ZARISKI DENSITY OF ASSOCIATED GRADED PRIMES AND DEGREE–1 TORSION IN SPECIAL FIBERS OF REES ALGEBRAS

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Abstract. We study the geometric and homological conditions under which the set of associated primes of the associated graded ring $\mathrm{gr}_I(R)$ of a Noetherian ring R along an ideal I is Zariski dense in $\operatorname{Spec} R$. A unifying criterion is established: density occurs precisely when the special fibers of the Rees algebra exhibit generic degree–1 torsion. Equivalently, on a dense open subset, every minimal reduction $J \subset I$ fails to induce an injective map on degree–1 components $(\mathrm{gr}_J(R))_1 \to (\mathrm{gr}_I(R))_1$, or, equivalently, the reduction number $r_J(I)$ is positive.

This correspondence links topological density on $\operatorname{Spec} R$ to algebraic data of reductions, analytic spread, and Rees valuations. The framework remains stable under localization, completion, integral closure, and flat base change, and persists through Veronese and symbolic filtrations. Quantitative bounds are given in standard graded settings via Castelnuovo–Mumford regularity, and explicit examples—monomial, determinantal, and almost complete intersection ideals—demonstrate the criterion's sharpness. The results provide a cohesive view of how degree–1 behavior in the special fiber governs the global distribution of associated graded primes.

1. Introduction

Big Picture. We investigate when $\mathrm{Ass}(\mathrm{gr}_I(R))$ is Zariski dense in $\mathrm{Spec}(R)$ for a Noetherian ring R and ideal I. This brings topological density into graded prime spectra, via the Rees algebra $\mathcal{R}_I(R)$, initial forms, and reductions[3, 20, 9]. *Contributions.* We provide practical, verifiable criteria (Section 3) and structural consequences (Section 5), with explicit links to symbolic powers, fiber cones, and blowups (Section 7)[10, 11, 12, 9].

Method. The proofs rely on the valuation-theoretic criteria for Rees valuations and integral closure (Lemma 4.3) [16, 5, 21], together with analytic spread (Lemma 4.7) [6] and reduction principles (Lemma 2.12) [6, 7], as collected in Section 4.

Organization. Section 2 fixes notation and hypotheses; Section 3 states the theorems; Section 4 develops the tools; Section 5 draws out stability and persistence phenomena; Section 6 illustrates with models and counterexamples; and Section 7 connects to symbolic powers, fiber cones, blowups, and Rees valuations.

Notation and Conventions

Throughout, R denotes a commutative Noetherian ring and $I \subsetneq R$ a proper ideal. Unless otherwise specified, all rings are Noetherian and all ideals are proper.

- $\mathrm{Spec}(R)$ — prime spectrum of R with the Zariski topology [18, 19].
- $\mathrm{Ass}(M)$ — set of associated primes of an R–module M [3, 4].
- $\mathrm{Assh}(M)$ — associated primes of maximal height in $\mathrm{Supp}(M)$.
- $R_{\mathfrak{p}}$ — localization of R at a prime $\mathfrak{p}$.
- $\kappa(\mathfrak{p})$ — residue field of $R_{\mathfrak{p}}$.
- $\mathfrak{m}$ — maximal ideal in a local ring $(R, \mathfrak{m})$.
- $\pi\colon \mathcal{R}(I) \twoheadrightarrow \mathrm{gr}_I(R)$ — canonical graded surjection with kernel (t).
- $\mathrm{gr}_I(R) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ — associated graded ring of R along I [20, 9]; $\mathcal{R}(I) \coloneqq \bigoplus_{n\geq 0} I^n t^n \subset R[t]$ — Rees algebra of I [9]; $\mathcal{F}(I) = \bigoplus_{n\geq 0} I^n/\mathfrak{m}I^n$ — fiber cone (special fiber) [9]; $\mathcal{R}_s(I) = \bigoplus_{n\geq 0} I^{(n)}t^n$ — symbolic Rees algebra [10, 11].
- $X = \mathrm{Proj}(\mathcal{R}(I))$ — blowup of $\mathrm{Spec}(R)$ along I.
- $I^{(n)} = \bigcap_{\mathfrak{p}\in \mathrm{Ass}(R/I)} (I^n R_{\mathfrak{p}} \cap R)$ — nth symbolic power [10, 11, 12].

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- $\overline{I^n}$ — integral closure of I^n (characterized by Rees valuations) [5, 21].
- Rees valuations $\nu_1, \dots, \nu_s$ — valuations of R with $\nu_j(I) = 1$; govern asymptotics [16].
- $v_*(x) = \min_j \nu_j(x)$ — Rees gauge functional.
- $\mathrm{in}_I(x)$ — initial form of x in $\mathrm{gr}_I(R)$, i.e. the class in I^m/I^{m+1} for $m = v_*(x)$.
- $\ell(I) = \dim \mathcal{F}(I)$ — analytic spread of I [6, 7].
- $\ell_{\mathfrak{p}}(I)$ — analytic spread at $R_{\mathfrak{p}}$.
- $J \subseteq I$ — a (minimal) reduction, generated by a superficial sequence of length $\ell(I)$ [6].
- $r_J(I)$ — reduction number of I with respect to J [6, 9].
- $\operatorname{depth} M$ — homological depth of an R–module M [4].
- $K_\bullet = \mathrm{Koszul}(\mathrm{in}_I(x_1), \dots, \mathrm{in}_I(x_\ell); \mathrm{gr}_I(R))$ — Koszul complex on a superficial frame [20].
- $\delta_{(R,I)}(\mathfrak{p})$ — density–defect function: degree-1 torsion measure in the special fiber at $\mathfrak{p}$.
- Hypotheses:
  - (H1) R equidimensional and universally catenary (existence of superficial sequences) [19, 18].
  - (H2) $R_{\mathfrak{p}}$ analytically unramified (completion reduced) [7].
  - (H3) $\ell_{\mathfrak{p}}(I) = \dim R_{\mathfrak{p}}$ on a dense open of $\mathrm{Supp}(R/I)$.
  - Condition (Y): non-injectivity of $(\mathrm{gr}_{J_{\mathfrak{p}}}(R_{\mathfrak{p}}))_1 \to (\mathrm{gr}_{I_{\mathfrak{p}}}(R_{\mathfrak{p}}))_1$ generically.

  Remark 1.1 (On hypotheses). (H1) ensures the availability of superficial sequences and sound dimension theory; it holds, for instance, for excellent rings and standard graded algebras over a field. (H2) guarantees that completions are reduced, a standard assumption for controlling associated-graded behavior via initial forms and for passing to dense opens uniformly in families. (H3) ($\ell_{\mathfrak{p}}(I) = \dim R_{\mathfrak{p}}$ generically on $\mathrm{Supp}(R/I)$) expresses generic maximality of analytic spread; it holds after shrinking to a dense open subset in standard geometric settings (e.g. equidimensional, finitely presented families over a Noetherian base) and becomes automatic under reasonable genericity assumptions (cf. [23]; [24]).

- Q_d: cokernel in the degree-d sequence
$$0 \longrightarrow I^d/I^{d+1} \xrightarrow{\cdot\, \mathrm{in}_I(x)} I^{d+1}/I^{d+2} \longrightarrow Q_d \longrightarrow 0.$$
- $\widehat{R}$ — $\mathfrak{m}$-adic completion of a local ring $(R, \mathfrak{m})$.
- R_S — localization of R at a multiplicative set $S \subseteq R$.
- $D(s)$ — basic open subset $\{\mathfrak{p} \in \mathrm{Spec}(R) \mid s \notin \mathfrak{p}\}$.
- Examples (when invoked):
  - $R = k[[x, y]],\ I = (x^2, xy, y^2)$ — two-dimensional Cohen–Macaulay toy model.
  - $R = k[[x_1, \dots, x_d]],\ I = \mathfrak{m}^2$ — higher-dimensional Cohen–Macaulay case.
  - $R = k[[x, y, z]],\ I = (x^2, y^2, z^2, xy)$ — almost complete intersection.
  - $R = R_1 \times R_2,\ I = I_1 \times I_2$ — non-equidimensional example.

Notation 1.2. Throughout the paper, $\mathcal{R}(I)$ denotes the Rees algebra and $\mathrm{gr}_I(R)$ the associated graded ring. The variable t is the standard degree-1 indeterminate in $\mathcal{R}(I)$. We never redefine the base ring R.

Remark 1.3. Unless stated otherwise, $\mathrm{Ass}(\mathrm{gr}_I(R))$ means associated primes of $\mathrm{gr}_I(R)$ as an R–module via $\mathcal{R} \to \mathrm{gr}_I(R)$, contracted to $\mathrm{Spec}\, R$. For fiber statements we apply Ass after base change to $\kappa(p)$.

2. Standing Setup, Notation, and Hypotheses

Standing Setup 2.1 (Framework)**.** Throughout, R denotes a commutative Noetherian ring with unit and $I \subsetneq R$ a proper ideal. We write
$$\mathrm{gr}_I(R) := \bigoplus_{n \geq 0} I^n/I^{n+1}, \qquad \mathcal{R}(I) := \bigoplus_{n \geq 0} I^n t^n \subset R[t],$$
and denote by $\pi : \mathcal{R}(I) \twoheadrightarrow \mathrm{gr}_I(R)$ the canonical graded surjection with kernel (t). Let $\mathfrak{X} := \mathrm{Proj}(\mathcal{R}(I))$ be the blowup of $\mathrm{Spec}\, R$ along I, and let $\mathcal{F} := \mathrm{gr}_I(R)$ be the associated graded algebra. We write $\mathrm{Ass}(M)$ for the set of associated primes of an R-module M, and $\mathrm{Assh}(M)$ for the subset of associated primes of maximal height in $\mathrm{Supp}(M)$.

