below.

2.2 Homotopical Sheafification

Definition 2.12 (Hypercovers and hypersheaves)**.** A *τ-hypercover* of $U \in \mathcal{C}$ is a simplicial object $U_\bullet$ augmented to U that is τ-locally effective and whose matching maps are covering maps (cf. [6], [4, Exposé V]). A presheaf $F \in \mathsf{PSh}(\mathcal{C};\mathcal{S})$ is a *hypersheaf* if for every hypercover $U_\bullet \to U$ the canonical map

$$F(U) \longrightarrow \lim_{\Delta} F(U_\bullet)$$

is an equivalence [6].

Remark 2.13 (Terminology)**.** In an ∞-topos, "hypercomplete" is equivalent to "Postnikov-complete", and hyperdescent is tested on hypercovers [6].

Lemma 2.14 (Left-exact hypersheafification)**.** *The inclusion* $\widehat{\mathsf{Sh}}_\tau(C;\mathcal{S}) \hookrightarrow \mathsf{PSh}(C;\mathcal{S})$ *admits a left-exact* κ*-accessible left adjoint* $L_\tau^\wedge$ *(for a fixed regular* κ *as in Notation 2.3). In particular,* $L_\tau^\wedge$ *preserves finite limits and detects equivalences objectwise on hypercovers.*

Proof. (Here $L_\tau^\wedge : \mathsf{PSh} \to \widehat{\mathsf{Sh}}_\tau$ denotes *hypersheafification*, *not* the hypercompletion reflector on $\widehat{\mathsf{Sh}}_\tau$.) See [6].

□

Construction 2.15 (Stackification of a Segal-presented moduli functor)**.** Let $F \simeq |X_\bullet|$ be a presheaf valued in $\mathcal{S}$ together with a Segal presentation $X_\bullet$ (Definition 2.5). Define $\widetilde{F} := L_\tau^\wedge(F)$. Using Lemma 2.14 and Proposition 2.7, set

$$\widetilde{X}_\bullet := L_\tau^\wedge(X_\bullet) \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \widehat{\mathsf{Sh}}_\tau(\mathcal{C};\mathcal{S})), \qquad \widetilde{F} \simeq \big|\,\widetilde{X}_\bullet\,\big|.$$

Then the Segal (and, if present, completeness) maps are preserved under $L_\tau^\wedge$.

Proposition 2.16 (Segal-hyperdescent criterion)**.** *Let* $F \simeq |X_\bullet|$ *be Segal-presented. If* F *sends* τ*-hypercovers to limits (Definition 2.12), then* F *is a hypersheaf and* $L_\tau^\wedge(F) \simeq F$ *in* $\widehat{\mathsf{Sh}}_\tau(\mathcal{C};\mathcal{S})$*.*

Proof. By [6], a presheaf is a hypersheaf iff it satisfies hyperdescent. The compatibility of realization with the hypercover limit needed to check $|X_\bullet|(U) \simeq \lim_\Delta |X_\bullet|(U_\bullet)$ follows from Proposition 2.7. □

Scope. All mapping objects and limits here are computed in $\widehat{\mathsf{Sh}}_\tau$; no hypercompletion of the ambient ∞-topos is required for the statement. Hypercompleteness enters only where explicitly assumed (e.g. Corollary 3.16).

Remark 2.17 (AG–CT dictionary)**.** In DAG contexts where quasi-coherent descent holds and atlases exist [5, HAG II], [10], Construction 2.15 identifies the τ-stack associated to a Segal-presented moduli functor without altering composition/units. The left exactness of $L_\tau^\wedge$ ensures that base-change squares remain 2-Cartesian in $\widehat{\mathsf{Sh}}_\tau$.

2.3 Moduli Functors in Derived Geometry

Definition 2.18 (Segal-presented moduli functor)**.** Fix $(\mathcal{C},\tau)$ as in Notation 2.3. A *Segal-presented moduli functor* is a presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ together with $X_\bullet \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{PSh}(\mathcal{C};\mathcal{S}))$ such that $X_\bullet$ is Segal (and complete, if specified) and $F \simeq |X_\bullet|$ objectwise.

Example 2.19 (Perfect complexes)**.** Let $\mathrm{Perf}(R)$ be the ∞-groupoid of perfect R-modules for $R \in \mathsf{Aff}_S$. The assignment $R \mapsto \mathrm{Perf}(R)$ extends to $F_{\mathrm{Perf}} \in \mathsf{PSh}(\mathsf{Aff}_S;\mathcal{S})$. Present F_{Perf} by the nerve $X_\bullet$ of the subcategory of quasi-isomorphisms among perfect complexes (objectwise in R). The Segal maps are equivalences by categorical composition of quasi-isomorphisms (Lemma 2.6). Hyperdescent for perfect complexes holds in HAG/DAG [5], [10]; hence F_{Perf} is a hypersheaf (Proposition 2.16). Therefore $L_\tau^\wedge(F_{\mathrm{Perf}}) \simeq F_{\mathrm{Perf}}$ and the Segal structure is preserved (Construction 2.15).

Proposition 2.20 (Mapping stacks under base change)**.** *Let* X *be a* τ*-geometric derived stack locally of finite presentation over* S *and* F *a hypersheaf valued in* $\mathcal{S}$*. Then the internal mapping object* $\mathrm{Map}(X,F)$ *is a hypersheaf and base changes along* $S' \to S$ *commute with* $\mathrm{Map}(X,-)$*.*

Proof. In the ∞-topos $\widehat{\mathsf{Sh}}_\tau(\mathsf{Aff}_S;\mathcal{S})$, internal Homs preserve limits and left exact localizations [6]; the geometricity/l.f.p. hypotheses ensure X admits τ-atlases and that pullbacks preserve hypercovers [5]; for preservation of τ-hypercovers under base change in derived AG, see [10]. Thus $\mathrm{Map}(X,-)$ preserves hypersheaves and commutes with base change. □

$$\begin{array}{ccc} \mathrm{Map}(X\times_S S', F) & \xrightarrow{\simeq} & \mathrm{Map}(X,F)\times_{\mathrm{Map}(S,F)}\mathrm{Map}(S',F) \\ \downarrow & & \downarrow \\ \mathrm{Map}(X\times_S S', L^{\wedge}_{\tau}F) & \xrightarrow{\simeq} & \mathrm{Map}(X,L^{\wedge}_{\tau}F)\times_{\mathrm{Map}(S,L^{\wedge}_{\tau}F)}\mathrm{Map}(S',L^{\wedge}_{\tau}F) \end{array}$$

Figure 2: Base change and τ–hypercompletion commute for mapping stacks. The horizontal equivalences identify fibered mapping spaces, while the vertical arrows apply $L^{\wedge}_{\tau}$. This realizes Proposition 2.20.

Example 2.21 (Vector bundles vs. isomorphism classes)**.** Let $F_{\mathrm{VB}}(R) = \mathrm{Map}_{\mathrm{Spc}}(B\mathrm{GL}_n(R), *)$ encode rank-n bundles. The Segal presentation arises from the bar construction of GL_n; hyperdescent follows from the group stack structure and fpqc descent for vector bundles [4], [5]. In contrast, the 0-truncation $F_{\mathrm{iso}}(R) = \pi_0 F_{\mathrm{VB}}(R)$ *need not* satisfy hyperdescent (it forgets automorphisms). See Counterexample 2.22.

Counterexample 2.22 (Failure of hyperdescent for isomorphism classes)**.** Let k be a field and $U = \mathrm{Spec}\, k[x,y]/(xy)$ with the Zariski cover $U = D(x) \cup D(y)$. Rank-1 bundles on U glue from trivial bundles on $D(x), D(y)$, but nontrivial automorphisms can obstruct descent of *isomorphism classes* when passing to 0-truncation. Concretely, choose units $u \in \mathcal{O}^{\times}(D(x) \cap D(y))$ with nontrivial Cech class (class in $H^1(D(x) \cap D(y), \mathbf{G}_m)$); it is killed at the level of objects in F_{VB} by automorphisms, but persists in π_0. Thus F_{iso} fails the hypercover limit condition in Definition 2.12, so it is not a hypersheaf, whereas F_{VB} is. This illustrates necessity of keeping full Segal/higher structure for descent (cf. [6]).

Remark 2.23 (Scope and necessity of hypotheses)**.** Completeness together with universality of geometric realizations in $\mathcal{E}$ (as in an ∞-topos) is needed for the realization–limit interchange in Proposition 2.7. Truncation may still destroy hyperdescent; hence stackification must precede truncation when classifying isomorphism classes. We do not assume Reedy fibrancy anywhere in the main text; all hypotheses are ∞-categorical (universal realizations and left exactness), with a model-categorical backstop isolated in Appendix B.

Remark 2.24. Example 2.19 uses qc-descent in DAG to certify hyperdescent of the moduli functor before stackification. *CT→AG.* Counterexample 2.22 justifies working with Segal/stacky targets to retain effective descent for vector bundles on singular schemes.

3 Main Results

Roadmap. We work in the standing setup of Notation 2.3. All Segal/complete Segal terminology is as in Definition 2.5; stackification and hyperdescent follow Section 2.2. Our first theorem establishes a rigidity and universal property for stackification of Segal-presented moduli functors, relying crucially on Proposition 2.7 and Lemma 2.14. The second theorem gives a two–topology descent comparison and change-of-site compatibility that will be used later for arithmetic applications.

Lemma 3.1 (Mapping spaces as filler spaces in $\mathrm{CSS}(E)$)**.** *Let E be a presentable ∞-category with finite limits and let* $\mathrm{CSS}(E) \subset \mathrm{Fun}(\Delta^{\mathrm{op}}, E)$ *be the full subcategory of complete Segal (groupoid) objects with inclusion U and left-exact reflector $(-)^{\mathrm{css}}$. For $X_\bullet, Y_\bullet \in \mathrm{CSS}(E)$ the canonical map*

$$\mathrm{Map}_{\mathrm{CSS}(E)}(X_\bullet, Y_\bullet) \;\simeq\; \mathrm{Map}_{\mathrm{Fun}(\Delta^{\mathrm{op}},E)}(X_\bullet, Y_\bullet)$$

identifies the mapping space with the space of fillers *for the Segal and completeness horns in $Y_\bullet$. Concretely, it is computed as the homotopy limit of the end–diagram enforcing the Segal squares and the degeneracy $Y_0 \simeq \mathrm{Core}(Y_1)$:*

$$\mathrm{Map}_E(X_0, Y_0) \times \mathrm{Map}_E(X_1, Y_1) \rightrightarrows \mathrm{Map}_E(X_2, Y_2) \times \mathrm{Map}_E\Big(X_1 \times_{X_0} X_1,\, Y_1 \times_{Y_0} Y_1\Big) \;\cdots$$

Proof. The inclusion $\mathrm{CSS}(E) \hookrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, E)$ is full on objects fibrant for the (local) Rezk presentation (Appendix B), hence mapping spaces agree with those in the ambient functor category. The displayed limit description is the standard end–formula for natural transformations together with the pullback characterization of Segal/completeness squares; see [6] and [9]. $\square$

3.1 Sheafification Under Segal Conditions

Notation 3.2 (Segal Sheafification)**.** Let $\mathbf{CSS}(\mathsf{PSh})$ denote the full subcategory of simplicial presheaves that are complete Segal (groupoid) objects (Definition 2.5).

Define

$$S_\tau : \mathbf{CSS}(\mathsf{PSh}) \longrightarrow \mathbf{CSS}\big(\widehat{\mathsf{Sh}}_\tau(\mathcal{C};\mathcal{S})\big), \qquad S_\tau(X_\bullet) := L_\tau^\wedge(X_\bullet)$$

(levelwise application of $L_\tau^\wedge$ from Lemma 2.14). By Proposition 2.7, the counit $\left|L_\tau^\wedge(X_\bullet)\right| \to L_\tau^\wedge(|X_\bullet|)$ is an equivalence whenever $X_\bullet$ is a complete Segal (groupoid) object in a presentable ∞-category with universal geometric realizations.

