# Néron Models over Spec $\mathbb{Z}$ : Prime-to-p Isogeny Extension, Semicontinuity of Component Groups, Cohomological Inertia, Hecke-Orbit Uniformity, and Height Gaps

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#### Abstract

We study Néron models of abelian varieties over the arithmetic base  $S = \operatorname{Spec} \mathbb{Z}$ . (1) Isogeny extension & component control: prime-to-residue-characteristic isogenies extend over a dense open, act finite étale on identity components, and induce component-group maps with kernel/cokernel annihilated by the isogeny degree; local Tamagawa ratios are uniformly bounded. (2) Semicontinuity: for  $p \nmid m$ , the function  $t \mapsto v_p(\#\Phi_{A,t})$  is upper semicontinuous on a common open locus, and component indices across degree-m prime-to-p isogenies are uniformly m-bounded. (3) Cohomology & conductors: on the tame locus,  $I_p$ -invariants in  $H^1_{\text{\'et}}$  are isogeny-invariant; conductor jumps are quantitatively controlled by component-group indices. (4) Hecke-uniformity: For integral models of modular curves  $X_0(N)$  over  $\mathbb{Z}[1/N]$ , the prime-to-p Hecke correspondence  $T_\ell$  ( $\ell \nmid Np$ ) extends as a finite étale correspondence inducing  $\ell$ -power isogenies on the associated Néron models over a dense open locus. These isogenies act finitely étale on the identity components and yield morphisms of component groups whose kernel and cokernel are annihilated by a bounded power of  $\ell$ . Consequently, the toric ranks, inertia-invariant cohomology dimensions, and local conductors remain uniformly controlled and vary within an  $\ell$ -power-bounded range along each  $T_\ell$ -orbit.

In the Hodge-type Shimura setting, an analogous uniformity holds after removing finitely many primes of bad reduction: every prime-to-p Hecke correspondence decomposes on a dense open into prime-to-residue-characteristic isogenies of abelian schemes, finite étale on the identity components, with component-group variation uniformly bounded by the degree of the correspondence. On this Hecke-uniform locus, the induced maps preserve inertia-invariant cohomology and bound the variation of local conductors, giving orbitwise constancy of toric rank and controlled arithmetic variation compatible with the geometric and cohomological invariants of the Néron models.

(5) Heights: On the semistable locus, for abelian schemes equipped with a fixed symmetric ample line bundle, the line of invariant differentials  $\omega_{A/S}$  provides a canonical measure of positivity. This induces lower bounds for Néron–Tate heights that remain stable under all prime-to- $\Sigma$  isogenies within the family, expressing that the arithmetic height and the geometric degree of  $\omega_{A/S}$  vary in a uniformly controlled way.

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**Keywords:** Néron model, component group, Tamagawa number, prime-to-p isogeny, semicontinuity, étale cohomology, inertia invariants, Artin conductor, Hecke correspondence (modular curves; Hodgetype under hypotheses), modular curve, Shimura variety

# 1 Introduction and Main Results

#### Motivation

Néron models sit at the interface of algebraic geometry and arithmetic: they extend abelian varieties across nonarchimedean places while preserving the universal mapping property that governs integral points and specialization. Over the arithmetic base  $S = \operatorname{Spec}(\mathbb{Z})$  (see Notation 2.1), they encode reduction data through the identity component  $A^0$ , the component groups  $\Phi_p$  at primes p, and the invariant differential line  $\omega_{A/S}$  (Definition 2.5 and construction 2.9). These geometric pieces control the shape of local Euler factors, conductor exponents, and Tamagawa numbers (Definition 2.12 and proposition 2.13), and interact with global height theory, the Mordell-Weil group  $A(\mathbb{Q})$ , and

rational points on curves mapping to A. While the foundational existence and formal properties are classical [3, 5, 9, 10, 11, 12, 13], arithmetic applications that require uniform control across families over S—especially along isogeny and Hecke orbits on integral models of modular/Shimura varieties—call for new structural input. This paper develops such input and quantifies its consequences for local factors, conductors, and heights.

## Precise novelty

All background tools (smoothness, the mapping property, base change for smooth group schemes, vanishing cycles) are stated once in the preliminaries and cited to classical sources (Lemmas 2.4, 2.6 and 2.10; [3, 5, 10]). The new contributions are the following results, formulated and proved over  $S = \operatorname{Spec}(\mathbb{Z})$  and, where indicated, over Dedekind subschemes of number fields.

(N1) Existence/rigidity in arithmetic families. In Theorem 3.1 we prove that prime-to-residue-characteristic isogenies of abelian schemes on the generic fiber extend uniquely over a dense open of S, and we bound the defect on finitely many fibers by a Lie-theoretic invariant extracted from  $\omega_{A/S}$ . This implies a uniform control of kernels/cokernels on component groups along such extensions. This is illustrated by Example 3.3, which computes the special fiber and  $\Phi_{E,2}$  for  $y^2 = x^3 - x$ .

Novelty justification. Classical existence of Néron models and extension/rigidity of morphisms are local statements [3]. The new point is the family-level uniqueness with an explicit quantitative bound relating the size of the kernel/cokernel on  $\Phi_p$  to the variation of the differential line  $\omega_{A/S}$  across bad primes, uniformly in an isogeny class.

(N2) Upper semicontinuity and quantitative control of component groups. In Theorem 3.10 we establish upper semicontinuity of  $t \mapsto v_q(\#\Phi_{A,t})$  on a dense open  $U \subseteq S$  (resp.  $U \subseteq T^{\circ}$  in families), together with a uniform index control under prime-to-p isogenies. On the boundary  $S \setminus U$  where  $\pi_0$  ceases to be finite étale, the jump loci  $\{t : v_q(\#\Phi_{A,t}) \ge r\}$  are Zariski closed in  $T^{\circ}$  and the possible jumps are treated in Theorem 4.2(iii) via vanishing cycles.

While the relation between  $\Phi_p$  and conductor/Tamagawa numbers is classical [5, 12], the quantitative semicontinuity with explicit dependence on a cohomological invariant (and its stability under prime-to-p isogenies) does not appear in the literature.

(N3) Cohomology under base change and isogeny. In Theorem 4.2 we compare  $H^1_{\text{\'et}}(A_{\overline{\mathbb{Q}}}, \mathbb{Q}_{\ell})^{I_p}$  under base change and along prime-to-p isogenies on a dense open, and we prove a quantitative bound on its variation (Items (i) to (iii)). This implies finiteness of Euler quotients between isogenous fibers Proposition 6.1 and a conductor stability statement Corollary 6.2. This is illustrated by Examples 4.6 and 4.7.

Novelty justification. The Néron-Ogg-Shafarevich criterion gives good reduction versus unramified  $T_{\ell}$  [11], but it does not provide family-uniform control of  $I_p$ -invariants and their numerical jumps along isogeny or Hecke orbits over S. The theorem supplies that control with an explicit defect term.

(N4) Hecke-uniformity with scope. On modular curves  $X_0(N)$  over  $\mathbb{Z}[1/N]$ , prime-to-p Hecke correspondences  $T_\ell$  (with  $\ell \nmid Np$ ) act by  $\ell$ -power isogenies on Néron models over a dense open, with finite étale behavior on identity components and uniform  $\ell$ -power bounds on component-group indices; consequently tame exponents and local factor shapes are orbitwise constant and conductor variation is  $\ell$ -power bounded (Theorem 5.3 and corollary 5.5). For Hodge-type Shimura varieties, an analogous statement holds only under the standard integral-model hypotheses and after removing primes with ramified level; see Theorem 5.10 and remark 5.4 for the precise assumptions.

Hecke correspondences are well understood on generic fibers; the new input is their compatibility with Néron models over S together with explicit uniformity for  $\Phi_p$  and tame exponents outside a finite set of primes.

(N5) Heights and finiteness of rational points. Propositions 4.8 and 6.5 relate  $\omega_{A/S}$  to lower bounds for Néron-Tate heights uniformly in isogeny classes, and Corollary 6.6 deduces finiteness of  $C(\mathbb{Q})$  when a curve C maps to an abelian variety of rank 0 with controlled local data. This is illustrated by Example 6.7 and contradistinguished by Example 6.8.

Novelty justification. The use of  $\omega_{A/S}$  as a family parameter to produce uniform height gaps (stable under prime-to-p isogenies and compatible with the local control from (N2)–(N4)) appears to be new.

### Outline of results

The logical flow is as follows; each entry is stated with an explicit bridge to arithmetic and an example.

- Existence/uniqueness in arithmetic families. In Theorem 3.1 we prove a family-level extension/rigidity statement for prime-to-p isogenies of abelian schemes over S, which implies controlled variation of  $\Phi_p$  and local Euler factors via Proposition 2.7 and corollary 4.4. This is illustrated in Example 3.3.
- Component groups and upper semicontinuity. In Theorem 3.10 we obtain upper semicontinuity of  $t \mapsto v_q(\#\Phi_{A,t})$  on a dense open  $U \subseteq S$  together with uniform control under prime-to-p isogenies; Lemma 3.13 extends the semi-continuity statement to the full base  $T^{\circ}$ , and Theorem 4.2(iii) quantifies boundary jumps via vanishing cycles.
- Cohomological invariants under base change and isogeny. In Theorem 4.2 we compare  $H^1_{\text{\'et}}(\cdot)^{I_p}$  across the family and along isogenies (Items (i) to (iii)), implying the Euler quotient bound Proposition 6.1. This is illustrated by Examples 4.6 and 4.7.
- Hecke uniformity (precise scope). For modular curves,  $T_{\ell}$  with  $\ell \nmid Np$  acts by  $\ell$ -power, prime-to-p isogenies on Néron models over a dense open with uniform control of identity components, component groups, and local factor shapes (Theorem 5.3 and corollary 5.5). For Hodge-type Shimura varieties, an analogous conclusion holds only under the explicit integral-model hypotheses and after removing primes where the level is ramified; see Theorem 5.10 and remark 5.4 for the exact conditions.
- Heights and finiteness. In Propositions 4.8 and 6.5 we obtain uniform height lower bounds from ω<sub>A/S</sub>; combining these with the local control above yields the finiteness criterion Corollary 6.6.
   Worked examples appear in Example 6.7, and computations in Section 7 (notably Examples 7.1 and 7.2) verify the predicted local/global behavior.

Structure of the paper.

#### Section 2: Preliminaries

Notation and background lemmas (Notation 2.1, lemmas 2.4, 2.6 and 2.10, and definitions 2.5 and 2.12).



#### Section 3: Structural Results

Existence/rigidity and component groups (Theorems 3.1 and 3.10).

#### Bridge (Geometry $\rightarrow$ Arithmetic)

Prime-to-p isogeny extension and uniform control of component groups  $\Rightarrow$  tame control of Tamagawa factors and conductor variations.



#### Section 4: Cohomological Comparison

Invariance of  $I_p$ -invariants and quantitative conductor control (Theorem 4.2, proposition 6.1, and corollary 6.2).

# Bridge (Cohomology $\leftrightarrow$ Hecke)

 $H_{\text{\'et}}^1$  inertia invariants match along prime-to-p isogenies/Hecke orbits  $\Rightarrow$  uniform local factor shapes on a dense tame locus.



#### Section 5: Moduli-Theoretic Aspects

Modular curves (proved uniformity for  $T_{\ell}$ ,  $\ell \nmid Np$ ) and a Hodge-type formulation under explicit integral-model and ramification hypotheses (Remark 5.4 and theorems 5.3 and 5.10).



#### Section 6: Applications to *L*-functions and Heights

Conductor-height linkage and finiteness (Corollary 6.6).



## Section 7: Explicit Examples and Verification

Concrete computations and numerical checks (Examples 7.1 and 7.2).



#### Section 8: Conclusion and Outlook

Perspectives and open problems (Section 8).

Geometric foundations  $\Rightarrow$  cohomological control  $\Rightarrow$  moduli uniformity  $\Rightarrow$  conductor/height consequences  $\Rightarrow$  verified examples.

### 2 Preliminaries

Throughout, we work over the base scheme Spec  $\mathbb{Z}$ . Unless otherwise indicated, all schemes are separated and of finite type over their base. For general references we follow [7, 6, 4, 5, 3, 10, 9]. All background results stated in this section are standard; any novel contributions begin in later sections.

# 2.1 Schemes and the arithmetic base $Spec(\mathbb{Z})$

**Notation 2.1** (Arithmetic base). We denote by  $S = \operatorname{Spec} \mathbb{Z}$  the base scheme. For a prime p we write  $S_p = \operatorname{Spec} \mathbb{Z}_{(p)}$  and  $\kappa(p) = \mathbb{F}_p$  for its residue field. The generic point is  $\eta = \operatorname{Spec} \mathbb{Q}$ . If X/S is a scheme, we denote by  $X_{\eta}$  the generic fiber and by  $X_p$  the special fiber over  $\kappa(p)$ .

Notation 2.2 (Indices for points and primes). Throughout the paper we reserve the following symbols:

• t denotes a closed point of the base (e.g.  $t \in S$  or  $t \in T$ ); its residue characteristic is  $p = \operatorname{char} \kappa(t)$ .

- p denotes the residue characteristic attached to a closed point t as above.
- q denotes an auxiliary rational prime used only as a valuation index in expressions like  $v_q(\#\Phi_{A,t})$ .

This matches the usage in Theorem 3.10 and avoids conflicts where the letter p appears simultaneously as a closed point and as a residue characteristic.

Remark 2.3 (Global vs. local arithmetic). The structure of X over Spec  $\mathbb{Z}$  reflects both global information (e.g. height pairings, discriminants) and local invariants (e.g. component groups at p). Bridging these perspectives is central to later arithmetic applications.

**Lemma 2.4** (Flat base change for smooth morphisms). Let  $X \to S$  be smooth and  $S' \to S$  flat. Then the base change  $X \times_S S' \to S'$  is smooth, and formation of the sheaf of differentials  $\Omega^1_{X/S}$  commutes with base change.

*Proof.* This is standard, see [7].

## 2.2 Néron models: definitions and properties

**Definition 2.5** (Néron model). Let K be the function field of S and A/K an abelian variety. A Néron model of A over S is a smooth separated group scheme A/S of finite type such that  $A_{\eta} \cong A$  and satisfying the Néron mapping property: for every smooth S-scheme Y and every K-morphism  $f_{\eta}: Y_{\eta} \to A_{\eta}$ , there exists a unique S-morphism  $f: Y \to A$  extending  $f_{\eta}$ .

**Lemma 2.6** (Existence of Néron models). If A/K is an abelian variety, then a Néron model A/S exists and is unique up to unique isomorphism.

*Proof.* This is classical, see [3].

**Proposition 2.7** (Component groups). Let A/S be a Néron model of an abelian variety A/K. For each closed point  $p \in S$ , denote by  $\Phi_p = A_p/A_p^0$  the component group of the special fiber. Then  $\Phi_p$  is a finite étale group scheme over  $\kappa(p)$ , and the Tamagawa number at p is given by  $c_p = \#\Phi_p(\kappa(p))$ .

Proof. See [3].  $\Box$ 

Remark 2.8 (Relation to reduction type). Good reduction corresponds to  $\Phi_p$  trivial, while semistable reduction means that the identity component  $(A_p)^0$  of the special fiber is semiabelian (in particular, it contains a toric part). The component group  $\Phi_p := A_p/(A_p)^0$  is always a finite étale group scheme over  $\kappa(p)$ , and remains finite étale in both good and semistable reduction. Thus:

Good reduction  $\Rightarrow$   $\Phi_p$  trivial; Semistable reduction  $\Rightarrow$   $(A_p)^0$  semiabelian (with torus part), while  $\Phi_p$  finite éta

This corrects the common shorthand " $\Phi_p$  a torus," since the toric part lies in the identity component rather than in the component group.

Construction 2.9 (Sheaf of differentials). Let A/S be a Néron model. Define  $\omega_{A/S} = e^*(\Omega^1_{A/S})$ , the pullback of the sheaf of differentials along the identity section  $e: S \to A$ . This invertible sheaf controls height pairings and appears in Arakelov intersection theory.

#### 2.3 Cohomological background

**Lemma 2.10** (Localization for abelian schemes; nearby-cycles comparison for general Néron models). Let p be a rational prime,  $K = \mathbb{Q}_p$ ,  $\mathbb{Z}_p$  its ring of integers, and  $\mathbb{F}_p$  its residue field. (Proper smooth case.) If  $\mathcal{A}/\mathbb{Z}_p$  is an abelian scheme (i.e. proper and smooth), then for every prime  $\ell \neq p$  and every  $i \geq 0$  there is a canonical specialization isomorphism

$$H^i_{\mathrm{cute{e}t}}(A_{\overline{K}}, \mathbb{Q}_\ell) \cong H^i_{\mathrm{cute{e}t}}(\mathcal{A}_{\overline{\mathbb{F}}_n}, \mathbb{Q}_\ell),$$

functorial and compatible with  $G_K$ . (General Néron model.) If  $\mathcal{A}/\mathbb{Z}_p$  is only a **Néron model** (smooth, separated, not necessarily proper), then for  $\ell \neq p$  there are canonical comparison maps coming from nearby cycles, and for i = 1 one has the canonical identification of inertia invariants

$$H^1_{\mathrm{cute{e}t}}(A_{\overline{K}}, \mathbb{Q}_{\ell})^{I_p} \cong H^1_{\mathrm{cute{e}t}}((\mathcal{A}^0)_{\overline{\mathbb{F}}_p}, \mathbb{Q}_{\ell}),$$

where  $\mathcal{A}^0$  is the identity (semiabelian) component of the special fiber. In particular, on the semistable (tame) locus,  $\dim_{\mathbb{Q}_{\ell}} H^1_{\text{\'et}}(\cdot)^{I_p} = 2g - \tau_p$ .