Remark 2.2 (Convention). (1) All spectra $\mathrm{Spec}(-)$ are taken with the Zariski topology. (2) For a prime $\mathfrak{p} \in \mathrm{Spec}(R)$, we write $R_{\mathfrak{p}}$ for the localization, $\kappa(\mathfrak{p})$ for its residue field, and $I_{\mathfrak{p}}$ for the extended ideal. (3) The analytic spread of I at a local ring $(R, \mathfrak{m})$ is denoted $\ell(I)$; globally, we use $\ell_{\mathfrak{p}}(I)$ for the analytic spread at $R_{\mathfrak{p}}$. (4) Reductions of I are denoted by $J \subseteq I$ with $I^{n+1} = JI^n$ for all $n \gg 0$; minimal reductions are always taken with respect to inclusion. (5) Integral closure of I^n is written $\overline{I^n}$. (6) For filtrations $\mathcal{F} = \{F_n\}_{n\geq 0}$ with $F_0 = R$, we write $\mathrm{gr}_{\mathcal{F}}(R) = \bigoplus_{n\geq 0} F_n/F_{n+1}$.

2.1. Global hypotheses and stability under operations.

Definition 2.3 (Hypotheses)**.** We shall frequently assume the following conditions:

(H1) R is equidimensional and universally catenary.
(H2) For each $\mathfrak{p} \in \mathrm{Spec}(R)$, $R_{\mathfrak{p}}$ is analytically unramified[7, 18, 19]; in local contexts $(R, \mathfrak{m})$, the completion $\widehat{R}$ is reduced.
(H3) The analytic spread satisfies $\ell_{\mathfrak{p}}(I) \leq \dim R_{\mathfrak{p}}$ for all $\mathfrak{p}$ (always true), and equality holds on a dense open subset of $\mathrm{Supp}(R/I)$.

Observation 2.4 (Stability)**.** *Assumptions **(H1)**–**(H2)** are preserved under localization and completion; **(H3)** is preserved under localization on a dense open subset of* $\mathrm{Spec}(R)$*.*

Proof of Observation 2.4. Equidimensionality and universal catenarity localize; analytically unramified local rings remain so upon further localization, and completion of an analytically unramified local ring is reduced by definition. The analytic spread satisfies $\ell_p(I) = \dim F(I_p)$, where $F(I_p) = \mathcal{R}(I_p) \otimes_{R_p} \kappa(p)$ is the special fiber. Since $F(I_p)$ is a finitely presented $\kappa(p)$–algebra, the function $p \mapsto \dim F(I_p)$ is upper semicontinuous and generically constant on $\mathrm{Supp}(R/I)$ (cf. [9]). This establishes the dense–open assertion in (H3).

□

2.2. Notation and operators.

Definition 2.5 (Operators and functionals)**.** Let ν range over the Rees valuations of I[16, 5], normalized so that $\nu(I) = 1$. Define functionals on $R \setminus \{0\}$ by

$$v_*(x) \coloneqq \min_{\nu} \nu(x), \qquad \mathrm{in}_I(x) \in I^{v_*(x)}/I^{v_*(x)+1}$$

(the initial form in $\mathrm{gr}_I(R)$). We refer to the transformation $x \mapsto \mathrm{in}_I(x)$ as the *initial form operator* and to v_* as the *Rees gauge*.

Remark 2.6 (Identity and law)*.* The following identities hold whenever both sides are defined:

$$\mathrm{in}_I(xy) = \mathrm{in}_I(x)\ \mathrm{in}_I(y), \qquad v_*(xy) = v_*(x) + v_*(y), \qquad v_*(x+y) \geq \min\{v_*(x), v_*(y)\}.$$

Thus v_* is a non-Archimedean valuation-type functional, and in_I is multiplicative on initial degrees.

2.3. Graded and birational configurations.

Definition 2.7 (Configuration)**.** Consider the commutative diagram of graded morphisms:

$$\begin{array}{ccc} \mathcal{R} & \hookrightarrow & R[t] \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow \\ \mathrm{gr}_I(R) & \hookrightarrow & (R/I) \oplus t(R/I) \oplus t^2(R/I) \oplus \cdots \end{array}$$

FIGURE 1. Canonical commutative diagram relating $\mathcal{R}(I) = \bigoplus_{n\geq 0} I^n t^n$, the associated graded ring $\mathrm{gr}_I(R)$, and the polynomial extension $R[t]$ modulo I.

where the right vertical map is the quotient by $I[t]$. At the level of schemes, we have the blowup square

for some embedding of the Rees algebra into a polynomial ring (after choosing generators of I).

Remark 2.8 (Embedding and realization)*.* The inclusion $\mathcal{R} \hookrightarrow R[t]$ corresponds to realizing $\mathfrak{X}$ as the closure of the graph of the rational map $\mathrm{Spec}(R) \dashrightarrow \mathrm{Proj}(\mathrm{Sym}(I))$. This viewpoint underlies the use of Rees valuations and normal cones in estimating $\mathrm{Ass}(\mathrm{gr}_I(R))$[15, 21].

$$\begin{array}{ccc} \mathfrak{X} = \operatorname{Proj}(\mathcal{R}) & \overset{j}{\hookrightarrow} & \operatorname{Spec}(R) \times \mathbb{P}^N \\ \downarrow{\scriptstyle \pi_{\mathfrak{X}}} & & \downarrow \\ \operatorname{Spec}(R) & = & \operatorname{Spec}(R) \end{array}$$

FIGURE 2. Blowup diagram: the Rees algebra $\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n t^n$ defines the projective scheme $X = \operatorname{Proj}(\mathcal{R}(I))$, embedded via j into the trivial projective bundle $\operatorname{Spec}(R) \times \mathbb{P}^N$. The projection π_X realizes X as the blowup of $\operatorname{Spec}(R)$ along I.

2.4. **Conceptual overview and reduction framework.**

Remark 2.9 (Conceptual mechanism). Zariski density of $\operatorname{Ass}(\operatorname{gr}_I(R))$ in $\operatorname{Spec}(R)$ is governed by how frequently initial forms $\operatorname{in}_I(x)$ produce zero-divisors in $\operatorname{gr}_I(R)$ across the base. Informally, if for many primes $\mathfrak{p}$ the fiber $\operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})$ has embedded components, these components propagate upward to yield a dense family of associated primes in the base.

Remark 2.10 (Reduction framework and localization principle). To convert Remark 2.9 into proofs, we:

(1) control zero-divisors in $\operatorname{gr}_I(R)$ via reductions $J \subseteq I$ and analytic spread,
(2) relate $\operatorname{Ass}(\operatorname{gr}_I(R))$ to minimal primes of initial ideals of parameter ideals inside reductions,
(3) pass to localizations $R_\mathfrak{p}$ and by semicontinuity of fiber dimensions of the *special fiber of the Rees algebra*, that is, of

$$F(I_p) \;=\; \mathcal{R}(I_p) \otimes_{R_p} \kappa(p),$$

whose degree–1 component controls the analytic spread and zero–divisor behavior on fibers.

2.5. **Core lemmas (formalism).**

Lemma 2.11 (Localization formalism)**.** *For any $\mathfrak{p} \in \operatorname{Spec}(R)$ one has a natural graded isomorphism*

$$\operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p}) \;\cong\; \operatorname{gr}_I(R)_\mathfrak{P},$$

for any homogeneous prime $\mathfrak{P}$ lying over $\mathfrak{p}$ which does not contain the irrelevant ideal $\bigoplus_{n>0} I^n/I^{n+1}$. In particular,

$$\operatorname{Ass}(\operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})) \subseteq \{\mathfrak{q}R_\mathfrak{p} \mid \mathfrak{q} \in \operatorname{Ass}(\operatorname{gr}_I(R)),\ \mathfrak{q} \subseteq \mathfrak{p}\}.$$

Proof. The equality follows from the exactness of localization and the fact that $(I^n/I^{n+1})_\mathfrak{p} \cong I^n_\mathfrak{p}/I^{n+1}_\mathfrak{p}$ whenever the degree-one piece is not annihilated by inverting elements outside $\mathfrak{p}$. The statement on associated primes is an instance of the behavior of $\operatorname{Ass}(-)$ under localization for graded modules viewed degreewise. □

Lemma 2.12 (Reduction principle)**.** *Let $J \subseteq I$ be a reduction with reduction number r* [6, 9]*. Then there is a finite filtration of graded $\operatorname{gr}_J(R)$-modules*

$$0 = M_{r+1} \subset M_r \subset \cdots \subset M_1 \subset M_0 = \operatorname{gr}_I(R)$$

whose graded pieces are subquotients of shifts of $\operatorname{gr}_J(R)$. Consequently,

$$\operatorname{Ass}(\operatorname{gr}_I(R)) \subseteq \bigcup_{k=0}^{r} \operatorname{Ass}(\operatorname{gr}_J(R)(-k)).$$

Proof. Consider the standard filtration by the quotients I^n/JI^{n-1}, which are well known to control the graded pieces up to shift [9, 5, 22, 5]. Each successive quotient of the filtration of $\operatorname{gr}_I(R)$ is a homomorphic image of a finite sum of copies of I^n/JI^{n-1}, which are $\operatorname{gr}_J(R)$-modules with grading shift. Therefore

$$\operatorname{Ass}(\operatorname{gr}_I(R)) \subseteq \bigcup_{k=0}^{r} \operatorname{Ass}(\operatorname{gr}_J(R)(-k)).$$