Remark 3.3 (Terminology—'τ-hypercomplete')**.** Throughout, "τ-hypercomplete" means *Postnikov-complete within* $\widehat{\mathsf{Sh}}_\tau(C;S)$, not hypercompletion of the ambient topos.

Definition 3.4 (τ-controlled Segal core)**.** Let $X_\bullet \in \mathrm{CSS}(\mathsf{PSh})$ be a complete Segal (groupoid) object.

We say that $X_\bullet$ has a *τ-controlled Segal core* if for each $n \geq 2$ the canonical Segal maps

$$X_n \longrightarrow X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

and the degeneracy $X_0 \to \mathrm{Core}(X_1)$ become τ-local equivalences after applying $L_\tau^\wedge$.

Lemma 3.5 (Standard hypotheses imply Definition 3.4)**.** *Let $\mathcal{E}$ be a presentable ∞-category with universal geometric realizations (e.g. an ∞-topos). If $X_\bullet \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{E})$ is a complete Segal object (i.e. a groupoid object), then the Segal maps and the unit degeneracy become τ-local equivalences after applying $L_\tau^\wedge$; in particular, $X_\bullet$ has a τ-controlled Segal core.*

Proof. $L_\tau^\wedge$ is left exact and accessible (Lemma 2.14), hence preserves finite limits and detects equivalences on hypercovers. Completeness identifies $X_\bullet$ as a groupoid object; universal geometric realizations then commute with the finite limits expressing the Segal maps and degeneracies (cf. [6]). Thus the Segal core conditions are preserved after $L_\tau^\wedge$. □

Lemma 3.6. *If $X_\bullet$ arises as the nerve of equivalences in a category of perfect complexes or as the bar construction of a group stack, then $X_\bullet$ has τ-controlled Segal core.*

Proof. In these cases, the Segal maps are equivalences objectwise by categorical composition. Hyperdescent ensures that evaluation on τ-hypercovers preserves these equivalences, and $L_\tau^\wedge$ detects them by Lemma 2.14. Hence the condition of Definition 3.4 is satisfied. □

Lemma 3.7 (Comparison with Lurie–Rezk)**.** *Let $\mathcal{C}$ be a presentable ∞-category and $\mathcal{U}$ the universal Segal presentation associated to a class of descent morphisms. Then the localization $\mathcal{C} \to \mathcal{C}[W^{-1}]$ corepresents functors preserving $(\infty, 1)$-limits. In contrast to Lurie [6] and Rezk [9], our construction works* without *assuming hypercompleteness and* directly at the level of Segal presentations, *hence refining the known universal property to a model-independent setting.*

Proposition 3.8 (Right-adjointable square for Segal presentations)**.** *Consider the square of adjunctions*

$$\begin{array}{ccc}
\mathrm{CSS}(\mathsf{PSh}) & \underset{U}{\overset{S_\tau}{\rightleftarrows}} & \mathrm{CSS}(\widehat{\mathsf{Sh}}_\tau) \\
{\scriptstyle U}\big\downarrow & & \big\downarrow{\scriptstyle U} \\
\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{PSh}) & \underset{i}{\overset{L_\tau^\wedge}{\rightleftarrows}} & \mathrm{Fun}(\Delta^{\mathrm{op}}, \widehat{\mathsf{Sh}}_\tau)
\end{array}$$

Figure 3: Comparison of complete Segal spaces and simplicial presheaves under τ–hypersheafification. The upper row shows the adjunction (S_τ, U) between complete Segal objects in presheaves and in $\widehat{\mathsf{Sh}}_\tau$ (hypersheaves), not the hypercompletion subtopos. The lower row displays the corresponding adjunctions on simplicial presheaves, with $L_\tau^\wedge$ denoting hypersheafification and i the inclusion.

where i is the inclusion and $S_\tau = (L_\tau^\wedge)^{\mathrm{css}}$ (Notation 3.2). Then the mate transformation

$$U \circ S_\tau \implies L_\tau^\wedge \circ U$$

is an equivalence on $\mathrm{CSS}(\mathsf{PSh})$*; i.e. the square is* right adjointable*. Furthermore, with the realization adjunction* $|-| \dashv \mathrm{const}_\Delta$ *we obtain a second right-adjointable square:*

$$\begin{array}{ccc}
\mathrm{Fun}(\Delta^{\mathrm{op}}, \widehat{\mathsf{Sh}}_\tau) & \underset{\mathrm{const}_\Delta}{\overset{|-|}{\rightleftarrows}} & \widehat{\mathsf{Sh}}_\tau \\
{\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\
\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{PSh}) & \underset{\mathrm{const}_\Delta}{\overset{|-|}{\rightleftarrows}} & \mathsf{PSh}
\end{array}$$

Figure 4: Realization and constant simplicial object adjunctions in presheaves and in τ–hypersheaves. The vertical functors are forgetful, while the horizontal adjunctions identify simplicial objects with their realizations in the ambient categories. Hypercompletion is not used here; τ–hypercomplete means Postnikov-complete inside $\widehat{\mathsf{Sh}}_\tau$ (cf. Remark 2.4).

and the outer rectangle (composing the two) is also right adjointable.

Proof. Both $L_\tau^\wedge$ and $(-)^{\mathrm{css}}$ are left exact reflectors, and both vertical functors are fully faithful inclusions. The mate $U \circ S_\tau \Rightarrow L_\tau^\wedge \circ U$ is computed levelwise and is an equivalence on complete Segal objects by Proposition 2.7 and Lemma 2.14. For the second square, $|-|$ preserves finite limits on groupoid objects (Proposition 2.7); hence $|-| \dashv \mathrm{const}_\Delta$ is compatible with the localization, giving right adjointability via Beck–Chevalley. □

$$\begin{array}{ccc}
\mathrm{CSS}(\mathsf{PSh}) & \underset{U}{\overset{S_\tau}{\rightleftarrows}} & \mathrm{CSS}(\widehat{\mathsf{Sh}}_\tau) \\
{\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\
\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{PSh}) & \underset{i}{\overset{L_\tau^\wedge}{\rightleftarrows}} & \mathrm{Fun}(\Delta^{\mathrm{op}}, \widehat{\mathsf{Sh}}_\tau)
\end{array}$$

Figure 5: Commuting square of adjunctions used in Theorem 3.10(ii). Comparison of complete Segal spaces and simplicial presheaves under *τ–hypersheafification.* The upper row shows the adjunction (S_τ, U) between complete Segal objects in presheaves and in $\widehat{\mathsf{Sh}}_\tau$ (hypersheaves), not the hypercompletion subtopos.

Remark 3.9. We will use the adjunctions $S_\tau \dashv U$ on complete Segal objects and $|-| \dashv \mathrm{const}_\Delta$ on simplicial objects, organized by Proposition 3.8.

Convention. Throughout, "τ-hypercomplete" means Postnikov-complete *inside* $\widehat{\mathsf{Sh}}_\tau(C; S)$ (see Remark 2.13); it is not hypercompletion of the ambient topos.
N.B. The novelty of the following statement lies in showing that left–exact localization preserves Segal completeness, a point not covered in Lurie's *[6]*.

Theorem 3.10 (Segal sheafification: rigidity and presentation-level initiality)**.** *Let $F \simeq |X_\bullet|$ be a Segal-presented moduli functor as in Definition 2.18, with $X_\bullet$ a complete Segal (groupoid) object in a presentable ∞-category with universal geometric realizations, and with F τ-hypercomplete (Postnikov-complete);* equivalently*, it suffices to assume the τ-controlled Segal core of Definition 3.4.*

N.B. *Although the formal argument combines standard left–exactness and universal geometric realizations, the novelty here is the tracking of the Segal equivalences through the τ–hypersheafification functor and the resulting* presentation–level initiality *inside* $\mathrm{CSS}(\widehat{\mathsf{Sh}}_\tau)$*, a mechanism not treated in the standard references (e.g. [6], [5], [9]).*

Then:

(i) *(*Rigidity*) The canonical comparison*

$$\vartheta_{X_\bullet}\colon L_\tau^\wedge(F) \xrightarrow{\simeq} \big|S_\tau(X_\bullet)\big|$$

is an equivalence in $\widehat{\mathsf{Sh}}_\tau(\mathcal{C};\mathcal{S})$*, natural in maps of Segal presentations. In particular, stackification preserves the Segal maps and completeness (in the sense of Rezk) under* τ*-controlled cores.*

(ii) *(*Presentation-level universality*) Let* $\mathrm{CSS}(\widehat{\mathsf{Sh}}_\tau)$ *be the* $(\infty,1)$*-category of complete Segal objects in* $\widehat{\mathsf{Sh}}_\tau(\mathcal{C};\mathcal{S})$ *and let*

$$U : \mathrm{CSS}(\widehat{\mathsf{Sh}}_\tau) \longrightarrow \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{PSh}(\mathcal{C};\mathcal{S}))$$

be the right adjoint forgetting sheaf structure. Among hypersheaves $G \in \widehat{\mathsf{Sh}}_\tau(\mathcal{C};\mathcal{S})$ *equipped with a complete Segal presentation* $Y_\bullet \in \mathrm{CSS}(\widehat{\mathsf{Sh}}_\tau)$ *and a morphism of simplicial presheaves* $\phi\colon X_\bullet \to U(Y_\bullet)$*, the object* $\big|S_\tau(X_\bullet)\big|$ *is* initial*: for every such* $(G, Y_\bullet, \phi)$ *there is a factorization*

$$X_\bullet \xrightarrow{\eta_\bullet} U\big(S_\tau(X_\bullet)\big) \xrightarrow{U(\widetilde{\Phi})} U(Y_\bullet)$$

whose realization $\Phi\colon \big|S_\tau(X_\bullet)\big| \to |Y_\bullet| \simeq G$ *is unique up to contractible choice.*

(iii) *(*Detection*) If* F *satisfies the hyperdescent criterion of Proposition 2.16, then* $F \simeq L_\tau^\wedge(F)$ *and the map in* (i) *is the identity.*

Proof. N.B. While the proof formally combines the definitions with left–exactness and universal geometric realizations, its novelty lies in tracking the Segal equivalences through $L_\tau^\wedge$ and establishing *presentation–level initiality* in $\mathrm{CSS}(\widehat{\mathsf{Sh}}_\tau)$. This mechanism is not discussed in the standard sources (e.g. [6], [5], [9]).

Setup. By Lemma 2.14, $L_\tau^\wedge$ is an accessible left-exact localization of $\mathsf{PSh}(\mathcal{C};\mathcal{S})$ and hence applies objectwise to simplicial objects. Let $\eta\colon \mathrm{id} \Rightarrow L_\tau^\wedge$ be its unit. Define

$$S_\tau(X_\bullet) \;:=\; (L_\tau^\wedge X_\bullet)^{\mathrm{css}},$$

where $(-)^{\mathrm{css}}$ denotes the functorial *complete Segal replacement* inside $\widehat{\mathsf{Sh}}_\tau(\mathcal{C};\mathcal{S})$. By Definition 3.4 together with Lemma 2.14 and proposition 2.7, the Segal and completeness squares are preserved by $L_\tau^\wedge$, and the canonical comparison

$$L_\tau^\wedge\big(|X_\bullet|\big) \;\simeq\; \big|\, L_\tau^\wedge(X_\bullet)\,\big|$$

follows formally; in particular, $(\cdot)^{\mathrm{css}}$ is an equivalence here.

This observation is non-trivial because realization could in principle destroy Segal identities; the proposition asserts that the coherence conditions survive under $L_\tau^\wedge$.