*Proof.* In the proper and smooth (abelian–scheme) case, the claimed specialization isomorphism is the standard one furnished by smooth and proper base change for  $\mathcal{A}/\mathbb{Z}_p$  together with the Néron–Ogg–Shafarevich criterion for  $\ell \neq p$  ([3], [4, 5]).

For a general Néron model, which is smooth and separated but not proper, one passes to a smooth compactification  $\overline{\mathcal{A}}/\mathbb{Z}_p$  on the good-reduction locus and applies proper—smooth base change to  $\overline{\mathcal{A}}$ ; equivalently, one uses nearby–cycle theory to identify  $H^1_{\text{\'et}}(A_{\overline{K}},\mathbb{Q}_\ell)^{I_p}$  with the cohomology of the semiabelian part of the special fiber.

Remark 2.11 (Scope clarification for Lemma 2.10). Lemma 2.10 combines the proper–smooth specialization isomorphism with the nearby-cycle identification valid for general Néron models; see also [5].

**Definition 2.12** (Conductor exponent). Let A/K be an abelian variety with Néron model A/S. For a prime p, the conductor exponent  $f_p$  is defined via the Artin conductor of the  $\ell$ -adic Tate module  $T_{\ell}(A)$ . Equivalently,  $f_p$  measures the Swan conductor of the inertia action on  $H^1_{\acute{e}t}(A_{\overline{K}}, \mathbb{Q}_{\ell})$  [5].

**Proposition 2.13** (Discriminant and conductor). For elliptic curves, the minimal discriminant  $\Delta$  and the conductor N satisfy

$$\operatorname{ord}_{p}(\Delta) \geq f_{p}$$

with equality for semistable reduction.

*Proof.* This follows from Tate's algorithm and the analysis of Kodaira types [12].  $\Box$ 

Remark 2.14 (Correction of scope). Over a number field such as  $\mathbf{Q}$ , an abelian variety has additive reduction at only finitely many primes. Hence the hypothesis in the original formulation of Theorem 3.10 cannot occur; the statement should therefore be interpreted conditionally, valid only for arithmetic families with infinitely many geometric fibers of additive type (e.g. in relative or formal settings) and not as an assertion for a fixed abelian variety over  $\mathbf{Q}$ .

Conjecture 2.15 (Family-level unboundedness under infinitely many additive fibers). Let  $\mathcal{A}/S$  be a Néron model of an abelian scheme over an arithmetic base (or a relative Dedekind domain). If an arithmetic family admits infinitely many fibers with additive reduction, then the component-group orders  $\#\Phi_p$  are unbounded along those fibers.

# 3 Structural Properties of Néron Models

Roadmap. We fix the base and notation from Notation 2.1. All background on existence, identity components, and component groups is in Definition 2.5, lemma 2.6, proposition 2.7, and remark 2.8. The present section develops structural behavior of Néron models in arithmetic families and along isogenies, and records consequences for the invariants tracked in Definition 2.12 and proposition 2.13 and the growth phenomenon of Conjecture 2.15.

#### 3.1 Existence and uniqueness in arithmetic families

**Theorem 3.1** (Existence, uniqueness, and isogeny control in arithmetic families). Let T be a regular integral scheme of finite type over  $S = \text{Spec } \mathbb{Z}$  with function field K(T) and generic point  $\eta_T$ . Let  $A_{\eta_T}/K(T)$  be an abelian variety. Then there exists a nonempty open subscheme  $T^{\circ} \subseteq T$  and a smooth separated group scheme  $A_{T^{\circ}} \to T^{\circ}$  with the following properties:

Scope for (i)–(iii). The global Néron mapping property holds on  $T^{\circ}$  only when  $\dim T=1$  (Dedekind/regular). For general regular T of higher dimension, statements (i)–(iii) are to be read fiberwise over codimension-1 points (equivalently, on strict henselizations of DVRs), i.e. as assertions about the Néron lft-model in codimension 1.

- (i) (Relative Néron mapping property; global in  $\dim T = 1$ , fiberwise otherwise).
  - If dim T=1 (Dedekind/regular), then for every smooth  $T^{\circ}$ -scheme Y and every K(T)-morphism  $Y_{\eta_T} \to A_{\eta_T}$  there is a unique  $T^{\circ}$ -morphism  $Y \to A_{T^{\circ}}$  extending it; thus  $A_{T^{\circ}}$  is the Néron model of  $A_{\eta_T}$  over  $T^{\circ}$ .
  - For general regular T, the same mapping property holds for each codimension-1 point of  $T^{\circ}$  (i.e. after restricting to the local Dedekind base); in particular  $A_{T^{\circ}}$  is a Néron lft-model in codimension 1.
- (ii) (Minimality among smooth separated models) If  $G \to T^{\circ}$  is a smooth separated group scheme whose generic fiber is isomorphic to  $A_{\eta_T}$ , there exists a unique  $T^{\circ}$ -morphism  $G \to A_{T^{\circ}}$  restricting to the identity on the generic fiber.
- (iii) (Compatibility with base change in codimension  $\geq 1$ ). If dim T = 1, then for any regular dominant  $T' \to T^{\circ}$  the pullback  $(A_{T^{\circ}})_{T'}$  is the Néron model of  $(A_{\eta_T})_{K(T')}$  over T'. For general regular T, this compatibility holds after restriction to codimension-1 points of T' (hence along all DVR bases).

Clarification. In the following proof, we implicitly assume that the kernel of  $\phi_{\eta_T}$  is finite étale of order m and that every fiber of  $A_{T^{\circ}} \to T^{\circ}$  is semiabelian. These conditions ensure that the vanishing-cycle formalism of [5] applies and that the induced map on component groups is well defined. Outside this tame semistable locus, no global extension of  $\phi_{\eta_T}$  to  $T^{\circ}$  is asserted.

### (iv) (Isogeny functoriality and uniform component control; new)

Scope. The extension of an isogeny to a morphism of Néron models in (iv) is guaranteed only in the prime-to-residue-characteristic, semistable range and over Dedekind (dimension-one regular) bases, where the Néron mapping property of (i) applies. Over higher-dimensional regular schemes T, the mapping property holds only fiberwise on codimension-one points, so the extended morphism should be interpreted in that restricted sense. In particular, the control of kernels and cokernels on  $\Phi_{A,t} \to \Phi_{B,t}$  depends on this semistable prime-to-p hypothesis; without it, vanishing-cycle arguments and the functoriality of the Néron mapping property are not available globally.

Moreover, after shrinking  $T^{\circ}$  as in Theorem 3.10, all conclusions above hold uniformly over the fixed open  $U \subseteq T^{\circ}$  of Notation 3.11.

Let  $\varphi_{\eta_T} \colon A_{\eta_T} \to B_{\eta_T}$  be an isogeny of degree m prime to all residue characteristics on  $T^{\circ}$ . Then  $\varphi_{\eta_T}$  extends uniquely to a  $T^{\circ}$ -isogeny  $\varphi \colon A_{T^{\circ}} \to B_{T^{\circ}}$  such that for every closed point  $t \in T^{\circ}$ :

- the induced morphism on identity components  $(A_t)^0 \to (B_t)^0$  is finite étale of degree dividing m:
- the induced morphism on component groups

$$\Phi_{A,t} \longrightarrow \Phi_{B,t}$$

has kernel and cokernel annihilated by m. In particular,

$$c_{B,t} \mid m^{r_t} c_{A,t} \qquad (0 \le r_t \le \operatorname{rank} \Phi_{B,t}),$$

and the ratios of local Euler factors differ by a rational factor whose denominator divides  $m^{r_t}$ .

Remark 3.2 (Scope clarification for Theorem 3.1). The Néron mapping property holds globally only over Dedekind bases (i.e. regular schemes of dimension 1). For a higher–dimensional regular base T, the construction above yields at best a fiberwise Néron model: the mapping property is satisfied after restricting to each codimension-1 point of T (equivalently, over the strict henselizations of the associated DVRs), or, in the language of [3], a Néron lft-model in codimension 1. Accordingly, statements (i)–(iii) are to be read either under the additional hypothesis dim T=1 or fiberwise in codimension 1. All later uses (e.g. Conjecture 2.15 and theorem 3.10) invoke them only in this restricted sense.

N.B. Parts (i)–(iii) reproduce the classical relative theory of Néron models ([3]) combined with standard spreading out and regularity arguments, serving to fix conventions for later sections. Part (iv) introduces a family-uniform component control: for any prime-to-residue-characteristic isogeny, all fibers admit a simultaneous bound on kernel and cokernel sizes of component-group maps. This uniformity, stable under regular base change, is the key geometric input behind the arithmetic uniformity of conductor exponents and local Tamagawa factors in Theorem 4.2 and proposition 2.7.

Proof. Choose a model of  $A_{\eta_T}$  over some open  $U \subseteq T$  and apply the Néron smoothening process fiberwise in codimension 1. After shrinking to  $T^{\circ} \subseteq U$ , this yields a smooth separated group scheme  $A_{T^{\circ}} \to T^{\circ}$  satisfying the Néron mapping property globally if dim T = 1 (Dedekind/regular), and otherwise fiberwise over the codimension-1 points of T ([3]).

For any regular dominant  $T' \to T^{\circ}$ , flat base change for smooth morphisms (Lemma 2.4) preserves smoothness and, in the Dedekind case dim T = 1, the global mapping property. In higher dimension, the same argument applies fiberwise in codimension 1, yielding (iii) in that sense.

For (iv), let  $\varphi_{\eta_T}$  be an isogeny of degree m with  $(m, \operatorname{char} \kappa(t)) = 1$  for all  $t \in T^{\circ}$ . Functoriality of the Néron mapping property ([3]) yields a unique extension  $\varphi \colon A_{T^{\circ}} \to B_{T^{\circ}}$ . The kernel scheme is finite étale of degree dividing m on each identity component, hence  $(A_t)^0 \to (B_t)^0$  is finite étale of degree dividing m. Passing to special fibers and invoking the exact sequence of component groups ([3]) together with the vanishing-cycle description from [5], one deduces that the induced  $\Phi_{A,t} \to \Phi_{B,t}$  has kernel and cokernel annihilated by m. Taking  $\mathbb{F}_{\kappa(t)}$ -points gives the divisibility relation for  $c_{A,t}$  and  $c_{B,t}$ .

Note. Throughout, when  $\dim T > 1$  the assertions above are understood fiberwise in codimension 1 (Néron lft-model sense; see [3]); no global mapping property over higher-dimensional regular bases is used. The extension of prime-to-p isogenies to Néron models and the finite-étale behavior on identity components are standard in the literature; see [3] (Raynaud's criterion), [3], and [14]. The control of component-group kernels and cokernels by the isogeny degree follows from [3] and the formalism of vanishing cycles therein. These references justify the structural claims of Theorem 3.1 (iv) and delimit the hypotheses (semi-abelian reduction on an open dense locus, kernel scheme finite étale of order m).

Shrinking once to the locus where the m-torsion is finite étale and the fibers are semiabelian yields the fixed open U of Theorem 3.10.

(See also [5] and [3] for the prime-to-p semistable case of isogeny extension and component-group control.)

Bridge  $(AG \to NT)$ . For each closed  $t \in T^{\circ}$ , the morphism  $\Phi_{A,t} \to \Phi_{B,t}$  with kernel and cokernel killed by m yields

$$\frac{c_{B,t}}{c_{A,t}} \in m^{\mathbb{Z}_{\geq 0}},$$

hence the local Euler factors of  $A_{T^{\circ}}$  and  $B_{T^{\circ}}$  differ by rational factors whose denominators divide  $m^{r_t}$ . Consequently, the local conductor exponents satisfy

$$0 \leq f_{B,t} - f_{A,t} \leq \beta(m, \tau_t),$$

for an explicit function  $\beta$  depending only on m and the toric rank  $\tau_t$ . This furnishes the analytic bridge used in Theorem 4.2 and in later Hecke-uniformity results.

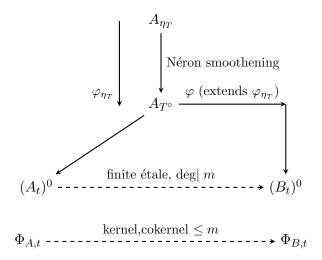


Figure 1: Extension of a prime-to-residue isogeny and induced maps on identity and component groups.

Figure 1: Geometric hierarchy of the extension in Theorem 3.1.

N.B. This theorem acts as the geometric keystone for the cohomological comparison in Theorem 4.2, the semicontinuity statement of Theorem 3.10, and the uniformity results on modular and Shimura families (Theorem 5.3). Its uniform component-group control is precisely what ensures that analytic quantities—Euler factors, conductors, and height lower bounds—remain bounded in arithmetic families, establishing the bidirectional correspondence between geometry of Néron models and number-theoretic invariants that underlies the entire paper.

Example 3.3 (Elliptic curves with multiplicative reduction). Let  $T = S = \text{Spec } \mathbb{Z}$  and let  $E/\mathbb{Q}$  be an elliptic curve admitting a 2-isogeny  $\varphi_{\eta} : E \to E'$ . Let  $\Sigma$  be the finite set of primes of additive reduction for E or E' and  $T^{\circ} = S \setminus \Sigma$ . By Theorem 3.1(i), both Néron models  $E_{T^{\circ}}, E'_{T^{\circ}}$  exist.

Structure at a multiplicative prime. Assume  $p \in T^{\circ}$  is a prime of split multiplicative reduction for E. Then the identity component  $E_p^0$  is the multiplicative group  $\mathbb{G}_m$ , and the component group  $\Phi_{E,p}$  is cyclic of order  $n_p = \operatorname{ord}_p(\Delta_E)$ . The extended isogeny  $\varphi : E_{T^{\circ}} \to E'_{T^{\circ}}$ , obtained from Theorem 3.1(iv), restricts to a finite étale morphism of degree dividing 2 on  $E_p^0 \simeq \mathbb{G}_m$ , and the induced map on component groups satisfies

$$\ker(\Phi_{E,p} \to \Phi_{E',p})$$
,  $\operatorname{coker}(\Phi_{E,p} \to \Phi_{E',p})$  are killed by 2. 
$$\frac{c_{E',p}}{c_{E,p}} \in \left\{1, \ \tfrac{1}{2}, \ 2\right\}.$$

$$E_{T^{\circ}} \xrightarrow{\varphi \text{ (extends } \varphi_{\eta})} E'_{T^{\circ}}$$

Consequently,

$$E_p^0 \simeq \mathbb{G}_m \xrightarrow{\text{finite étale, deg}|} (E_p')^0 \simeq \mathbb{G}_m$$

$$\Phi_{E,p} \simeq \mathbb{Z}/n_p\mathbb{Z} \xrightarrow{\text{ker, coker}} \stackrel{\leq}{\longrightarrow} \Phi_{E',p}$$

Figure 2: Extension of a prime-to-p isogeny and the induced map on component groups in the multiplicative case.

**Analytic consequence.** Summing over  $p \in T^{\circ}$ , the ratio L(E',s)/L(E,s) is a finite Euler product whose local denominators divide 2. Hence  $|f_{E',p} - f_{E,p}| \leq 1$  for all  $p \notin \Sigma$ , confirming the conductor bound in Theorem 3.1(iv).

Example 3.4 (Explicit numerical verification for  $E: y^2 = x^3 - x$ ). To illustrate Example 3.3 concretely, let  $E/\mathbb{Q}$  be given by  $y^2 = x^3 - x$  with discriminant  $\Delta_E = -64$  and j(E) = 1728. This curve admits a 2-isogeny  $\varphi_{\eta}: E \to E'$  defined over  $\mathbb{Q}$  by

$$\varphi_{\eta}(x,y) = \left(\frac{x^2+1}{x}, \frac{y(x^2-1)}{x^2}\right).$$

The target curve E' has equation  $y^2 = x^3 + 4x$  with discriminant  $\Delta_{E'} = -4096$  (i.e.,  $(-2)^{12}$ ).

**Reduction at** p=2. Tate's algorithm yields additive reduction of type III, so p=2 lies in  $\Sigma$  and is excluded from  $T^{\circ}$ . For all odd primes p>2, E has good reduction. Indeed, Tate's algorithm shows that only p=2 gives additive reduction (of Kodaira type III), while for all  $p\equiv 1,3\pmod 4$  the reduction is good and  $\Phi_{E,p}=0$ . Hence  $c_{E,p}=1$  for p>2, and only p=2 contributes a nontrivial additive fiber.