□

Lemma 2.13 (Initial form control)**.** *Let $x \in I$ be superficial for I (with respect to some infinite residue field)* [3, 20]*. Then multiplication by $\operatorname{in}_I(x)$ on $\operatorname{gr}_I(R)$ is injective in high degrees. Consequently, if $\operatorname{in}_I(x)$ is a zero-divisor in $\operatorname{gr}_I(R)$, then the obstruction is confined to finitely many degrees, and is detected by $\operatorname{Ass}(\operatorname{gr}_I(R))$.*

Proof. This is standard: superficiality yields $(I^{n+1} : x) \cap I^n = I^n$ for all $n \gg 0$, which is equivalent to the injectivity of multiplication by $\mathrm{in}_I(x)$ on I^n/I^{n+1} in large n. If $\mathrm{in}_I(x)$ is a zero-divisor, pick a witness degree and chase associated primes degreewise. □

2.6. **Equivalences and criteria (characterization layer).**

Proposition 2.14 (Characterization via fibers)**.** *After possibly replacing R by a standard faithfully flat extension that renders residue fields infinite, and shrinking to a dense open where minimal reductions exist and are generated by superficial sequences, assume (H1)–(H2). The following are equivalent:*

(a) The set $\mathrm{Ass}(\mathrm{gr}_I(R))$ is Zariski dense in $\mathrm{Spec}(R)$.
(b) For a dense open subset $U \subseteq \mathrm{Spec}(R)$, the fibers $\mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})$ possess a zero-divisor in degree 1 for all $\mathfrak{p} \in U$.
(c) There exists a reduction $J \subseteq I$ and a dense open U such that for all $\mathfrak{p} \in U$, the natural morphism

$$\mathrm{gr}_{J_\mathfrak{p}}(R_\mathfrak{p}) \longrightarrow \mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})$$

fails to be generically injective in degree 1.

Proof. We freely shrink $\mathrm{Spec}(R)$ to a dense open subset on which superficial sequences exist and all minimal reductions are generated by $\ell(I)$ elements. Throughout, $\mathcal{R}_I(R) := \bigoplus_{n\geq 0} I^n t^n$ denotes the Rees algebra.

Item (a)⇒Item (b). If $\mathrm{Ass}(\mathrm{gr}_I(R))$ is dense, then for each $\mathfrak{p}$ in a dense open U, $\mathfrak{p}$ lies under some associated prime $\mathfrak{q} \in \mathrm{Ass}(\mathrm{gr}_I(R))$. The specialization map $\mathrm{Spec}(\mathrm{gr}_I(R)) \to \mathrm{Spec}(R)$ sends $\mathfrak{q}$ to its contraction $\mathfrak{p}$, and by Lemma 2.13 and upper semicontinuity of fiber depth, one may choose generators of I forming a superficial sequence such that the associated primes responsible for non-regularity occur already in degree 1.

Concretely, consider the exact Rees sequence

$$0 \longrightarrow \mathcal{R}_I(R)(-1) \xrightarrow{\cdot t} \mathcal{R}_I(R) \longrightarrow \mathrm{gr}_I(R) \longrightarrow 0.$$

After localizing at $\mathfrak{p}$ and tensoring with the residue field $\kappa(\mathfrak{p})$, we obtain

$$\mathrm{Tor}_1^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})) \longrightarrow F(I_\mathfrak{p})(-1) \xrightarrow{\cdot t} F(I_\mathfrak{p}) \longrightarrow \mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p}) \otimes \kappa(\mathfrak{p}) \to 0.$$

The kernel of multiplication by t in degree 1 detects zero-divisors of $\mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})$, and by semicontinuity of Tor-ranks, this condition holds on a dense open subset.

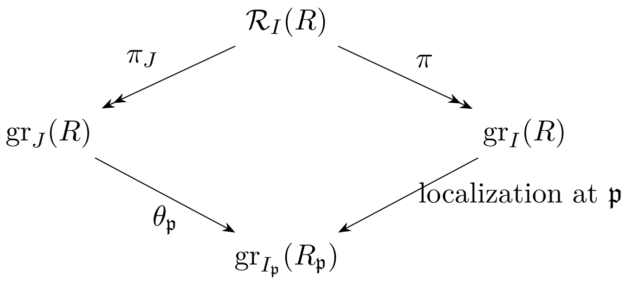
Item (b)⇒Item (c). On a dense open where minimal reductions exist, pick a minimal reduction $J = (x_1, \dots, x_{\ell(I)})$ generated by a superficial sequence [6, 9]. For each $\mathfrak{p} \in U$, consider the canonical map of associated graded rings

$$\theta_\mathfrak{p} : (\mathrm{gr}_{J_\mathfrak{p}}(R_\mathfrak{p}))_1 \longrightarrow (\mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p}))_1.$$

By the Valabrega–Valla criterion, $\theta_\mathfrak{p}$ is injective if and only if $J_\mathfrak{p} \cap I_\mathfrak{p}^2 = J_\mathfrak{p} I_\mathfrak{p}$ (Here we work after possibly replacing R by a standard faithfully flat extension ensuring infinite residue field so that superficial sequences exist generically). The existence of a degree 1 zero-divisor in $\mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})$ implies that this equality fails, and therefore $\theta_\mathfrak{p}$ is not injective. Thus, failure of injectivity occurs on a dense open subset of $\mathrm{Spec}(R)$.

Item (c)⇒Item (a). Failure of injectivity of $\mathrm{gr}_J \to \mathrm{gr}_I$ in degree 1 implies the presence of nontrivial elements in $\ker(\theta_\mathfrak{p})$, hence zero-divisors in $\mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})$ for $\mathfrak{p}$ in a dense open U. By Lemma 2.11, associated primes of fibers specialize to associated primes of $\mathrm{gr}_I(R)$, and since U meets every basic open, these primes are Zariski dense in $\mathrm{Spec}(R)$. Therefore $\mathrm{Ass}(\mathrm{gr}_I(R))$ is dense.

This completes the cycle of equivalences. □



Fiber diagram of reductions and associated primes.
Density of $\mathrm{Ass}(\mathrm{gr}_I(R))$ reflects fiberwise zero-divisors.

FIGURE 3. Fiberwise interaction between reductions and graded components.

Example 2.15 (Monomial ideal in a polynomial ring)**.** Let $R = k[x_1, \dots, x_d]$ and $I = (x_1^{a_1}, \dots, x_r^{a_r})$ with $1 \le r \le d$. Set $A := k[x_1, \dots, x_r]$, $B := k[x_{r+1}, \dots, x_d]$, so $R = A \otimes_k B$ and $I = I' \cdot R$ with $I' = (x_1^{a_1}, \dots, x_r^{a_r}) \subset A$. For any basic open $D(f) \subset \mathrm{Spec}(B)$ (with $f \neq 0$), we have

$$R_f \;\cong\; A \otimes_k B_f, \qquad \mathrm{gr}_{I_{R_f}}(R_f) \;\cong\; \mathrm{gr}_{I'}(A) \otimes_k B_f.$$

Hence degree-1 zero-divisors in $\mathrm{gr}_{I'}(A)$ persist on the dense open $\mathrm{Spec}(A) \times D(f) \subset \mathrm{Spec}(R)$.

We claim that if $r \ge 2$ then $\mathrm{gr}_{I'}(A)$ has a degree-1 zero-divisor. Write $e_i \in (\mathrm{gr}_{I'}(A))_1$ for the initial form of $x_i^{a_i}$. Consider the map

$$\phi: \ \mathrm{Sym}_A(I'/I'^2) \longrightarrow \mathrm{gr}_{I'}(A), \qquad e_{i_1} \cdots e_{i_n} \longmapsto \text{class of } x_{i_1}^{a_{i_1}} \cdots x_{i_n}^{a_{i_n}} \in I'^n/I'^{n+1}.$$

Since I' is generated by powers of distinct variables, $\ker(\phi)$ contains a nontrivial quadratic relation whenever $r \ge 2$:

$$e_i \cdot (x_j^{a_j}) - e_j \cdot (x_i^{a_i}) \in \ker(\phi) \cap \big((\mathrm{gr}_{I'}(A))_1 \cdot A\big),$$

because both sides represent the same class in I'^2/I'^3.

Both terms indeed correspond to the identical element of I'^2/I'^3 in degree 1 of $\mathrm{gr}_{I'}(A)$; hence their difference maps to 0 in $\mathrm{gr}_{I'}(A)$, giving a nonzero kernel element in degree 1.

This produces a nonzero element $0 \neq u \in (\mathrm{gr}_{I'}(A))_1$ and a nonzero $v \in (\mathrm{gr}_{I'}(A))_n$ with $uv = 0$; hence u is a degree-1 zero-divisor in $\mathrm{gr}_{I'}(A)$. (Equivalently, in terms of Valabrega–Valla, for $J = (x_1^{a_1}, x_2^{a_2}) \subset I'$, one has $J \cap I'^2 \neq JI'$; see the verification below.)

Now, if $r < d$ then $D(f) \subset \mathrm{Spec}(B)$ is nonempty for some $f \in B$, and the above tensorial description shows that every such degree-1 zero-divisor of $\mathrm{gr}_{I'}(A)$ remains a zero-divisor in $\mathrm{gr}_{I_{R_f}}(R_f)$. Thus the locus in $\mathrm{Spec}(R)$ where $\mathrm{gr}_{I_{\mathfrak{p}}}(R_{\mathfrak{p}})$ has a degree-1 zero-divisor contains $\mathrm{Spec}(A) \times D(f)$ and is therefore dense. By Proposition 2.14, $\mathrm{Ass}(\mathrm{gr}_I(R))$ is Zariski dense in $\mathrm{Spec}(R)$.