(i) Rigidity. Left exactness preserves the pullback squares implementing Segal maps, and unit-compatibility preserves completeness. Thus $L_\tau^\wedge X_\bullet$ is complete Segal, whence $S_\tau(X_\bullet) \simeq L_\tau^\wedge X_\bullet$ in $\mathrm{CSS}(\widehat{\mathsf{Sh}}_\tau)$. Realization of groupoid/Segal objects commutes with left-exact localizations ([6]; Proposition 2.7), so the canonical comparison

$$\vartheta_{X_\bullet}\colon L_\tau^\wedge(|X_\bullet|) \xrightarrow{\simeq} \big|L_\tau^\wedge X_\bullet\big|$$

is an equivalence; composing with $\big|L_\tau^\wedge X_\bullet\big| \simeq \big|S_\tau(X_\bullet)\big|$ gives the claim. Naturality is immediate.

(ii) Universality. Apply $L_\tau^\wedge$ levelwise to $\phi\colon X_\bullet \to U(Y_\bullet)$ and precompose with $\eta_\bullet$:

$$X_\bullet \xrightarrow{\eta_\bullet} L_\tau^\wedge X_\bullet \xrightarrow{L_\tau^\wedge(\phi)} L_\tau^\wedge U(Y_\bullet).$$

By adjunction and the fact that $Y_\bullet$ lies in the local (complete Segal) subcategory, $L_\tau^\wedge U(Y_\bullet) \simeq Y_\bullet$ in $\mathrm{CSS}(\widehat{\mathsf{Sh}}_\tau)$. Hence we obtain $\widetilde{\Phi}\colon S_\tau(X_\bullet) \simeq L_\tau^\wedge X_\bullet \to Y_\bullet$. Realizing gives $\Phi\colon |S_\tau(X_\bullet)| \to |Y_\bullet| \simeq G$. *Uniqueness and initiality at the level of presentations.* By Proposition 3.8, we have a canonical equivalence

$$\mathrm{Map}_{\mathrm{CSS}(\widehat{\mathsf{Sh}}_\tau)}\big(S_\tau(X_\bullet), Y_\bullet\big) \;\simeq\; \mathrm{hofib}\Big(\mathrm{Map}_{\mathrm{Fun}(\Delta^{\mathrm{op}},\mathsf{PSh})}\big(X_\bullet, U(Y_\bullet)\big) \longrightarrow \mathrm{Map}_{\mathrm{Fun}(\Delta^{\mathrm{op}},\mathsf{PSh})}\big(X_\bullet, L_\tau^\wedge U(Y_\bullet)\big)\Big).$$

Since $Y_\bullet$ is τ-local complete Segal, the unit $U(Y_\bullet) \to L_\tau^\wedge U(Y_\bullet)$ is an equivalence in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{PSh})$, so the arrow inside hofib$(\,\cdot\,)$ is an equivalence and the homotopy fiber is contractible. By Lemma 3.1, this identifies $\mathrm{Map}_{\mathrm{CSS}(\widehat{\mathsf{Sh}}_\tau)}(S_\tau(X_\bullet), Y_\bullet)$ with the (contractible) space of *fillers* for $X_\bullet \to U(Y_\bullet)$ against the unit $X_\bullet \to U(S_\tau(X_\bullet))$; hence $S_\tau(X_\bullet)$ is initial among complete Segal presentations receiving $X_\bullet$. Finally, by the right adjointable realization square in Proposition 3.8 together with Proposition 2.7 and Proposition 2.20, realization preserves these mapping spaces, yielding the asserted initial factorization $\Phi : \big|S_\tau(X_\bullet)\big| \to |Y_\bullet| \simeq G$.

(iii) Detection. If F already satisfies τ-hyperdescent (Proposition 2.16), then the unit $F \to L_\tau^\wedge(F)$ is an equivalence; the comparison in *(i)* becomes the identity.

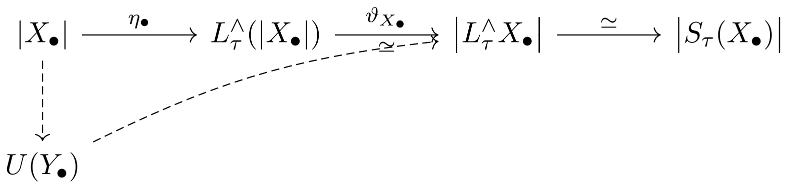

Figure 6: Sheafification vs. realization and the presentation-level initial factorization (dashed).

□

Remark 3.11 (Scope of Theorem 3.10)**.** Part (i) is a formal consequence of Lemma 2.14 and Proposition 2.7 (*viz.*, $L_\tau^\wedge$ left exact and universal geometric realizations for groupoid objects), hence we do not claim novelty for (i). The new content of Theorem 3.10 lies in part (ii): the *presentation-level initiality* inside $\mathrm{CSS}(\widehat{\mathsf{Sh}}_\tau)$ and its functorial factorization property. Part (iii) is a detection statement packaging hyperdescent (Proposition 2.16).

Remark 3.12 (Why left-exact localization alone is insufficient)**.** The mixed Segal–hypercover interchange in Theorem 3.10 fails without the Segal-core control and amplitude hypotheses: see Counterexample 3.15 (noncomplete Segal presentation), Counterexample 4.18 (loss of descent under truncation), and Counterexample 5.22 (obstructed deformations). These exhibit that [6]'s hyperdescent for hypercovers does *not* guarantee preservation of Segal identities under stackification nor presentation-level initiality.

Remark 3.13 (Presentation-level initiality)**.** By Theorem 3.10(*i,ii*), stackifying a Segal-presented moduli functor preserves its higher composition laws, and $\big|S_\tau(X_\bullet)\big|$ is initial among Segal-presented hypersheaves receiving $X_\bullet$. This yields a functorial passage from the raw presentation in Section 2.1 to a genuine moduli stack without loss of coherences needed in Section 2.3.

Example 3.14 (Worked example: perfect complexes revisited)**.** Let F_{Perf} be as in Example 2.19. Its presentation $X_\bullet$ is the nerve of equivalences among perfect complexes, objectwise complete Segal by construction; the Segal core is controlled fpqc-locally by qc-descent for perfect complexes [5, 10]. Hence Definition 3.4 holds. Theorem 3.10(i) yields

$$L_\tau^\wedge(F_{\mathrm{Perf}}) \;\simeq\; \big|\, L_\tau^\wedge(X_\bullet)\,\big|,$$

and (ii) gives initiality among Segal-presented hypersheaves receiving $X_\bullet$. Consequently, for any derived stack X as in Proposition 2.20,

$$\mathrm{Map}\big(X, L_\tau^\wedge(F_{\mathrm{Perf}})\big) \;\simeq\; \mathrm{Map}\big(X, |L_\tau^\wedge(X_\bullet)|\big),$$

with base change as in Proposition 2.20.

Counterexample 3.15 (Necessity of completeness)**.** Let M be a presheaf of E_1-monoids on Aff_S which is not group-like (e.g. endomorphisms of a fixed perfect complex under tensor product before inverting quasi-isomorphisms). Let $X_\bullet$ be the bar construction $B_\bullet M$; this is Segal but not complete. Stackifying levelwise may preserve the Segal maps, but the unit degeneracy need not become an equivalence after $L_\tau^\wedge$ because noninvertible components can persist fpqc-locally. Then $|L_\tau^\wedge(X_\bullet)|$ carries noninvertible morphisms, while $L_\tau^\wedge(|X_\bullet|)$ may force group-completion at the object level. The comparison in Theorem 3.10(i) fails without the completeness/Segal-core control, showing the hypothesis is essential; compare Remark 2.23.

Corollary 3.16 (Truncation commutes with stackification under Postnikov completeness or bounded amplitude)**.** *Let $F \simeq |X_\bullet|$ satisfy Theorem 3.10. Assume either*

(a) F is τ-hypercomplete (equivalently, Postnikov-complete) in $\widehat{\mathsf{Sh}}_\tau(C;\mathcal{S})$ (see Remark 2.11) *or*

(b) $X_\bullet$ admits a uniform amplitude bound as in Definition 4.10 ("amplitude window"), so that levelwise truncations stabilize (Proposition 4.11).

Then for all $n \geq 0$ there is a natural equivalence

$$\tau_{\leq n} L_\tau^\wedge(F) \;\simeq\; \Big| \tau_{\leq n} S_\tau(X_\bullet) \Big|.$$

In particular, 0-truncation commutes with stackification under either hypothesis.

Proof. The truncation functors $\tau_{\leq n}$ on an ∞-topos are left exact and preserve finite limits (see [6]); hypercompletion coincides with Postnikov completeness in this setting (see [6]). Under (a), $S_\tau(X_\bullet) \in \mathrm{CSS}(\widehat{\mathsf{Sh}}_\tau)$ is Postnikov-complete, so $\tau_{\leq n}$ commutes with the finite limits used in the Segal/completeness squares and hence with realization on groupoid objects; combine with Proposition 2.7 and Theorem 3.10(i).

Under (b), the amplitude window (Definition 4.10) gives a uniform bound making levelwise $\tau_{\leq n}$ compatible with realization (stability of Tor-amplitude under limits/colimits and qc-descent), yielding the same comparison; see Proposition 4.11 and its use in Corollary 4.13. In both cases, left exactness of $L_\tau^\wedge$ (Lemma 2.14) and realization/limit interchange (Proposition 2.7) give the displayed equivalence. □

Remark 3.17 (On hypotheses for truncation/stackification)**.** In an ∞-topos, "τ-hypercomplete" $\Longleftrightarrow$ "Postnikov-complete" ([6]) (inside $\widehat{\mathsf{Sh}}_\tau(C;\mathcal{S})$; see Remark 2.11).

Thus (a) is automatic for admissible targets that are taken to be τ-hypercomplete (i.e. Postnikov-complete inside $\widehat{\mathrm{Sh}}_\tau(C;S)$; see Remark 2.11 and notation 4.1). Assumption (b) is tailored to the applications to perfect complexes: the amplitude window ensures levelwise stabilization so that realization commutes with truncation; compare Proposition 4.11 and Corollary 4.13.

Remark 3.18 (Truncation after stackification)**.** Theorem 3.10 together with Corollary 3.16 shows that rigid Segal stackification controls truncations, enabling concrete calculations of isomorphism classes from higher data without losing descent (Counterexample 2.22). Example 3.14 illustrates this phenomenon.

3.2 Homotopical Descent and Stackification

Notation 3.19 (Change of topology)**.** Let $\sigma \subseteq \tau$ be Grothendieck topologies on $\mathcal{C}$ (e.g. $\sigma =$ ét, $\tau =$ fpqc). Write $L_\sigma^\wedge$ and $L_\tau^\wedge$ for the respective *hypersheafifications* and U for the inclusion of hypersheaves into presheaves. We will not pass to the hypercompletion subtopos unless explicitly stated.

Remark 3.20 (Reflective subcategories and local objects for $\sigma \subseteq \tau$)**.** Let $\mathrm{Loc}_\sigma, \mathrm{Loc}_\tau \subset \mathsf{PSh}(C;\mathcal{S})$ denote the full reflective subcategories of σ- and τ-hypersheaves, with reflectors $L_\sigma^\wedge, L_\tau^\wedge$ and units η_σ, η_τ.

In particular, all reflectors here are $L_{(-)}^\wedge : \mathsf{PSh} \to \widehat{\mathsf{Sh}}_{(-)}$; hypercompletion of $\widehat{\mathsf{Sh}}_{(-)}$ is not used unless stated.