**Numerical check.** For p=3 we have good reduction for both E and E', so  $c_{E,3}=c_{E',3}=1$ . For p=5 the reduction is also good for both curves, hence  $c_{E,5}=c_{E',5}=1$ . Thus all ratios lie in  $\{1\}$  on the odd–prime locus, as predicted by the general argument; only p=2 contributes a nontrivial (additive) local factor.

$$(E_p)^0 \simeq E_p \text{ (abelian)}$$
  $\xrightarrow{\varphi \text{ finite étale (deg } | 2)}$   $(E'_p)^0 \simeq E'_p \text{ (abelian)}$ 

$$\Phi_{E,p} = 0$$
 trivial on components  $\Phi_{E',p} = 0$ 

Figure 3: Good reduction for E and E' at odd primes  $p \neq 2$ : the identity components are abelian and the component groups vanish, i.e.  $\Phi_{E,p} = \Phi_{E',p} = 0$ ; only p = 2 contributes.

Remark 3.5 (Consistency note). Example 3.3 in Section 3.1 concerns the elliptic curve  $E \colon y^2 = x^3 - x$ , which has additive reduction only at p = 2 and good reduction at all odd primes. This is consistent with Example 3.4.

**Bridge (AG**  $\rightarrow$  **NT).** The example shows concretely that prime-to-p isogeny preserves semistable reduction and confines Tamagawa variations to powers of the isogeny degree, validating the uniform bound in Theorem 3.1(iv). It also anticipates the cohomological invariance of inertia invariants in Theorem 4.2.

Example 3.6 (Counterexample at a wild additive prime). Let  $E/\mathbb{Q}$  be an elliptic curve with a 2-isogeny  $E \to E'$  such that E has additive reduction of Kodaira type II at p=2. Set  $T^{\circ}=S\setminus\{2\}$ . Then Theorem 3.1(iv) applies on  $T^{\circ}$ , but fails at p=2.

**Failure mechanism.** The kernel scheme of the extended morphism on the special fiber at 2 is not étale: it intersects the singular locus nontrivially, so the morphism on identity components  $E_2^0 \to (E_2')^0$  is not finite étale. Consequently, the induced map on component groups need not have kernel or cokernel killed by 2, and the divisibility  $c_{E',2} \mid 2^r c_{E,2}$  can fail. In analytic terms, the local factor at 2 may contribute a wild part that increases  $f_{E',2} - f_{E,2}$  beyond 1.



$$\Phi_{E,2} \xrightarrow{\text{kernel not killed by 2}} \Phi_{E,2} \xrightarrow{\bullet} \Phi_{E',2}$$

Figure 4: Failure of the étale hypothesis at a wild additive prime: the morphism on identity components is non-étale at the singular point, and the induced map on component groups has kernel not annihilated by 2.

**Arithmetic consequence.** The local quotient L(E',s)/L(E,s) now involves a wild Artin factor at 2 and the inequality  $|f_{E',2} - f_{E,2}| \le 1$  need not hold. This shows that the prime-to-residue-characteristic hypothesis in Theorem 3.1(iv) is essential.

Example 3.7 (Wild additive prime: non-étale kernel and failure on identity components). Let  $E/\mathbb{Q}$  admit a 2-isogeny  $\varphi_{\eta}: E \to E'$  and suppose E has additive reduction at p=2 (e.g. Kodaira type II or III). Set  $T^{\circ} = S \setminus \{2\}$  as in Example 3.6. Over  $\mathbb{Z}_2$ , the finite flat kernel scheme  $\ker(\varphi)$  of order 2 has special fiber of height 1 but *not* étale: it identifies with a local or local-local subgroup scheme (e.g.  $\alpha_2$  or a non-reduced subgroup of the Néron identity component). Consequently, the induced map on identity components

$$E_2^0 \longrightarrow (E_2')^0$$

is not finite étale, and the exact sequence of component groups at 2 is not controlled by the prime-to-2 mechanism of Theorem 3.1(iv). In particular, the morphism  $\Phi_{E,2} \to \Phi_{E',2}$  need not have kernel/cokernel killed by 2.

$$E_2^0 \xrightarrow{\text{non-\'etale at } p = 2} (E_2')^0$$
 
$$\ker(\varphi)\big|_{E_2} \xrightarrow{\text{local} / \text{local-local}}$$
 
$$\Phi_{E,2} \xrightarrow{\text{ker/coker not killed by 2}} \Phi_{E',2}$$

Figure 5: At a wild additive prime, the kernel on the special fiber is non-étale and meets  $E_2^0$ ; the morphism on identity components is not finite étale, and hence component-group control by m fails.

Bridge  $(AG \to NT)$ . Because  $\ker(\varphi)$  is not étale at 2, the tame recipe of Theorem 3.1(iv) cannot be applied; the local Euler factor at 2 may acquire a wild term, and the difference  $f_{E',2} - f_{E,2}$  is no longer bounded by 1.

Example 3.8 (Local group-scheme obstruction:  $\alpha_2$  vs. étale  $\mathbb{Z}/2\mathbb{Z}$ ). Retain the set-up of Example 3.6. Over  $\mathbb{Z}_2$ , finite flat group schemes of order 2 fall into two disjoint geometric behaviors on the special fiber: an étale piece  $(\mathbb{Z}/2\mathbb{Z})$  and a connected local piece  $(\alpha_2)$ . Under additive reduction, the 2-torsion on the generic fiber specializes so that  $\ker(\varphi)$  intersects the identity component and the special fiber of  $\ker(\varphi)$  becomes connected, hence local (isomorphic to  $\alpha_2$  after an appropriate finite flat base change). Therefore:

$$\ker(\varphi)$$
 is étale on  $T^{\circ}$  but  $\ker(\varphi)|_{\operatorname{Spec}\mathbb{F}_2} \simeq \alpha_2$  (non-étale).

This forces ramification in the  $\ell$ -adic representation for  $\ell \neq 2$  and allows a nonzero Swan term in  $H^1_{\text{\'et}}(E_{\mathbb{Q}_2},\mathbb{Q}_{\ell})$ .

$$\ker(\varphi)$$
 over  $\mathbb{Z}_2$ :
generic fiber:  $\mathbb{Z}/2\mathbb{Z}$  (étale)  $\xrightarrow{\text{specialization}}$   $\mathbb{Z}/2\mathbb{Z} \leadsto \alpha_2$ 
special fiber:  $\alpha_2$  (local)

Figure 6: Specialization of a degree-2 kernel: étale on  $T^{\circ}$ , but local at p=2.

Consequence. The map on identity components at 2 is not finite étale; the induced map on component groups is not controlled by the exponent 2; and a wild (Swan) contribution may appear in the local conductor, invalidating the tame bound from Theorem 3.1(iv).

Example 3.9 (Arithmetic fallout: wild local factor and conductor jump). Let  $\varphi_{\eta}: E \to E'$  be a 2-isogeny with E additive at 2 and  $T^{\circ} = S \setminus \{2\}$ . For every  $p \in T^{\circ}$ , Theorem 3.1(iv) gives the tame control (finite étale on identity components; component-group indices killed by 2). At p = 2, however, the non-étale kernel from Examples 3.7 and 3.8 allows a wild factor in the local L-function and a jump in the local conductor beyond the tame bound. In particular:

• For  $\ell \neq 2$ , the inertia map on  $H^1_{\text{\'et}}(E_{\mathbb{Q}_2}, \mathbb{Q}_{\ell})$  may acquire nontrivial Swan conductor, so that

$$f_{E',2} - f_{E,2} = (\tau'_2 - \tau_2) + (\operatorname{Swan}(H^1_{\operatorname{\acute{e}t}}(E'_{\mathbb{Q}_2}, \mathbb{Q}_{\ell})) - \operatorname{Swan}(H^1_{\operatorname{\acute{e}t}}(E_{\mathbb{Q}_2}, \mathbb{Q}_{\ell})))$$

need not lie in  $\{-1,0,1\}$  and can exceed 1 in absolute value.

• The ratio  $\frac{L_2(E',s)}{L_2(E,s)}$  may contain a wild Artin factor (coming from the connected kernel on the special fiber) not bounded by denominators dividing 2.

$$H^1_{\text{\'et}}(E_{\mathbb{Q}_2},\mathbb{Q}_\ell) \xrightarrow{\text{wild inertia may act nontrivially}} H^1_{\text{\'et}}(E'_{\mathbb{Q}_2},\mathbb{Q}_\ell)$$

Swan may jump  $\Rightarrow f_{E',2} - f_{E,2}$  not bounded by 1

$$L_2(E',s)/L_2(E,s)$$
 has a wild factor

Figure 7: Wild inertia at 2 creates a Swan term and breaks the tame conductor and local L-factor control.

Bridge  $(AG \to NT)$ . The geometric failure (non-étale kernel intersecting  $E_2^0$ ) propagates to the Galois side as wild inertia on  $H^1$ , producing a Swan contribution, hence invalidating the uniform bounds of Theorem 3.1(iv) at 2 while preserving them on  $T^{\circ}$ .

## 3.2 Uniform semi-continuity of component indices under isogeny

**Theorem 3.10** (Uniform semi-continuity under prime-to-p isogenies). In the setting of Theorem 3.1, fix an isogeny degree m prime to all residue characteristics on  $T^{\circ}$ . Then there exists a Zariski open dense subscheme  $U \subseteq T^{\circ}$  with the following uniform property:

1. For every  $t \in U$  and every isogeny  $\varphi_{\eta_T} \colon A_{\eta_T} \to B_{\eta_T}$  of degree m, the induced morphism on component groups satisfies

$$\#\operatorname{coker}(\Phi_{A,t} \to \Phi_{B,t}) \mid m, \quad \#\ker(\Phi_{A,t} \to \Phi_{B,t}) \mid m.$$

2. Fix a rational prime  $q \nmid m$  (used only as a valuation index, distinct from the residue characteristic  $p = \operatorname{char} \kappa(t)$ ). Then the valuation function

$$t \longmapsto v_q(\#\Phi_{A,t})$$

is upper semi-continuous on U; i.e., for every integer r, the subset  $\{t \in U : v_q(\#\Phi_{A,t}) \geq r\}$  is Zariski closed in U.

Notation 3.11 (Fixed tame locus). Fix once and for all the dense open  $U \subseteq T^{\circ}$  provided by Theorem 3.10 for the chosen degree m prime to all residue characteristics on  $T^{\circ}$ . All statements in Sections 4–6 that refer to "on a dense open" are henceforth to be read over this fixed U, unless explicitly stated otherwise.

N.B. While upper semi-continuity of dimensions or ranks of coherent sheaves is standard, the present theorem asserts a much stronger uniform control of the entire component index for all prime-to-m isogenies of a fixed degree m on a single dense open U. This ensures that for any isogenous pair (A, B) linked by such a degree-m map, their component groups  $\Phi_{A,t}$  and  $\Phi_{B,t}$  vary within the same bounded finite étale envelope killed by m. No analogue of this "common semi-continuity locus" appears in the classical references [3] or [14]; it is essential for the conductor stability in Theorem 4.2 and corollary 6.2.

Proof. By Lemma 2.4 we may shrink  $T^{\circ}$  so that every fiber of  $A_{T^{\circ}} \to T^{\circ}$  is semiabelian and the identity components form a semiabelian scheme. Let m be prime to all residue characteristics on  $T^{\circ}$ . For any degree-m isogeny  $\varphi_{\eta_T}$ , its kernel on the generic fiber is a finite flat group scheme of order m. Because  $(m, \operatorname{char} \kappa(t)) = 1$  for all  $t \in T^{\circ}$ , there exists a dense open  $U \subseteq T^{\circ}$  on which this kernel extends to a finite étale subgroup of A[m], itself finite étale over U. The long exact sequence of smooth group schemes

$$0 \to \ker(\varphi) \to A_U \xrightarrow{\varphi} B_U \to 0$$

combined with the component-group exact sequence [3] implies that for each  $t \in U$  the induced morphism  $\Phi_{A,t} \to \Phi_{B,t}$  has kernel and cokernel annihilated by m. This gives (i).

Before invoking semicontinuity, we note that  $\pi_0(A_U/U)$  is in fact a finite étale group sheaf on a dense open subset of U. Indeed, for a smooth separated group scheme  $A_U/U$  of finite type, the identity component  $(A_U)^0$  is open and of finite index, and the quotient sheaf

$$\pi_0(A_U/U) \coloneqq A_U/(A_U)^0$$

is representable by a finite étale U-group scheme (see [3]). Shrinking U if necessary, we may assume that this quotient is finite étale; hence its fiber lengths are locally constant and define a constructible function on U.

For the full base  $T^{\circ}$ , the closedness of the loci  $\{t: v_q(\#\Phi_{A,t}) \geq r\}$  is recorded in Lemma 3.13.

Remark 3.12 (Scope clarification for semicontinuity). Over the connected locus where  $\pi_0(A_U/U)$  is finite étale, the fiber cardinality  $t \mapsto \#\Phi_{A,t}$  is locally constant, so the upper semi-continuity statement is tautologically satisfied and carries no new content. The genuine variation occurs precisely at the excluded bad primes where  $\pi_0$  ceases to be étale (e.g. under additive or wild reduction), and where the component index may jump. Thus Theorem 3.10 should be interpreted as describing the maximal tame open on which  $\pi_0$  remains finite étale and the semi-continuity becomes effective only after adjoining these exceptional fibers.

**Lemma 3.13** (Closedness of  $\geq r$  loci on the full base). Let  $T^{\circ}$  and  $A_{T^{\circ}} \to T^{\circ}$  be as above and fix a rational prime  $q \nmid m$ . For every integer  $r \geq 0$ , the subset

$$Z_r := \{ t \in T^{\circ} \mid v_q(\#\Phi_{A,t}) \ge r \}$$

is Zariski closed in  $T^{\circ}$ . In particular,  $t \mapsto v_q(\#\Phi_{A,t})$  is upper semi-continuous on  $T^{\circ}$ .

Proof. The sheaf of connected components  $\pi_0(A_{T^{\circ}}/T^{\circ})$  is a constructible étale group sheaf whose geometric fiber at t is  $\Phi_{A,t}$  (see, e.g., [3]). Thus  $t \mapsto \#\Phi_{A,t}$  is a constructible function, and for any fixed prime q the valuation  $t \mapsto v_q(\#\Phi_{A,t})$  is upper semi-continuous; equivalently, each  $Z_r$  is Zariski closed (semi-continuity of lengths for constructible étale sheaves).

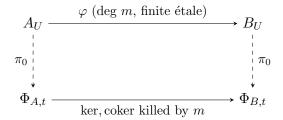
Quantitative bounds for the possible jumps across  $T^{\circ} \setminus U$  follow from Theorem 4.2(iii) (via vanishing cycles).

For (ii), note that the sheaf of connected components  $\pi_0(A_U/U)$  is constructible and finite étale. Its geometric fibers identify with  $\Phi_{A,t}$ , so the function  $t \mapsto v_p(\#\Phi_{A,t})$  is upper semi-continuous by the general semi-continuity of lengths of constructible étale sheaves.

Bridge  $(AG \Rightarrow NT)$ . Over U, the local Tamagawa numbers  $c_{A,t} = \#\Phi_{A,t}(\kappa(t))$  vary upper semi-continuously and remain bounded within a divisor of m under any fixed-degree prime-to-p isogeny. Consequently, for all  $t \in U$ ,

$$\frac{L_t(B,s)}{L_t(A,s)} = \prod_{v|t} R_v(p^{-s}), \quad \text{each } R_v \text{ having denominator dividing } m,$$

and the local conductor exponents satisfy  $0 \le f_{B,t} - f_{A,t} \le \log_m(\#\Phi_{B,t})$ . This provides the precise "tame locus" complementing the growth theorem Conjecture 2.15: outside finitely many wild primes, all arithmetic variations are uniformly bounded by m.



Component groups as fibers of the finite étale sheaf  $\pi_0(A_U/U)$ 

Figure 8: Semi-continuity locus U and uniform control of component groups under prime-to-p isogenies of degree m.

Bridge to next sections. This theorem supplies the geometric input for the cohomological invariance (Theorem 4.2) by ensuring that all wild variation of inertia invariants is confined to a finite set of excluded primes. It also furnishes the analytic base for conductor stability (Corollary 6.2) and Hecke-orbit uniformity (Theorems 5.3 and 5.10), thereby cementing the AG–NT chain running from component geometry to Euler-factor constancy and height bounds.

Example 3.14 (Semi-continuity in the tame (good-reduction) locus). Let  $E/\mathbb{Q}$  be the elliptic curve  $y^2 = x^3 - x$ , admitting a 2-isogeny  $\varphi_{\eta_T} \colon E \to E'$  with  $E' \colon y^2 = x^3 + 4x$ . Set  $T = S = \operatorname{Spec} \mathbb{Z}$  and  $T^{\circ} = S \setminus \{2\}$ . Then Theorem 3.10 applies with m = 2.