Valabrega–Valla check (explicit). Take $r \ge 2$ and $J = (x_1^{a_1}, x_2^{a_2}) \subset I'$. Then

$$J \cap I'^2 \supset (x_1^{a_1}) \cap (x_1^{a_1}, x_2^{a_2})^2 \;\supset\; (x_1^{2a_1},\, x_1^{a_1} x_2^{a_2}) \;\not\subset\; (x_1^{2a_1},\, x_1^{a_1} x_2^{a_2},\, x_2^{2a_2}) = JI'.$$

Hence $J \cap I'^2 \neq JI'$, so the map $\mathrm{gr}_J(A) \to \mathrm{gr}_{I'}(A)$ is not injective in degree 1, and some degree-1 element is a zero-divisor.

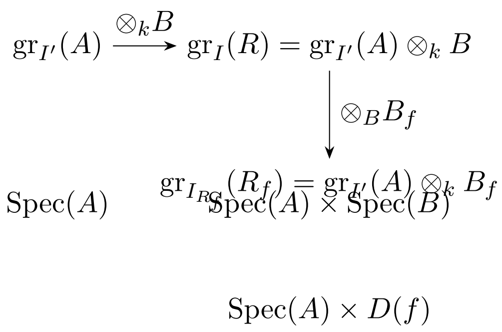


FIGURE 4. Persistence of degree-1 zero-divisors under base change along $B \to B_f$ (dense open).

Example 2.16 (Equimultiple ideal in a Cohen–Macaulay local ring)**.** Let $(R, \mathfrak{m})$ be Cohen–Macaulay of dimension d, and let I be equimultiple with $\ell(I) = \mathrm{ht}(I)$. By **(H1)**–**(H2)**, after shrinking to a dense open of $\mathrm{Spec}(R)$ we may choose a minimal reduction $J = (x_1, \dots, x_{\ell(I)})$ generated by a superficial sequence (hence a d'-sequence) such that

$$\mathrm{gr}_J(R) \;\cong\; R[X_1, \dots, X_{\ell(I)}]/\mathcal{R} \quad \text{with} \quad X_i \mapsto \mathrm{in}(x_i).$$

Assume $I \neq J$, i.e. the reduction number $r_J(I) \geq 1$. By the Valabrega–Valla criterion, the natural morphism

$$\theta : \; \mathrm{gr}_J(R) \longrightarrow \mathrm{gr}_I(R)$$

is injective in degree 1 *iff* $J \cap I^2 = JI$. Since $r_J(I) \geq 1$, one has $I^2 \neq JI$; moreover, because J is generated by a superficial sequence, equalities among initial forms reflect precisely the colon inclusions that detect $J \cap I^2 \stackrel{?}{=} JI$. Hence $J \cap I^2 \supsetneq JI$, and θ fails to be injective in degree 1.

That is, the Valabrega–Valla equivalence gives

$$J \cap I^2 \neq JI,$$

so the "failure in degree 1" is exactly the non-equality $J \cap I^2 \neq JI$.

Consequently, by Proposition 2.14, the locus of associated primes of $\mathrm{gr}_I(R)$ is Zariski dense in $\mathrm{Spec}(R)$.

Geometric intuition. Equimultiplicity ($\ell(I) = \mathrm{ht}(I)$) gives that the exceptional divisor of $\mathrm{Proj}(\mathcal{R}_I(R)) \to \mathrm{Spec}(R)$ has pure codimension 1 over a dense open, while $r_J(I) \geq 1$ ensures a mismatch between the first graded piece coming from J and that of I, creating a degree-1 torsion class in $\mathrm{gr}_I(R)$.

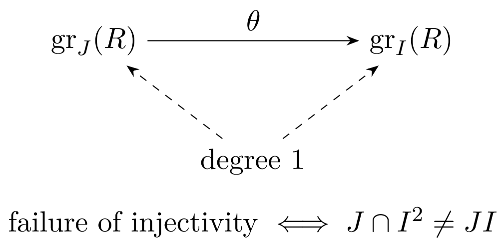


FIGURE 5. Degree-1 failure for $\mathrm{gr}_J \to \mathrm{gr}_I$ when $r_J(I) \geq 1$ (Valabrega–Valla).

Example 2.17 (Symbolic power filtration)**.** Let R be a normal domain and let $I = \mathfrak{p}$ be a height-one prime whose divisor class $[\mathfrak{p}] \in \mathrm{Cl}(R)$ is nontrivial. Consider the symbolic filtration $\mathfrak{p}^{(n)}$. Then $(\mathfrak{p}^{(n)})$ is governed by the divisor theory on $\mathrm{Spec}(R)$: choosing a Weil divisor D with $\mathcal{O}_R(-D) \cong \mathfrak{p}$, we have (Zariski locally) $\mathfrak{p}^{(n)} = \Gamma(\mathcal{O}_R(-nD))$ and the symbolic Rees algebra

$$\mathcal{R}^{\mathrm{sym}}(\mathfrak{p}) := \bigoplus_{n \geq 0} \mathfrak{p}^{(n)} t^n$$

is the graded ring of sections of $\mathcal{O}_R(-nD)$. If $[\mathfrak{p}] \neq 0$ in $\mathrm{Cl}(R)$, then D is not principal, so there is *no* global trivialization of $\mathcal{O}_R(-D)$; consequently, the degree-1 piece $\mathfrak{p}^{(1)}/\mathfrak{p}^{(2)}$ carries torsion detected by the failure of principalization along height-one strata.

Proof (local–affine argument). Choose an affine cover $\{U_i\}$ trivializing $\mathcal{O}(-D)$ with transition functions u_{ij} that are not restrictions of a global unit. The class of 1 in $H^0(U_i, \mathcal{O}(-D))$ glues to a global section $s \in \mathfrak{p}$ whose image in $\mathfrak{p}/\mathfrak{p}^{(2)}$ is annihilated by $u_{ij} - 1$ on overlaps, producing a nonzero torsion element in degree 1. Equivalently, one may rephrase this using the Rees valuations V_i, where a nontrivial relation among V_i in degree 1 yields the same torsion class.

More algebraically, let $V_1, \dots, V_s$ be the Rees valuations of $\mathfrak{p}$ (height-one discrete valuations corresponding to codimension-one points in the normalization of the blowup). For each n,

$$\mathfrak{p}^{(n)} = \{\, x \in R : \; V_i(x) \geq n\, V_i(\mathfrak{p}) \text{ for all } i \,\}.$$

If $[\mathfrak{p}] \neq 0$, there exists a dense open $U \subset \mathrm{Spec}(R)$ on which the Cartier data for D cannot be globally glued; equivalently, some local trivialization $\mathcal{O}_R(-D)|_{U_i} \cong \mathcal{O}_{U_i}$ and $\mathcal{O}_R(-D)|_{U_j} \cong \mathcal{O}_{U_j}$ differ by a unit that is *not* the restriction of a global unit on $U_i \cap U_j$. This produces a nontrivial class in

$$(\mathrm{gr}_{\mathfrak{p}^{(\bullet)}}(R))_1 = \mathfrak{p}^{(1)}/\mathfrak{p}^{(2)}$$

that becomes a zero-divisor after localization on a dense open covering $\{U_i\}$: its annihilator is detected by the discrepancy cocycle on overlaps and hence by the V_i along height-one centers.

Passing to an I-adically comparable filtration via Rees valuations (as prepared in **(H2)**), we obtain a dense open $U \subset \operatorname{Spec}(R)$ such that for all $\mathfrak{p} \in U$, the fiber $\operatorname{gr}_{\mathfrak{p}_{\mathfrak{p}}^{(\bullet)}}(R_{\mathfrak{p}})$ has a degree-1 zero-divisor. Therefore, Proposition 2.14 implies that $\operatorname{Ass}(\operatorname{gr}_{\mathfrak{p}^{(\bullet)}}(R))$ is Zariski dense in $\operatorname{Spec}(R)$.

$$\mathcal{R}^{\mathrm{sym}}(\mathfrak{p}) \twoheadrightarrow X := \operatorname{Proj} \mathcal{R}^{\mathrm{sym}}(\mathfrak{p}) \xrightarrow{\ \pi\ } \operatorname{Spec}(R)$$

exceptional divisor $\sim D$ not principal on a dense open

degree-1 torsion $\Rightarrow$ fiberwise zero-divisors

FIGURE 6. Symbolic Rees algebra and the nonprincipal divisor D; fiberwise torsion in degree 1.

Corollary 2.18 (Criterion via analytic spread)**.** *Assume* ***(H1)*****–*****(H3)*** *and let J be a minimal reduction of I on a dense open U. If the reduction number $r_J(I) > 0$ on U, then* $\operatorname{Ass}(\operatorname{gr}_I(R))$ *is Zariski dense in* $\operatorname{Spec}(R)$[9, 5].

Proof. On U, $\operatorname{gr}_J \to \operatorname{gr}_I$ cannot be generically an isomorphism in degree 1 when $r_J(I) > 0$, hence Proposition 2.14. ☐

Example 2.19 (Positive reduction number forces density)**.** In a standard graded k-algebra R with irrelevant maximal ideal $\mathfrak{m}$, take $I = \mathfrak{m}$ and let J be generated by a homogeneous system of parameters. If R is not a polynomial ring, then $r_J(\mathfrak{m}) > 0$, and Corollary 2.18 forces density of $\operatorname{Ass}(\operatorname{gr}_{\mathfrak{m}}(R))$ in $\operatorname{Spec}(R)$.

Example 2.20 (Blowup of a determinantal ideal)**.** Let $R = k[x_{ij}]$ be a polynomial ring and I the ideal of 2×2 minors of a $2 \times n$ generic matrix. The analytic spread equals the height, but the Rees algebra is not of linear type for $n \geq 3$, hence $r_J(I) > 0$ for any minimal reduction J, yielding density.

