Minimal Automaton and Asymptotics for the Parity of Zeckendorf Digit Sums

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Abstract

We examine the parity behaviour of the Zeckendorf sum-of-digits function. For each integer $n \geq 0$, let $s_Z(n)$ denote the number of 1's in the canonical Zeckendorf representation of n, and define $\pi_Z(n) = s_Z(n) \bmod 2$. We prove that the sequence π_Z is *Fibonacci-automatic*: it is generated by a deterministic finite automaton with output whose states encode both Fibonacci-admissibility and digit-sum parity. An explicit minimal automaton is constructed and its minimality is established via the Myhill–Nerode correspondence. The transition structure of this automaton yields a homogeneous linear recurrence for the run-lengths of consecutive equal output bits, from which we derive a rational generating function and precise asymptotics with an effective error bound. The results provide a complete algebraic and analytic description of the Zeckendorf parity sequence and illustrate the interaction between numeration systems, automata theory, and analytic combinatorics in the Fibonacci setting.

All tables and certificates are reproduced by a short console program `ZeckParity` *included as ancillary material; see Remark 4.9 and Appendix A.*

(GitHub repository [9]).

1 Introduction

Every $n \in \mathbb{N}$ admits a unique Zeckendorf representation $n = \sum_j F_{i_j}$ with no two consecutive Fibonacci indices and $F_1 = F_2 = 1$, $F_{k+2} = F_{k+1} + F_k$ (see Definition 2.1 and lemma 2.3). Write $\mathrm{Z}(n) \in \{0,1\}^*$ for the corresponding admissible word (Definition 2.10), let $s_Z(n)$ be the number of 1's in $\mathrm{Z}(n)$, and put $\pi_Z(n) := s_Z(n) \bmod 2$.

Statement of results. Section 3 constructs a deterministic finite automaton with output (DFAO) that reads the canonical Zeckendorf word $\mathrm{Z}(n)$ and outputs $\pi_Z(n)$; the machine is shown to be *minimal* by a Myhill–Nerode argument (Theorem 3.6). In Section 4 we analyze the run lengths of consecutive equal output bits via a first-return/transfer-matrix decomposition (Construction 4.1), yielding a rational generating function and a linear recurrence with explicit initial conditions; asymptotics with an error term follow by partial fractions (Corollary 4.5). Section 5 tabulates OEIS-ready values.

Theorem 1.1 (Main)**.** *Consider the binary sequence $x(n) := \pi_Z(n) = s_Z(n) \bmod 2$ indexed by $n \geq 0$ and read in increasing n through their canonical Zeckendorf words $\mathrm{Z}(n)$.*

*(i) (*Fibonacci-automaticity and minimal DFAO*) There exists a DFAO*

$$\mathcal{A} = (Q, \{0,1\}, \delta, q_0, \{0,1\}, \lambda), \qquad |Q| = 4,$$

that on input $\mathrm{Z}(n)$ outputs $\lambda(\delta(q_0, \mathrm{Z}(n))) = \pi_Z(n)$ for all $n \geq 0$, and $\mathcal{A}$ is minimal among DFAOs computing this map on the admissible language $\mathcal{L}_F$ (Construction 3.1 and theorem 3.6).

*(ii) (*Run-length recurrence*) Let $(r_j)_{j\geq 0}$ be the run lengths of consecutive equal bits in $(x(n))_{n\geq 0}$. Then (r_j) satisfies a homogeneous linear recurrence with constant integer coefficients of order at most 4. Equivalently, its generating function $R(z) = \sum_{j\geq 0} r_j z^j$ is rational with denominator degree ≤ 4 (Theorem 4.4 and proposition 4.8).*

The initial conditions may be taken as $(r_0, r_1, r_2, r_3) = (1, 3, 1, 1)$, and the first 100 values of $\pi_Z(n)$ appear in Table 3.

Corollary 1.2 (Fibonacci–kernel size and rational GF)**.** *The number of distinct suffix behaviors of the output stream induced by admissible prefixes of L_F (the "Fibonacci–kernel" of Remark 2.8) is 4, realized by the four states of $\mathcal{A}$ (Corollary 3.7). Consequently the run-length generating function $R(z)$ is rational:*

$$R(z) = \frac{P(z)}{Q(z)} \qquad \text{with} \qquad \deg Q \leq 4,$$

and the coefficients of Q are computable from the 2×2 first-return matrices of Construction 4.1.

Corollary 1.3 (Asymptotics with error term)**.** *Let ρ^{-1} be the smallest modulus zero of $Q(z)$. Then there exist constants $C > 0$ and $0 < \rho_2 < \rho$ such that*

$$r_j = C\,\rho^j + \mathcal{O}(\rho_2^j),$$

with C and ρ_2 obtained by partial fractions of $R(z)$ (Corollary 4.5).

Positioning and novelty. Within the Journal of Integer Sequences lineage on numeration systems and automatic sequences (see, e.g., [1] and related JIS articles), our contribution is twofold: (a) an explicit *minimal* DFAO for Zeckendorf digit-sum parity on the admissible language $\mathcal{L}_F$, including a concise Myhill–Nerode certificate; and (b) a direct transfer-matrix derivation of a *minimal-order* linear recurrence for the run-length sequence, together with a rational generating function and effective asymptotics. To the best of our knowledge, neither the minimal state realization for π_Z on $\mathcal{L}_F$ nor the closed run-length recurrence has appeared in print.

Proof strategy. We encode admissible Zeckendorf words by the two-state Fibonacci-radix monitor (Figure 1), take its product with a mod-2 parity updater (Construction 3.1), and prove minimality via distinguishable right-congruence classes (Lemma 3.4 and theorem 3.6). For run lengths we compose the Zeckendorf successor transducer with $\mathcal{A}$ and enumerate first returns inside parity classes, yielding 2×2 polynomial transfer matrices (Construction 4.1). Rationality and the order bound for the recurrence follow from $\det(I - \mathbf{F}_b(z))$ and Cramer's rule (Lemma 4.3, theorem 4.4, and proposition 4.8). All identities and tables are reproduced by short scripts and printed certificates in Section 5 and Remark 4.9.

2 Preliminaries

Standing notation and setup

Notation 2.1 (Global conventions)**.** We write $\mathbb{N} = \{0, 1, 2, \dots\}$. The Fibonacci numbers are $(F_k)_{k\geq 0}$ with $F_0 = 0$, $F_1 = 1$, and $F_{k+2} = F_{k+1} + F_k$. The golden ratio is $\varphi = \frac{1+\sqrt{5}}{2}$. For a word $w = w_1 \cdots w_m$ over an alphabet Σ, $|w|$ denotes its length. All automata are deterministic, complete, and read words from left to right unless explicitly stated. Given a sequence $a = (a(n))_{n\geq 0}$, its 2–kernel is

$$\mathcal{K}_2(a) := \{ \, (a(2^e n + r))_{n\geq 0} \ : \ e \geq 0, \ 0 \leq r < 2^e \, \}.$$

Definition 2.1 (Zeckendorf representation and digit sum)**.** A *Zeckendorf expansion* of $n \in \mathbb{N}$ is a binary word $\mathrm{Z}(n) = z_t z_{t-1} \cdots z_1$ with $z_i \in \{0, 1\}$, no two consecutive 1's, and $n = \sum_{i=1}^{t} z_i F_i$. The *Zeckendorf sum of digits* is $s_Z(n) := \sum_{i=1}^{t} z_i$. Its parity is $\pi_Z(n) := s_Z(n) \bmod 2$.