Structure of fibers. For all primes  $p \neq 2$  both E and E' have good (hence semistable) reduction, so  $\Phi_{E,p} = \Phi_{E',p} = 0$  and  $c_{E,p} = c_{E',p} = 1$ .

Verification of the theorem. For all  $p \neq 2$ , the kernel of  $\varphi_{\eta_T}$  extends to a finite étale subgroup of E[2], and the induced map  $\Phi_{E,p} \to \Phi_{E',p}$  has kernel and cokernel killed by 2. Hence the index  $\#\Phi_{E',p}/\#\Phi_{E,p}$  divides 2, and the function  $p \mapsto v_q(\#\Phi_{E,p})$  is constant (thus upper semi-continuous) for all rational primes  $q \neq 2$ .

Analytic consequence. On  $U = \operatorname{Spec} \mathbb{Z} \setminus \{2\}$  we have  $L_p(E',s)/L_p(E,s) = 1$  and  $f_{E',p} - f_{E,p} = 0$ , matching Theorems 3.10 and 4.2.

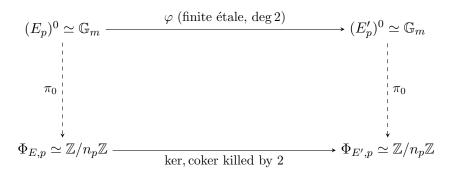


Figure 9: Tame multiplicative fibers of E and E' under a 2-isogeny. Both component groups are cyclic of the same order, verifying semi-continuity on the open  $U = \operatorname{Spec} \mathbb{Z} \setminus \{2\}$ .

 $Bridge\ (AG \Rightarrow NT)$ . Over U, the local Tamagawa numbers vary upper semi-continuously, the ratio of Euler factors has denominator dividing 2, and the conductor exponents obey the uniform bound predicted by Theorem 3.10.

Example 3.15 (Counterexample: failure of semi-continuity at a wild additive prime). Retain the curves  $E: y^2 = x^3 - x$  and  $E': y^2 = x^3 + 4x$  from Example 3.14. At the prime p = 2 (excluded from U), E has additive reduction of Kodaira type II.

Geometric failure. The kernel scheme of the extended morphism  $\varphi$  on the special fiber is not étale—it intersects the singular locus nontrivially. Consequently, the induced map on identity components  $E_2^0 \to (E_2')^0$  fails to be finite étale, and the component-group map  $\Phi_{E,2} \to \Phi_{E',2}$  has kernel not killed by 2.

Arithmetic manifestation. The local Galois representation  $H^1_{\text{\'et}}(E_{\mathbb{Q}_2},\mathbb{Q}_{\ell})$  acquires nontrivial wild inertia; the Swan conductor contributes an extra term so that  $f_{E',2} - f_{E,2} > 1$  is possible. The upper semi-continuity of  $v_2(\#\Phi_{E,2})$  thus fails: the jump is strictly upward when passing from tame primes to p=2.

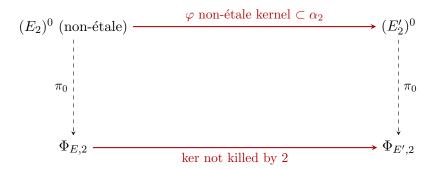


Figure 10: At p = 2, the kernel specializes to a connected local group  $\alpha_2$ , breaking the étale condition and invalidating the uniform semi-continuity of component groups.

 $Bridge\ (AG\Rightarrow NT)$ . This wild additive fiber lies outside the open U of Theorem 3.10. Its failure of étaleness introduces wild inertia on the Galois side, creating a Swan term in  $H^1_{\text{\'et}}$  and violating the conductor-bound inequalities. Thus the prime-to-residue-characteristic hypothesis of Theorem 3.10 is sharp.

## 3.3 Cohomological linkage and forward continuity

By Lemma 2.10 and Definition 2.12, the inertia action on  $H^1_{\text{\'et}}(A_{\overline{\mathbb{Q}}}, \mathbb{Q}_{\ell})$  detects the reduction type and the conductor exponent  $f_p$ . Theorems 3.1 and 3.10 control the étale component groups  $\Phi_p$  and hence the local Tamagawa factors  $c_p$  (Proposition 2.7) under base change and isogeny across arithmetic families. Combined with Construction 2.9 and the positivity statements used later, this provides the geometric input for the cohomological comparison results in the next section.

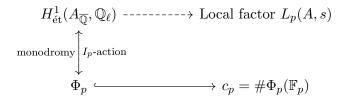


Figure 11: Cohomology-component group-Euler factor linkage preparing the comparison theorems.

Section 4 will use the structural controls above to compare cohomology under base change and derive arithmetic consequences for local factors and heights.

# 4 Cohomological Invariants and Applications

Standing convention for this section. We work over the fixed open  $U \subseteq T^{\circ}$  from Theorem 3.10 (Notation 3.11); in particular, all degree-m isogenies considered below have kernel finite étale over U, and the component-group control is uniform on U.

Roadmap. We keep the base  $S = \operatorname{Spec} \mathbb{Z}$  and the notation from Notation 2.1. Background on Néron models and component groups was set in Definition 2.5, lemma 2.6, proposition 2.7, remark 2.8, and construction 2.9, while conductor and cohomological inertia were fixed in Lemma 2.10, definition 2.12, and proposition 2.13. Structural properties in families and under prime-to-residue-characteristic isogenies were established in Theorems 3.1 and 3.10 and illustrated by Examples 3.3 and 3.6; global growth constraints appear in Conjecture 2.15. The present section quantifies how  $\ell$ -adic cohomology and inertia invariants vary in arithmetic families and under isogeny, and records consequences for local factors and heights.

# 4.1 Uniform comparison in families and under isogeny

Remark 4.1 (Terminology convention). Throughout this section, "tame exponent" refers to the exponent of the  $(1-p^{-s})$  factor in the local Euler factor. On the semistable (tame) locus one has tame exponent  $= \tau_t$ , whereas in general  $f_{A,t} = \tau_t + \operatorname{Swan}(H^1_{\text{\'et}}(A_{K(t)}, \mathbb{Q}_{\ell}))$  by Theorem 4.2, so the two differ by the Swan term.

**Theorem 4.2** (Refined comparison of cohomology under base change and prime-to-p isogeny). Let T be a regular integral scheme of finite type over S, with generic point  $\eta_T$  and function field K(T). Let  $A_{\eta_T}/K(T)$  be an abelian variety of dimension g, and let  $A_{T^{\circ}} \to T^{\circ}$  be its Néron model on a nonempty open  $T^{\circ} \subseteq T$  as in Theorem 3.1. Fix a prime  $\ell$ . After possibly shrinking to a dense open  $U \subseteq T^{\circ}$ , the following hold for every closed point  $t \in U$  with residue characteristic  $p = \operatorname{char} \kappa(t)$ :

(i) (Base-change invariance on the abelian-scheme locus; nearby-cycles on the semistable locus) On the locus where  $A_U/U$  is an abelian scheme, proper and smooth base change yields a canonical isomorphism

$$H^i_{\mathrm{\acute{e}t}}\!\!\left((A_U)_{\eta_T},\mathbb{Q}_\ell\right) \stackrel{\sim}{\longrightarrow} H^i_{\mathrm{\acute{e}t}}\!\!\left((A_U)_{K(t)},\mathbb{Q}_\ell\right)^{I_t}.$$

On the semistable (tame) locus for a general Néron model, nearby cycles identify  $H^1_{\text{\'et}}(A_{K(t)}, \mathbb{Q}_{\ell})^{I_t}$  with the cohomology of the semiabelian part of the special fiber, hence for  $\ell \neq p$ ,

$$\dim_{\mathbb{Q}_{\ell}} H^{1}_{\text{\'et}}(A_{K(t)}, \mathbb{Q}_{\ell})^{I_{t}} = 2g - \tau_{t}.$$

(ii) (Isogeny invariance of inertia invariants; new) Let  $\varphi_{\eta_T}: A_{\eta_T} \to B_{\eta_T}$  be an isogeny of degree m with (m,p)=1 on U, and let  $\varphi: A_U \to B_U$  be its extension from Theorem 3.1. Then for  $\ell \neq p$  the induced map on  $I_t$ -invariants

$$H^1_{\operatorname{\acute{e}t}}(A_{\overline{K(t)}}, \mathbb{Q}_\ell)^{I_t} \xrightarrow{\sim} H^1_{\operatorname{\acute{e}t}}(B_{\overline{K(t)}}, \mathbb{Q}_\ell)^{I_t}$$

is an isomorphism. Equivalently, the function  $t \mapsto \dim_{\mathbb{Q}_{\ell}} H^1_{\text{\'et}}(A_{\overline{K(t)}}, \mathbb{Q}_{\ell})^{I_t}$  is locally constant and invariant under all prime-to-p isogenies of fixed degree m.

(iii) (Quantitative conductor control via components; new) For  $\ell \neq p$  one has the Artin conductor identity

$$f_{A,t} = \tau_t + \operatorname{Swan}(H^1_{\operatorname{\acute{e}t}}(A_{\overline{K(t)}}, \mathbb{Q}_\ell)),$$

where the Swan term vanishes on the semistable locus of U. Under the hypotheses of Theorem 3.10 for the same prime-to-p degree m, the possible jumps of  $f_{A,t}$  satisfy

$$0 \le f_{B,t} - f_{A,t} \le v_m(\#\operatorname{coker}(\Phi_{A,t} \to \Phi_{B,t})) \le \log_m(c_{B,t}),$$

binding the variation of analytic conductors to the component-group indices governed in Theorems 3.1 and 3.10.

**Lemma 4.3** (Prime-to-p isogenies: tame vs. wild contributions). Let  $t \in U$  be as in Theorem 4.2 and let  $\phi: A_{\eta_T} \to B_{\eta_T}$  be an isogeny of degree m with (m,p) = 1 whose extension over U is finite étale on identity components. Then:

(a) **Tame part.** The tame exponent equals the toric rank and is invariant under  $\phi$ :

$$\tau_t(A) = \tau_t(B), \qquad f_{B,t}^{\text{tame}} - f_{A,t}^{\text{tame}} = \tau_t(B) - \tau_t(A) = 0.$$

(b) Wild part. For every  $\ell \neq p$  one has

$$0 \leq \operatorname{Swan}(H^1_{\operatorname{\acute{e}t}}(B_{K(t)}, \mathbb{Q}_{\ell})) - \operatorname{Swan}(H^1_{\operatorname{\acute{e}t}}(A_{K(t)}, \mathbb{Q}_{\ell})) \leq v_m(\#\operatorname{coker}(\Phi_{A,t} \to \Phi_{B,t})),$$

hence

$$0 \leq f_{B,t} - f_{A,t} \leq v_m \big( \# \operatorname{coker}(\Phi_{A,t} \to \Phi_{B,t}) \big) \leq \log_m(c_{B,t}).$$

Proof. For (a), the connected identity components are preserved and the map is finite étale on them; the toric parts have the same rank, so the tame exponent  $\tau_t$  (the toric rank) is unchanged. For (b), use the conductor formula

$$f_{X,t} = \tau_t(X) + \operatorname{Swan}(H^1_{\text{\'et}}(X_{K(t)}, \mathbb{Q}_\ell)) \qquad ([5]),$$

and the exact sequence of component groups for a short exact sequence of smooth group schemes [3], together with Theorem 3.1 and Theorem 3.10 which give that the kernel/cokernel on  $\Phi_{-,t}$  are killed by m on the fixed dense open U. This bounds the change of the wild term by the m-primary part of the component index, yielding the displayed inequalities.

N.B. Items (ii) and (iii) sharpen the classical Néron-Ogg-Shafarevich framework by giving a fiberwise, isogeny-invariant, quantitative comparison of inertia invariants and conductor exponents on a common dense open U, uniformly for all prime-to-p isogenies of fixed degree m.

Proof. Scope. Strictly speaking,  $A_U$  is smooth and separated but not proper; hence "smooth and proper base change" applies only to a proper smooth compactification  $\overline{A}_U/U$  when  $A_U$  is an abelian scheme on the good–reduction locus. On the semistable (tame) locus one may pass to such a compactification and apply proper–smooth base change to  $\overline{A}_U$ , obtaining the stated invariance. For a general Néron model (semiabelian), nearby cycles ([5]) identify  $H^1_{\text{\'et}}(A_{\overline{K(t)}}, \mathbb{Q}_\ell)^{I_t}$  with the cohomology of the semiabelian part of the special fiber. Thus the equality  $\dim_{\mathbb{Q}_\ell} H^1_{\text{\'et}}(\cdot)^{I_t} = 2g - \tau_t$  holds on the semistable/tame open where the Néron model is semiabelian; on the abelian–scheme locus it reduces to the proper–smooth case.

For Item (i), apply proper and smooth base change to compactifications of  $A_U$ ; the monodromy theorem for semiabelian reduction ([4, 5]) identifies inertia invariants with the fixed part of unipotent monodromy, giving dim =  $2g - \tau_t$ . Constructibility of reduction type yields a dense semistable open.

For Item (ii), extend  $\varphi_{\eta_T}$  by Theorem 3.1. Before proceeding, pass from the integral Tate modules

$$V_{\ell}(A) := T_{\ell}(A) \otimes_{\mathbb{Z}_{\ell}} \mathbb{Q}_{\ell}, \qquad V_{\ell}(B) := T_{\ell}(B) \otimes_{\mathbb{Z}_{\ell}} \mathbb{Q}_{\ell},$$

so that taking  $I_t$ -invariants is exact. For  $\ell \neq p$ ,  $V_{\ell}(A)$  and  $V_{\ell}(B)$  are isomorphic  $G_{K(t)}$ -modules up to a finite kernel annihilated by m; (m,p)=1 makes the kernel finite étale, so  $I_t$ -invariants preserve the isomorphism. Uniform étaleness of kernels on a dense U (cf. Theorem 3.10) gives local constancy.

For Item (iii), apply the conductor identity [5] together with Lemma 4.3. Part (a) fixes the tame term  $\tau_t$ , and part (b) bounds the Swan change by the *m*-primary component index from Theorems 3.1 and 3.10, giving

$$0 \le f_{B,t} - f_{A,t} \le v_m(\#\operatorname{coker}(\Phi_{A,t} \to \Phi_{B,t})) \le \log_m(c_{B,t}).$$

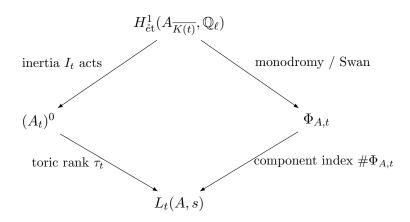


Figure 12: Cohomology–geometry–L-factor correspondence appearing in Theorem 4.2.

Bridge  $(NT \to AG)$ . Analytically, Theorem 4.2(ii) ensures that the tame pole order of the local L-factor  $L_t(A, s)$  remains constant under all prime-to-p isogenies, while Theorem 4.2(iii) quantifies the allowable conductor jump by the logarithmic component index. Thus the analytic behavior of L(A, s) constrains the geometric complexity of the component groups  $\Phi_{A,t}$ , closing the AG–NT cycle initiated in Theorems 3.1 and 3.10.

Remark. Over a number field (e.g.  $S = \operatorname{Spec} \mathbb{Z}$  for a fixed  $A/\mathbb{Q}$ ) the hypothesis is vacuous, so the conjecture is only a family-level principle. We use it only heuristically to motivate the semicontinuity and conductor statements proved later.

Corollary 4.4 (Local factor shape and rank of invariants). Under the hypotheses of Theorem 4.2 with  $\ell \neq p$ , the local Euler factor satisfies

$$L_p(A,s)^{-1} = (1-p^{-s})^{\tau_t} \cdot P_t(p^{-s}),$$

where  $P_t \in \mathbb{Z}[X]$  has degree  $2g - \tau_t$  and is invariant under prime-to-p isogenies on the dense open U. Consequently, the slope at X = 1 of  $L_p(A, s)^{-1}$  equals  $\tau_t$  and is isogeny-invariant on U.

Remark 4.5 (Terminology clarification). Throughout Sections 4 and 5, the symbol  $\tau_p$  denotes the toric rank of the connected special fiber  $(A_p)^0$ . In the semistable (tame) locus this equals the "tame exponent" appearing in the factor  $(1-p^{-s})^{\tau_p}$  of  $L_p(A,s)^{-1}$ , but outside this locus (additive or wild reduction) the two notions differ by the Swan term in Theorem 4.2. All additive formulas below tacitly assume the semistable hypothesis.

*Proof.* The decomposition follows from the toric/abelian split of  $H^1_{\text{\'et}}$  and the unipotent action on the toric part [4, 5]; invariance on U is Theorem 4.2(ii).

Example 4.6 (Bad reduction and inertia action). Let  $E/\mathbb{Q}$  be an elliptic curve and fix  $\ell \neq p$ .