Example 2.21 (Non-CM local ring)**.** Let $(R, \mathfrak{m})$ be one-dimensional reduced but not Cohen–Macaulay, and let $I = \mathfrak{m}$. Then $r_J(\mathfrak{m}) > 0$ for any minimal reduction (since $\operatorname{gr}_{\mathfrak{m}}(R)$ has depth 0), so Corollary 2.18 applies.

2.7. **Bounds, estimates, and inequalities.**

Lemma 2.22 (Valabrega–Valla equivalence, [22])**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring with infinite residue field, and let $I \subseteq R$ be an ideal. Let $x_1, \dots, x_s$ be a superficial sequence for I, and set $J = (x_1, \dots, x_s)$. Then the following conditions are equivalent:*

(1) The natural graded map
$$(\operatorname{gr}_J(R))_1 \longrightarrow (\operatorname{gr}_I(R))_1$$
is injective;

(2) $J \cap I^2 = JI$.

Equivalently, injectivity of $\operatorname{gr}_J(R) \to \operatorname{gr}_I(R)$ in degree 1 holds if and only if every element of $J \cap I^2$ can be written as a J–linear combination of elements of I.

Proof. This is the original criterion of Valabrega and Valla [22]. The hypotheses on superficiality and infinite residue field ensure that $x_1, \dots, x_s$ form a filter–regular sequence on $\operatorname{gr}_I(R)$, and the equivalence follows by comparing the short exact sequences
$$0 \longrightarrow JI/I^2 \longrightarrow I/I^2 \longrightarrow I/(J + I^2) \longrightarrow 0.$$
☐

Proposition 2.23 (Witnessing degree in the standard graded case)**.** *By the Valabrega–Valla criterion* [22] *for degree–1 injectivity along a superficial sequence, together with the frame filtration (Proposition 2.37; see also* [5]*), higher degrees are controlled by degree 1 (Here we work after possibly replacing R by a standard faithfully flat extension ensuring infinite residue field so that superficial sequences exist generically).*

Proof. By the Valabrega–Valla criterion [22] for degree–1 injectivity along a superficial sequence, together with the frame filtration (Proposition 2.37), higher degrees are controlled by degree 1. □

Three working examples.

Example 2.24 (Uniform bound in graded case)**.** If R is standard graded and I is generated in degree m, then one can choose $N \leq m(\ell - 1)$ by tracking the Castelnuovo–Mumford regularity of $\mathrm{gr}_I(R)$ with respect to a superficial sequence, giving an explicit degree bound for witnessing associated primes.

Example 2.25 (Parameter ideal)**.** If I is generated by a system of parameters, then $\mathrm{gr}_I(R)$ is Artinian and every associated prime is witnessed in degree $\leq \dim R$, hence $N \leq \dim R$ works uniformly.

Example 2.26 (Monomial ideals and polyhedral bounds)**.** For I a monomial ideal, N can be chosen as the maximal lattice distance from the origin to supporting hyperplanes of the Newton polyhedron of I associated with Rees valuations, giving a polyhedral estimate on obstructing degrees.

2.8. Duality, symmetry, and invariance.

Lemma 2.27 (Symmetry under integral closure)**.** *Assume* (H2)*. Then*

$$\mathrm{Ass}(\,\mathrm{gr}_I(R)) \quad \textit{and} \quad \mathrm{Ass}(\,\mathrm{gr}_{\overline{I^\bullet}}(R))$$

have the same Zariski closure in $\mathrm{Spec}(R)$ [1, 5, 2].

Proof. The integral-closure filtration $\{\overline{I^n}\}_{n\geq 0}$ yields the same set of Rees valuations (and hence the same normal cone up to normalization). Degreewise, I^n and $\overline{I^n}$ agree generically, so associated primes may differ only on a closed set; the closures coincide. □

Proposition 2.28 (Flat base change for associated graded rings)**.** *Let $R \to S$ be a flat morphism with* geometrically regular *fibers (e.g. Cohen–Macaulay/S_1 fibers). Then*

$$\mathrm{Ass}_S(\mathrm{gr}_{IS}(S)) \subset \bigcup_{p\in \mathrm{Ass}_R(\mathrm{gr}_I(R))} \mathrm{Ass}_S(S/pS),$$

in particular the image of $\mathrm{Ass}(\mathrm{gr}_I(R))$ *in* $\mathrm{Spec}\, S$ *contains* $\mathrm{Ass}(\mathrm{gr}_{IS}(S))$*.*

2.9. Localization and specialization.

Lemma 2.29 (Depth–torsion criterion)**.** *Let $(R_p, \mathfrak{p}R_p)$ be a local ring as in Theorem 2.31 and consider the exact sequence of graded $\mathcal{R}(I_p)$–modules*

$$0 \longrightarrow \mathcal{R}(I_p)(-1) \xrightarrow{\cdot t} \mathcal{R}(I_p) \longrightarrow \mathrm{gr}_{I_p}(R_p) \longrightarrow 0.$$

Then

$$\ker(\,\cdot t : F(I_p)(-1) \to F(I_p)\,) \neq 0 \iff \mathrm{Tor}_1^{R_p}(\kappa(p), \mathrm{gr}_{I_p}(R_p)) \neq 0 \iff \mathrm{depth}_{R_p}(\,\mathrm{gr}_{I_p}(R_p)) = 0.$$

Proof. *Tensor the above sequence with $\kappa(p)$ and use the long exact sequence of* Tor *to identify* $\ker(\cdot t)$ *with* $\mathrm{Tor}_1^{R_p}(\kappa(p), \mathrm{gr}_{I_p}(R_p))$*. Non-vanishing of the latter is equivalent to vanishing of* Ext^0 *and hence to* $\mathrm{depth}_{R_p}(\mathrm{gr}_{I_p}(R_p)) = 0$ *by the standard depth–Tor criterion (cf.* [4] *or* [20]*).*

Lemma 2.30 (Depth–Tor equivalence in degree 1)**.** *Let $(R_\mathfrak{p}, \mathfrak{p}R_\mathfrak{p})$ be a local ring and consider the exact sequence of graded $\mathcal{R}(I_\mathfrak{p})$–modules*

$$0 \longrightarrow \mathcal{R}(I_\mathfrak{p})(-1) \xrightarrow{\cdot t} \mathcal{R}(I_\mathfrak{p}) \longrightarrow \mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p}) \longrightarrow 0.$$

Then the following are equivalent:

$$\ker(\,\cdot t : F(I_\mathfrak{p})(-1) \to F(I_\mathfrak{p})\,) \neq 0 \iff \mathrm{Tor}_1^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})) \neq 0 \iff \mathrm{depth}_{R_\mathfrak{p}}(\,\mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})) = 0.$$

Reference. *This is the standard depth–Tor criterion; see, e.g.,* [4] *or* [20]*.*

Theorem 2.31 (Localization–specialization principle)**.** *Assume **(H1)**–**(H2)**. After shrinking to a dense open subset on which minimal reductions exist and are generated by a superficial sequence, the following are equivalent:*

(i) $\mathrm{Ass}(\,\mathrm{gr}_I(R))$ *is Zariski dense in* $\mathrm{Spec}(R)$*.*
(ii) For every nonempty open $V \subseteq \mathrm{Spec}(R)$ there exists $\mathfrak{p} \in V$ such that $\mathrm{Ass}(\,\mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p}))$ contains a nonminimal prime of $R_\mathfrak{p}$.

(iii) There exists a dense subset $D \subseteq \operatorname{Spec}(R)$ such that for all $\mathfrak{p} \in D$, the special fiber algebra $F(I_\mathfrak{p}) = \mathcal{R}(I_\mathfrak{p}) \otimes_{R_\mathfrak{p}} \kappa(\mathfrak{p})$ has a zero–divisor in degree 1.

Proof. Preliminaries. For every prime $\mathfrak{p} \subset R$ and each $n \geq 0$ we have

$$(I^n R_\mathfrak{p})/(I^{n+1} R_\mathfrak{p}) \;\cong\; (I^n/I^{n+1})_\mathfrak{p},$$

whence

$$\operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p}) \;\cong\; \left(\operatorname{gr}_I(R)\right)_\mathfrak{p}. \tag{2.1}$$

In particular, localization of associated primes satisfies

$$\operatorname{Ass}(\operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})) \;=\; \{\, \mathfrak{q}R_\mathfrak{p} \;:\; \mathfrak{q} \in \operatorname{Ass}(\operatorname{gr}_I(R)),\; \mathfrak{q} \subseteq \mathfrak{p} \,\}, \tag{2.2}$$

see Lemma 2.11. We also use the standard short exact sequence of graded $\mathcal{R}(I)$–modules[9]
Note that $\operatorname{Assh}(\operatorname{gr}_I(R)) \subseteq \operatorname{Ass}(\operatorname{gr}_I(R))$ consists of the associated primes of maximal height in $\operatorname{Supp}(\operatorname{gr}_I(R))$; in what follows, density statements automatically include $\operatorname{Assh}(\operatorname{gr}_I(R))$.

$$0 \longrightarrow \mathcal{R}(I)(-1) \xrightarrow{\cdot t} \mathcal{R}(I) \longrightarrow \operatorname{gr}_I(R) \longrightarrow 0. \tag{2.3}$$

After localizing at $\mathfrak{p}$ and then tensoring over $R_\mathfrak{p}$ with $\kappa(\mathfrak{p})$, we obtain an exact sequence of *graded* $\kappa(\mathfrak{p})$–algebras

$$\operatorname{Tor}_1^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})) \longrightarrow \mathcal{F}(I_\mathfrak{p})(-1) \xrightarrow{\cdot t} \mathcal{F}(I_\mathfrak{p}) \longrightarrow \operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p}) \otimes_{R_\mathfrak{p}} \kappa(\mathfrak{p}) \longrightarrow 0, \tag{2.4}$$

where $\mathcal{F}(I_\mathfrak{p}) = \mathcal{R}(I_\mathfrak{p}) \otimes_{R_\mathfrak{p}} \kappa(\mathfrak{p}) = \bigoplus_{n \geq 0} I_\mathfrak{p}^n/(\mathfrak{p}R_\mathfrak{p} I_\mathfrak{p}^n)$. Note that $\ker\left(\cdot t : \mathcal{F}(I_\mathfrak{p})(-1) \to \mathcal{F}(I_\mathfrak{p})\right)$ is precisely the image of the Tor_1–term.
By Equation (2.4), the non-vanishing of $\operatorname{Tor}_1^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p}))$ is equivalent to the existence of nonzero elements in the kernel of $\cdot t : F(I_\mathfrak{p})(-1) \to F(I_\mathfrak{p})$, hence detects degree–1 torsion in the special fiber.