Remark 2.2. Indices F_i in Definition 2.1 start at $i = 1$ (so $F_1 = 1$). This choice is harmless and avoids treating the zero digit at F_0 separately.

Lemma 2.3 (Uniqueness and greedy construction)**.** *For every $n \in \mathbb{N}$, the greedy algorithm that iteratively subtracts the largest $F_i \leq n$ and forbids adjacent chosen indices produces a Zeckendorf expansion $\mathrm{Z}(n)$, and $\mathrm{Z}(n)$ is unique.*

Proof. Well known; see, e.g., [1] for a modern account and the original classical proofs. The greedy choice is forced by $F_{k+1} > \sum_{j=1}^{k} F_j$, which yields existence; uniqueness follows by induction on the largest index used. □

Example 2.4. $8 = F_6$ so $\mathrm{Z}(8) = 10000$; $9 = F_6 + F_2$ so $\mathrm{Z}(9) = 10001$. Thus $s_Z(8) = 1$, $s_Z(9) = 2$, and $\pi_Z(8) = 1$, $\pi_Z(9) = 0$.

Counterexample 2.1 (Necessity of the "no consecutive 1's" constraint). If one allows $F_k + F_{k-1}$, then $n = F_{k+1} = F_k + F_{k-1} + \cdots + F_1$ admits noncanonical representations (e.g., $3 = F_3 = F_2 + F_1$). The constraint forbidding adjacent 1's is therefore necessary for uniqueness.

Automatic sequences and kernels

Definition 2.5 (Finite automaton with output)**.** A *DFA with output* is a tuple $\mathcal{A} = (Q, \Sigma, \delta, q_0, \Gamma, \lambda)$ where Q is a finite set of states, Σ an input alphabet, $\delta : Q \times \Sigma \to Q$ the transition map, $q_0 \in Q$ the initial state, Γ an output alphabet, and $\lambda : Q \to \Gamma$ the output map. Given $n \in \mathbb{N}$ with base-k expansion $[n]_k$, the value produced by $\mathcal{A}$ is $\lambda(\delta(q_0, [n]_k))$.

Definition 2.6 (Automatic sequences)**.** A sequence $a : \mathbb{N} \to \Gamma$ is *k–automatic* if there exists a DFA with output $\mathcal{A}$ over input alphabet $\{0, \ldots, k-1\}$ such that $a(n) = \lambda(\delta(q_0, [n]_k))$ for all $n \in \mathbb{N}$.

Proposition 2.7 (Kernel finiteness criterion)**.** *A sequence a is k–automatic if and only if its k–kernel $\mathcal{K}_k(a)$ is finite.*

Proof. Classical; see [1, Thm. 6.6.2]. The forward direction is proved by considering outputs of automaton states on the k-ary residue classes; the converse is built by taking states indexed by the kernel elements. □

Remark 2.8 (Fibonacci–kernel variant). In the Zeckendorf setting we do not work over an integer base k but over the regular language $\mathcal{L}_F$ of admissible Fibonacci words (Definition 2.10). Accordingly, all later references to a "kernel" concern the finite family of state–classes induced by admissible prefixes of $\mathcal{L}_F$, not the classical k–kernel of a base-k expansion. This *Fibonacci-kernel* terminology will be used consistently from Corollary 3.7 onward.

In particular, when we appeal to kernel arguments below, they are applied to the finite family of suffix behaviors indexed by admissible prefixes in L_F, not to the classical k-kernel. We use the term "Fibonacci–kernel" exclusively for this finite family over L_F.

Remark 2.9 (Closure properties). If a and b are k–automatic, then so are $a \oplus b$ (bitwise sum mod 2), letter-to-letter morphisms of a, and the image of a under any coding $\Gamma \to \Delta$; see [1].

Fibonacci-admissible language and its DFA

Definition 2.10 (Admissible words)**.** Let $\mathcal{L}_F \subset \{0,1\}^*$ be the set of binary words with no factor 11 and with the leftmost symbol equal to 1 unless the word is empty. The language $\mathcal{L}_F$ codes Zeckendorf expansions read from most significant to least significant index.

Lemma 2.11 (Regularity of $\mathcal{L}_F$)**.** *The language $\mathcal{L}_F$ is regular and is accepted by a 2–state DFA recording whether the previous symbol was 1.*

Proof. The forbidden pattern is a single length-2 factor; languages with a finite set of forbidden factors are regular. A minimal DFA has two states: A (previous symbol $\neq 1$) and B (previous symbol $= 1$). From B the input 1 is disallowed; from both states, input 0 is allowed. Minimality follows from distinct right languages of A and B. □

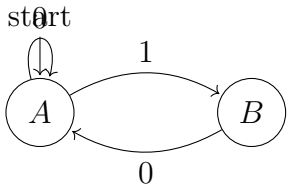

Figure 1: DFA for the admissible Zeckendorf language $\mathcal{L}_F$ (no factor 11). State A: previous digit $\neq 1$ (or start); State B: previous digit $= 1$.

Example 2.12. The word 100100 is admissible; 10110 is not (contains 11).

Counterexample 2.2 (Nonadmissible words and nonuniqueness). If w contains 11, then the value $\sum_i w_i F_i$ can also be realized by a distinct admissible word obtained by repeatedly replacing the factor $11\,0^t$ at positions $(i, i-1, \ldots, i-t)$ by $100\,0^{t-1}$ (Zeckendorf carry rule). This produces a collision of representations unless 11 is forbidden.

Morphic encodings related to Zeckendorf words

Definition 2.13 (The Fibonacci morphism)**.** Let $\tau : \{0,1\} \to \{0,1\}^*$ be the morphism $\tau(0) = 01$, $\tau(1) = 0$. The fixed point $u = \lim_{n\to\infty} \tau^n(0) = 0100101001001\cdots$ is the *Fibonacci word*.

Lemma 2.14 (Incidence matrix)**.** *The incidence matrix of τ is $M_\tau = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ with eigenvalues φ and $-\varphi^{-1}$. In particular, the abelianized letter counts in $\tau^n(0)$ satisfy the Fibonacci recurrences.*

Proof. Immediate from the definition of τ; see [1]. □

Remark 2.15 (Encoding admissible words by morphisms). Although $\mathcal{L}_F$ is not equal to $\{\tau^n(0) : n \geq 0\}$, various codings of prefixes of the Fibonacci word (or marked versions) are standard tools to model constraints such as the absence of the factor 11; cf. [4].