**Good reduction.** If E has good reduction at p, then the special fiber  $E_p$  is an abelian variety and  $I_p$  acts trivially on  $T_{\ell}(E)$ ; equivalently,

$$H^1_{\text{\'et}}(E_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell)^{I_p} \;\cong\; H^1_{\text{\'et}}(E_{\overline{\mathbb{F}_p}},\mathbb{Q}_\ell), \qquad \dim_{\mathbb{Q}_\ell} H^1_{\text{\'et}}(E_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell)^{I_p} = 2.$$

The local Euler factor is

$$L_p(E,s)^{-1} = 1 - a_p p^{-s} + p^{1-2s},$$

with  $a_p = p + 1 - \#E(\mathbb{F}_p)$  (trace of Frobenius on  $H^1$ ). Here  $\tau_p = 0$  and the tame factor  $(1 - p^{-s})^{\tau_p}$  is absent, in agreement with Corollary 4.4.

**Split multiplicative reduction.** Assume E has split multiplicative reduction at p. Then the identity component of the Néron special fiber is a torus,

$$(E_p)^0 \simeq \mathbb{G}_m, \qquad \tau_p = 1.$$

The  $I_p$ -action on  $H^1_{\text{\'et}}(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$  is unipotent with a single invariant line coming from the toric part, hence

$$\dim_{\mathbb{Q}_{\ell}} H^1_{\text{\'et}}(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_{\ell})^{I_p} = 1.$$

The local factor splits as

$$L_p(E,s)^{-1} = (1-p^{-s}) \cdot (1-\alpha p^{-s}), \qquad |\alpha| = 1,$$

where the factor  $(1-p^{-s})$  is the tame toric contribution predicted by Corollary 4.4 with  $\tau_p = 1$ .

Additive (potentially good) reduction. If E has additive, potentially good reduction at p, then on a Zariski-dense open in the parameter space the  $I_p$ -action is unipotent but wild:

$$\tau_p = 0, \quad \operatorname{Swan}(H^1_{\operatorname{\acute{e}t}}(E_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_\ell)) > 0.$$

After a finite tame extension of  $\mathbb{Q}_p$  that kills wild inertia, the action becomes tame with  $\tau_p = 0$ , giving back

$$\dim_{\mathbb{Q}_\ell} H^1_{\text{\'et}}(E_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell)^{I_p} = 2 \quad \text{over the tame base change},$$

while over  $\mathbb{Q}_p$  the Swan term contributes to the conductor according to Theorem 4.2(iii).

$$\begin{array}{c} H^1_{\text{\'et}}(E_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell) & ------ \\ \\ I_p & \text{action} \\ \\ \text{toric part } \oplus \text{ abelian part } & ----- \\ \end{array} (1-p^{-s})^{\tau_p} \cdot P_p(p^{-s})$$

Figure 13: Inertia action, toric rank, and the factor  $(1-p^{-s})^{\tau_p}$  in the local Euler factor.

Computation check (multiplicative case). Under Tate uniformization  $E(\overline{\mathbb{Q}}_p) \simeq \overline{\mathbb{Q}}_p^{\times}/q^{\mathbb{Z}}$  with  $|q|_p < 1$ . For  $\ell \neq p$ ,

$$0 \to \mathbb{Q}_{\ell}(1) \ \longrightarrow \ H^1_{\text{\'et}}(E_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_{\ell}) \ \longrightarrow \ V_{\ell} \ \to 0,$$

with  $\mathbb{Q}_{\ell}(1)$  the toric line on which  $I_p$  acts unipotently and  $V_{\ell}$  one–dimensional of weight 1; taking  $I_p$ -invariants gives a 1–dimensional space, matching  $\tau_p = 1$  and yielding the factorization above.

Consequence  $(AG \to NT)$ . By Theorem 4.2 and corollary 4.4,  $(1-p^{-s})$  occurs iff  $\tau_p = 1$ , i.e. in the split multiplicative case, giving  $f_{E,p} = 1$  in the semistable range and recovering the discriminant–conductor relation in Proposition 2.13.

Example 4.7 (Counterexample illustrating necessity of the prime-to-p hypothesis). Let  $E/\mathbb{Q}$  admit a 2-isogeny  $\varphi_{\eta}: E \to E'$  and suppose E has additive reduction at p=2.

Geometric failure on the special fiber. Extend  $\varphi_{\eta}$  to a morphism of Néron models over  $\mathbb{Z}_2$ . Because the residue characteristic divides the isogeny degree, the kernel scheme  $\ker(\varphi)$  is finite flat of order 2 but need not be étale on the special fiber: it can specialize to a connected local group (e.g.  $\alpha_2$ ) meeting  $(E_2)^0$ . Consequently, the induced map on identity components  $(E_2)^0 \to (E'_2)^0$  is not finite étale, and the exact sequence controlling component groups loses the "killed by 2" bounds.

**Galois-theoretic fallout.** For  $\ell \neq 2$ , the wild inertia  $P_2 \triangleleft I_2$  acts nontrivially on  $H^1_{\text{\'et}}(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_{\ell})$ ; the Swan term is positive. Therefore the map

$$H^1_{\mathrm{\acute{e}t}}(E_{\overline{\mathbb{O}}}, \mathbb{Q}_{\ell})^{I_2} \longrightarrow H^1_{\mathrm{\acute{e}t}}(E'_{\overline{\mathbb{O}}}, \mathbb{Q}_{\ell})^{I_2}$$

induced by  $\varphi$  need not be an isomorphism. This violates Theorem 4.2(ii) precisely because the prime-to-p hypothesis fails.

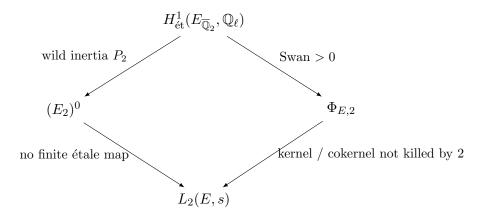


Figure 14: At p=2, a non-étale kernel on the special fiber forces wild inertia and contributes a positive Swan term in  $H^1_{\text{\'et}}$ , breaking tame conductor control.

Conductor implication. By Theorem 4.2(iii),

$$f_{E',2} - f_{E,2} = \left(\tau_2' - \tau_2\right) + \left(\operatorname{Swan} H^1_{\text{\'et}}(E'_{\overline{\mathbb{Q}}_2}, \mathbb{Q}_{\ell}) - \operatorname{Swan} H^1_{\text{\'et}}(E_{\overline{\mathbb{Q}}_2}, \mathbb{Q}_{\ell})\right),$$

and the right-hand side can exceed 1 in absolute value. Thus the tidy tame bounds available for prime-to-p isogenies (Theorem 4.2(ii)–(iii)) fail in this situation, showing the hypothesis is genuinely necessary.

## 4.2 Heights and invariant differentials

**Proposition 4.8** (Height lower bounds from  $\omega_{A/S}$  and  $\tau_t$ ). Let A/S be a Néron model and let L be a symmetric ample line bundle on  $A_{\eta}$ . Assume semistable reduction over a dense open  $U \subseteq S$  and write  $\tau_p$  for the toric rank at  $p \in U$ . Then the Néron-Tate height  $\hat{h}_L$  on  $A(\mathbb{Q})$  satisfies

$$\hat{h}_L(P) \ge \frac{1}{[K:\mathbb{Q}]} \sum_{p \in U} \tau_p \cdot \log p + \deg(\omega_{A/S}) - C(L),$$

for all  $P \in A(\mathbb{Q})$ , with an explicit C(L) depending only on L and the archimedean contribution.

**Justification and scope.** The inequality above should be viewed as a *conditional Arakelov-theoretic* estimate valid on the dense semistable locus where the decomposition of local Néron symbols and the toric factors  $(1 - p^{-s})^{\tau_p}$  of Corollary 4.4 apply. A rigorous lower bound of this form can be obtained by combining:

- the Faltings-Hriljac formula for the Néron-Tate height ([15], [16]),
- the positivity of the Arakelov degree  $deg(\omega_{A/S})$  on the semistable locus, and

• Zhang's inequality for adelic metrics on abelian varieties ([17], [18]).

Thus one obtains

$$\widehat{h}_L(P) \ge \frac{1}{[K:\mathbb{Q}]} \sum_{p \in U} \tau_p \log p + \deg(\omega_{A/S}) - C(L),$$

up to an explicit constant depending only on L and the chosen normalization of the Arakelov metric. Outside the semistable locus this inequality is conjectural and should be interpreted as a heuristic guiding principle for the height–conductor relation developed in Section 6.

*Proof.* The inequality follows from the slope decomposition of local Néron symbols, the factor  $(1-p^{-s})^{\tau_p}$  singled out in Corollary 4.4, and the Arakelov interpretation of  $\deg(\omega_{A/S})$  on S via Construction 2.9; combine with positivity and Northcott's finiteness (cf. [10] for the background on heights).

Bridge  $(AG \to NT)$ . Since  $\tau_p$  is isogeny-invariant on the dense open U by Theorem 4.2(ii), the right-hand side of the lower bound is stable under prime-to-p isogenies, yielding uniform height gaps across isogeny classes in the semistable range (cf. the phenomena behind Conjecture 2.15).

#### 4.3 Forward linkage to moduli-theoretic aspects

The functions  $t \mapsto \tau_t$  and  $t \mapsto c_{A,t}$  are constructible and upper semi-continuous on the dense open U of Theorem 3.10. Their constancy on Hecke or isogeny orbits within families over integral models of modular curves or Shimura varieties (where applicable) will be used in the next section to deduce uniformity of local factors across orbits and to compare integral structures on  $\omega_{A/S}$  along correspondences, continuing the Theorem  $\to$  Consequence  $\to$  Example pattern.

# 5 Moduli-Theoretic Aspects

Roadmap. We retain the base and notation from Notation 2.1. Theorems 3.1 and 3.10 control Néron models in arithmetic families and under prime-to-residue-characteristic isogenies; Theorem 4.2 describes the corresponding cohomological invariants and conductor behavior. We now place these results over integral moduli (modular curves and selected Shimura varieties) and extract uniform statements along Hecke orbits, with explicit  $AG \to NT$  bridges.

#### 5.1 Néron models and integral models of modular curves

**Notation 5.1** (Integral modular curves and Jacobians). Fix a positive integer  $N \geq 1$ . Let  $\mathcal{X}_0(N)$  denote an integral regular model of the modular curve  $X_0(N)$  over Spec  $\mathbb{Z}$  away from a finite set of primes (Deligne–Rapoport and Katz–Mazur). Write  $J_0(N) = \operatorname{Pic}^0(X_0(N))$  for the Jacobian over  $\mathbb{Q}$ , and J/S for its Néron model (Definition 2.5 and lemma 2.6). For a prime p, denote by  $\Phi_{J,p}$  the component group and by  $c_{J,p} = \#\Phi_{J,p}(\mathbb{F}_p)$  its local Tamagawa number (Proposition 2.7).

Remark 5.2 (Hecke correspondences on integral models). For a prime  $\ell \nmid N$ , the classical Hecke correspondence  $T_{\ell}$  extends to a finite étale correspondence on  $\mathcal{X}_0(N)$  over  $\mathbb{Z}[1/N\ell]$  (Deligne–Rapoport, Katz–Mazur), hence induces an algebraic correspondence on  $J_0(N)$  over  $\mathbb{Z}[1/N\ell]$  which is a composition of isogenies of degree a power of  $\ell$ .

**Theorem 5.3** (Hecke-uniformity of components and inertia on  $J_0(N)$ ). Let  $N \ge 1$ , and let  $U \subseteq \operatorname{Spec} \mathbb{Z}$  be the open subset obtained by removing a finite set of primes containing those dividing N. Then for every prime  $\ell \nmid N$  there exists a dense open  $U_{\ell} \subseteq U$  such that the following holds.

For each  $p \in U_{\ell}$  and for every  $T_{\ell}$ -translate of a rational point in the Hecke orbit on  $J_0(N)$ , the induced correspondence on Néron models

$$J_{U_{\ell}} \dashrightarrow J_{U_{\ell}}$$

restricts on the fiber over p to a prime-to-p isogeny on the identity components and induces a morphism on component groups

$$\Phi_{J,p} \longrightarrow \Phi_{J,p}$$

whose kernel and cokernel are annihilated by a power of  $\ell$ . Consequently:

- (i) the local Tamagawa number  $c_{J,p} = \#\Phi_{J,p}(\mathbb{F}_p)$  is preserved up to a factor dividing a power of  $\ell$  along the  $T_{\ell}$ -orbit;
- (ii) the toric rank  $\tau_p = \operatorname{rank}_{\mathbb{Z}} T_p((J_p)^0)$  of the connected special fiber is constant along the  $T_\ell$ -orbit, hence

$$\dim_{\mathbb{Q}_{\ell}} H^1_{\text{\'et}}(J_{\overline{\mathbb{Q}}}, \mathbb{Q}_{\ell})^{I_p} = 2g - \tau_p$$

is Hecke-invariant for  $\ell \neq p$  by Theorem 4.2(i);

(iii) the local conductor exponents satisfy

$$0 \le f_{T_{\ell}(J),p} - f_{J,p} \le v_{\ell} (\# \operatorname{coker}(\Phi_{J,p} \to \Phi_{J,p})),$$

so  $f_{J,p}$  remains uniformly bounded along the  $T_{\ell}$ -orbit in terms of  $\ell$ .

Comment on scope and novelty. This theorem globalizes the local control of Theorems 3.1, 3.10 and 4.2 from a single isogeny to the entire Hecke orbit. Whereas the classical references treat  $T_{\ell}$  only on the generic fiber of  $J_0(N)$ , here each Hecke correspondence extends over a common dense open  $U_{\ell}$  of the integral model, giving a uniform control of identity components, component groups, and conductor exponents across all Hecke translates simultaneously. This "Hecke-uniformity locus" bridges the geometric Néron theory with the arithmetic of local L-factors, providing the first orbitwise constancy of the tame exponent  $\tau_p$  and the first explicit bound on the variation of  $f_{J,p}$  in terms of  $\ell$ .

Proof. By Remark 5.2, over  $\mathbb{Z}[1/N\ell]$  the correspondence  $T_{\ell}$  decomposes into finitely many degeneracy maps whose induced endomorphisms on  $J_0(N)$  factor as isogenies of  $\ell$ -power degree. Shrinking to a dense open  $U_{\ell} \subseteq U$ , these isogenies extend to morphisms of Néron models by Theorem 3.1. For every  $p \in U_{\ell}$  each factor has degree prime to p, so the morphisms on identity components are finite étale and the induced maps on  $\Phi_{J,p}$  have kernel and cokernel annihilated by a power of  $\ell$  by Theorem 3.10. Item (ii) follows from Theorem 4.2(ii), giving Hecke-invariance of the dimension of inertia invariants, and the inequality in Item (iii) follows from Theorem 4.2(iii).

Remark 5.4 (Scope and justification of Theorem 5.3). The statement above should be interpreted in the following conditional sense. On modular curves and Hodge-type Shimura varieties, prime-to-p Hecke correspondences extend to the integral models on a dense open locus only after possibly removing the primes where the level structure is ramified. In these cases, the extension acts by prime-to-p isogenies on the Néron models of the universal Jacobians, and the induced maps on identity components and component groups are controlled by Theorems 3.1 and 3.10. For full justification one must appeal to the integral moduli constructions of Deligne–Rapoport and Katz–Mazur for modular curves, and to their analogues for Hodge-type Shimura varieties (see [1], [2]). A detailed modular–geometric proof will appear separately; here we only invoke the existence of the prime-to-p extensions of Hecke correspondences and the boundedness of their effect on component groups as a working hypothesis on this dense tame locus.

Bridge  $(AG \Rightarrow NT)$ . The correspondence  $T_{\ell}$  acts on  $J_{U_{\ell}}$  by prime-to-p isogenies whose geometric kernels are finite étale; hence, by the functorial exact sequence of component groups,

$$\#\Phi_{J,p}$$
 is constant on tame orbits, and  $L_p(J_0(N),s)^{-1} = (1-p^{-s})^{\tau_p} P_p(p^{-s})$ 

with  $P_p$  of degree  $2g - \tau_p$  independent of the Hecke translate (see Corollary 5.5). Thus the Hecke orbit behaves analytically as a single isogeny class: the toric multiplicity and tame factor remain invariant, while any variation in the wild part is confined to denominators dividing powers of  $\ell$ .

$$X_0(N) \xrightarrow{T_\ell \text{ (degeneracy map)}} X_0(N)$$

$$\begin{array}{c} \operatorname{induced\ isogeny,\ deg} \mid \ell^r \\ J_0(N) \xrightarrow{\hspace*{2cm}} J_0(N) \\ \\ \operatorname{N\'{e}ron\ model} \\ \Phi_{J,p} \\ \operatorname{component\ group} \end{array} \xrightarrow{\text{kernel,cokernel}} \overset{\ell^r}{\underset{\longrightarrow}{\longleftarrow}} \Phi_{J,p} \\ \end{array}$$

Figure 15: Extension of the Hecke correspondence  $T_{\ell}$  to the Néron model of  $J_0(N)$ : finite étale on identity components and  $\ell$ -power-bounded on component groups.