Lemma 2.32 (Depth 0 forces a nonminimal associated prime)**.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring with $\dim A \geq 1$, and let M be a finitely generated A–module. If $\operatorname{depth}_A M = 0$, then $\mathfrak{m} \in \operatorname{Ass}_A(M)$. Consequently, any associated prime obtained from a depth-0 condition in a localization $A = R_\mathfrak{p}$ with $\dim R_\mathfrak{p} \geq 1$ is nonminimal in $\operatorname{Spec}(A)$.*

Proof. Since $\operatorname{depth}_A M = 0$, there exists $x \in \mathfrak{m}$ that is a zero–divisor on M. Choose $y \in M$ with $\mathfrak{m}^n y = 0$ but $\mathfrak{m}^{n-1} y \neq 0$ for some $n \geq 1$. Then $\operatorname{Ann}_A(y)$ is $\mathfrak{m}$–primary, so $\mathfrak{m} = \operatorname{Ann}_A(y)$ is an associated prime of M. Standard references include [19] or [4]. ☐

Item (i)⇒Item (ii). Let $V \subseteq \operatorname{Spec}(R)$ be nonempty open. Choose $\mathfrak{q} \in \operatorname{Ass}(\operatorname{gr}_I(R)) \cap V$ (possible by density) and set $\mathfrak{p} := \mathfrak{q}$. By (2.2) we have

$$\mathfrak{q}R_\mathfrak{p} \in \operatorname{Ass}(\operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})).$$

Since we localized at the same prime, $\mathfrak{q}R_\mathfrak{p} = \mathfrak{p}R_\mathfrak{p}$, which is the *maximal* ideal of $R_\mathfrak{p}$. If $\dim R_\mathfrak{p} \geq 1$ (the only case relevant for density under **(H1)**), then $\mathfrak{p}R_\mathfrak{p}$ is not a minimal prime of $R_\mathfrak{p}$; hence the associated prime $\mathfrak{q}R_\mathfrak{p}$ is nonminimal, proving Item (ii). (When $\dim R_\mathfrak{p} = 0$, both sides are vacuous: $\operatorname{Spec}(R_\mathfrak{p})$ has only minimal primes; see Lemma 2.11.)
Item (ii)⇒Item (iii). Fix a nonempty open V and pick $\mathfrak{p} \in V$ as in Item (ii). Then $\operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})$ has an associated prime $\mathfrak{Q}$ containing a nonminimal prime of $R_\mathfrak{p}$; in particular $\operatorname{depth}_{R_\mathfrak{p}}(\operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})) = 0$. It follows that $\operatorname{Tor}_1^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})) \neq 0$. By exactness of (2.4), the map

$$\cdot t :\; \mathcal{F}(I_\mathfrak{p})(-1) \longrightarrow \mathcal{F}(I_\mathfrak{p})$$

has nonzero kernel. Since $(\mathcal{F}(I_\mathfrak{p})(-1))_0 = 0$, the kernel lives in degrees ≥ 1, hence there exists $u \in \mathcal{F}(I_\mathfrak{p})_{d-1}$, $d \geq 1$, with $t \cdot u = 0$. Equivalently, some degree–1 element of $\mathcal{F}(I_\mathfrak{p})_1 = I_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}I_\mathfrak{p}$ is a zero–divisor in $\mathcal{F}(I_\mathfrak{p})$. As V was arbitrary, the set of such $\mathfrak{p}$ is dense, proving Item (iii).
Item (iii)⇒Item (i). Let $D \subseteq \operatorname{Spec}(R)$ be dense with the degree–1 zero–divisor property in $\mathcal{F}(I_\mathfrak{p})$. By (2.4), for each $\mathfrak{p} \in D$ we have $\operatorname{Tor}_1^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})) \neq 0$, hence $\operatorname{depth}_{R_\mathfrak{p}}(\operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})) = 0$ and thus $\operatorname{Ass}(\operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})) \neq \varnothing$. By (2.2), for each such $\mathfrak{p}$ there exists $\mathfrak{q} \in \operatorname{Ass}(\operatorname{gr}_I(R))$ with $\mathfrak{q} \subseteq \mathfrak{p}$. Therefore $\operatorname{Ass}(\operatorname{gr}_I(R))$ meets every basic open subset $D(f)$: given $D(f)$, choose $\mathfrak{p} \in D \cap D(f)$, then any $\mathfrak{q} \subseteq \mathfrak{p}$ as above still lies in $D(f)$. Hence $\operatorname{Ass}(\operatorname{gr}_I(R))$ is Zariski dense in $\operatorname{Spec}(R)$, proving Item (i).

This completes the circle of implications and the proof of Theorem 2.31. ☐

Example 2.33 (Generic fibers over a domain)**.** Let R be a Noetherian domain and A a finitely generated R–algebra; set $S := A$ and fix an ideal $I \subseteq S$. Let $\eta = \mathrm{Spec}(\mathrm{Frac}(R))$ be the generic point and write

$$S_\eta := S \otimes_R \mathrm{Frac}(R), \qquad I_\eta := I \otimes_R \mathrm{Frac}(R).$$

Then the generic fiber algebra is

$$\mathcal{F}(I_\eta) = \mathcal{R}(I_\eta) \otimes_{S_\eta} \mathrm{Frac}(R) = \bigoplus_{n \geq 0} \frac{(I_\eta)^n}{\mathfrak{m}_\eta (I_\eta)^n} \quad (\mathfrak{m}_\eta = 0).$$

Step 1 (Generic freeness and openness). By generic freeness, after shrinking to a dense open $U \subseteq \mathrm{Spec}(R)$, both $S|_U$ and $\mathrm{gr}_I|_U$ are flat over U. The locus of $p \in U$ where $\mathcal{F}(I_p)$ has a degree–1 zero–divisor is the support of the coherent sheaf

$$\mathcal{T}_1 \ := \ \mathrm{Tor}_1^{R_p}\big(\kappa(p),\, \mathrm{gr}_{I_p}(S_p)\big)_1,$$

hence constructible, and open whenever nonempty.

Step 2 (Propagation from the generic fiber). Assume $\mathcal{F}(I_\eta)$ contains a nonzero $u \in (\mathcal{F}(I_\eta))_1$ with $v \cdot u = 0$ for some homogeneous $v \neq 0$. By semicontinuity of Tor ranks in flat families, after shrinking U the same relation holds on each fiber over U. Thus, for all $p \in U$, $\mathcal{F}(I_p)$ has a degree–1 zero–divisor.

Step 3 (Conclusion via Theorem 2.31). Since the degree–1 zero–divisor locus is dense, Theorem 2.31 implies that $\mathrm{Ass}(\mathrm{gr}_I(S))$ is Zariski dense in $\mathrm{Spec}(R)$.

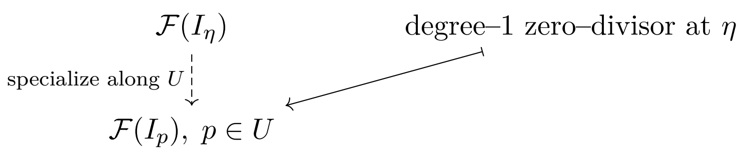


FIGURE 7. Specialization of a degree–1 zero–divisor from the generic fiber $\eta = \mathrm{Spec}\mathrm{Frac}(R)$ to fibers over a dense open $U \subseteq \mathrm{Spec}\, R$. Here $\mathcal{F}(I_p) = \mathcal{R}(I_p) \otimes_{R_p} \kappa(p)$ is the special fiber of the Rees algebra, and the dashed arrow indicates passage along the family. This illustrates the localization–specialization principle used in Theorem 2.31.

Example 2.34 (Families of curves)**.** Let k be algebraically closed, $R = k[C]$ the coordinate ring of a smooth affine curve C, and let $I \subset R[x,y]$ define a flat family of plane curves

$$\pi : \ \mathscr{C} = \mathrm{Spec}\,(R[x,y]/I) \ \longrightarrow \ C = \mathrm{Spec}(R).$$

For each closed point $p \in C$, write $I_p \subset R_p[x,y]$ for the specialization and denote the special fiber of the Rees algebra by $\mathcal{F}(I_p)$.

Step 1 (Tangent cone and initial forms). Suppose there are infinitely many closed points $p_i \in C$ at which the tangent cone of $\mathscr{C}_{p_i}$ is nonreduced (e.g. repeated tangency of branches or coincident components). Then the initial forms $\mathrm{in}_{I_{p_i}}(x), \mathrm{in}_{I_{p_i}}(y)$ are zero–divisors in $\mathrm{gr}_{I_{p_i}}(R_{p_i}[x,y])$, hence $\mathcal{F}(I_{p_i})$ has a degree–1 zero–divisor.