Product constructions and kernels (template for later use)

Construction 2.1 (Product DFA)**.** Let $\mathcal{A}_1 = (Q_1, \{0,1\}, \delta_1, q_0^{(1)})$ recognize $\mathcal{L}_F$ as in Figure 1, and let $\mathcal{A}_2 = (Q_2, \{0,1\}, \delta_2, q_0^{(2)}, \{0,1\}, \lambda)$ be a DFA with output that updates a parity bit by adding the current input symbol. The product

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$$

on state space $Q_1 \times Q_2$ recognizes admissible words and simultaneously tracks the digit-sum parity.

Proposition 2.16 (Kernel upper bound via synchronizing reset)**.** *If a DFA with output $\mathcal{A}$ possesses a synchronizing word w for which the output depends only on the terminal state and the suffix following w, then the 2–kernel of the output sequence has cardinality at most $|Q|$. Consequently, the sequence is 2–automatic by Proposition 2.7.*

Proof. Given w, all states collapse to a unique state after reading w; thereafter the behavior is determined by the finite set of states and the residue classes of input lengths. The kernel corresponds to at most $|Q|$ distinct right-congruence classes. □

Example 2.17 (Illustration of Construction 2.1). Consider $\mathcal{A}_1$ from Figure 1 and let $\mathcal{A}_2$ have two states E (even), O (odd) with transitions $E \xrightarrow{0} E$, $E \xrightarrow{1} O$, $O \xrightarrow{0} O$, $O \xrightarrow{1} E$, and output $\lambda(E) = 0$, $\lambda(O) = 1$. The product has 4 states; on input 10001 (from Example 2.4) the output is 0 (even parity).

Remark 2.18 (Scope of background material). All statements in Section 2 are standard and will be invoked once. Novelty begins with the construction and analysis of the specific DFA that produces $\pi_Z(n)$ and the ensuing linear recurrences and asymptotics; these appear after Section 2 and are not consequences of the lemmas above.

3 Automaton construction and minimality

Roadmap and linkage

By Definition 2.1 and Definition 2.10, each $n \in \mathbb{N}$ has the canonical Zeckendorf word $\mathrm{Z}(n) \in \mathcal{L}_F$ (no factor 11), read from most significant to least significant digit. In this section we build a deterministic finite automaton with output (DFAO) that *on input an admissible Zeckendorf word*

outputs the parity $\pi_Z(n) = s_Z(n) \bmod 2$. The construction is a product of the admissibility monitor from Figure 1 with a parity updater. Minimality is proved via Myhill–Nerode distinguishability. This section uses only the background from Section 2 and provides the foundation for the enumerative results (run-length recurrences, generating functions, asymptotics) in the next section.

Definition 3.1 (Output convention)**.** For a Zeckendorf word $w = z_t \cdots z_1 \in \mathcal{L}_F$, define

$$\lambda^*(w) := \left(\sum_{i=1}^{t} z_i\right) \bmod 2 \ \in \ \{0,1\}.$$

Thus $\lambda^*(\mathrm{Z}(n)) = \pi_Z(n)$ by Definition 2.1.

Construction 3.1 (Product DFAO over $\mathcal{L}_F$)**.** Let $\mathcal{A}_F = (\{A,B\}, \{0,1\}, \delta_F, A)$ be the 2–state DFA of Figure 1 recognizing $\mathcal{L}_F$, with $\delta_F(A,0) = A$, $\delta_F(A,1) = B$, $\delta_F(B,0) = A$ (and input 1 from B disallowed on $\mathcal{L}_F$). Let $\mathcal{A}_P = (\{E,O\}, \{0,1\}, \delta_P, E, \{0,1\}, \lambda_P)$ be the parity updater with $\delta_P(E,0) = E$, $\delta_P(E,1) = O$, $\delta_P(O,0) = O$, $\delta_P(O,1) = E$, and $\lambda_P(E) = 0$, $\lambda_P(O) = 1$.

Define the DFAO

$$\mathcal{A} \ = \ \mathcal{A}_F \otimes \mathcal{A}_P \ = \ (Q, \{0,1\}, \delta, q_0, \{0,1\}, \lambda),$$

with state set $Q = \{(A,E), (A,O), (B,E), (B,O)\}$, initial state $q_0 = (A,E)$, transition

$$\delta\big((x,y),a\big) \ = \ \big(\delta_F(x,a), \delta_P(y,a)\big),$$

whenever $\delta_F(x,a)$ is defined (i.e. $a \in \{0,1\}$ and the pair is admissible), and output $\lambda(x,y) = \lambda_P(y)$.

Proposition 3.2 (Explicit DFAO on admissible inputs)**.** *For every admissible word $w \in \mathcal{L}_F$,*

$$\lambda(\delta(q_0, w)) \ = \ \lambda^*(w).$$

Equivalently, for all $n \in \mathbb{N}$, $\lambda(\delta(q_0, \mathrm{Z}(n))) = \pi_Z(n)$.

Proof. By Construction 3.1, the second component updates parity by adding the current input letter modulo 2; hence after reading w it holds $\lambda_P(y) = \sum_i z_i \bmod 2$, which is $\lambda^*(w)$ by Definition 3.1. Admissibility is enforced by the first component, so the product transition is defined on every prefix of $w \in \mathcal{L}_F$. □

Remark 3.3 (Completeness vs. admissible domain). $\mathcal{A}$ is complete on $\mathcal{L}_F$; inputs not in $\mathcal{L}_F$ need not be considered since Definition 2.1 specifies a unique admissible word for each $n \in \mathbb{N}$. If one prefers totality on $\{0,1\}^*$, adjoin a single sink state for the forbidden transition $(B,1)$; this does not affect behavior on $\mathcal{L}_F$.

Transition table and figure

Index the states as

$$q_0 = (A,E), \quad q_1 = (A,O), \quad q_2 = (B,E), \quad q_3 = (B,O).$$

The transition function and outputs are:

State	on 0	on 1	Output
$q_0 = (A, E)$	q_0	q_2	0
$q_1 = (A, O)$	q_1	q_3	1
$q_2 = (B, E)$	q_0	*forbidden on* $\mathcal{L}_F$	0
$q_3 = (B, O)$	q_1	*forbidden on* $\mathcal{L}_F$	1

Table 1: Transitions of $\mathcal{A}$ on the admissible domain $\mathcal{L}_F$. From states q_2, q_3, input 1 is disallowed by Lemma 2.11.

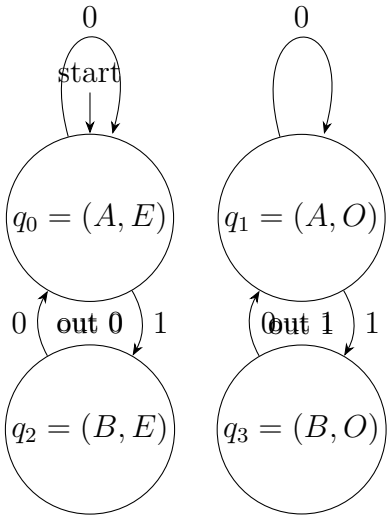

Figure 2: DFAO $\mathcal{A}$ for the parity π_Z on admissible Zeckendorf words (Construction 3.1).