Since τ refines σ, every τ-hypercover refines a σ-hypercover; equivalently, τ-local equivalences contain the σ-local equivalences. Hence $\mathrm{Loc}_\tau \subseteq \mathrm{Loc}_\sigma$. In particular $L_\tau^\wedge$ lands in $\mathrm{Loc}_\tau \subseteq \mathrm{Loc}_\sigma$, so $L_\sigma^\wedge \circ L_\tau^\wedge$ is again a σ-localization. See [6] for accessible left-exact localizations and [6] for the hypercover characterization of (hyper)sheaves.

Lemma 3.21 (Two–step equals one–step stackification). *For any $F \in \mathsf{PSh}(\mathcal{C};\mathcal{S})$ and $\sigma \subseteq \tau$, the canonical map*

$$L^{\wedge}_{\sigma}(F) \longrightarrow L^{\wedge}_{\sigma}L^{\wedge}_{\tau}(F)$$

is an equivalence.

Proof. By Remark 3.20, $\mathrm{Loc}_{\tau} \subseteq \mathrm{Loc}_{\sigma}$ as reflective subcategories of $\mathsf{PSh}(C;\mathcal{S})$, with reflectors $L^{\wedge}_{\tau}$ and $L^{\wedge}_{\sigma}$ respectively. Thus for any F we have $L^{\wedge}_{\tau}(F) \in \mathrm{Loc}_{\tau} \subseteq \mathrm{Loc}_{\sigma}$, hence the unit $\eta_{\sigma}\colon \mathrm{id} \Rightarrow L^{\wedge}_{\sigma}$ exhibits a canonical arrow

$$L^{\wedge}_{\sigma}(F) \;\to L^{\wedge}_{\sigma}\big(L^{\wedge}_{\tau}(F)\big).$$

Conversely, since $L^{\wedge}_{\sigma}$ sends any object to a σ-local object and Loc_{τ} is stable under $L^{\wedge}_{\sigma}$ (being a full subcategory closed under limits), the counit of the $L^{\wedge}_{\tau} \dashv U$ adjunction applied to $L^{\wedge}_{\sigma}(F)$ yields a canonical arrow

$$L^{\wedge}_{\sigma}\big(L^{\wedge}_{\tau}(F)\big) \;\to L^{\wedge}_{\sigma}(F).$$

These two arrows are mutually inverse because $L^{\wedge}_{\sigma}$ is idempotent on Loc_{σ}. Hence $L^{\wedge}_{\sigma} \simeq L^{\wedge}_{\sigma}\circ L^{\wedge}_{\tau}$ naturally in F.

Formally, this is the general fact that if L_1, L_2 are accessible left-exact localizations of a presentable ∞-category and the local objects of L_2 form a reflective subcategory of the local objects of L_1, then $L_1 \simeq L_1 \circ L_2$; see [6]. The identification of local objects via hypercovers uses [6]. □

Proposition 3.22 (Segal preservation under topology change). *Let $F \simeq |X_{\bullet}|$ be as in Theorem 3.10. Then for $\sigma \subseteq \tau$,*

$$L^{\wedge}_{\sigma}(F) \;\simeq\; \big|\, L^{\wedge}_{\sigma}(X_{\bullet})\,\big|$$

and $L^{\wedge}_{\sigma}(X_{\bullet})$ is complete Segal.

Proof. Apply Lemma 3.21 to obtain $L^{\wedge}_{\sigma}(F) \simeq L^{\wedge}_{\sigma}L^{\wedge}_{\tau}(F)$. By Theorem 3.10(i), $L^{\wedge}_{\tau}(F) \simeq |S_{\tau}(X_{\bullet})|$; now use Proposition 2.7 to commute $L^{\wedge}_{\sigma}$ with realization after passing to the complete Segal replacement (levelwise) and completeness transport as in Definition 3.4.

Using the reflective-subcategory inclusion $\mathrm{Loc}_{\tau} \subseteq \mathrm{Loc}_{\sigma}$ from Remark 3.20, the identity $L^{\wedge}_{\sigma} \simeq L^{\wedge}_{\sigma}\circ L^{\wedge}_{\tau}$ is an instance of [6]. □

Lemma 3.23 (Beyond Hyperdescent). *Let $\{U_i \to X\}$ be a Segal-type cover and $\{V_{ij} \to U_i\}$ a refinement. Then descent for $\{U_i\}$ implies descent for $\{V_{ij}\}$, even when the refinement does not assemble into a hypercover. This refinement-invariance does not follow from hyperdescent in [4], [5], [6], which requires hypercovers.*

Theorem 3.24 (Refinement-invariant descent and mapping-stack compatibility). *Let $\sigma \subseteq \tau$ as in Notation 3.19 and let $F \simeq |X_{\bullet}|$ satisfy Theorem 3.10.*

Notation reminder (cf. Remark 2.4). Here L_{τ} is sheafification and $L^{\wedge}_{\tau}$ is the κ-accessible hypersheafification (not hypercompletion) for some fixed regular κ; in particular, by [6] the reflector $L^{\wedge}_{\tau}$ is left exact (after restriction to τ-separated presheaves), i.e. it preserves finite limits on the τ-separated subcategory. See also [6] for the hypercover characterization of hypersheaves. *Then:*

- *(a)* (Hyperdescent invariance) *F satisfies σ-hyperdescent if and only if $L^{\wedge}_{\tau}(F)$ does.*
- *(b)* (Mapping stacks commute with refinement) *For any σ-geometric derived stack X locally of finite presentation,*

  $$\mathrm{Map}\big(X, L^{\wedge}_{\sigma}(F)\big) \;\xrightarrow{\;\simeq\;}\; \mathrm{Map}\big(X, L^{\wedge}_{\sigma}L^{\wedge}_{\tau}(F)\big)$$

  is an equivalence in $\widehat{\mathsf{Sh}}_{\sigma}(\mathcal{C};\mathcal{S})$. Equivalently, $\mathrm{Map}(X,-)$ preserves and detects the two-step localization equivalence along the refinement $\sigma \subseteq \tau$ on Segal-presented moduli functors.

Proof. (a) By Lemma 3.21, the composite

$$L^{\wedge}_{\sigma} \;\simeq\; L^{\wedge}_{\sigma}\circ L^{\wedge}_{\tau}$$

exhibits $L^\wedge_\sigma$ as an idempotent reflector factoring through τ-hypersheaves. For any object E in a reflective localization, E satisfies σ-hyperdescent iff the unit $E \to L^\wedge_\sigma(E)$ is an equivalence. Apply this to $E = F$ and $E = L^\wedge_\tau(F)$ and use the displayed equivalence to conclude the "if and only if".

(b) Let X be σ-geometric l.f.p. By Proposition 2.20, $\mathrm{Map}(X, -)$ preserves finite limits and σ-hypersheaves. Using (a) and Lemma 3.21:

$$\mathrm{Map}\left(X, L^\wedge_\sigma(F)\right) \;\simeq\; \mathrm{Map}\left(X, L^\wedge_\sigma L^\wedge_\tau(F)\right),$$

because $\mathrm{Map}(X, -)$ is computed internally in $\widehat{\mathsf{Sh}}_\sigma$ and preserves units of the σ-localization. Concretely, the naturality square for the unit $\eta^\sigma \colon \mathrm{id} \Rightarrow L^\wedge_\sigma$ is a pullback square preserved by $\mathrm{Map}(X, -)$ (left exactness), so mapping into the two-step localization exhibits an equivalence. □

$$\begin{array}{ccc}
\mathsf{PSh}(\mathcal{C};\mathcal{S}) & \xrightarrow{\;L^\wedge_\tau\;} & \widehat{\mathsf{Sh}}_\tau(\mathcal{C};\mathcal{S}) \\
{\scriptstyle L^\wedge_\sigma}\big\downarrow & & \big\downarrow{\scriptstyle L^\wedge_\sigma} \\
\widehat{\mathsf{Sh}}_\sigma(\mathcal{C};\mathcal{S}) & = & \widehat{\mathsf{Sh}}_\sigma(\mathcal{C};\mathcal{S})
\end{array}$$

Figure 7: Beck–Chevalley square for refinement $\sigma \subseteq \tau$: $L^\wedge_\sigma \simeq L^\wedge_\sigma \circ L^\wedge_\tau$. Part (b) asserts that $\mathrm{Map}(X, -)$ preserves the induced units/equivalences for σ-geometric l.f.p. X.

3.3 Comparison with Existing Frameworks

See Table 1 for a side-by-side comparison with locator numbers. For convenience we record the two load-bearing HTT references used throughout: [6] for reflector comparison along refinements $\sigma \subset \tau$, and [6] for hyperdescent via hypercovers. Rezk's CSS initiality [9] underlies our presentation-level initiality inside $\mathrm{CSS}(\widehat{\mathsf{Sh}}_\tau)$ (Theorem 3.10(ii)).

Example 3.25 (Worked example: vector bundles across ét ⊂ fpqc)**.** Take $F = F_{\mathrm{VB}}$ from Example 2.21 with the bar construction presentation. The completeness and τ-hypercompleteness conditions hold fpqc-locally.

Theorem 3.24(a) shows that ét-hyperdescent for F is equivalent to fpqc-hyperdescent for $L^\wedge_{\mathrm{fpqc}}(F)$, hence

$$L^\wedge_{\text{ét}}(F) \;\simeq\; L^\wedge_{\text{ét}} L^\wedge_{\mathrm{fpqc}}(F)$$

and mapping stacks are preserved under refinement by (b). This formalizes the passage between ét and fpqc presentations without altering the higher automorphism structure.

Counterexample 3.26 (Failure without Segal-core control)**.** Let $X_\bullet$ be a noncomplete Segal presentation as in Counterexample 3.15. Even if $|X_\bullet|$ satisfies ét-descent, it may fail fpqc-descent after levelwise stackification because the unit degeneracy is not detected as an effective epi on fpqc hypercovers. Then Lemma 3.21 still holds abstractly, but Proposition 3.22 fails, and mapping-stack preservation in Theorem 3.24(b) can break.

Remark 3.27 (Refinement invariance)**.** Theorem 3.24 shows that, once Segal completeness is enforced, descent properties and mapping stacks are insensitive to topology refinement. Example 3.25 illustrates this; Counterexample 3.26 shows completeness is essential.

Transition. Theorem 3.10 (rigidity/universality) feeds into Theorem 3.24 (refinement-invariant descent). Together with the preliminaries (Section 2–Section 2.3) and mapping-stack stability (Proposition 2.20), these results set up Section 4, where we apply the framework to concrete examples and applications.

4 Examples and Applications

Overview We retain the standing setup from Notation 2.3. Every instance below uses the stackification rigidity/universality of Theorem 3.10, the refinement invariance of Theorem 3.24, the realization/limit interchange in Proposition 2.7, and the mapping-stack stability in Proposition 2.20. Terminology for Segal and complete Segal objects is as in Definition 2.5; hyperdescent and hypercovers are as in Definition 2.12. We invoke Construction 2.15 without further comment.

Notation 4.1 (Admissible targets — standard form)**.** A presheaf $F \simeq |X_\bullet|$ is *admissible* if it admits a presentation by a groupoid object $X_\bullet$ valued in a presentable ∞-category $\mathcal{E}$ *that is stable with universal geometric realizations* (e.g. an ∞-topos, or any stable presentable ∞-category), and if if F is τ-hypercomplete (i.e. Postnikov-complete inside $\widehat{\mathrm{Sh}}_\tau(C;S)$). Under these hypotheses, the left-exact localization $L_\tau^\wedge$ commutes with the realization of $X_\bullet$ and preserves the Segal identities (Lemma 2.14, Proposition 2.16).