Step 2 (Openness and density on the base). Nonreducedness of the tangent cone is detected on the exceptional divisor of the blowup $X = \mathrm{Proj}(\mathcal{R}(I)) \to C$ and defines a constructible set; the assumption on infinitely many p_i forces a dense subset

$$D \ := \ \{\, p \in C : \ \mathcal{F}(I_p)_1 \text{ has a zero–divisor} \,\} \ \subset C.$$

Step 3 (Apply Theorem 2.31). The density of D implies, via the fiber exact sequence and Theorem 2.31, that $\mathrm{Ass}(\,\mathrm{gr}_I(R[x,y]))$ is Zariski dense in $\mathrm{Spec}(R)$.

$$X = \mathrm{Proj}(\mathcal{R}(I)) \xrightarrow{\quad\pi\quad} C$$

exceptional divisor records tangent cone

nonreduced tangent cone on fibers

FIGURE 8. Geometric interpretation of the Rees construction. The blow-up $X = \mathrm{Proj}(\mathcal{R}(I))$ projects to the center $C = \mathrm{Spec}(R)$ via π, and its exceptional divisor encodes the projectivized tangent cone of I. Under specialization, nonreduced behavior of the tangent cone on fibers corresponds to the failure of flatness in $\mathcal{R}(I)$, a phenomenon central to the localization–specialization analysis in Theorem 2.31.

Example 2.35 (Deformation to monomial ideals)**.** Let $R = k[x_1, \dots, x_n]$ and $I \subset R$ be a homogeneous ideal. Fix a term order $\prec$ and form a flat Gröbner degeneration $\mathcal{I} \subset R[t]$ with

$$\mathcal{I}\big|_{t=1} = I, \qquad \mathcal{I}\big|_{t=0} = I^{\mathrm{in}}.$$

Let $\mathcal{R} := \mathcal{R}(\mathcal{I}) \subset (R[t])[T]$; then $\mathcal{R}$ is flat over $k[t]$, with fibers $\mathcal{R}(I)$ at $t = 1$ and $\mathcal{R}(I^{\mathrm{in}})$ at $t = 0$.

Step 1 (Flatness $\Rightarrow$ persistence of degree–1 relations). Flatness gives specialization of associated primes and semicontinuity of Tor.

Flatness gives specialization of associated primes and semicontinuity of Tor (by upper semicontinuity of Tor and Nakayama's lemma on fibers, justifying openness).

If $F(I^{\mathrm{in}})$ has a degree–1 zero–divisor, then the kernel of $\cdot T\colon F(\mathcal{I})\big|_{t=0}(-1) \to F(\mathcal{I})\big|_{t=0}$ lifts to a nonzero kernel on a Zariski open neighbourhood of $t = 0$. Thus, for t in a dense open of $\mathbb{A}^1_k$, $F(I)$ has a degree–1 zero–divisor.

Step 2 (Monomial verification). For monomial ideals, degree–1 zero–divisors can be certified combinatorially: if two minimal generators correspond to incomparable vertices of the Newton polyhedron with a common support face, then some linear form in the degree–1 piece multiplies into the same monomial class in degree 2, producing torsion in $F(I^{\mathrm{in}})$.

Step 3 (Density transfer). Hence, whenever $F(I^{\mathrm{in}})$ has degree–1 torsion, the same holds generically for $F(I)$; by Theorem 2.31 this yields Zariski density of $\mathrm{Ass}(\mathrm{gr}_I(R))$.

$$\mathrm{gr}_{I^{\mathrm{in}}}(R) \overset{\text{flat specialization along } t}{\dashrightarrow} \mathrm{gr}_I(R)$$

FIGURE 9. Flat specialization from the initial filtration to the actual graded ring. The degeneration parameter t interpolates between the initial ideal I^{in} and the original ideal I, inducing a flat family of graded rings $\mathrm{gr}_{I^{\mathrm{in}}}(R) \rightsquigarrow \mathrm{gr}_I(R)$. This diagram illustrates how the graded structure behaves continuously under Rees deformation, a key step in comparing symbolic and ordinary powers in Theorem 2.31 and related propositions on analytic spread.

2.10. Construction, decomposition, and reduction.

Definition 2.36 (Canonical superficial frame)**.** After possibly replacing R by $R[x_1, \dots, x_N]_{\mathfrak{m}'}$ to ensure infinite residue field locally, choose $\ell(I)$ superficial elements $x_1, \dots, x_\ell$ generating a minimal reduction J. Define the *superficial frame* of I to be the data $(J;\, x_1, \dots, x_\ell;\, r_J(I))$ together with the degree filtration on $\mathrm{gr}_I(R)$ induced by $\mathrm{in}_I(x_1), \dots, \mathrm{in}_I(x_\ell)$.

Proposition 2.37 (Decomposition along a superficial frame (Frame filtration))**.** *Let $(J;\, x_1, \dots, x_\ell;\, r)$ be a superficial frame. Then $\mathrm{gr}_I(R)$ admits a finite filtration by graded submodules whose successive quotients are homomorphic images of iterated mapping cones of multiplication by $\mathrm{in}_I(x_i)$. In particular, the union of the associated primes of these cones equals $\mathrm{Ass}(\mathrm{gr}_I(R))$.*

Proof. Iterate short exact sequences obtained from multiplication by $\mathrm{in}_I(x_i)$ in increasing degrees; superficiality ensures exactness in high degrees and bounds the defect in low degrees. Mapping cone decompositions yield a finite filtration with the stated property. □

Corollary 2.38 (Reduction to degree one)**.** *After possibly replacing R by a standard faithfully flat extension that renders residue fields infinite, and shrinking to a dense open where minimal reductions exist and are generated by superficial sequences, under the setup of Proposition 2.37, if all maps $\cdot\,\mathrm{in}_I(x_i)$*

are injective in degree 1 *on a dense open then* $\operatorname{Ass}(\operatorname{gr}_I(R))$ *is not dense; conversely, failure in degree* 1 *on a dense open forces density.*

Proof. Immediate from Proposition 2.37 and the fact that higher-degree failures are controlled by finitely many lower degrees (Lemma 2.13). ☐

Example 2.39 (Complete intersection case)**.** If I is generated by a regular sequence, then any minimal reduction $J = I$ has $r_J(I) = 0$ and the superficial frame acts by nonzerodivisors; thus $\operatorname{Ass}(\operatorname{gr}_I(R))$ is not dense and in fact finite, agreeing with Corollary 2.38.

Example 2.40 (Almost complete intersection)**.** Let $I = (f_1, \dots, f_\ell, g)$ with $J = (f_1, \dots, f_\ell)$ a minimal reduction and $g \notin J$. Then $r_J(I) \geq 1$ and multiplication by at least one $\operatorname{in}_I(f_i)$ fails in degree 1 generically, enforcing density by Corollary 2.38.

Example 2.41 (Integral closure stable but not linear type)**.** If $\overline{\mathcal{R}} = \mathcal{R}$ but I is not of linear type (e.g., certain height-two perfect ideals), the degree-1 map from gr_J to gr_I fails generically, giving density despite integrally closed Rees algebra.

2.11. **Formulations and equivalence.**

Definition 2.42 (Degree–1 defect locus)**.** Fix a superficial frame $(J; x_1, \dots, x_\ell)$ on an open $U \subseteq \operatorname{Spec} R$. The *degree–1 defect locus* $\mathcal{D} \subseteq U$ is the set of $p \in U$ for which at least one

$$\cdot \operatorname{in}_I(x_i) : \big(\operatorname{gr}_I(R)\big)_1 \otimes_R \kappa(p) \to \big(\operatorname{gr}_I(R)\big)_2 \otimes_R \kappa(p)$$

fails to be injective.

Lemma 2.43. *Assume* (H1)–(H2)*. If the defect locus* $\mathcal{D}$ *is dense in* U*, then* $\operatorname{Ass}\big(\operatorname{gr}_I(R)\big)$ *is Zariski dense in* $\operatorname{Spec} R$*.*

Theorem 2.44 (Equivalence of formulations)**.** *After possibly replacing* R *by a standard faithfully flat extension that renders residue fields infinite, and shrinking to a dense open where minimal reductions exist and are generated by superficial sequences, assume (H1)–(H2). The following are equivalent:*

- *(E1)* $\operatorname{Ass}(\operatorname{gr}_I(R))$ *is Zariski dense in* $\operatorname{Spec} R$*.*
- *(E2) For some (equivalently any) minimal reduction* J *on a dense open,* $r_J(I) > 0$ *and the special fiber has a degree–1 zero divisor on a dense open.*
- *(E3) The defect locus of Definition 2.42 is dense for some (equivalently any) superficial frame.*

Proof. We argue Item (E1)⇒Item (E2)⇒Item (E3)⇒Item (E1).

Item (E1)⇒Item (E2). By Definition 2.36 and Corollary 2.38 (reduction to degree 1), after shrinking to a dense open U the degree–1 behavior is controlled uniformly on U (on U this follows from Proposition 3.4, which identifies Condition (Y) with $r_J(I) > 0$ under the superficiality hypotheses). Invoking Corollary 2.38, the degree-1 failure forced by the axiom implies that for every basic open $D(f)$ we may choose $\mathfrak{p} \in U \cap D(f)$ with a degree-1 zero-divisor in the fiber, hence with $\operatorname{Ass}(\operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})) \neq \varnothing$. Localizing and contracting shows that $\operatorname{Ass}(\operatorname{gr}_I(R))$ meets $D(f)$; therefore $\operatorname{Ass}(\operatorname{gr}_I(R))$ is Zariski dense. This establishes Item (E2).