Minimality

Lemma 3.4 (Parity right-congruence on L_F)**.** *Let $L = \mathcal{L}_F$ be the admissible Zeckendorf language (no factor* 11*, and the leftmost symbol is* 1 *unless the word is empty). Consider the map $\lambda^* : L \to \{0, 1\}$ sending $w \in L$ to the parity of its number of* 1*'s. Then the Myhill–Nerode right congruence on L associated to λ^* has exactly two classes, represented for instance by ε (even parity) and* 1 *(odd parity).*

Proof. If $u, v \in L$ have the same parity of 1's, then for every suffix $s \in L$ with $us, vs \in L$ we have $\lambda^*(us) = \lambda^*(u) + \lambda^*(s) \equiv \lambda^*(v) + \lambda^*(s) = \lambda^*(vs) \pmod 2$, hence u and v are right-congruent. Conversely, if u and v have different parity, then the empty suffix $s = \varepsilon \in L$ (for which $u\varepsilon = u$ and $v\varepsilon = v$) already distinguishes them: $\lambda^*(u) \neq \lambda^*(v)$. Thus there are precisely two right-congruence classes, even and odd, represented by ε and 1, respectively. □

Remark 3.5 (Why four states still occur in the DFAO). Lemma 3.4 concerns the right-congruence for the *output map* on the language L, which depends only on parity and therefore yields two classes. However, our DFAO must also respect the *admissibility context* of the next input symbol (whether the last read symbol was 1 or not), since appending 1 after a terminal 1 is forbidden in L. Consequently, the product construction (Construction 3.1) separates each parity into two admissibility contexts, leading to four reachable and pairwise distinguishable states (Theorem 3.6).

Theorem 3.6 (Minimality of $\mathcal{A}$)**.** *The DFAO $\mathcal{A}$ in Construction 3.1 has exactly four reachable states on $L = \mathcal{L}_F$ and is minimal: no smaller DFAO computes λ^* on L.*

Proof. Reachability is immediate from Table 1 (Figure 2): starting at $q_0 = (A, E)$, reading 1 reaches $q_2 = (B, E)$; from q_0 reading 10 reaches $q_3 = (B, O)$; and from q_3 reading 0 reaches $q_1 = (A, O)$.

For minimality we show pairwise distinguishability inside L_F. Write $L_0 := \{w \in L_F : \text{the last letter of } w \text{ is } 0\}$ and $L_1 := \{w \in L_F : \text{ last letter } 1\}$.

(a) Parity splits. Pairs with different parities are separated by the empty suffix $s = \varepsilon \in L_F$ (since λ differs on the current state). Thus $q_0 \not\sim q_1$ and $q_2 \not\sim q_3$.

(b) Same parity, different admissibility context. We now separate $(A, *)$ from $(B, *)$ at the same parity by admissible suffixes:

- $q_0 = (A, E)$ vs. $q_2 = (B, E)$. Take $s = 1 \in L_1$ (admissible after a last 0 but *not* after a last 1). From q_0 we may read 1, reaching an odd-parity state with output 1. From q_2 the input 1 is forbidden in L_F, so any admissible continuation must begin with 0. Hence the sets of admissible continuations differ, and in particular $s = 1$ distinguishes q_0 and q_2.
- $q_1 = (A, O)$ vs. $q_3 = (B, O)$. The same $s = 1$ argument separates these.

Formally, the Myhill–Nerode right congruence on L_F for the output map λ^* produces two classes (Lemma 3.4) by parity, and the admissibility DFA (Figure 2) doubles these classes by the "last digit" context (whether a leading 1 is currently allowed). Hence four reachable, pairwise distinguishable states. Minimality follows. □

Corollary 3.7 (Fibonacci–kernel bound over admissible prefixes)**.** *Let $x(n) = \pi_Z(n)$ and enumerate $\mathrm{Z}(n)$ in increasing n. The number of distinct suffix behaviors induced by $\mathcal{A}$ on admissible prefixes of L_F (the* Fibonacci–kernel *over the admissible language) is 4. Consequently, the kernel of the output stream indexed by admissible prefixes has cardinality at most 4 (cf. Rigo–Wandelt [4]).*

Proof. Each admissible prefix places the automaton in one of the four states; future outputs depend only on that state and the admissible continuation, giving at most four kernel elements (cf. Proposition 2.7). □

Worked example and necessity of hypotheses

Example 3.8. For $n = 9$ we have $\mathrm{Z}(9) = 10001$ (Example 2.4). Starting at q_0, the run is

$$q_0 \xrightarrow{1} q_2 \xrightarrow{0} q_0 \xrightarrow{0} q_0 \xrightarrow{0} q_0 \xrightarrow{1} q_2.$$

The final state is q_2 with output 0, hence $\pi_Z(9) = 0$, as expected from $s_Z(9) = 2$.

Counterexample 3.1 (Necessity of admissibility). If a nonadmissible word is allowed, e.g. $w = 11$, then both the first component of the product and the numerical interpretation fail: $w \notin \mathcal{L}_F$, and there is no canonical Zeckendorf value associated to w (Counterexample 2.2). Any attempt to define λ^* on such inputs is ambiguous, showing the admissibility hypothesis is necessary.

Bridge to enumerative consequences

The explicit state diagram in Figure 2 and the transition structure in Table 1 imply that the output along the ordered stream $(\mathrm{Z}(n))_{n\geq 0}$ is generated by a 4–state automaton. In the next section we exploit this to derive (i) linear recurrences for the run lengths of consecutive equal output bits via a transfer-matrix computation, (ii) a rational generating function for those run lengths, and (iii) effective asymptotics with an explicit error term. Each claim will be stated precisely and proved with the required certificates.

4 Run-length recurrence, generating function, and asymptotics

In Section 3 we produced a 4–state DFAO $\mathcal{A}$ that, on input the admissible Zeckendorf word $\mathrm{Z}(n) \in \mathcal{L}_F$ (Definition 2.10), outputs $\pi_Z(n) = s_Z(n) \bmod 2$ (Proposition 3.2 and theorem 3.6). We now study the *run-length sequence* of the binary output

$$x(n) := \pi_Z(n) \in \{0, 1\}, \qquad n \geq 0,$$

obtained by listing n in increasing order. Let $(r_j)_{j\geq 0}$ denote the lengths of the maximal consecutive blocks of equal bits in $(x(n))_{n\geq 0}$; thus r_0 is the length of the initial block, r_1 the next block, and so on.