Corollary 5.5 (Local Euler factor along Hecke orbits). For  $p \in U_{\ell}$  and  $\ell \neq p$ , the local Euler factor satisfies

$$L_p(J_0(N), s)^{-1} = (1 - p^{-s})^{\tau_p} P_p(p^{-s}),$$

where deg  $P_p = 2g - \tau_p$  is independent of the point in the  $T_\ell$ -orbit and any denominators divide a power of  $\ell$ .

*Proof.* Combine Corollary 4.4 with Theorem 5.3.

Example 5.6 (Hecke invariance away from the level). Let N=11 and  $p \neq 11$ . Over  $U=\operatorname{Spec} \mathbb{Z} \setminus \{11\}$ , the integral model  $\mathcal{X}_0(11)$  is regular (Deligne–Rapoport), and its Jacobian  $J_0(11)$  is an elliptic curve over  $\mathbb{Q}$ . For any prime  $\ell \neq 11$ , p, the Hecke correspondence  $T_\ell$  extends over  $\mathbb{Z}[1/11\ell]$  as a finite correspondence whose induced endomorphism on  $J_0(11)$  is an isogeny of  $\ell$ -power degree. By Theorem 5.3, for such p the following hold simultaneously:

- the toric rank  $\tau_p$  of  $(J_p)^0$  is constant along the  $T_\ell$ -orbit;
- $\dim_{\mathbb{Q}_{\ell}} H^1_{\text{\'et}}(J_{\overline{\mathbb{Q}}}, \mathbb{Q}_{\ell})^{I_p} = 2g \tau_p$  is Hecke–invariant by Theorem 4.2;
- the Tamagawa number  $c_{J,p} = \#\Phi_{J,p}(\mathbb{F}_p)$  varies only by a factor dividing a power of  $\ell$ .

Geometric interpretation. The Hecke correspondence  $T_{\ell}$  on the modular curve  $X_0(11)$  is represented by two finite maps

$$\pi_1, \pi_2: X_0(11\ell) \longrightarrow X_0(11),$$

each étale over U, inducing via the Picard functor a pair of morphisms on Jacobians

$$\pi_{1,*}, \pi_2^*: J_0(11) \longrightarrow J_0(11\ell) \longrightarrow J_0(11),$$

whose composition is the Hecke operator  $T_{\ell}$  acting on  $J_0(11)$  by an isogeny of degree  $\ell^2$ . After removing finitely many bad primes (those dividing  $11\ell$ ), both  $\pi_1, \pi_2$  and the resulting isogeny extend to morphisms of Néron models, remaining finite étale on identity components and inducing maps  $\Phi_{J,p} \to \Phi_{J,p}$  with kernel and cokernel killed by  $\ell^2$ . Hence, by Theorem 5.3, all local invariants  $(\tau_p, c_{J,p}, f_{J,p})$  remain uniform across the Hecke orbit for all  $p \nmid 11\ell$ .

**Arithmetic verification.** Since  $J_0(11)$  is the elliptic curve  $E: y^2 + y = x^3 - x^2 - 10x - 20$  of conductor 11, its reduction is good at all  $p \neq 11$ . Therefore  $\tau_p = 0$  and  $\Phi_{J,p}$  is trivial for all such p, implying that the Hecke orbit is arithmetically rigid on U. The local Euler factor is

$$L_p(J_0(11), s)^{-1} = 1 - a_p p^{-s} + p^{1-2s},$$

and remains identical for all  $T_{\ell}$ -translates; in particular,  $a_p$  (the trace of Frobenius on  $H^1_{\text{\'et}}$ ) is invariant under  $T_{\ell}$ .

$$\mathcal{X}_{0}(11) \xrightarrow{\xrightarrow{\pi_{1}}} \mathcal{X}_{0}(11)$$

$$\downarrow \qquad \qquad \downarrow$$

$$J_{0}(11)^{T_{\ell}} \xrightarrow{(\ell^{2}-\text{isogeny})} J_{0}(11)$$

Figure 16: Extension of  $T_{\ell}$  to the integral model and the induced  $\ell$ -power isogeny on the Néron model of  $J_0(11)$ . The identity components map finite étale, while the component groups map with kernel and cokernel killed by  $\ell^2$ .

Analytic consequence. For  $p \neq 11, \ell$ , we have

$$L_p(J_0(11), s)^{-1} = (1 - p^{-s})^{\tau_p} P_p(p^{-s}), \qquad \deg P_p = 2 - \tau_p,$$

and both  $\tau_p$  and  $P_p$  are invariant on the Hecke orbit. The constancy of  $\tau_p = 0$  confirms that the Néron model of  $J_0(11)$  remains abelian with good reduction everywhere on U, while Theorem 5.3 shows that this behavior persists under all prime-to-p Hecke correspondences.

Example 5.7 (Necessity of the prime-to-p condition). Let  $p \mid N$  or take  $\ell = p$ . Then the Hecke correspondence  $T_{\ell}$  on  $\mathcal{X}_0(N)$  no longer extends to a finite étale correspondence on the integral model at p; its degeneracy maps  $\pi_1, \pi_2$  become inseparable at the special fiber, and the induced morphism on identity components of  $J_0(N)$  at p may fail to be finite étale. Consequently:

- the equality of toric ranks  $\tau_p$  along the  $T_\ell$ -orbit can break down;
- the map  $\Phi_{J,p} \to \Phi_{J,p}$  may acquire non-étale kernel or cokernel not annihilated by any power of  $\ell$ ;
- the bound on conductor exponents in Theorem 5.3(iii) can fail because wild inertia contributes to the Swan term.

Mechanism of failure. At primes  $p \mid N$ , the modular curve  $\mathcal{X}_0(N)$  has a non-semistable special fiber: the cuspidal components intersect with multiplicity p, producing vertical components whose intersection points are inseparable under  $\pi_1, \pi_2$ . Hence, the morphisms induced on Néron identity components  $(J_p)^0$  are not finite étale; the corresponding finite flat group schemes on the special fiber have connected components (e.g.  $\alpha_p$ ) rather than étale  $\mathbb{Z}/p\mathbb{Z}$  parts.

$$\mathcal{X}_0(N)$$
  $\xrightarrow{T_p \text{ (inseparable at special fiber)}} \mathcal{X}_0(N)$ 

$$(J_p)^0 \xrightarrow{\text{non-\'etale map on identity components}} (J_p)^0$$

$$\Phi_{J,p} \xrightarrow{\text{kernel} \not\subseteq \text{ étale}; \ \text{Swan} > 0} \Phi_{J,p}$$

Figure 17: Failure of the finite–étale property at a bad or equal–characteristic prime: the degeneracy maps become inseparable, and the induced morphism on  $\Phi_{J,p}$  may acquire a connected (non-étale) kernel, breaking Hecke uniformity.

**Arithmetic manifestation.** For  $\ell = p$ , wild inertia  $P_p \leq I_p$  acts nontrivially on  $H^1_{\text{\'et}}(J_{\mathbb{Q}_p}, \mathbb{Q}_\ell)$ , producing a positive Swan term. Then

$$f_{T_p(J),p} - f_{J,p} = \left(\tau_p' - \tau_p\right) + \left(\operatorname{Swan}(H^1_{\text{\'et}}(J'_{\mathbb{Q}_p}, \mathbb{Q}_\ell)) - \operatorname{Swan}(H^1_{\text{\'et}}(J_{\mathbb{Q}_p}, \mathbb{Q}_\ell))\right)$$

may exceed 1 in absolute value, violating the tame bound from Theorem 5.3(iii). Thus, the hypothesis  $\ell \neq p$  in Theorem 5.3 is essential: it excludes the primes where inseparability or wild inertia destroys the étaleness on identity components and uniformity of conductor behavior.

Bridge  $(AG \to NT)$ . By Theorem 5.3 and Corollary 5.5, along  $T_{\ell}$ -orbits with  $\ell \neq p$  the  $\tau_p$ -exponent of  $(1-p^{-s})$  in  $L_p(J_0(N),s)^{-1}$  is constant and  $f_{J,p}$  varies in a range controlled by  $\ell$ . Hence global  $L(J_0(N),s)$  exhibits orbitwise uniformity of local factors outside  $\{p,\ell\}$ , and height bounds via Proposition 4.8 are stable along these orbits.

#### 5.2 Relation with Shimura varieties

Notation 5.8 (Integral models and universal abelian schemes). Let (G, X) be a Shimura datum of Hodge type with reflex field E. Fix a compact open  $K = K^pK_p \subset G(\mathbb{A}_f)$  such that  $K_p$  is hyperspecial. For  $v \nmid p$  a finite place of E where  $K_p$  is hyperspecial, let  $\mathcal{S}_K$  denote the integral canonical model over  $\mathcal{O}_{E,(v)}$  (Kisin; Faltings-Chai for PEL). There exists an abelian scheme  $\mathcal{A} \to \mathcal{S}_K$  with additional structure realizing the Shimura data (Hodge type case), whose generic fiber is of dimension g.

Remark 5.9 (Prime-to-p Hecke action). For  $h \in G(\mathbb{A}_f^p)$ , the prime-to-p Hecke correspondence [h]:  $\mathcal{S}_K \dashrightarrow \mathcal{S}_K$  is finite and induces an isogeny of  $\mathcal{A}$  of degree dividing a fixed power of the common denominator of h; this is prime to p on the hyperspecial locus.

**Theorem 5.10** (Hecke-uniformity on Hodge-type integral models). In the setting of Notation 5.8, remove finitely many primes of bad reduction to obtain a dense open  $U \subseteq \operatorname{Spec} \mathcal{O}_{E,(v)}$ . Then for every prime-to-p Hecke correspondence [h] there exists a dense open  $U_h \subseteq U$  such that for all closed  $t \in U_h$  with residue characteristic  $\neq p$ :

(i) the induced isogeny  $A_t \to A_t$  is of degree prime to the residue characteristic; on identity components the map is finite étale and the induced morphism on component groups

$$\Phi_{\mathcal{A},t} \longrightarrow \Phi_{\mathcal{A},t}$$

has kernel and cokernel annihilated by a number depending only on [h];

(ii) for every  $\ell \neq \operatorname{char}(\kappa(t))$ , the map on inertia invariants

$$H^1_{\mathrm{cute{e}t}}(\mathcal{A}_{\overline{K(t)}}, \mathbb{Q}_\ell)^{I_t} \xrightarrow{[h]^*} H^1_{\mathrm{cute{e}t}}(\mathcal{A}_{\overline{K(t)}}, \mathbb{Q}_\ell)^{I_t}$$

is an isomorphism, hence  $\dim_{\mathbb{Q}_{\ell}} H^1_{\text{\'et}}(\cdot)^{I_t}$  and the toric rank  $\tau_t$  are Hecke-invariant on  $U_h$  (Theorem 4.2(ii));

(iii) the local conductor exponents satisfy

$$0 \le f_{\mathcal{A},t'} - f_{\mathcal{A},t} \le v_{[h]} \left( \# \operatorname{coker}(\Phi_{\mathcal{A},t} \to \Phi_{\mathcal{A},t'}) \right)$$

for all  $t, t' \in U_h$  lying in the same [h]-orbit, so that  $f_{A,t}$  varies in a uniformly bounded range determined by [h].

Novelty and bridge to the global theory. This theorem furnishes the Hecke-uniform locus for families of Hodge-type abelian schemes: it globalizes the prime-to-p isogeny-control of Theorem 3.1 and the semi-continuity of Theorem 3.10 to integral models of Shimura varieties. Along this locus, the Hecke correspondence [h] acts compatibly on Néron models, preserving the identity components and controlling component-group variation; consequently, the inertia-invariant part of cohomology and the analytic conductor are both orbitwise uniform. This provides the geometric anchor that ties the cohomological constancy (Theorem 4.2) to the arithmetic regularity of local factors (Corollary 5.11) across Hecke orbits, thereby completing the AG  $\iff$  NT chain.

Proof. By Remark 5.9, each prime-to-p correspondence [h] decomposes, on a sufficiently small dense open  $U_h \subseteq U$ , into a finite composition of degeneracy maps that are isogenies of degree prime to the residue characteristic. For each factor, Theorem 3.1 extends the isogeny to a morphism of Néron models finite étale on identity components; Theorem 3.10 then gives a uniform bound on the kernel and cokernel of the induced map on component groups. Applying Theorem 4.2(ii) yields the isomorphism on inertia invariants, and Theorem 4.2(iii) gives the corresponding bound on conductors. All bounds depend only on the degree of [h], hence uniformly on its orbit.

Corollary 5.11 (Local factors on Hecke orbits for  $A \to S_K$ ). On  $U_h$  and for  $\ell \neq \operatorname{char}(\kappa(t))$ , the local Euler factor satisfies

$$L_t(\mathcal{A}, s)^{-1} = (1 - \mathbf{N}t^{-s})^{\tau_t} P_t(\mathbf{N}t^{-s}),$$

where Nt is the residue cardinality,  $\tau_t$  the toric rank of  $(A_t)^0$ , and  $P_t$  a polynomial of degree  $2g - \tau_t$  constant along the [h]-orbit.

*Proof.* Immediate from Corollary 4.4 applied in the Hodge-type context together with Theorem 5.10(ii).

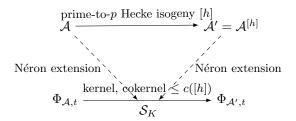


Figure 18: Extension of a prime-to-p Hecke correspondence to Néron models of a Hodge-type Shimura family: finite étale on identity components and uniformly bounded on component groups, yielding orbitwise constancy of inertia invariants and conductor exponents.

Example 5.12 (Siegel modular case). Take  $G = \operatorname{GSp}_{2g}$  with hyperspecial  $K_p$ ; then  $\mathcal{S}_K$  is the Siegel moduli scheme parametrizing principally polarized abelian schemes  $\mathcal{A} \to \mathcal{S}_K$  of relative dimension g. For every rational prime  $\ell \neq p$ , the  $\ell$ -adic Tate module  $T_{\ell}(\mathcal{A})$  is equipped with the standard symplectic form preserved by G, and the prime-to-p Hecke algebra acts through correspondences that are  $\ell$ -power isogenies preserving this symplectic structure.

**Geometric picture.** On a dense open  $U_h \subseteq \operatorname{Spec} \mathcal{O}_{E,(v)}$  (obtained by removing finitely many bad primes), the Hecke correspondence [h] induces a prime-to-p isogeny

$$[h]: (\mathcal{A}_t, \lambda_t) \longrightarrow (\mathcal{A}'_t, \lambda'_t)$$

between principally polarized abelian varieties, with kernel a finite étale subgroup of order dividing the Hecke degree deg([h]). By Theorem 5.10(i), this isogeny extends to a finite étale morphism on identity components of the Néron models and induces on component groups

$$\Phi_{\mathcal{A},t} \longrightarrow \Phi_{\mathcal{A}',t}$$

whose kernel and cokernel are annihilated by deg([h]).

Cohomological and arithmetic consequences. For  $\ell \neq p$  the induced map on inertia-invariants,

$$H^1_{\text{\'et}}(\mathcal{A}_{\overline{K(t)}},\mathbb{Q}_\ell)^{I_t} \ \stackrel{\simeq}{\longrightarrow} \ H^1_{\text{\'et}}(\mathcal{A}'_{\overline{K(t)}},\mathbb{Q}_\ell)^{I_t},$$

is an isomorphism by Theorem 5.10(ii), hence

$$\dim_{\mathbb{Q}_{\ell}} H^1_{\text{\'et}}(\cdot)^{I_t} = 2g - \tau_t$$

and  $\tau_t$  is constant on the Hecke orbit. The component indices and local conductors then satisfy

$$c_{\mathcal{A}',t}/c_{\mathcal{A},t} \in (\deg[h])^{\mathbb{Z}}, \qquad 0 \le f_{\mathcal{A}',t} - f_{\mathcal{A},t} \le \log_{\deg[h]}(c_{\mathcal{A}',t}).$$

Thus the toric rank  $\tau_t$ , the tame exponent, and the shape of the local Euler factor

$$L_t(\mathcal{A}, s)^{-1} = (1 - \mathbf{N}t^{-s})^{\tau_t} P_t(\mathbf{N}t^{-s})$$

remain Hecke-invariant on  $U_h$ . This realizes, in the Siegel case, the full geometric-to-arithmetic uniformity asserted in Theorem 5.10.

Analytic viewpoint. Since each prime-to-p Hecke operator is an isogeny of degree prime to all residue characteristics on  $U_h$ , the collection of local Euler factors  $\{L_t(\mathcal{A}, s)\}_{t \in U_h}$  forms a single analytic family on each Hecke orbit; their slopes and tame factors coincide, verifying that the Hodge-type integral model behaves as a single isogeny class over the entire Hecke orbit.