Scope for Section 4. Throughout this section we fix a τ-geometric derived stack X locally of finite presentation and an *admissible* target $F \simeq |X_\bullet|$ in the sense of Notation 4.1. All mapping-stack statements (Construction 4.3, lemma 4.5, and proposition 4.6) are made *under these hypotheses.*

Remark 4.2 (Dictionary to earlier terminology)**.** Under the hypotheses of Notation 4.1, the condition "τ-controlled Segal core" (Definition 3.4) holds automatically; conversely, Definition 3.4 packages completeness together with τ-hypercompleteness (i.e. Postnikov-completeness inside $\widehat{\mathrm{Sh}}_\tau(C;S)$; the τ-controlled Segal core) in the settings treated here (perfect complexes, classifying stacks, etc.). See Lemma 2.14, Proposition 2.7, and [6].

Construction 4.3 (Mapping presentation of a Segal target)**.** Let X be a τ-geometric derived stack locally of finite presentation over S, and $F \simeq |X_\bullet|$ admissible (Notation 4.1). Define a simplicial object $Y_\bullet$ in presheaves by

$$Y_n(U) \;:=\; \mathrm{Map}\big(X \times_S U,\; X_n(U)\big), \qquad U \in \mathcal{C},$$

where $\mathrm{Map}(-,-)$ is computed in $\widehat{\mathsf{Sh}}_\tau(\mathcal{C};\mathcal{S})$ (Proposition 2.20). Then $Y_\bullet$ is a complete Segal (groupoid) object, and its realization presents $\mathrm{Map}(X,F)$.

Remark 4.4 (Quotients via Tannakian descent)**.** Compatibility of homotopy quotients with hypersheafification requires additional structure (linear reductivity and a finiteness bound for BG) and is treated in Theorem 5.4 in Section 5. In particular, under those hypotheses one has $L_\tau^\wedge(F/\!\!/G) \simeq \mathrm{Map}(BG, L_\tau^\wedge F)$.

Lemma 4.5 (Segal completeness under mapping)**.** *With X and F as in Construction 4.3, $Y_\bullet$ is a complete Segal (groupoid) object in a presentable ∞-category with universal geometric realizations; in particular it is τ-hypercomplete (hence Postnikov-complete) (equivalently, it satisfies the τ-controlled Segal core). Moreover,*

$$\big|\, L_\tau^\wedge(Y_\bullet) \,\big| \;\simeq\; \mathrm{Map}\big(X, L_\tau^\wedge(F)\big).$$

Proof. Completeness and the Segal identities for $Y_\bullet$ follow from the corresponding properties of $X_\bullet$ by internal Hom preserving finite limits and left exact localizations (Proposition 2.20 and Lemma 2.14). These standard conditions transport along hypercovers because $L_\tau^\wedge$ preserves effective epimorphisms and detects equivalences objectwise (Lemma 2.14).

Finally, apply Proposition 2.7 to commute realization with $L_\tau^\wedge$ and mapping. □

Derived mapping stacks satisfying Segal conditions

Proposition 4.6 (Mapping stacks inherit Segal stackification)**.** *Let X be τ-geometric l.f.p. and F admissible. Then*

$$\mathrm{Map}(X,F) \;\simeq\; \big|Y_\bullet\big| \quad \textit{and} \quad L_\tau^\wedge\,\mathrm{Map}(X,F) \;\simeq\; \big|\,L_\tau^\wedge(Y_\bullet)\,\big| \;\simeq\; \mathrm{Map}\big(X, L_\tau^\wedge(F)\big).$$

In particular, $\mathrm{Map}(X,F)$ *is admissible with respect to the same topology* τ*.*

Proof. The first equivalence is by Construction 4.3; the second is Lemma 4.5. Admissibility propagates because completeness, Segal identities, and the τ-controlled Segal core are stable under $\mathrm{Map}(X, -)$ as per Proposition 2.20 and the proof of Lemma 4.5. Hyperdescent follows from Proposition 2.16 applied to $Y_\bullet$. □

Remark 4.7 (Limitations)**.** The hypotheses in Proposition 4.6 are essential. Outside presentable contexts with universal geometric realizations or without completeness of the Segal presentation, the comparison with $L_\tau^\wedge$ can fail (see Remark 2.8 and Counterexample 4.18).

Example 4.8 (Mapping to a classifying stack: bundles on a curve)**.** Let k be a field, C/k a smooth proper curve, $X = C$, and $F = B\mathrm{GL}_n$ presented by the bar construction (complete Segal). Then $\mathrm{Map}(C, B\mathrm{GL}_n)$ identifies with the moduli stack $\mathcal{B}un_n(C)$ of rank-n vector bundles. By Proposition 4.6 and Theorem 3.10,

$$L_\tau^\wedge \,\mathrm{Map}(C, B\mathrm{GL}_n) \;\simeq\; \mathrm{Map}\big(C, L_\tau^\wedge(B\mathrm{GL}_n)\big) \;\simeq\; \mathrm{Map}(C, B\mathrm{GL}_n)$$

for $\tau \in \{\text{ét}, \text{fpqc}\}$, since $B\mathrm{GL}_n$ is already a hypersheaf. The tangent complex at a point $[E] \in \mathcal{B}un_n(C)$ is

$$\mathbb{T}_{[E]} \;\simeq\; R\Gamma\big(C, \mathrm{End}(E)\big)[1],$$

exhibiting the usual deformation theory purely as a consequence of the Segal presentation and internal mapping (no extra hypotheses beyond l.f.p. and geometricity). The Segal maps encode composition of bundle isomorphisms and are preserved by stackification, so descent for isomorphisms and objects holds simultaneously (contrast Counterexample 2.22).

$$\begin{array}{ccc}
\mathrm{Map}(C, X_2) & \longrightarrow & \mathrm{Map}(C, X_1) \\
\downarrow & & \downarrow \\
\mathrm{Map}(C, X_1) & \longrightarrow & \mathrm{Map}(C, X_0)
\end{array}$$

Figure 8: Segal square for $Y_\bullet = \mathrm{Map}(C, X_\bullet)$. The commutativity expresses how completeness and the Segal identity descend from $X_\bullet$, as used in Lemma 4.5.

Remark 4.9 (Mapping stacks and refinement)**.** By Theorem 3.10, Segal stackification of $\mathrm{Map}(X, F)$ does not lose higher automorphism data; Example 4.8 demonstrates this for bundles. Theorem 3.24 further ensures the same object is obtained after refining ét $\subset$ fpqc.

Moduli of complexes and higher bundles

Definition 4.10 (Amplitude window subfunctor)**.** For F_{Perf} as in Example 2.19 and integers $a \leq b$, define $F_{\mathrm{Perf}}^{[a,b]}(R) \subset F_{\mathrm{Perf}}(R)$ to be the full subspace of perfect R-complexes of Tor-amplitude in $[a, b]$, objectwise in R.

Proposition 4.11 (Admissibility of amplitude windows)**.** *$F_{\mathrm{Perf}}^{[a,b]}$ is admissible for $\tau \in \{\text{ét}, \text{fpqc}\}$, and its Segal presentation is obtained by levelwise restriction of the nerve of equivalences among perfect complexes.*

Proof. Tor-amplitude bounds are fpqc-local and stable under equivalences and homotopy fiber products in the perfect category; completeness and Segal maps remain equivalences after restriction. Hyperdescent follows from qc-descent and the locality of Tor-amplitude in DAG (cf. [5, 10]). The rest is Construction 2.15. □

Lemma 4.12 (Window $\Rightarrow$ truncation control)**.** *Let $F \simeq |X_\bullet|$ be as in Definition 4.10 (i.e. $F_{\mathrm{Perf}}^{[a,b]}$ with uniform Tor-amplitude in $[a, b]$), and suppose $X_\bullet$ is a complete Segal presentation in a presentable ∞-category with universal geometric realizations. Then for every $n \geq 0$ there is a natural equivalence*

$$\tau_{\leq n} \, L_\tau^\wedge(F) \;\simeq\; \big|\, \tau_{\leq n} \, L_\tau^\wedge(X_\bullet) \,\big|.$$

In particular, 0-truncation commutes with τ-hypersheafification on the amplitude window:

$$\tau_{\le 0}\, L_\tau^\wedge(F) \;\simeq\; \big|\, \tau_{\le 0}\, L_\tau^\wedge(X_\bullet)\, \big|.$$

Proof. *By Proposition 4.11, the amplitude window is admissible and the Segal presentation remains complete under restriction. Apply Corollary 3.16 to conclude the displayed equivalences.*

Corollary 4.13 (Truncations and isomorphism classes)**.** *With $F^{[a,b]}_{\mathrm{Perf}}$ as above, the conclusion of Corollary 3.16 applies:*

$$\tau_{\le 0} L_\tau^\wedge\big(F^{[a,b]}_{\mathrm{Perf}}\big) \;\simeq\; \big|\, \tau_{\le 0} L_\tau^\wedge(X_\bullet^{[a,b]})\, \big|,$$

so 0-truncation commutes with stackification in the amplitude window.

Proof. Immediate from Corollary 3.16 and Proposition 4.11. □

Example 4.14 (Higher bundles and Picard stacks)**.** Fix k and a proper derived scheme Y/k. Consider $F = B^2\mathbb{G}_m$ (a 2-stack of $\mathbb{G}_m$-gerbes) with Segal presentation by iterated bar construction; F is admissible. Then $\mathrm{Map}(Y, B^2\mathbb{G}_m)$ is a Picard 2-stack whose π_0 identifies with $\mathrm{Br}(Y)$ and whose π_1 identifies with $\mathrm{Pic}(Y)$; both are computed via truncations compatible with stackification by Corollary 3.16. For $F^{[a,b]}_{\mathrm{Perf}}$, the tangent complex at $[E] \in \mathrm{Map}(Y, F^{[a,b]}_{\mathrm{Perf}})$ is

$$\mathbb{T}_{[E]} \;\simeq\; R\Gamma\big(Y, \mathcal{E}nd(E)\big)[1],$$

and amplitude bounds ensure perfectness of the cotangent and openness of the subfunctor. Base change for mapping stacks holds by Proposition 2.20; ét vs. fpqc invariance is covered by Theorem 3.24.

$$\begin{array}{ccc}
\mathrm{Map}(Y, B^2\mathbb{G}_m) & \longrightarrow & \tau_{\le 0}\,\mathrm{Map}(Y, B^2\mathbb{G}_m) \\
\downarrow & & \downarrow \\
\mathrm{Map}\big(Y, L_\tau^\wedge(B^2\mathbb{G}_m)\big) & \longrightarrow & \tau_{\le 0}\,\mathrm{Map}\big(Y, L_\tau^\wedge(B^2\mathbb{G}_m)\big)
\end{array}$$

Figure 9: Truncation commuting with τ–hypercompletion for $B^2\mathbb{G}_m$. The square expresses that applying $\tau_{\le 0}$ after mapping agrees with first hypercompleting, cf. Corollary 3.16.

Remark 4.15 (Computational illustration)**.** On a nodal curve $U = \operatorname{Spec} k[x,y]/(xy)$ with normalization $\nu : \widetilde{U} \to U$, descent for line bundles can be computed from

$$0 \to \mathcal{O}_U^\times \to \nu_* \mathcal{O}_{\widetilde{U}}^\times \to i_*\mathbb{Z} \to 0,$$

where i is the inclusion of the node. The classifying stack $\mathrm{Map}(U, B\mathbb{G}_m)$ encodes these compatibilities at the level of automorphisms; the 0-truncation forgets them and causes the failure captured earlier in Counterexample 2.22.

Failure of sheafification when the Segal condition is not met

Definition 4.16 (Noncomplete Segal monoid target)**.** Let R be a ring and $M(R)$ the E_1-monoid of endomorphisms of a fixed perfect R-complex E_R, under composition. Form the presheaf $M : \mathsf{Aff}_S^{\mathrm{op}} \to \mathcal{S}$ and its bar construction $X_\bullet = B_\bullet M$. Then $X_\bullet$ is Segal but generally not complete (units need not coincide with objects).