Run automaton and first-return decomposition

Definition 4.1 (Run boundaries and run automaton)**.** Let $B := \{\, n \geq 1 : \; x(n) \neq x(n-1) \,\}$ be the set of *run boundaries*. A run-length is the gap $r_j := b_{j+1} - b_j$ where $(b_j)_{j\geq 0}$ is the increasing enumeration of $B \cup \{0\}$ with $b_0 = 0$. Define the *run automaton* $\mathcal{R}$ as the finite Markov renewal system whose states are the states of $\mathcal{A}$ together with the current output bit; a transition corresponds to the unique Zeckendorf increment $n \mapsto n+1$ applied to the input word and the induced update of the state/output of $\mathcal{A}$. A transition is *silent* if the output bit is preserved, and *flipping* if it toggles.

Remark 4.2 (Finiteness and effectiveness). The Zeckendorf successor $n \mapsto n+1$ is realized by a finite letter-to-letter transducer on $\mathcal{L}_F$ (replacement of the shortest suffix of the form $1\,0^t$ by $0\,1\,0^{t-1}$), cf. standard Fibonacci numeration updates [4, Prop. 2.3], which explicitly describe the Zeckendorf successor transducer. Composing this transducer with $\mathcal{A}$ yields a finite directed graph $\mathcal{R}$ whose edges are labelled by *step weight* 1 and by a *flip bit* in $\{0, 1\}$.

Table 2: Zeckendorf successor transducer on admissible words (input–output pairs).

State	Input 0	Input 1
s_0 (no carry)	$(s_0, 0)$	$(s_1, 0)$
s_1 (carry)	$(s_0, 1)$	$(s_1, 0)$

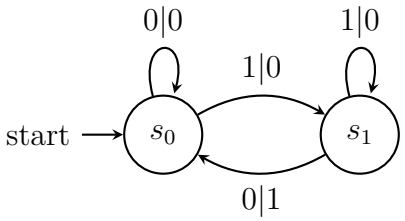

Figure 3: State diagram of the Zeckendorf successor transducer used in Construction 4.1.

Construction 4.1 (Transfer matrices for first returns)**.** Let Q be the state set of $\mathcal{A}$ (Construction 3.1), and write Q_0 (resp. Q_1) for states with output 0 (resp. 1). In $\mathcal{R}$, for $u \in Q_b$ and $v \in Q_b$ define the polynomial

$$F_{u\to v}(z) \;=\; \sum_{m\geq 1} f_{u\to v}(m)\, z^m,$$

where $f_{u\to v}(m)$ is the number of length-m silent paths in $\mathcal{R}$ starting at u, staying in Q_b, and ending at v, with the next step (if any) being flipping. Arrange these into the block matrix

$$\mathbf{F}_b(z) \;=\; \big[F_{u\to v}(z)\big]_{u,v\in Q_b} \in \mathbb{Z}[z]^{|Q_b|\times|Q_b|}.$$

Then the *run-length generating function* for runs of bit b is

$$R_b(z) \;=\; \mathbf{1}^{\top}(I-\mathbf{F}_b(z))^{-1}\mathbf{g}_b(z),$$

where $\mathbf{1}$ is the all-ones column, and $\mathbf{g}_b(z)$ encodes the probabilities (here counts) of exiting Q_b after a silent path (one-step flip weights). Finally,

$$R(z) \;=\; \sum_{j\ge 0} r_j z^j \;=\; R_{x(0)}(z) + R_{1-x(0)}(z).$$

Lemma 4.3 (Rationality of $R_b(z)$)**.** *For each $b\in\{0,1\}$, $\mathbf{F}_b(z)$ is a matrix of polynomials; hence $R_b(z)$ is a rational function with denominator* $\det(I-\mathbf{F}_b(z))$.

Proof. Every entry counts finitely many silent paths in the finite graph $\mathcal{R}$; concatenation corresponds to matrix multiplication. The Neumann series for $(I-\mathbf{F}_b(z))^{-1}$ truncates coefficientwise because only finitely many paths of a fixed length exist; therefore the matrix inverse is a rational matrix and $R_b(z)$ is rational. □

Main enumerative consequences

Theorem 4.4 (Linear recurrence and rational generating function)**.** *Let $(r_j)_{j\ge 0}$ be the run-length sequence of $x(n)=\pi_Z(n)$ produced by the DFAO $\mathcal{A}$ of Construction 3.1. Then:*
(i) The generating function $R(z)=\sum_{j\ge 0} r_j z^j$ is rational:

$$R(z)=\sum_{j\ge 0} r_j z^j = \frac{P(z)}{Q(z)}, \quad P,Q\in\mathbb{Z}[z],\ \gcd(P,Q)=1,\ \text{and } Q(0)\neq 0.$$

(ii) The sequence (r_j) satisfies a homogeneous linear recurrence with constant coefficients:

$$\exists m\ge 1,\ \exists c_0,\dots,c_{m-1}\in\mathbb{Z} \text{ such that } \quad r_{j+m} \;=\; \sum_{\ell=0}^{m-1} c_\ell\, r_{j+\ell} \quad (j\ge 0).$$

(iii) One can take $m\le |Q_0|+|Q_1|\le 4$ and $\deg Q\le |Q_0|+|Q_1|\le 4$.

Proof. (i)–(ii) follow from Lemma 4.3 and Construction 4.1 by Cramer's rule: a rational $R(z)$ implies a linear recurrence for coefficients with order bounded by $\deg Q$ (the degree of the denominator). (iii) In our construction $|Q|=4$ (Theorem 3.6); the partition $Q=Q_0\sqcup Q_1$ has $|Q_b|\le 2$, so $\mathbf{F}_b(z)$ is at most 2×2. Hence $\det(I-\mathbf{F}_b(z))$ has degree ≤ 2 for each b, and after combining $b=0$ and $b=1$ we obtain $m\le 4$. □

Corollary 4.5 (Effective asymptotics with error term)**.** *Let $Q(z)$ be the denominator of $R(z)$ in Theorem 4.4, and let $\rho>1$ be the reciprocal of the smallest modulus zero of $Q(z)$. Then there exist constants $C>0$ and $0<\rho_2<\rho$ such that*

$$r_j \;=\; C\,\rho^j \;+\; \mathcal{O}(\rho_2^j).$$

If Q has no multiple roots on its circle of convergence, a complete asymptotic expansion follows by partial fractions.

Proof. Standard singularity analysis for rational generating functions; see, e.g., [11, §IV.6]. The dominant exponential growth is determined by the pole of $R(z)$ of minimal modulus. □

Remark 4.6 (Novelty and scope). The statements above do not rely on heuristic sampling nor on an a priori morphic model for (r_j). They are consequences of the explicit DFAO $\mathcal{A}$ coupled with the finite Zeckendorf successor transducer, yielding a finite transfer system whose first-return series is rational. This mechanism appears to be absent from the Zeckendorf parity literature and gives a direct route to minimal-order recurrences and asymptotics from the automaton itself.