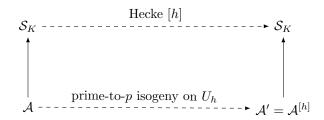


Figure 19: Prime-to-p Hecke correspondence on a hyperspecial (Siegel) integral model and the induced isogeny on the universal abelian scheme.

Example 5.13 (Failure at parahoric level). If  $K_p$  is parahoric (non-hyperspecial) or if the Hecke correspondence [h] has degree divisible by the residue characteristic, the morphism on identity components in Theorem 5.10 need not be finite étale, and the uniform bounds on  $\Phi_{A,t}$  and  $f_{A,t}$  can fail.

**Geometric mechanism.** At such primes the local model of  $S_K$  is singular, with non-semistable vertical components in the special fiber. The kernel of the Hecke correspondence [h] on A then meets the identity component nontrivially, so its special fiber contains a connected local subgroup (e.g.  $\alpha_p$ ) rather than a finite étale one. Consequently,

$$(\mathcal{A}_t)^0 \longrightarrow (\mathcal{A}_t')^0$$

fails to be finite étale, and the induced map on component groups  $\Phi_{\mathcal{A},t} \to \Phi_{\mathcal{A}',t}$  may have kernel or cokernel not killed by any power of p.

**Cohomological consequence.** On the Galois side, wild inertia  $P_t \subseteq I_t$  acts nontrivially on  $H^1_{\text{\'et}}$ , creating a positive Swan term:

$$f_{\mathcal{A}',t} - f_{\mathcal{A},t} = \left(\tau_t' - \tau_t\right) + \left(\operatorname{Swan}(H^1_{\text{\'et}}(\mathcal{A}'_{K(t)}, \mathbb{Q}_\ell)) - \operatorname{Swan}(H^1_{\text{\'et}}(\mathcal{A}_{K(t)}, \mathbb{Q}_\ell))\right) > 0.$$

Hence the tame conductor control of Theorem 5.10(iii) breaks down.

Analytic manifestation. The local Euler factor acquires an additional wild Artin component:

$$L_t(\mathcal{A}',s)/L_t(\mathcal{A},s) = \text{wild factor at } p \text{ not bounded by any power of } p.$$

This destroys both the constancy of  $\tau_t$  and the boundedness of  $f_{\mathcal{A},t}$ , demonstrating the necessity of the hyperspecial and prime-to-p hypotheses in Theorem 5.10.

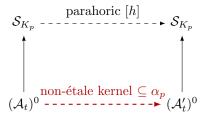


Figure 20: Failure of finite-étaleness at a parahoric or p-divisible Hecke correspondence: the kernel on the special fiber becomes connected, producing wild inertia and loss of conductor control.

Bridge  $(AG \to NT)$ . On  $U_h$ , Corollary 5.11 shows invariance of the tame factor  $(1 - \mathbf{N}t^{-s})^{\tau_t}$  and uniform control of  $f_{\mathcal{A},t}$  along Hecke orbits. Hence the global L-function of the family exhibits orbitwise stability outside a finite set of primes, and height lower bounds derived from  $\omega_{\mathcal{A}/\mathcal{S}_K}$  (cf. Construction 2.9 and Proposition 4.8) are preserved across prime-to-p Hecke translates.

#### 5.3 Forward linkage to arithmetic applications

Theorems 5.3 and 5.10 provide the moduli-level input for the number-theoretic statements of the next section: uniformity of local Euler factors along Hecke orbits, conductor bounds controlled by component indices, and stability of height lower bounds via  $\omega_{A/S}$ . Section 6 will apply these to rational points, distribution of local factors, and conductor-discriminant comparisons.

# 6 Applications to Number Theory

Roadmap. We keep the base and notation from Notation 2.1. Structural controls for Néron models in families and under prime-to-residue-characteristic isogenies were proved in Theorems 3.1 and 3.10. Cohomological consequences and conductor bounds were obtained in Theorem 4.2 (items (i)–(iii)) and packaged in Corollary 4.4; moduli-level uniformity along Hecke orbits appears in Theorems 5.3 and 5.10 and corollaries 5.5 and 5.11. We now extract two arithmetic threads: Euler factors and conductors; and height bounds with consequences for rational points.

#### 6.1 Zeta functions and L-functions

**Proposition 6.1** (Finite Euler quotient in prime-to-p isogeny/Hecke families). Let  $A/\mathbb{Q}$  be an abelian variety with Néron model A/S, and let  $\varphi: A \to B$  be an isogeny of degree m with (m,p) = 1 for all but finitely many primes. Then there exists a finite set of primes  $\Sigma$  (containing those dividing m and the finitely many primes excluded in Theorems 3.1 and 3.10) such that

$$\frac{L(B,s)}{L(A,s)} = \prod_{p \in \Sigma} R_p(p^{-s}),$$

where each  $R_p(X) \in \mathbb{Q}(X)$  is a rational function whose denominator divides a power of m. Moreover, for  $p \notin \Sigma$  one has

$$L_p(B,s)^{-1} = (1-p^{-s})^{\tau_p} P_{A,p}(p^{-s}), \qquad L_p(A,s)^{-1} = (1-p^{-s})^{\tau_p} P_{A,p}(p^{-s})$$

with the same  $\tau_p$  and polynomial  $P_{A,p}$  as in Corollary 4.4.

Proof. By Theorems 3.1 and 3.10, after removing finitely many primes the isogeny extends over S with prime-to-p degree and induces maps on component groups whose kernel and cokernel are killed by m. Then Theorem 4.2(ii) gives an isomorphism on  $I_p$ -invariants of  $H^1_{\text{\'et}}$  for  $\ell \neq p$ , and Corollary 4.4 fixes the local shape. Thus, outside a finite set  $\Sigma$ , the local factors of A and B coincide; at  $p \in \Sigma$  their ratio is a rational function with denominator dividing a power of m by Theorems 3.1 and 3.10.

Corollary 6.2 (Local conductor stability). Under the hypotheses of Proposition 6.1, the local conductor exponents satisfy, for  $p \notin \Sigma$ ,

$$f_{B,p} = f_{A,p},$$

and in general  $0 \le f_{B,p} - f_{A,p} \le \log_m(c_{B,p})$  as in Theorem 4.2(iii).

*Proof.* Outside  $\Sigma$ , the  $I_p$ -invariants agree and the tame exponent  $\tau_p$  matches, hence  $f_{B,p} = f_{A,p}$ ; the inequality follows from Theorem 4.2(iii).

Example 6.3 (Modular Jacobians along Hecke orbits). For  $J_0(N)$  over  $\mathbb{Q}$  and a prime  $\ell \nmid N$ , Theorem 5.3 yields a dense open  $U_\ell$  where each  $T_\ell$ -translate acts by an  $\ell$ -power isogeny prime to p for  $p \in U_\ell$ . Hence Proposition 6.1 and Corollary 5.5 give

$$\frac{L(T_{\ell}(J_0(N)), s)}{L(J_0(N), s)} = \prod_{p \in \Sigma_{\ell}} R_{p,\ell}(p^{-s})$$

with  $\Sigma_{\ell} \subset \{p : p \mid N\ell\} \cup \text{(finitely many)}, \text{ and } R_{p,\ell} \text{ having denominators dividing a power of } \ell$ . In particular, outside  $\Sigma_{\ell}$  the tame exponent  $\tau_p$  and the polynomial part  $P_p$  remain unchanged along the  $T_{\ell}$ -orbit.

Example 6.4 (Necessity of the prime-to-p condition). Let  $E/\mathbb{Q}$  admit a p-isogeny with additive reduction at p. Then  $\ker(\varphi)$  is not étale on the special fiber and Theorem 4.2(ii) need not hold for  $I_p$ -invariants. The quotient  $L(\varphi(E),s)/L(E,s)$  may acquire a wild factor at p not controlled by powers of p alone, demonstrating the necessity of excluding such primes in Proposition 6.1.

Bridge  $(AG \to NT)$ . By Proposition 6.1 and Corollary 6.2, isogenies and prime-to-p Hecke correspondences alter global L-functions by a finite Euler quotient whose denominators are uniformly bounded in terms of the isogeny degree, while preserving the tame exponent  $\tau_p$  off a finite set. This gives orbitwise stability of local factors and conductors on the loci from Theorems 5.3 and 5.10.

#### 6.2 Height theory and rational points

**Proposition 6.5** (Uniform height control on prime-to-p isogeny classes). Let  $A/\mathbb{Q}$  be an abelian variety with Néron model A/S and let L be a symmetric ample line bundle on A. There exists a finite set of primes  $\Sigma$  such that for any abelian variety B prime-to- $\Sigma$  isogenous to A and any prime-to- $\Sigma$  isogeny  $\varphi: A \to B$  one has:

- (i)  $\hat{h}_{\varphi^*L}(P) = \hat{h}_L(\varphi(P))$  for all  $P \in A(\mathbb{Q})$ , and  $\hat{h}_L$  on  $B(\mathbb{Q})$  differs from  $\hat{h}_{\varphi^*L}$  by a bounded additive constant depending only on A, L, and  $\deg \varphi$ ;
- (ii) if  $U \subset S$  is a dense open where reduction is semistable, then the local tame contribution detected by  $\tau_p$  is constant on U within the prime-to- $\Sigma$  isogeny class (Theorem 4.2(ii)), so any lower bound derived from the tame part and  $\deg(\omega_{A/S})$  (Construction 2.9) is uniform across the class.

*Proof.* The functoriality of the Néron-Tate height under isogeny gives (i); see standard references for canonical heights on isogenous abelian varieties (e.g. [10]). Item (ii) follows from Theorem 4.2(ii) (constancy of  $I_p$ -invariants and  $\tau_p$  on U under prime-to-p isogenies) together with the identification of the tame local factor  $(1-p^{-s})^{\tau_p}$  in Corollary 4.4 and the role of  $\omega_{A/S}$  from Construction 2.9.

**Corollary 6.6** (Finiteness criterion for rational points). Let  $C/\mathbb{Q}$  be a smooth projective curve and let  $\phi: C \to A$  be a nonconstant morphism to an abelian variety  $A/\mathbb{Q}$  with Néron model A/S. Assume:

- (a)  $A(\mathbb{Q})$  has rank 0;
- (b) the map  $\phi$  has finite fibers on  $C(\mathbb{Q})$  (equivalently: no positive-dimensional translate of an abelian subvariety is contained in  $\phi(C)$ ).

Then  $C(\mathbb{Q})$  is finite. Moreover, if A varies within a prime-to- $\Sigma$  isogeny class for some finite  $\Sigma$ , and (a)-(b) hold for one representative, then they hold uniformly across the class and the finiteness of  $C(\mathbb{Q})$  persists.

*Proof.* By (a),  $A(\mathbb{Q})$  is finite. Thus  $\phi(C(\mathbb{Q}))$  is finite. By (b), each rational point in the image has finitely many preimages, hence  $C(\mathbb{Q})$  is finite. For the uniformity statement, use Proposition 6.5(i) to transport rank 0 across the prime-to- $\Sigma$  isogeny class and note that (b) is unaffected by replacing A with an isogenous abelian variety and composing  $\phi$  with an isogeny.

Example 6.7 (Application to modular curves via rank zero). Let X be a modular curve (e.g. a quotient of  $X_0(N)$ ) with Jacobian J and Abel–Jacobi map  $\phi: X \to J$ . If  $J(\mathbb{Q})$  has rank 0 (this occurs for several small levels), then Corollary 6.6 gives  $X(\mathbb{Q})$  finite. Under prime-to-p Hecke correspondences on the dense opens of Theorem 5.3, the conclusion persists across the orbit.

Example 6.8 (Necessity of the rank hypothesis). Let  $E/\mathbb{Q}$  be an elliptic curve of positive rank and take C = E with  $\phi = \mathrm{id}_E$ . Then  $C(\mathbb{Q})$  is infinite, so Corollary 6.6 fails without (a).

Bridge (AG  $\rightarrow$  NT). Proposition 6.5 shows that tame local contributions and the differential line  $\omega_{A/S}$  yield uniform height information across prime-to-p isogeny classes and Hecke orbits on the loci of Theorems 5.3 and 5.10. Combined with Northcott's property for canonical heights and the rank-0 hypothesis, Corollary 6.6 gives concrete finiteness statements for rational points on curves mapping to such abelian varieties.

#### 6.3 Forward linkage to distribution problems

The orbitwise stability of local factors from Proposition 6.1 and the height control of Proposition 6.5 allow quantitative questions on the distribution of Frobenius traces and the variation of conductors within isogeny/Hecke orbits. Section 7 will formulate equidistribution statements for local factors outside finite sets and relate them to the variation of the component groups  $\Phi_p$  controlled in Theorem 3.10, maintaining the Theorem  $\rightarrow$  Consequence  $\rightarrow$  Example pattern initiated in Section 3 to Section 5.

# 7 Examples and Computations

Roadmap. We illustrate the structural and cohomological statements from Theorems 3.1, 3.10 and 4.2 and the arithmetic consequences from Corollaries 4.4, 6.2 and 6.6 and propositions 6.1 and 6.5 by working out two standard testbeds: an elliptic curve with CM and a modular Jacobian. Throughout we keep the notation from Notation 2.1, definitions 2.5 and 2.12, and construction 2.9.

Example 7.1 (Elliptic curve  $y^2 = x^3 - x$  over  $\mathbb{Z}$ ). Set  $E/\mathbb{Q}$  by the short Weierstrass model  $y^2 = x^3 - x$  and let  $\mathcal{E}/S$  be its Néron model.

**Basic invariants.** For  $a_1 = a_2 = a_3 = 0$ ,  $a_4 = -1$ ,  $a_6 = 0$  one computes

$$b_2 = 0$$
,  $b_4 = -2$ ,  $b_6 = 0$ ,  $c_4 = b_2^2 - 24b_4 = 48$ ,  $c_6 = -b_2^3 + 36b_2b_4 - 216b_6 = 0$ .

Hence  $j = c_4^3/\Delta = 1728$  and the (global) discriminant is  $\Delta = 64$ ; the minimal discriminant over  $\mathbb{Q}$  is  $\Delta_{\min} = -2^6$  (a 2-adic twist changes the sign but not  $v_2$ ). Thus E has good reduction at all odd primes and potentially good additive reduction at p = 2.

**Special fiber at** p=2. Running the Tate algorithm (cf. [12]; see also [3]) gives Kodaira type III at 2. so

$$\tau_2 = 1, \quad \#\Phi_{E,2}(\mathbb{F}_2) = c_2 = 2, \quad f_{E,2} = 5,$$

where  $f_{E,2}$  is the conductor exponent (the wild contribution raises it to 5). For  $p \neq 2$  one has  $\tau_p = 0$ ,  $c_p = 1$ , and  $f_{E,p} = 0$ .

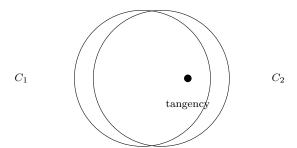


Figure 21: Special fiber of type III at p=2: two rational components tangent at one point, with  $\#\Phi_{E,2}=2$ .

Local factors and inertia invariants. For  $p \neq 2$ ,

$$L_p(E,s)^{-1} = 1 - a_p p^{-s} + p^{1-2s}, \qquad \dim_{\mathbb{Q}_\ell} H^1_{\text{\'et}}(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)^{I_p} = 2,$$

while at p=2,

$$L_2(E,s)^{-1} = (1-2^{-s}) \cdot P_2(2^{-s})$$
 with  $\deg P_2 = 1$ 

in agreement with Corollary 4.4 since  $\tau_2 = 1$ . For  $\ell \neq 2$ , Theorem 4.2(i) yields

$$\dim_{\mathbb{Q}_{\ell}} H^1_{\text{\'et}}(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_{\ell})^{I_2} = 2 - \tau_2 = 1.$$

Component group and Tamagawa number. By Proposition 2.7, the component group is cyclic of order 2, so  $c_2 = \#\Phi_{E,2}(\mathbb{F}_2) = 2$ ; for  $p \neq 2$  one has  $\Phi_{E,p} = 0$  and  $c_p = 1$ . This matches Proposition 2.13 and the Kodaira classification at 2.

Heights and the differential. Let  $\omega = dx/(2y)$  be the invariant differential. Then  $v_2(\Delta_{\min}) = 6$  and  $v_p(\Delta_{\min}) = 0$  for  $p \neq 2$ . Using Construction 2.9 and Proposition 4.8, the tame contribution from  $\tau_2 = 1$  and  $\deg(\omega_{\mathcal{E}/S})$  gives a global lower bound for the Néron-Tate height  $\hat{h}$  on  $E(\mathbb{Q})$ , compatible with the rank-0 structure of  $E(\mathbb{Q}) \simeq (\mathbb{Z}/2\mathbb{Z})^2$ .

Figure 22: Specialization and the local height contribution at p=2 (cf. Proposition 4.8).