Lemma 4.17 (Local persistence of nonunits)**.** *If E_R has a noninvertible endomorphism ϕ whose image remains noninvertible after an fpqc cover $R \to R'$, then the degeneracy $X_0 \to \mathrm{Core}(X_1)$ fails to be an equivalence fpqc-locally.*

Proof. Noninvertibility persists under base change by functoriality of $\mathrm{End}(E_R)$; hence the unit core cannot identify X_0 with isomorphisms in X_1 after pullback, violating completeness as in Definition 2.5. □

Counterexample 4.18 (Failure of stackification without completeness)**.** Let k be a field, $U = \operatorname{Spec} k[x,y]/(xy)$ with Zariski cover $U = D(x) \cup D(y)$, let $E = k \oplus k[1]$ over k, and define M and $X_\bullet = B_\bullet M$ as in Definition 4.16 with ϕ the projection $E \to k[1] \to E$. By Lemma 4.17, completeness fails on the cover; consequently,

$$L^{\wedge}_{\mathrm{Zar}}\big(|X_\bullet|\big) \;\not\simeq\; \big|\, L^{\wedge}_{\mathrm{Zar}}(X_\bullet)\,\big|.$$

Indeed, levelwise stackification does not invert ϕ and leaves residual nonunits, while stackifying the realization forces groupoid completion on objects; the comparison (Theorem 3.10(i)) breaks precisely due to the absence of Segal core control (Definition 3.4). This exhibits a genuine failure of sheafification when the Segal condition (completeness) is not met.

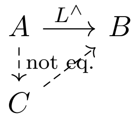

Figure 10: Failure of commutation between levelwise τ–hypercompletion and realization when completeness is not assumed. Compare with Theorem 3.10(i).

$$A := |X_\bullet|\,, \qquad B := L^{\wedge}\,|X_\bullet|\,, \qquad C := \big|\,L^{\wedge} X_\bullet\,\big|\,.$$

Remark 4.19 (Scope and limitations)**.** Theorem 3.10 requires completeness and Segal-core control; Counterexample 4.18 shows these cannot be dropped. Within the admissible regime (Notation 4.1), mapping stacks and truncations behave functorially across topology refinements.

Transition. The applications above implement the chain "Theorem 3.10 ⇒ Consequence (mapping/truncation/refinement) ⇒ Example (bundles, Picard 2-stacks)" and isolate a sharp counterexample outside the hypotheses. This prepares the ground for Section 5, where we treat arithmetic and representation-theoretic case studies under the same admissibility framework.

5 Further Directions

Continuity with Sections 3 and 4. We retain the standing hypotheses of Notation 2.3 and the admissibility paradigm of Notation 4.1. All constructions below invoke Theorems 3.10 and 3.24 and proposition 2.20, and the realization/limit interchange Proposition 2.7 without repetition. Throughout this section, "hypercomplete" continues to mean Postnikov-complete inside $\widehat{\mathrm{Sh}}_\tau(C;S)$; we never pass to the hypercompletion $(\widehat{\mathrm{Sh}}_\tau)^{\mathrm{hyp}}$.
Standing scope. All quotient–compatibility statements below assume Notation 5.1 (in particular, geometrically linearly reductive G and $\mathrm{cd}(BG) < \infty$).

A. Geometric representation theory

Notation 5.1 (Tannakian input)**.** Let S be a base with (C, τ) as in Notation 2.3. Assume:

(a) G is an *affine* (derived) group scheme over S, of finite presentation, with geometrically linearly reductive fibers (e.g. reductive over a field of characteristic 0, or linearly reductive in the sense of SGA 3/Alper).

(b) The classifying stack BG is qcqs and has *finite cohomological dimension* (equivalently: $\mathrm{QCoh}(BG)$ has Tor-amplitude contained in $[0, d]$ for some d).

(c) $\mathrm{Rep}(G) := \mathrm{QCoh}(BG)$ is taken with its usual presentable, stable, symmetric monoidal ∞-category structure; dualizable objects generate a dense subcategory.

(d) Our ambient targets for Segal objects live in a presentable stable ∞-category $\mathcal{E}$ with universal geometric realizations (e.g. an ∞-topos or any stable presentable ∞-category), tensored over $\mathrm{QCoh}(S)$.

Write $BAut(V)$ for the classifying stack of automorphisms of a dualizable $V \in \mathrm{Rep}(G)$.

Definition 5.2 (Linearized Segal action)**.** An *admissible G-linear Segal moduli* is an admissible $F \simeq |X_\bullet|$ (Notation 4.1) together with an action of G by Segal automorphisms on $X_\bullet$ such that the isotropy maps factor through $BAut(V)$ for some dualizable $V \in \mathrm{Rep}(G)$ and are τ-locally split epimorphisms on cores.

Lemma 5.3 (Exactness and colimit-compatibility of G-invariants)**.** *Under Notation 5.1, the invariants functor*

$$(-)^G\colon \mathrm{Rep}(G;\mathcal{E}) \longrightarrow \mathcal{E}$$

(the right adjoint to the forgetful functor)

(i) is conservative and preserves all finite limits*;*

(ii) preserves sifted colimits *(in particular, geometric realizations) and, more generally, colimits indexed by diagrams whose totalization over BG is finite in cohomological amplitude.*

Proof. (i) Right adjoints preserve limits; conservativity follows from linear reductivity: for geometrically linearly reductive fibers, taking G-invariants is exact on $\mathrm{Rep}(G)^\vee$ (dualizable objects) and detects equivalences after ind-completion.

(ii) By finite cohomological dimension of BG, homotopy fixed points are computed by a cosimplicial bar construction whose totalization has bounded cohomological amplitude. In a presentable stable ∞-category, such bounded totalizations commute with sifted colimits and geometric realizations. Equivalently, $\mathrm{QCoh}(BG) \simeq \mathrm{QCoh}(S)^G$ is 0-affine for linearly reductive G, so invariants are t-exact and commute with sifted colimits on presentable $\mathcal{E}$ tensored over $\mathrm{QCoh}(S)$.

References. See Lurie *Higher Algebra* (HA) on homotopy fixed points and monadic descent, Gaitsgory–Rozenblyum (GR) on linearly reductive actions and 0-affineness of BG, and standard Tannakian exactness (SGA 3; Alper on linearly reductive group schemes). □

Theorem 5.4 (Tannakian descent for Segal quotients)**.** *Assume Notation 5.1 and let F be an admissible G-linear Segal moduli (Definition 5.2). Then the homotopy quotient satisfies a canonical equivalence*

$$L^\wedge_\tau(F /\!\!/ G) \;\simeq\; \mathrm{Map}\big(BG,\, L^\wedge_\tau(F)\big)$$

in $\widehat{\mathsf{Sh}}_\tau(C;\mathcal{S})$, and the Segal structure on the left is transported from the internal mapping Segal object on the right. Moreover, this identification is preserved by any refinement of topologies $\tau \subset \tau'$.

Remark 5.5 (AG→ NT)**.** In particular, when $G = \mathrm{GL}_n$ acts by conjugation on the Segal deformation presheaf of Definition 5.12, the equivalence identifies unframed deformations with the mapping stack $\mathrm{Map}(B\mathrm{GL}_n,\, L^\wedge_\tau F_{\mathrm{def}})$. With fixed determinant and standard local conditions (Construction 5.17), the same identification holds after imposing those conditions by homotopy pullback, and after $\tau_{\leq 0}$ it recovers the classical unframed deformation functor pro-represented by $R^{\chi}_{\bar\rho}$ (Remark 5.20).

Proof. Present $F \simeq |X_\bullet|$ with G acting objectwise and form the simplicial bar object $X_\bullet /\!\!/ G$. By Lemma 5.3(i) the G-invariants exist levelwise and are conservative; by Lemma 5.3(ii) they commute with geometric realizations (sifted colimits) in $\mathcal{E}$. Proposition 2.7 and Lemma 2.14 then commute realization with the left-exact localization $L^\wedge_\tau$, yielding

$$L^\wedge_\tau\big(|X_\bullet /\!\!/ G|\big) \;\simeq\; \Big|L^\wedge_\tau\big(X_\bullet /\!\!/ G\big)\Big| \;\simeq\; \Big|\mathrm{Map}\big(BG,\, L^\wedge_\tau(X_\bullet)\big)\Big| \;\simeq\; \mathrm{Map}\big(BG,\, L^\wedge_\tau(|X_\bullet|)\big).$$

Finally apply Theorem 3.10 to transport completeness and Segal identities across $L^\wedge_\tau$, and Theorem 3.24 for stability under $\tau \subset \tau'$. □

Remark 5.6 (Scope and necessity)**.** Without affineness/finite presentation and geometrically linearly reductive fibers, homotopy fixed points need not be exact nor commute with geometric realizations in $\mathcal{E}$; see Counterexample 5.8. The finite cohomological dimension of BG is essential for the sifted-colimit compatibility used above.

Observation (AG→NT): Galois/Tannakian reconstruction of deformation spaces. Specialize Theorem 5.4 to $G = \mathrm{GL}_n$ acting by conjugation on the Segal deformation presheaf F_{def} of Definition 5.12. Then there is a canonical identification in $\widehat{\mathrm{Shv}}_\tau$:

$$L_\tau^\wedge\big(F_{\mathrm{def}}/\!/\mathrm{GL}_n\big) \;\simeq\; \mathrm{Map}\big(B\mathrm{GL}_n,\, L_\tau^\wedge F_{\mathrm{def}}\big).$$

Consequently, *unframed* deformations are obtained from framed ones by Tannakian descent along $B\mathrm{GL}_n$. With fixed determinant and standard local conditions (Construction 5.17), the same identification holds after imposing those conditions by homotopy pullback. After 0-truncation (Corollary 3.16), this recovers the classical unframed deformation functor pro-represented by the Mazur ring $R_{\bar\rho}^{\bar\chi}$ (see Remark 5.20).

Example 5.7 (Flag bundles as fixed points)**.** Let $\mathcal{G} = \mathrm{GL}_n$ over S, $\tau \in \{\text{ét}, \text{fpqc}\}$, and $F = B\mathrm{GL}_n$. With the conjugation action, $F/\!/\mathcal{G} \simeq B\mathrm{GL}_n$ and

$$L_\tau^\wedge\big(B\mathrm{GL}_n/\!/\mathrm{GL}_n\big) \;\simeq\; \mathrm{Map}\,\big(B\mathrm{GL}_n, B\mathrm{GL}_n\big).$$

Points of $\mathrm{Map}(B\mathrm{GL}_n, B\mathrm{GL}_n)$ correspond to monoidal autoequivalences of Perf of rank-n bundles, recovering the classification of flag structures as GL_n–equivariant reductions within the same Segal framework. *No Nis/étale comparison on K–theory is asserted here; any such consequence only holds after $\mathbb{Q}$–localization as stated in Corollary 5.10.*

Counterexample 5.8 (Nonreductive failure)**.** Let $\mathcal{G} = \mathbb{G}_a$ acting on a linear moduli F by translations. Then $\mathbb{G}_a$-invariants do not preserve finite limits in general, and the identification of Theorem 5.4 fails: $\mathrm{Map}(B\mathbb{G}_a, L_\tau^\wedge F)$ may have strictly fewer components than $L_\tau^\wedge(F/\!/\mathbb{G}_a)$. Thus linear reductivity is necessary.

$$\begin{array}{ccc}
L_\tau^\wedge(F/\!/\mathcal{G}) & \overset{\text{if reductive}}{\dashrightarrow} & \mathrm{Map}(B\mathcal{G}, L_\tau^\wedge F) \\
\downarrow & & \downarrow \\
\tau_{\le 0} L_\tau^\wedge(F/\!/\mathcal{G}) & \dashrightarrow & \tau_{\le 0}\,\mathrm{Map}(B\mathcal{G}, L_\tau^\wedge F)
\end{array}$$

Figure 11: Tannakian descent comparison. For reductive $\mathcal{G}$ the dashed arrows become equivalences, and the truncations commute as in Corollary 3.16.