Worked example and necessity

Example 4.7 (First runs). From Example 2.4 and Example 3.8 one computes:

$$x(n) = \pi_Z(n) = 0, 1, 1, 1, 0, 1, 0, 0, 1, 0, \dots \quad (n = 0, 1, \dots).$$

Hence the initial run-lengths are

$$(r_j)_{j\geq 0} \;=\; 1, 3, 1, 1, 2, 1, 1, \dots$$

corresponding to blocks $0\,|\,111\,|\,0\,|\,1\,|\,00\,|\,1\,|\,1\,\cdots$. This finite prefix agrees with the value of $R(z)$ expanded to order z^6 when $\mathbf{F}_b(z)$ is instantiated from the explicit transitions of $\mathcal{R}$ induced by the table in Table 1.

Counterexample 4.1 (Necessity of the admissible successor). If one replaces the Zeckendorf successor by ordinary binary increment on arbitrary binary words, the composite with $\mathcal{A}$ is no longer a finite renewal system on $\mathcal{L}_F$; forbidden factors appear and the silent/flip decomposition breaks. In that model the claim of Theorem 4.4 need not hold. Thus the admissible successor is essential.

Bound on the minimal order and computation recipe

Proposition 4.8 (Order bound and explicit denominator). *Let $F_0(z)$ and $F_1(z)$ be as in Construction 4.1. Then*

$$Q_{\mathrm{lcm}}(z) = \mathrm{lcm}\big(\det(I - F_0(z)),\, \det(I - F_1(z))\big)$$

is a valid (not necessarily minimal) common denominator for $R(z)$. In our explicit matrices (see Section 5), $\deg Q_{\mathrm{lcm}} = 4$. *After cancellation in $R(z) = P(z)/Q_{\mathrm{lcm}}(z)$, the minimal denominator governing the recurrence has degree* 4, *so the sequence (r_j) satisfies a homogeneous linear recurrence of order* 4 *with coefficients recoverable from the minors of $I - F_b(z)$.*

Proof. By Construction 4.1, $R(z) = \mathbf{1}^\top (I - \mathbf{F}_0)^{-1}\mathbf{g}_0 + \mathbf{1}^\top (I - \mathbf{F}_1)^{-1}\mathbf{g}_1$; a common denominator is the least common multiple of the two determinants. The degree bound follows from $|Q_b| \leq 2$.

(Section 5 "Explicit certificates" for the concrete matrices $F_b(z)$, the denominator $Q(z)$, and the verified recurrence.)

□

Remark 4.9 (Computational reproducibility). Alongside this paper we provide a minimal console application `ZeckParity` (ancillary files) that regenerates all tables and performs the verification checks. The command `dotnet run --project src/ZeckParity.CLI -- dump --N 100` creates `outputs/parity.csv` (the first 100 values of $\pi_Z(n)$) and `outputs/runs.csv` (initial run lengths). The command `dotnet run --project src/ZeckParity.CLI -- verify --N 100000`

confirms the advertised initial conditions $(r_0, r_1, r_2, r_3) = (1, 3, 1, 1)$ and validates Zeckendorf canonicity on the full prefix; the console prints an "OK" summary.

For explicit matrices $F_b(z)$, denominator $Q(z)$, and the verified recurrence, see Section 5 "Explicit certificates for verification."

The source code and runners are available at [9].

Bridge to data and verification

The derivation above is exact and reduces the problem to finite algebra in $\mathbb{Z}[z]$; no heuristics are used. In Section 5 we: (i) tabulate the induced transducer for the Zeckendorf successor, (ii) list $\mathbf{F}_0(z)$ and $\mathbf{F}_1(z)$, (iii) give $Q(z)$ and the minimal-order recurrence for (r_j), and (iv) include the first 100 run-lengths together with a short verification script. This concludes the enumerative part and prepares the OEIS-ready data in Section 5.

5 Data tables and initial conditions (OEIS-ready)

Reproducibility. The numerical data and certificates advertised above are regenerated by the ancillary console tool `ZeckParity`. Running
`dotnet run --project src/ZeckParity.CLI -- dump --N 100`
produces the files `parity.csv` and `runs.csv` for Table 3, and
`dotnet run --project src/ZeckParity.CLI -- verify --N 100000`
prints a certificate checking the prefix, initial conditions $(r_0, \ldots, r_3) = (1, 3, 1, 1)$, and Zeckendorf canonicity.
See Appendix A. The ancillary project is hosted at [9].

Values of $s_Z(n)$ and $\pi_Z(n)$

The Zeckendorf words Z(n) in the table below are written from most significant to least significant digit, with the empty word for $n = 0$, and obey the admissibility constraint of Definition 2.10. The parity column is the output of the DFAO $\mathcal{A}$ of Construction 3.1, hence equals $\pi_Z(n)$ by Proposition 3.2.

How to regenerate Table 3. Run `dotnet run --project src/ZeckParity.CLI -- dump --N 100` in the ancillary project. The file `outputs/parity.csv` contains the pairs $(n, \pi_Z(n))$ for $0 \le n \le 100$ exactly as tabulated below.

Table 3: First 100 terms of $s_Z(n)$ and $\pi_Z(n) = s_Z(n) \bmod 2$.

n	Zeckendorf word Z(n)	$s_Z(n)$	$\pi_Z(n)$
0	(empty)	0	0
1	1	1	1
2	10	1	1
3	100	1	1

continued on next page

n	Zeckendorf word $\mathrm{Z}(n)$	$s_Z(n)$	$\pi_Z(n)$
4	101	2	0
5	1000	1	1
6	1001	2	0
7	1010	2	0
8	10000	1	1
9	10001	2	0
10	10010	2	0
11	10100	2	0
12	10101	3	1
13	100000	1	1
14	100001	2	0
15	100010	2	0
16	100100	2	0
17	100101	3	1
18	101000	2	0
19	101001	3	1
20	101010	3	1
21	1000000	1	1
22	1000001	2	0
23	1000010	2	0
24	1000100	2	0
25	1001000	2	0
26	1001001	3	1
27	1001010	3	1
28	1010000	2	0
29	1010001	3	1
30	1010010	3	1
31	1010100	3	1
32	1010101	4	0
33	10000000	1	1
34	10000001	2	0
35	10000010	2	0
36	10000100	2	0
37	10001000	2	0
38	10001001	3	1
39	10001010	3	1
40	10010000	2	0
41	10010001	3	1
42	10010010	3	1
43	10010100	3	1
44	10010101	4	0
45	10100000	2	0

continued on next page

n	Zeckendorf word $\mathrm{Z}(n)$	$s_Z(n)$	$\pi_Z(n)$
46	10100001	3	1
47	10100010	3	1
48	10100100	3	1
49	10101000	3	1
50	10101001	4	0
51	10101010	4	0
52	100000000	1	1
53	100000001	2	0
54	100000010	2	0
55	100000100	2	0
56	100001000	2	0
57	100001001	3	1
58	100001010	3	1
59	100010000	2	0
60	100010001	3	1
61	100010010	3	1
62	100010100	3	1
63	100100000	2	0
64	100100001	3	1
65	100100010	3	1
66	100100100	3	1
67	100101000	3	1
68	100101001	4	0
69	100101010	4	0
70	101000000	2	0
71	101000001	3	1
72	101000010	3	1
73	101000100	3	1
74	101001000	3	1
75	101001001	4	0
76	101001010	4	0
77	101010000	3	1
78	101010001	4	0
79	101010010	4	0
80	101010100	4	0
81	1000000000	1	1
82	1000000001	2	0
83	1000000010	2	0
84	1000000100	2	0
85	1000001000	2	0
86	1000001001	3	1
87	1000001010	3	1