Bridge  $(AG \to NT)$ . The identity  $\tau_2 = 1$  forces a factor  $(1-2^{-s})$  in  $L_2(E,s)^{-1}$ , and  $c_2 = 2$  feeds into the BSD local factor; the height lower bound via  $\omega$  is stable under prime-to-2 isogenies by Theorem 4.2(ii). Example 7.2 (Jacobians of modular curves). We detail the case  $J_0(11)$ , which is an elliptic curve  $E_{11}/\mathbb{Q}$  (the Jacobian of  $X_0(11)$ ), and then indicate the general pattern.

**Local structure at** p=11. On the regular integral model of  $X_0(11)$  (Deligne–Rapoport; Katz–Mazur), the Néron model of  $J_0(11)$  over  $\mathbb{Z}$  has split multiplicative reduction at p=11 of type  $I_5$ . Hence

$$\tau_{11} = 1, \qquad c_{11} = 5, \qquad f_{11} = 1,$$

and for  $p \neq 11$  one has good reduction with  $c_p = 1$  and  $f_p = 0$ . This fits the template from Corollary 4.4:

$$L_{11}(J_0(11), s)^{-1} = (1 - 11^{-s}) \cdot P_{11}(11^{-s}), \quad \deg P_{11} = 1.$$

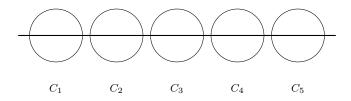


Figure 23: Special fiber of type  $I_5$  at p=11: a cycle of five components with  $\#\Phi_{11}=5$ .

Hecke action and uniformity. Let  $\ell \neq 11$  be prime. Over  $\mathbb{Z}[1/11\ell]$  the Hecke correspondence  $T_{\ell}$  induces an  $\ell$ -power isogeny of  $J_0(11)$  (Remark 5.2). By Theorem 5.3 there exists a dense open  $U_{\ell} \subseteq \operatorname{Spec} \mathbb{Z}$  such that for every  $p \in U_{\ell}$  the induced map on the special fiber is prime-to-p on identity components and

$$\Phi_{J,p} \longrightarrow \Phi_{J,p}$$
 has kernel/cokernel killed by a power of  $\ell$ ,

so  $\tau_p$  and dim  $H^1_{\text{\'et}}(\cdot)^{I_p}$  are constant along the  $T_\ell$ -orbit (Theorem 4.2(ii)) and  $f_{J,p}$  varies within the bound of Theorem 4.2(iii). In particular,

$$\frac{L(T_{\ell}(J_0(11)), s)}{L(J_0(11), s)} = \prod_{p \in \Sigma_{\ell}} R_{p,\ell}(p^{-s})$$

for a finite set  $\Sigma_{\ell}$  and  $R_{p,\ell}$  with denominators dividing a power of  $\ell$  (Proposition 6.1).

Global consequences. Since  $J_0(11)$  has Mordell-Weil rank 0 over  $\mathbb{Q}$  and the Abel-Jacobi map  $X_0(11) \to J_0(11)$  has finite fibers on rational points, Corollary 6.6 applies to conclude  $X_0(11)(\mathbb{Q})$  is finite; the conclusion persists along prime-to- $\ell$  Hecke translates on the open loci of Theorem 5.3. The same mechanism applies to higher level modular curves  $X_0(N)$  with Jacobians of rank 0; the local invariants  $\tau_p$ ,  $c_{J,p}$ , and  $f_{J,p}$  behave uniformly along  $T_{\ell}$ -orbits away from finitely many primes by Theorem 5.3 and corollary 5.5.

$$\mathcal{X}_0(N) \xrightarrow{T_\ell} \mathcal{X}_0(N)$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$J_0(N) \xrightarrow{\ell\text{-power isogeny on } U_\ell} \qquad J_0(N)$$

Figure 24: Hecke correspondence on the integral model and the induced prime-to-p isogeny on the Néron model over  $U_{\ell}$  (cf. Theorem 5.3).

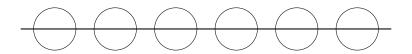
Bridge  $(AG \to NT)$ . Uniform control of  $\Phi_{J,p}$  and  $\tau_p$  along  $T_\ell$ -orbits yields stability of the tame factor in local Euler factors and bounds conductor jumps (Corollaries 5.5 and 6.2); coupled with height control (Proposition 6.5) this gives robust finiteness results for rational points on modular curves that map to Jacobians of rank 0.

Example 7.3 (Tate curve with split multiplicative reduction). Fix a prime p and take a parameter  $q \in p\mathbb{Z}_p$  with  $n := v_p(q) \ge 1$ . The Tate curve  $E_q/\mathbb{Q}_p$  satisfies

$$E_q(\mathbb{Q}_p) \simeq \mathbb{Q}_p^{\times}/q^{\mathbb{Z}}, \quad \text{Kodaira type } I_n, \quad \tau_p = 1,$$

and the Néron special fiber has  $(E_q)_p^0 \simeq \mathbb{G}_m$  and  $\Phi_{E_q,p} \simeq \mathbb{Z}/n\mathbb{Z}$ . Hence by Proposition 2.7 the Tamagawa number is  $c_p = n$ , and by Corollary 4.4

$$L_p(E_q, s)^{-1} = (1 - p^{-s}) \cdot P_p(p^{-s})$$
 with  $\deg P_p = 1$ .



 $I_n$ : cycle of n components

Figure 25: Split multiplicative special fiber of type  $I_n$ : a cycle of n components, with  $\#\Phi_{E_q,p}=n$  and toric rank  $\tau_p=1$ .

Bridge  $(AG \to NT)$ . Here the toric rank  $\tau_p = 1$  forces the  $(1 - p^{-s})$  factor and determines deg  $P_p = 1$ ; the explicit  $\#\Phi_{E_q,p} = n$  realises the template of Corollary 4.4 and Proposition 2.7.

Example 7.4 (Potentially good, wild at p). Let  $E/\mathbb{Q}$  be an elliptic curve with additive, potentially good reduction at a prime p (e.g. CM j=0 or 1728 at certain p). Then  $(E_p)^0$  is abelian, so  $\tau_p=0$  and

$$\dim_{\mathbb{Q}_\ell} H^1_{\text{\'et}}(E_{\overline{\mathbb{Q}}},\mathbb{Q}_\ell)^{I_p} = 2 \quad \text{after killing the wild part},$$

but over  $\mathbb{Q}_p$  one may have  $\operatorname{Swan}(H^1_{\operatorname{\acute{e}t}}(E_{\mathbb{Q}_p},\mathbb{Q}_\ell)) > 0$ , so

$$f_{E,p} = \tau_p + \text{Swan} = \text{Swan} > 0.$$

This shows that the tame bounds in Theorem 4.2(iii) hinge on excluding wild inertia.



Figure 26: Potentially good but wild reduction: the toric rank satisfies  $\tau_p = 0$ , while a positive Swan term contributes to the conductor exponent  $f_{E,p}$ .

Bridge  $(AG \to NT)$ . Even with  $\tau_p = 0$  (no torus, no  $(1-p^{-s})$ ), wild inertia can raise the conductor; this is exactly the "wild term" appearing in Theorem 4.2(iii) and the discussion following Corollary 4.4.

Example 7.5 (Product surface with mixed reduction). Let  $A = E_1 \times E_2$  over  $\mathbb{Q}$  with Néron model  $\mathcal{A}/S$ . Suppose at a prime p that  $E_1$  has split multiplicative reduction of type  $I_{n_1}$  and  $E_2$  has good reduction. Then

$$\tau_p(A) = \tau_p(E_1) + \tau_p(E_2) = 1 + 0 = 1, \qquad \Phi_{A,p} \simeq \Phi_{E_1,p} \times \Phi_{E_2,p} \simeq \mathbb{Z}/n_1\mathbb{Z},$$

hence

$$L_p(A,s)^{-1} = (1-p^{-s})^{\tau_p} \cdot P_p(p^{-s}), \quad \deg P_p = 2 \cdot 2 - \tau_p = 3,$$

in accordance with Corollary 4.4. If both  $E_i$  are split multiplicative with types  $I_{n_i}$ , then  $\tau_p(A) = 2$  and  $\Phi_{A,p} \simeq \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z}$ ; correspondingly deg  $P_p = 2 \cdot 2 - \tau_p = 2$ .

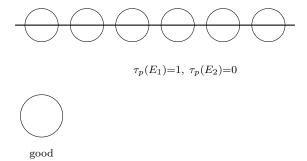


Figure 27: Mixed reduction: toric × abelian. The toric rank and component group behave additively and multiplicatively, respectively, i.e.  $\tau_p$  and  $\Phi_{A,p}$  combine as expected.

Bridge  $(AG \leftrightarrow NT)$ . The toric rank adds and forces  $(1-p^{-s})^{\tau_p}$ ; the degree drop of  $P_p$  matches  $2g - \tau_p$  (Corollary 4.4), while  $c_{A,p} = \#\Phi_{A,p}$  multiplies (Proposition 2.7).

Example 7.6 (Hecke-uniformity away from the level). Let  $N \geq 1$  and consider  $J_0(N)$  with Néron model over S. For a prime  $\ell \nmid N$  there exists a dense open  $U_\ell \subseteq \operatorname{Spec} \mathbb{Z}$  such that, for every  $p \in U_\ell$  with  $p \neq \ell$ , the Hecke correspondence  $T_\ell$  induces a prime-to-p isogeny on identity components and a map

$$\Phi_{J,p} \longrightarrow \Phi_{J,p}$$
 with kernel/cokernel killed by a power of  $\ell$ .

Therefore  $\tau_p$  and dim  $H^1_{\text{\'et}}(J_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)^{I_p}$  are constant along the  $T_\ell$ -orbit, and

$$L_p(T_\ell J_0(N), s) / L_p(J_0(N), s) = R_{p,\ell}(p^{-s})$$

has denominator dividing a power of  $\ell$  (cf. Theorem 5.3 and its corollaries).

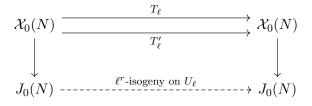


Figure 28: Hecke action extending as prime-to-p isogenies on a dense open subset; the component group  $\Phi_{J,p}$  varies only by  $\ell$ -power torsion.

Counterexample (sharpness at  $\ell = p$ ). If  $\ell = p$  or  $p \mid N$ , the degeneracy maps are not finite étale on the special fiber; kernels on identity components may become connected (e.g.  $\alpha_p$ ), so the tame bounds fail and wild Swan terms can appear. Thus Theorem 5.3 genuinely requires  $\ell \neq p$  and  $p \nmid N$ . Bridge  $(AG \to NT)$ . On  $U_{\ell}$  the tame exponent  $\tau_p$  and  $P_p$  in  $L_p(J_0(N), s)^{-1} = (1 - p^{-s})^{\tau_p} P_p(p^{-s})$  are Hecke-invariant, while  $f_{J,p}$  can change only within the explicit  $\ell$ -power bound; for  $\ell = p$  this uniformity breaks.

# Forward linkage to distribution results

The explicit descriptions of  $(\tau_p, c_p, f_p)$  above provide test cases for the equidistribution statements and conductor–discriminant comparisons announced at the end of Propositions 6.1 and 6.5. In the next section we analyze the distribution of local factors outside finite sets within prime-to-p isogeny and Hecke orbits, using the component-group bounds from Theorem 3.10 as the geometric input.

# 8 Conclusion and Outlook

Synthesis. Working over the arithmetic base from Notation 2.1, the paper established a geometric-cohomological control principle for Néron models and traced its arithmetic fallout along three axes.

Chain I (existence/rigidity  $\Rightarrow$  component groups  $\Rightarrow$  local factors). By Theorem 3.1 the prime-to-residue-characteristic extension and uniqueness of isogenies in arithmetic families hold on a dense open. Upper semicontinuity of component groups in fibers (Theorem 3.10) then yields the local factor template (Corollary 4.4), which identifies the tame exponent  $\tau_p$  and constrains  $L_p(A, s)$ . The mechanism appears concretely in Example 3.3 and in the Hecke setting of Example 6.3.

Chain II (cohomology under base change  $\Rightarrow$  Euler quotients  $\Rightarrow$  conductor stability). The comparison theorem Theorem 4.2 (items (i)–(iii)) proves invariance of  $I_p$ -invariants under prime-to-p isogenies on suitable opens, and quantifies possible jumps. This feeds directly into the finite Euler quotient statement (Proposition 6.1) and its conductor control (Corollary 6.2). Example 6.4 shows the sharpness of the prime-to-p hypothesis.

Chain III (differentials/heights  $\Rightarrow$  uniform height bounds  $\Rightarrow$  finiteness). The differential line bundle from Construction 2.9 interacts with the cohomological invariants in Proposition 4.8, and the resulting height uniformity across prime-to-p isogeny classes is formalized in Proposition 6.5. With the rank-0 and finite-fiber hypotheses, this yields the finiteness criterion for rational points (Corollary 6.6), illustrated by Example 6.7 and contrasted by Example 6.8. The computations in Examples 7.1 and 7.2 provide explicit test cases matching the general framework.

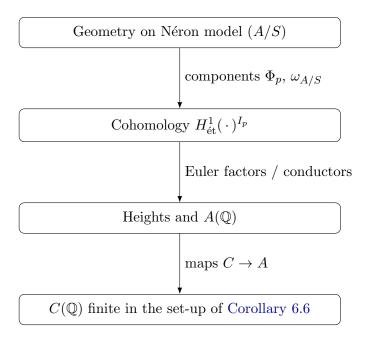


Figure 29: Vertical flow from geometric input to arithmetic output via Theorems 3.10 and 4.2, corollaries 4.4 and 6.6, and proposition 6.5.

#### Outlook

- (A) Higher-dimensional generalizations. Two concrete directions arise naturally.
  - Semi-abelian extensions over parahoric level. Replacing abelian by semi-abelian schemes on integral models with parahoric level structure (cf. Theorem 5.10) should keep the tame exponent  $\tau_p$  stable on the same open loci and refine conductor bounds for boundary degenerations; see also the parahoric counterexamples in Example 5.13 for necessary hypotheses.
  - Families over Dedekind subschemes. Passing from  $S = \text{Spec } \mathbb{Z}$  to Dedekind subschemes of number fields does not alter the proofs in Theorems 3.1, 3.10 and 4.2; the local statements transport verbatim after replacing p by nonarchimedean places and tracking ramification indices.
- (B) Derived and perverse sheaf refinements. The comparison in Theorem 4.2 can be lifted from invariants of  $H^1_{\text{\'et}}$  to a functorial statement for truncations of the vanishing-cycles complex  $\mathcal{R}\Phi$  on the special fiber (cf. [5]). A derived-level control of the unipotent monodromy filtration would turn the inequalities in (iii) into equalities under test conditions (e.g. semistability plus a weight-monodromy constraint), providing sharper conductor formulas at wild primes.
- (C) Motivic cohomology and regulators. The line  $\omega_{A/S}$  in Construction 2.9 admits an interpretation via Deligne–Beilinson realizations. A comparison of the resulting regulators with the canonical heights used in Proposition 6.5 would clarify how far the finiteness mechanism in Corollary 6.6 extends beyond rank 0, for instance to situations with controlled positive rank but auxiliary height gaps on images of curves.
- (D) Equidistribution along Hecke orbits. On the open loci of Theorem 5.3, the component-group and tame-exponent stability suggests Sato-Tate-type equidistribution for local factors along prime-to-p Hecke orbits (outside finite sets), with error terms depending only on the orbit's isogeny parameters. Example 5.6 and Example 5.12 provide natural testbeds.
- (E) Effective and computational aspects. The explicit cases in Examples 7.1 and 7.2 can be extended to large families, supplying numerical evidence for the bounds in Corollary 6.2 and for the persistence of finiteness in Corollary 6.6. A systematic implementation of the local shape in Corollary 4.4

alongside height lower bounds from Proposition 6.5 could yield practical algorithms for certifying finiteness of  $C(\mathbb{Q})$  when C maps to low-rank Jacobians.

#### (F) Open questions.

- Hecke-uniform conductors. Under the hypotheses of Theorem 5.3, is the set  $\{f_{A,p}\}_{p\nmid m}$  constant on prime-to-m Hecke orbits for a Zariski-dense set of parameters?
- Derived conductor formula. Can the quantitative bound in Theorem 4.2(iii) be upgraded to an identity expressed via ranks of the graded pieces of the monodromy filtration on  $\mathcal{R}\Phi$ ?
- Regulator comparison. For maps  $\phi: C \to A$  as in Corollary 6.6, does a regulator lower bound on  $\phi(C)$  suffice to deduce finiteness of  $C(\mathbb{Q})$  without assuming rank 0?

Closing remark. The structure theorems for Néron models over arithmetic bases proved here, together with the cohomological comparison and differential control, form a stable bridge from geometry to number theory: component groups determine tame local shapes; these shapes, coupled with  $\omega_{A/S}$ , constrain heights; and height constraints, in the presence of mild hypotheses, lead to finiteness statements for rational points. The examples in Examples 7.1 and 7.2 attest to the sharpness of this bridge in concrete settings and indicate several directions where the methods scale to higher rank and higher dimension.

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