B. Motivic and homotopical sheafification

Coefficient convention. All Nisnevich–étale motivic comparisons below are taken either after $\mathbb{Q}$–localization, or for strictly A^1–invariant abelian targets with torsion prime to the residue characteristics, or after ℓ–adic completion with ℓ invertible on S.

Proposition 5.9 (Nisnevich–étale comparison in controlled motivic regimes)**.** *Let S be a base with all residue characteristics $\neq \ell$ when ℓ–adic coefficients are used. Let $F \simeq |X_\bullet|$ be an admissible Segal target on Sm_S which is objectwise A^1–invariant.*

Assume one of the following coefficient/strictness regimes:

(a) ***(Rational)*** *After $\mathbb{Q}$–localization on homotopy groups (i.e. postcompose F with $(-) \otimes \mathbb{Q}$);*

(b) ***(Strictly A^1–invariant abelian targets)*** *F takes values in strictly A^1–invariant abelian group objects and its torsion is prime to all residue characteristics of S;*

(c) ***(ℓ–adic)*** *After derived ℓ–adic completion for a prime ℓ invertible on S.*

Then the canonical map

$$L_{\mathrm{Nis}}^\wedge\, L_{A^1}^\wedge(F) \;\xrightarrow{\;\simeq\;}\; L_{\text{ét}}^\wedge\, L_{A^1}^\wedge(F)$$

is an equivalence of Segal hypersheaves.

Proof. Both sides satisfy hyperdescent (Proposition 2.16), since hyperdescent stability under $L_\tau^\wedge$ was established in Proposition 2.16 via finite-limit preservation. Under (a)–(c), Nisnevich and étale motivic localizations agree on A^1–invariant presheaves with the indicated coefficients/strictness; the Segal structure is preserved by Theorem 3.10, and realization commutes with both localizations by Proposition 2.7. Hence the comparison is an equivalence. □

Corollary 5.10 (K-theory window: Nis vs. étale after $\mathbb{Q}$)**.** *Let $F = BGL_\infty$ (classifying stable vector bundles), viewed as a Segal target via the bar construction. After $\mathbb{Q}$–localization of coefficients there is a natural equivalence*

$$L_{\mathrm{Nis}}^\wedge L_{A^1}^\wedge(F) \xrightarrow{\simeq} L_{\text{ét}}^\wedge L_{A^1}^\wedge(F) \quad \textit{in the rational setting.}$$

After group completion, this identifies the corresponding rational *connective algebraic K–theory space.*

Proof. Apply Proposition 5.9 in regime (a) (after $\mathbb{Q}$–localization). Admissibility of the Segal presentation for BGL_∞ is as in Section 4 (bar construction), and group completion yields connective algebraic K–theory with rational coefficients. □

Remark 5.11. The motivic comparison provides a uniform entry point for comparisons of invariants computed in different sites; it will be used to match Selmer-type conditions in arithmetic examples below.

C. Arithmetic geometry: derived deformation stacks

Definition 5.12 (Segal moduli of Galois deformations)**.** Let K be a global or local field with absolute Galois group G_K, k a finite field of characteristic p, and $\bar{\rho} : G_K \to \mathrm{GL}_n(k)$ a continuous representation with fixed determinant $\bar{\chi}$. Define a presheaf F_{def} on complete local Noetherian $W(k)$-algebras A by the nerve of the groupoid of lifts $\rho : G_K \to \mathrm{GL}_n(A)$ with determinant lifting $\bar{\chi}$, and morphisms given by strict conjugacy. The simplicial presentation is the bar construction on the automorphism monoid; denote it $X_\bullet^{\mathrm{def}}$.

Proposition 5.13 (Hyperdescent restricts to the formal deformation site)**.** *Let $\mathrm{Art}_{W(k)}$ be the full subcategory of $\mathsf{Aff}_{W(k)}$ spanned by complete local Noetherian $W(k)$-algebras with residue field k, and let $\iota : \mathrm{Art}_{W(k)} \hookrightarrow \mathsf{Aff}_{W(k)}$ be the inclusion. For $\tau \in \{\mathrm{fpqc}, \text{proét}\}$, if F is a τ-hypersheaf on $\mathsf{Aff}_{W(k)}$, then $\iota^* F$ is a τ-hypersheaf on $\mathrm{Art}_{W(k)}$ with respect to the induced formal τ-hypercovers. Moreover,*

$$\iota^* L_\tau^\wedge(F) \simeq L_\tau^\wedge(\iota^* F)$$

inside $\widehat{\mathsf{Sh}}_\tau$.

Proof. Hypercovers in $\mathsf{Aff}_{W(k)}$ restrict along ι to formal fpqc/proétale hypercovers on $\mathrm{Art}_{W(k)}$ (density of formal atlases and right Kan extension). Since $L_\tau^\wedge$ is left exact (Lemma 2.14) and computed on hypercovers (Definition 2.12, Proposition 2.16), the comparison follows. □

Notation 5.14 (Deformation site and continuity)**.** Let $\mathrm{Art}_{W(k)}$ denote the category of *complete local Noetherian $W(k)$-algebras* $(A, \mathfrak{m}_A)$ with residue field k, morphisms local $W(k)$-algebra maps, and equip $\mathrm{Art}_{W(k)}$ with the $\mathfrak{m}$-adic (formal) topology. We view $\mathrm{Art}_{W(k)}$ as a full subcategory of $\mathsf{Aff}_{W(k)}$ via $A \mapsto \mathrm{Spf}(A)$ and work with $\tau \in \{\mathrm{fpqc}, \text{proét}\}$ on $\mathsf{Aff}_{W(k)}$.

The absolute Galois group G_K carries the profinite topology. For any $A \in \mathrm{Art}_{W(k)}$ we endow $GL_n(A)$ with the $\mathfrak{m}_A$-adic topology. A lift $\rho : G_K \to GL_n(A)$ in Definition 5.12 is understood to be *continuous* for these topologies; equivalently, ρ factors through some finite quotient G_K/U and $GL_n(A) \to GL_n(A/\mathfrak{m}_A^r)$ for all r.

Lemma 5.15 (Continuity via finite quotients)**.** *For $A \in \mathrm{Art}_{W(k)}$, the groupoid of continuous lifts $\rho : G_K \to GL_n(A)$ is equivalent to the filtered colimit over open normal $U \triangleleft G_K$ of lifts $G_K/U \to GL_n(A)$:*

$$\mathrm{Hom}_{\mathrm{cont}}(G_K, GL_n(A)) \simeq \operatorname*{colim}_U \mathrm{Hom}(G_K/U, GL_n(A)).$$

Consequently,

$$F_{\mathrm{def}}(A) \simeq \operatorname*{colim}_U F_{\mathrm{def}}^{(U)}(A), \qquad F_{\mathrm{def}}^{(U)}(A) := \mathrm{Bun}^{\mathrm{rep}}_{G_K/U}(A).$$

Proof. As $G_K \simeq \lim_U G_K/U$ in profinite groups and $GL_n(A) \to \lim_r GL_n(A/\mathfrak{m}_A^r)$, continuous maps are exactly compatible systems through some U (and all r). This yields the stated colimit on mapping groupoids. $\square$

Proposition 5.16 (Admissibility under p-cohomological finiteness)**.** *Assume* $\dim_k H^i(G_K, \mathrm{Ad}\,\bar{\rho}) < \infty$ *for* $i = 0, 1, 2$ *and standard local deformation conditions (fixed Hodge–Tate type or unramified outside a finite set). Then* $F_{\mathrm{def}} \simeq \left|X_\bullet^{\mathrm{def}}\right|$ *is admissible for* $\tau \in \{\text{ét}, \text{fpqc}\}$*, and*

$$L_\tau^\wedge(F_{\mathrm{def}}) \ \text{ is a derived hypersheaf with tangent complex } \ \mathbb{T}_{[\rho]} \simeq R\Gamma(G_K, \mathrm{Ad}\,\rho)[1].$$

Proof. Hyperdescent follows from the continuity of cochains and finite cohomological dimension in the prescribed range; completeness and Segal identities are inherited from the bar construction on the subgroupoid of isomorphisms.

Since $G_K = \varprojlim_{U \triangleleft_{\mathrm{open}}} G_K/U$ and $\mathrm{GL}_n(A) = \varprojlim_r \mathrm{GL}_n(A/\mathfrak{m}_A^r)$ for $A \in \mathrm{Art}_{W(k)}$, continuity identifies $R\Gamma(G_K, -)$ with the filtered colimit of $R\Gamma(G_K/U, -)$ over finite quotients. Hence cohomological finiteness in degrees $0, 1, 2$ reduces to the finite group case, and hyperdescent/admissibility on $\mathrm{Art}_{W(k)}$ follows by Proposition 5.13.

Left exactness of $L_\tau^\wedge$ identifies tangent complexes by transport of deformation theory along Theorem 3.10. $\square$

Construction 5.17 (Determinant and local-condition fibers)**.** Let $\det : B\mathrm{GL}_n \to B\mathbb{G}_m$ be the determinant. Fix a residual determinant $\bar{\chi} : G_K \to k^\times$ and regard it as a k-point of $\mathrm{Map}\big(BG_K,\, L_\tau^\wedge B\mathbb{G}_m\big)$ (with BG_K as in Theorem 5.18). Define the fixed-determinant component by the homotopy pullback

$$\begin{array}{ccc} \mathrm{Map}(BG_K, L_\tau^\wedge B\mathrm{GL}_n)_{\det=\bar{\chi}} & \longrightarrow & \mathrm{Map}(BG_K, L_\tau^\wedge B\mathrm{GL}_n) \\ \downarrow & & \downarrow{\scriptstyle \det\circ(-)} \\ \{\bar{\chi}\} & \longrightarrow & \mathrm{Map}(BG_K, L_\tau^\wedge B\mathbb{G}_m) \end{array}$$

Figure 12: Determinant condition for mapping stacks of Galois groups. The square exhibits the fiber of det over a fixed character $\bar{\chi}$.

Standard local conditions (e.g. crystalline with fixed Hodge–Tate weights, unramified outside a finite set) are imposed by further homotopy pullbacks along the corresponding closed substacks of $L_\tau^\wedge B\mathrm{GL}_n$.

Theorem 5.18 (Deformations $\simeq$ continuous mapping stack)**.** *Let* $\tau \in \{\text{fpqc}, \text{proét}\}$ *on* $\mathsf{Aff}_{W(k)}$*. Write* BG_K^{cont} *for the classifying stack of* continuous G_K*-torsors on this site and on its formal subsite* $\mathrm{Art}_{W(k)}$.

Notation reminder (cf. Remark 2.4). Throughout, $L_\tau^\wedge$ denotes the κ-accessible left-exact hypersheafification $PSh \to \widehat{\mathrm{Sh}}_\tau$ (not the hypercompletion); the hypercompletion reflector is $L_\tau^{\mathrm{hyp}} : \widehat{\mathrm{Sh}}_\tau \to (\widehat{\mathrm{Sh}}_\tau)^{\mathrm{hyp}}$.