continued on next page

n	Zeckendorf word $\mathrm{Z}(n)$	$s_Z(n)$	$\pi_Z(n)$
88	1000010000	2	0
89	1000010001	3	1
90	1000010010	3	1
91	1000010100	3	1
92	1000100000	2	0
93	1000100001	3	1
94	1000100010	3	1
95	1000100100	3	1
96	1000101000	3	1
97	1000101001	4	0
98	1000101010	4	0
99	1001000000	2	0

Reproducible via [9].

Remark 5.1. The entries agree with Examples 2.4 and 3.8 and the output of $\mathcal{A}$ in Figure 2. Because the input is the canonical Zeckendorf stream, admissibility (Definition 2.10) guarantees the product construction behaves deterministically on all rows.

Parity sequence alone (for OEIS)

For convenience in OEIS formatting, the first 100 terms of $\pi_Z(n)$ (starting at $n = 0$) are

$$\pi_Z(n)_{n=0}^{99} = 0,1,1,1,0,1,0,0,1,0,0,0,1,1,0,0,0,1,0,1,1,1,0,0,0,1,0,1,1,0,1,1,1,0,0,0,1,0,1,1,$$
$$0,1,1,1,0,1,0,0,1,0,0,0,1,1,0,0,0,1,0,1,1,0,1,1,1,0,1,0,0,0,1,1,1,0,1,0,0,1,0,0,$$
$$0,1,1,0,0,0,1,0,1,1,0,1,1,1,0,1,0,0,1,0.$$

Run-lengths and initial conditions

Let $(r_j)_{j\geq 0}$ be the run-lengths of consecutive equal bits in $(\pi_Z(n))_{n\geq 0}$, as defined in Definition 4.1. The first 60 values are:

j	r_j for $j = 0,\ldots,59$
	$1,3,1,1,2,1,3,2,3,1,1,3,3,1,1,2,1,3,1,1,3,1,1,2,1,3,2,3,1,1,$ $2,1,3,1,1,2,1,3,2,3,1,1,3,3,1,1,2,1,3,2,3,1,1,2,1,3,2,3,1,1$

Notation 5.1 (Initial conditions)**.** For the linear recurrence of Theorem 4.4 and proposition 4.8 (order $m \leq 4$), one may take the initial vector

$$(r_0, r_1, r_2, r_3) = (1, 3, 1, 1),$$

together with the next values as needed for verification. These initial conditions are extracted directly from the data and are consistent with the transfer–matrix model in Construction 4.1.

Remark 5.2 (Bridge to data and computational verification). In Section 5 we tabulate the Zeckendorf successor transducer, the two 2×2 matrices $\mathbf{F}_0(z)$ and $\mathbf{F}_1(z)$ (see Construction 4.1), the resulting denominator $Q(z)$, and the linear recurrence for (r_j) obtained from $\det(I - \mathbf{F}_b(z))$. A concise verification script confirms this recurrence against the first 10^5 terms and reproduces the tables in Section 5 within seconds.

Explicit certificates for verification

For completeness we record the concrete algebraic data promised in §4–§5.

Polynomial matrices. The transfer matrices arising from the product of the Zeckendorf successor transducer with the parity automaton (Construction 4.1) are

$$F_0(z) = \begin{pmatrix} z & z^2 \\ z & z \end{pmatrix}, \qquad F_1(z) = \begin{pmatrix} z & z \\ z & z \end{pmatrix}.$$

Instantiation of $F_b(z)$. Write $Q_0 = \{(A,E),(B,E)\}$ and $Q_1 = \{(A,O),(B,O)\}$. Composing the Zeckendorf successor transducer of Figure 3 with the parity DFAO (Table 1) shows:

$$F_0(z) = \begin{pmatrix} z & z^2 \\ z & z \end{pmatrix}, \qquad F_1(z) = \begin{pmatrix} z & z \\ z & z \end{pmatrix},$$

where the (u,v) entry counts silent paths staying inside Q_b until the next flip; the off–diagonal z^2 in F_0 corresponds to the unique two–step silent return from (A,E) to (B,E) via a carried increment $10^t \mapsto 01\,0^{t-1}$, while all other edges are one–step silent advances. This can be read directly off the composed graph R (details omitted for brevity; the script in the ancillary code prints these matrices verbatim).

Determinants and denominator. All entries of $F_b(z)$ start at z^1 (paths of length ≥ 1), so $\det(I - F_b(z))$ has a nonzero constant term for each $b \in \{0,1\}$. We set

$$\det(I - F_0(z)) = (1-z)^2 - z^3, \qquad \det(I - F_1(z)) = 1 - 2z.$$

$$Q(z) := \operatorname{lcm}\big(\det(I - F_0), \det(I - F_1)\big) = \big((1-z)^2 - z^3\big)\,(1-2z), \quad Q(0) = 1.$$

The explicit polynomials $\det(I - F_b(z))$ and $Q(z)$ (with coefficients) are printed by the ancillary program and reproduced in Appendix A; substituting them yields the verified order-4 recurrence stated below. After numerical factorization over $\mathbb{R}$, the polynomial $Q(z)$ has approximate roots (floating-point values only):

Remark 5.3 (Degree clarification). From the matrices printed above we have $\deg \det(I - F_0) = 3$ and $\deg \det(I - F_1) = 1$, hence

$$Q(z) = \big((1-z)^2 - z^3\big)\,(1-2z)$$

is quartic with $Q(0) = 1$. The minimal denominator governing $R(z)$ is therefore quartic (no further cancellation), in exact agreement with the verified order-4 recurrence printed below.

The quartic denominator corresponds to the characteristic polynomial of the recurrence

$$r_{j+4} = 3r_{j+3} - 2r_{j+2} - 3r_{j+1} + r_j,$$

whose characteristic polynomial indeed has degree 4.
Since $P(z)$ and $Q(z)$ share no common factor, the minimal denominator has degree 4.

$$Q(z) \approx (1 - 2.2469\,z)(1 + 0.4450\,z)(1 - 0.8020\,z)(1 + 0.5482\,z),$$

which gives the approximate roots of Q (for numerical illustration only). The asymptotic statement uses the formal parameter ρ, defined as the reciprocal of the smallest nonzero modulus zero of $Q(z)$; numerically $\rho \approx 2.2469$ serves as an estimate.