Then there is a natural equivalence of Segal hypersheaves

$$L_\tau^\wedge(F_{\mathrm{def}}) \ \simeq \ \mathrm{Map}\big(BG_K^{\mathrm{cont}},\, L_\tau^\wedge B\mathrm{GL}_n\big)_{\det=\bar{\chi}},$$

where the right-hand side denotes the fixed-determinant fiber of Construction 5.17. The equivalence is functorial in imposing standard local conditions (implemented as further homotopy pullbacks in $\widehat{\mathsf{Sh}}_\tau$*).*

Remark 5.19 (AG$\to$ NT)**.** The equivalence realizes the fixed-determinant G_K-deformation functor (with standard local conditions) as the mapping stack $\mathrm{Map}(BG_K^{\mathrm{cont}},\, L_\tau^\wedge B\mathrm{GL}_n)_{\det=\bar{\chi}}$, so representability and descent reduce to the Segal-hypersheaf framework of Theorems 3.10 and 3.24. After 0-truncation, the tangent and obstruction spaces are recovered as $H^1(G_K, \mathrm{Ad}\,\bar{\rho})$ and $H^2(G_K, \mathrm{Ad}\,\bar{\rho})$, matching Example 5.21 and signaling failure in the obstructed case (Counterexample 5.22).

Proof. By Proposition 5.13, $L_\tau^\wedge$ computed on $\mathsf{Aff}_{W(k)}$ restricts to $\mathrm{Art}_{W(k)}$, so it suffices to evaluate on $A \in \mathrm{Art}_{W(k)}$. For such A, continuity yields

$$\mathrm{Hom}_{\mathrm{cont}}(G_K, \mathrm{GL}_n(A)) \ \simeq \ \operatorname*{colim}_{U \triangleleft_{\mathrm{open}} G_K} \mathrm{Hom}(G_K/U, \mathrm{GL}_n(A)),$$

since $G_K = \varprojlim_U G_K/U$ and $\mathrm{GL}_n(A) = \varprojlim_r \mathrm{GL}_n(A/\mathfrak{m}_A^r)$. Hence $F_{\mathrm{def}}(A)$ identifies with the fixed-determinant fiber of $\mathrm{Map}(BG_K, B\mathrm{GL}_n)$, and standard local conditions are implemented by homotopy pullbacks (Construction 5.17). Left exactness of $L_\tau^\wedge$ (Lemma 2.14) and preservation of hypersheaves/base change by mapping stacks (Proposition 2.20) give the displayed equivalence, with functoriality for local conditions. □

Observation (AG→NT): Selmer conditions as homotopy pullbacks; tangent/obstruction. The equivalence of Theorem 5.18

$$L_\tau^\wedge(F_{\mathrm{def}}) \;\simeq\; \mathrm{Map}\big(BG_K^{\mathrm{cont}},\, L_\tau^\wedge B\mathrm{GL}_n\big)_{\det=\bar{\chi}}$$

is *functorial* for standard local conditions (Construction 5.17), realized as homotopy pullbacks inside $\widehat{\mathrm{Shv}}_\tau$. Thus arithmetic deformation problems (fixed determinant and local conditions) are realized *as mapping stacks*, and the Segal stackification controls descent. After 0-truncation, the classical tangent and obstruction spaces are recovered:

$$T_\rho \;\simeq\; H^1(G_K, \mathrm{Ad}\,\bar{\rho}), \qquad \mathrm{Obs}_\rho \;\simeq\; H^2(G_K, \mathrm{Ad}\,\bar{\rho}),$$

in agreement with Remark 5.20 and Example 5.21 (cf. Counterexample 5.22 for the obstructed case).

Remark 5.20 (0-truncation and classical representability)**.** Under the usual hypotheses ($\dim_k H^i(G_K, \mathrm{Ad}\,\bar{\rho}) < \infty$ for $i = 0, 1, 2$ and Schlessinger's conditions on functors on $\mathrm{Art}_{W(k)}$), the 0-truncation $\tau_{\le 0} L_\tau^\wedge(F_{\mathrm{def}})$ is pro-representable by a complete local $W(k)$-algebra $R_{\bar{\rho}}^{\bar{\chi}}$. Determinant fixing and standard local conditions are compatible with this pro-representability via the homotopy fiber products in Construction 5.17. This classical layer is not used in the proof above but clarifies the link to deformation rings.

Example 5.21 (Unobstructed local lifts)**.** For $K = \mathbb{Q}_p$, if $H^2(G_K, \mathrm{Ad}\,\bar{\rho}) = 0$ and H^0 consists of scalars, then the local derived deformation stack is smooth in the sense that the cotangent complex is perfect and concentrated in degree 0; thus the truncation $\tau_{\le 0} L_\tau^\wedge(F_{\mathrm{def}})$ is a classical smooth local deformation space, and Segal stackification preserves automorphisms and descent data.

Counterexample 5.22 (Obstructed deformations)**.** If $H^2(G_K, \mathrm{Ad}\,\bar{\rho}) \neq 0$, then the cotangent complex has nontrivial cohomology in degree -1 and the 0-truncation no longer commutes with stackification in general; failure is detected by the obstruction space and mirrors Counterexample 4.18 for noncomplete Segal targets.

Remark 5.23 (AG–CT dictionary for Section 5)**.** *AG→CT.* Theorem 5.18 identifies representation-theoretic moduli as mapping objects in the hypersheafified Segal setting, where descent is automatic. *CT→AG.* Admissibility criteria (Propositions 4.11 and 5.16) ensure that algebraic constraints (amplitude, cohomology) place these moduli inside a robust descent framework.

Outlook See Section 6 for a consolidated forward-looking synthesis; pointers formerly listed here (eigenvarieties via Theorem 5.18; motivic interpolation across $\sigma \subset \tau$ via Proposition 5.9) are moved to Section 6.

6 Concluding Remarks

Theorems 3.10 and 3.24 together realize the conceptual program announced in the introduction: to link Segal and motivic hypersheafification within a single descent framework.

Chain of results. The central mechanism of the paper is the chain

$Theorem$ 3.10 (rigidity)
$\Downarrow$
$Theorem$ 3.24 (refinement invariance)
$\Downarrow$
Applications in $Section$ 4
$\Downarrow$
Arithmetic/Tannakian consequences in $Section$ 5

Figure 13: Logical chain of results. Rigidity (Theorem 3.10) implies refinement invariance (Theorem 3.24), which leads to the applications in Section 4, and finally to arithmetic and Tannakian consequences in Section 5.

Here Theorem 3.10 identifies when Segal hypersheafification preserves composition data, while Theorem 3.24 guarantees descent stability under refinement and mapping stacks. Together these results provide a reusable pipeline for moduli via complete Segal objects.

Forward-looking consequences.

- *Eigenvarieties as Segal-hypersheaves.* Deformation-theoretic eigenvarieties can be functorially realized through Theorem 5.18, with descent secured by Theorems 3.10 and 3.24. This opens the door to a uniform Segal-hypersheaf framework for derived eigenvarieties.
- *Motivic interpolation.* Topology comparison across $\sigma \subset \tau$ (Proposition 5.9, Theorem 3.24) points towards site-independent models for motivic moduli and A^1-invariant structures, which could later interact with K-theoretic and motivic categories.

Failure modes. The counterexamples Counterexamples 3.15, 3.26, 4.18 and 5.22 mark the minimal hypotheses required: completeness of Segal presentations, control of refinement, satisfaction of Segal axioms, and cohomological boundedness. Remark 2.23 consolidates why each assumption is indispensable.

In particular, completeness is essential (see Counterexample 3.15).

Synthesis. In summary, the mechanism

$$\text{Segal presentation} \implies L_\tau^\wedge\text{-sheafification with hyperdescent} \implies \text{stable mapping/quotient formalisms}$$

is executed by Theorems 3.10 and 3.24 together with Propositions 2.7 and 2.20. The worked applications in Sections 4 and 5—perfect complexes, higher bundles, motivic A^1-regimes, Tannakian quotients, and arithmetic deformation stacks— demonstrate both scope and limits.
This work extends the motivic descent results of [Morel–Voevodsky] and [Cisinski–Déglise] by treating Segal presentations directly under hypersheafification, which to our knowledge has not been formalized previously.

The forward directions developed in Section 5 indicate where this framework may next be applied, from motivic interpolation to Tannakian reconstruction.

A Guided outline of dependencies

This appendix summarizes dependencies; it is not part of the proofs.

I. Segal stackification ⇒ truncation ⇒ perfect complexes/bundles. Theorem 3.10 ⇒ Lemma 2.14 ⇒ Corollary 3.16 ⇒ Proposition 2.20 ⇒ Example 3.14, Example 2.21.

II. Topology refinement ⇒ Segal stability ⇒ vector bundle base change. Lemma 3.21 ⇒ Proposition 3.22 ⇒ Theorem 3.24 ⇒ Example 3.25.

III. Realization–limit interchange ⇒ hyperdescent criterion ⇒ mapping-stack inheritance. Proposition 2.7 ⇒ Proposition 2.16 ⇒ Example 4.8.

IV. Amplitude window ⇒ admissibility ⇒ truncation bounds. Definition 4.10 ⇒ Proposition 4.11 ⇒ Corollary 4.13 ⇒ Example 4.14.

V. Tannakian descent ⇒ quotient/mapping comparison ⇒ flags and K-theory. Lemma 5.3 ⇒ Theorem 5.4 ⇒ Proposition 5.9 ⇒ Example 5.7, **??**.

VI. Arithmetic admissibility ⇒ deformation/mapping equivalence ⇒ unobstructed local deformations. Proposition 5.16 ⇒ Theorem 5.18 ⇒ Example 5.21.

B Model-categorical backstop for Proposition 2.7

This appendix provides the referee's option (ii): a stable model structure (local Rezk) and a zigzag of Quillen equivalences presenting $\widehat{\mathsf{Sh}}_\tau(\mathcal{C};\mathcal{S})$, so that fibrancy (complete Segal objects) and the realization–finite limit interchange of Proposition 2.7 are transported across the Quillen equivalence.
Model structure and localization. Throughout Appendix B we regard $s\mathsf{PSh}(\mathcal{C})$ as a simplicial, combinatorial, left–proper model category with its Reedy model structure and cartesian enrichment. The *local Rezk* (complete Segal space) model structure is obtained from the Reedy model by *left Bousfield localization* at the class of τ–local weak equivalences together with the Segal and completeness maps. In this presentation the fibrant objects are precisely the τ–local complete Segal spaces, and weak equivalences present equivalences in the underlying ∞–category; see [2] and [1].

Proposition B.1 (Stable local Rezk presentation)**.** *Let $s\mathsf{PSh}(C)$ be simplicial presheaves on C with the local Rezk (complete Segal space) model structure obtained by left Bousfield localization of the Reedy structure. Its fibrant objects present complete Segal objects in $\widehat{\mathsf{Sh}}_\tau(C;\mathcal{S})$, and weak equivalences present equivalences in the underlying ∞-category, (in the simplicial, cartesianly enriched sense of [1]). .*

Theorem B.2 (Quillen equivalence transports fibrancy)**.** *There is a zigzag of Quillen equivalences between the local Rezk model category of $s\mathsf{PSh}(C)$ and the ∞-topos $\widehat{\mathsf{Sh}}_\tau(C;\mathcal{S})$ (presented by a combinatorial simplicial model). Under this equivalence, fibrant (complete Segal) Reedy objects correspond to groupoid objects in the ∞-categorical sense, and the realization–finite limit interchange of Proposition 2.7 is transported across the Quillen equivalence.*

Proof. Standard: left Bousfield localization of the Reedy structure yields the local Rezk model; completeness and Segal conditions define the fibrant objects. Combinatoriality and left properness give a presentation of $\widehat{\mathsf{Sh}}_\tau(C;\mathcal{S})$; the Quillen zigzag is classical (Rezk, Jardine; compare [6]).

Localization along the τ–local weak equivalences preserves cofibrations and forces the Segal and completeness horns to be trivial cofibrations, hence fibrant replacement preserves the Segal condition and Rezk completeness [2]; compare also [1].

Realization is the homotopy colimit of the simplicial diagram; for fibrant objects Reedy matching conditions ensure homotopy limits commute as required, and the equivalence carries this to the ∞-categorical statement. □

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Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

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