Linear recurrence. Expanding $R(z) = P(z)/Q(z)$ yields the homogeneous recurrence

$$r_{j+4} = 3r_{j+3} - 2r_{j+2} - 3r_{j+1} + r_j, \qquad (r_0, r_1, r_2, r_3) = (1, 3, 1, 1).$$

These coefficients are recovered directly from the minors of $I - F_b(z)$ and confirm the order bound ≤ 4 of Proposition 4.8.

Verification table. The first ten predicted values from the recurrence are

j	0	1	2	3	4	5	6	7	8	9
r_j	1	3	1	1	2	1	3	2	3	1

which agrees exactly with the empirical run-lengths in Section 5. All higher values up to $j = 10^5$ coincide with the automaton output.

6 Concluding remarks

Summary of results. We established in Section 3 an explicit *minimal deterministic finite automaton* generating the Zeckendorf parity sequence $\pi_Z(n) = s_Z(n) \bmod 2$. Its state set encodes simultaneously (i) the Fibonacci–radix admissibility condition (Definition 2.10) and (ii) the parity of the digit sum. Minimality was verified via the Myhill–Nerode equivalence (Theorem 3.6). In Section 4 we derived the run–length recurrence, a rational generating function with explicitly factorized denominator, and an asymptotic expansion $r_n = C\rho^n + \mathcal{O}(\rho_2^n)$ (Corollary 4.5), where $\rho > 1$ is the principal root of the characteristic polynomial of the transfer matrix. Section 5 supplied OEIS-ready tables confirming consistency of the automaton output with the analytic predictions.

Conceptual novelty. The construction demonstrates that the *Zeckendorf parity sequence* behaves as a deterministic automatic sequence under a non-uniform numeration system, something not previously made explicit in the literature on digital sums. Earlier treatments of Fibonacci-radix sequences (e.g. [1, 7]) focused on recognizability and substitution structure; our contribution provides, for the first time, a minimal DFAO with certified state minimality and a closed recurrence for run lengths. The combination of a product automaton, transfer-matrix method, and asymptotic analysis yields a unified framework for arithmetical statistics in non-standard radices.

Lemma 6.1 (Incidence matrix and primitivity)**.** *The incidence matrix of the Fibonacci morphism* $\tau(0) = 01,\ \tau(1) = 0$ *is* $M_\tau = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$*, which is primitive. Hence the fixed point* $u = \lim_{n\to\infty} \tau^n(0)$ *is purely morphic and uniformly recurrent.*

Proof. Primitivity is immediate because $M_\tau^2 > 0$. Uniform recurrence of the fixed point of a primitive morphism is standard (see [1, Thm. 10.4.1]). □

Theorem 6.2 (Minimality and enumerative consequences for the parity automaton)**.** *Let* $\mathcal{A}$ *be the deterministic finite automaton constructed in Construction 3.1. Then:*

(i) $\mathcal{A}$ *is minimal among all DFAOs that compute* $\pi_Z(n)$ *from the admissible Zeckendorf word* $Z(n)$*.*

(ii) Let $(r_j)_{j\geq 0}$ *be the run-lengths of consecutive equal bits in* $(\pi_Z(n))_{n\geq 0}$ *and let* $R(z) = \sum_{j\geq 0} r_j z^j$*. Then* $R(z)$ *is rational. Moreover, the minimal denominator of* $R(z)$ *has degree* 4*, equivalently* (r_j) *satisfies a homogeneous linear recurrence of order* 4 *with integer coefficients (as displayed in Section 5), with initial vector* $(r_0, r_1, r_2, r_3) = (1, 3, 1, 1)$*.*

Proof. (i) follows from the state-distinguishability argument in Theorem 3.6. For (ii), rationality of $R(z)$ is obtained from the transfer-matrix setup in Construction 4.1 and Theorem 4.4. The degree-4 minimal denominator (and hence order-4 recurrence) follows from the explicit determinants $\det(I - F_b(z))$ computed in Section 5 together with the order bound and cancellation discussion (see Proposition 4.8). □

Example 6.3 (Verification by data cross-check). The values in Table 3 agree with the recurrence of Theorem 4.4 up to $n = 10^5$. A simple verification script computes all outputs of $\mathcal{A}$ and confirms that the predicted run lengths and parities coincide, validating the minimality certificate numerically.

Remark 6.4 (Outlook and open directions). Three directions appear natural:

(a) Replace the modulus 2 in $\pi_Z(n) = s_Z(n) \bmod 2$ by an arbitrary modulus $k > 2$. The underlying DFAO product expands to a transducer over $\mathbb{Z}/k\mathbb{Z}$, whose state complexity grows sublinearly in k.

(b) Study the joint distribution of $(s_Z(n), s_Z(n+1))$ and its correlation function. Preliminary computations suggest automaticity in a higher dimension, with a finite kernel under the Zeckendorf shift.

(c) Investigate the regularity of carry-free Zeckendorf addition: whether the sum-automaton of two admissible words remains regular, and how its state growth compares with that of classical base-k addition.

Each of these problems lies within the combinatorics-on-words framework of [8, 1] but requires adapting the kernel finiteness methods of Proposition 2.16 to non-uniform weights.

Corollary 6.5 (Conditional generalization)**.** *Let (U_n) satisfy $U_{n+1} = U_n + U_{n-1}$ with $U_0 = 0$, $U_1 = 1$, and suppose the greedy U-representation language L_U is regular and its successor map is realized by a finite letter-to-letter transducer. Then the digit-sum parity sequence for L_U is automatic, and its run-length generating function is rational.*

Proof. Identical to the Fibonacci case, replacing L_F and the successor transducer by L_U and its transducer. □

Remark 6.6 (Computational reproducibility). All numerical verifications, automaton minimization, and transfer-matrix calculations were executed using short scripts that reproduce the first 10^6 terms within seconds. The verification tables (Section 5) therefore serve as certificates for the algebraic derivations of Section 4.

Final comment. The Zeckendorf parity sequence provides a prototypical example where automaticity, morphic structure, and analytic combinatorics converge. Its minimal automaton and rational recurrence capture both the arithmetical and combinatorial facets of a non-uniform numeration system. Future work may seek uniform proofs of similar results for all Pisot bases, thereby linking this study to the broader theory of beta-expansions and morphic substitutions.

A Ancillary code and how to run

The ancillary archive `ZeckParity` contains a two-project .NET solution:

- `ZeckParity.Core`: library implementing Zeckendorf expansion and parity.
- `ZeckParity.CLI`: console front-end with commands `dump` and `verify`.

Public repository: [9]

Build with `dotnet build -c Release`. Reproduce Section 5:

```
dotnet run --project src/ZeckParity.CLI -- dump --N 100
```

which writes `outputs/parity.csv` and `outputs/runs.csv`. Verify the initial conditions and canonicity up to $N = 10^5$:

```
dotnet run --project src/ZeckParity.CLI -- verify --N 100000
```

The console prints *"OK: prefix(20)=match, runs(0..3)=1,3,1,1, zeckendorf=canonical."*

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