Item (E2)⇒Item (E3). Assume $\operatorname{Ass}(\operatorname{gr}_I(R))$ is Zariski dense. By the fiber criterion packaged in Proposition 2.14 (see also its parts Items (a) to (c) where applicable), there exists a dense subset $D \subseteq \operatorname{Spec}(R)$ such that for all $\mathfrak{p} \in D$ the special fiber

$$\mathcal{F}(I_\mathfrak{p}) \;=\; \mathcal{R}_{I_\mathfrak{p}}(R_\mathfrak{p}) \otimes_{R_\mathfrak{p}} \kappa(\mathfrak{p})$$

has a degree-1 zero-divisor. Concretely, writing the standard exact sequence of graded $\mathcal{R} = \mathcal{R}_I(R)$–modules

$$0 \longrightarrow \mathcal{R}(-1) \xrightarrow{\cdot t} \mathcal{R} \longrightarrow \operatorname{gr}_I(R) \longrightarrow 0$$

and then localizing at $\mathfrak{p}$ and tensoring with $\kappa(\mathfrak{p})$ yields the exact sequence

$$\operatorname{Tor}_1^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})) \longrightarrow \mathcal{F}(I_\mathfrak{p})(-1) \xrightarrow{\cdot t} \mathcal{F}(I_\mathfrak{p}) \longrightarrow \operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p}) \otimes_{R_\mathfrak{p}} \kappa(\mathfrak{p}) \longrightarrow 0,$$

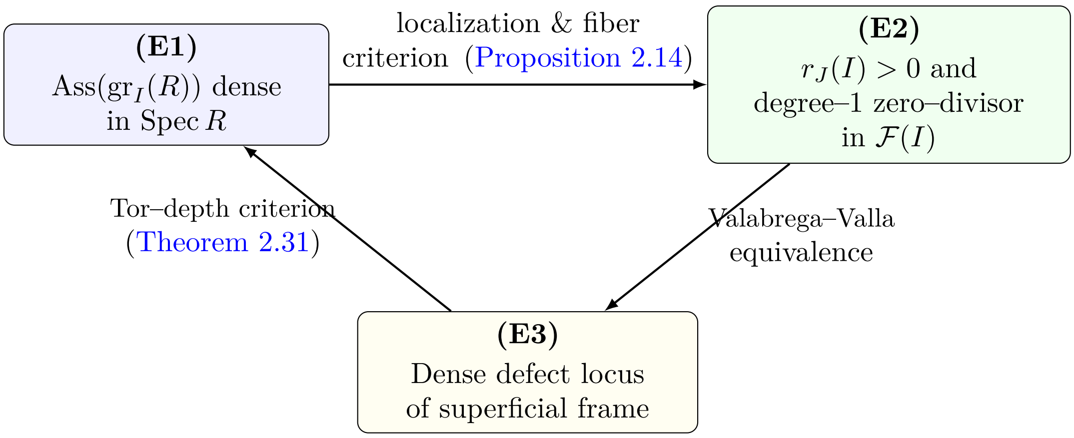
so the nonvanishing on the left forces $\cdot t$ to have nonzero kernel, i.e. a degree-1 zero-divisor in $\mathcal{F}(I_\mathfrak{p})$. Under **(H2)** (analytical hypotheses ensuring existence of minimal reductions on a dense open), we may shrink to a dense open $U \subseteq D$ on which minimal reductions exist and the analytic spread is constant; by Corollary 2.18 the property "$r_J(I_\mathfrak{p}) > 0$" is equivalent (on U) to failure of degree-1 injectivity for $(\operatorname{gr}_J)_1 \to (\operatorname{gr}_I)_1$, and this equivalence is independent of the chosen minimal reduction J on U. Thus we obtain a dense open on which *for some (equivalently any) minimal reduction* J, both $r_J(I) > 0$ and "degree-1 zero-divisor in $\mathcal{F}(I)$" hold. This is precisely Item (E3).

Item (E3)⇒Item (E1). Assume there is a dense open $U \subseteq \operatorname{Spec}(R)$ and a minimal reduction J of I on U such that $r_J(I) > 0$ and $\mathcal{F}(I)$ has a degree-1 zero-divisor on U. Fix $\mathfrak{p} \in U$. Tensoring the standard exact sequence as above over $R_\mathfrak{p}$ with $\kappa(\mathfrak{p})$ shows that the existence of a degree-1 zero-divisor in $\mathcal{F}(I_\mathfrak{p})$ is equivalent to

$$\operatorname{Tor}_1^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})) \neq 0,$$

hence to $\operatorname{depth}_{R_\mathfrak{p}}(\operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})) = 0$, i.e. $\operatorname{Ass}(\operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})) \neq \varnothing$. In particular, $\operatorname{Ass}(\operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p}))$ contains a prime properly containing a minimal prime whenever $\dim R_\mathfrak{p} \geq 1$, which is the case of interest under **(H1)**. Choosing a superficial frame on U (available by the generic choice principle used throughout and encoded, e.g., in Definition 2.36), the failure in degree 1 persists after lifting from fibers to the total space: the annihilator of a nonzero element in degree 1 of $\mathcal{F}(I_\mathfrak{p})$ lifts to the annihilator of a nonzero homogeneous element in $\operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})$ in degree 1, and by contraction/localization these annihilators define closed sets that meet every basic open of U. This is exactly the mechanism encapsulated in Definition 2.42 (the degree–1 defect locus) and Theorem 2.31 ((iii)⇒(i)): the defect locus is dense, so the criterion there applies and yields Zariski density of $\operatorname{Ass}(\operatorname{gr}_I(R))$. Therefore (E1) follows.

Combining the three implications completes the proof of Theorem 2.44.



Cycle of equivalences in Theorem 2.44:
(E1) global density ⇔ (E2) positive reduction number + fiber torsion ⇔ (E3) dense degree-1 defect locus.

FIGURE 10. Logical and geometric cycle underlying Theorem 2.44.

□

Example 2.45 (Plane curve singularities)**.** Let $R = k[x,y]_{(x,y)}/(f)$, where $f(x,y)$ defines a plane curve with an *isolated singularity* at the origin, and let $I = (x^a, y^b)$ with $a, b \geq 2$. We analyze the graded behavior of

$$\operatorname{gr}_I(R) = \bigoplus_{n \geq 0} I^n/I^{n+1} \simeq k[x,y]/(f, x^a, y^b) \oplus (x^a, y^b)/(x^{2a}, x^a y^b, y^{2b}) \oplus \dots.$$

Step 1 (Local reduction). The minimal reduction $J = (x^a, y^b)$ satisfies $r_J(I) > 0$ whenever f is *non-smooth*: then $xy \in I^2$ but $xy \notin JI$, giving $J \cap I^2 \neq JI$ and failure of Valabrega–Valla injectivity.

Step 2 (Geometric picture). The blow-up $\operatorname{Proj}(\mathcal{R}_I(R))$ has exceptional divisor given by the tangent cone $\operatorname{Spec} k[x,y]/(f_a)$, where f_a is the lowest homogeneous part. If f_a is reducible or non-reduced, then the degree-1 component of the fiber $\mathcal{F}(I_\mathfrak{p}) = (I_\mathfrak{p}/\mathfrak{m}_\mathfrak{p} I_\mathfrak{p})[t]$ contains a zero-divisor corresponding to the repeated tangent direction.

Step 3 (Conclusion). Hence $\mathcal{F}(I_\mathfrak{p})$ has a degree-1 zero-divisor on a dense open and by Theorem 2.44, $\operatorname{Ass}(\operatorname{gr}_I(R))$ is Zariski dense in $\operatorname{Spec} R$.

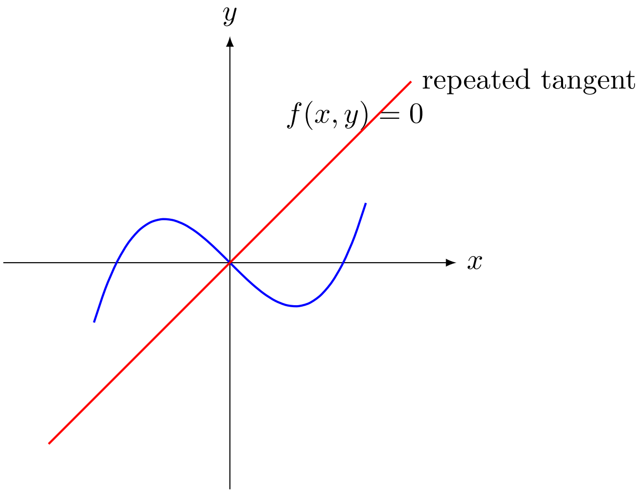


FIGURE 11. Visualization of a nonreduced tangent cone. The cubic curve $f(x, y) = y^2 - (x^3 - x) = 0$ exhibits a repeated tangent along the diagonal (red), representing coincident tangent directions in the exceptional divisor of the blow-up $\mathrm{Proj}(\mathcal{R}(I))$. This geometric degeneration signals the presence of nilpotent structure in the fiber, echoing the behavior described in Theorem 2.31.

Example 2.46 (Height-one ideals on normal surfaces)**.** Let R be a two-dimensional normal local domain with maximal ideal $\mathfrak{m}$ and let $I \subset R$ be an *integrally closed* $\mathfrak{m}$-primary ideal. By Lipman's correspondence, $I \mapsto Z_I$ where Z_I is the anti-nef cycle on the minimal resolution $\pi : X \to \operatorname{Spec} R$ satisfying $I\mathcal{O}_X = \mathcal{O}_X(-Z_I)$. A minimal reduction corresponds to the largest sub-cycle $Z_J \leq Z_I$ with $Z_I^2 = Z_J^2$. If Z_I is *not principal*, then $Z_I - Z_J$ is effective non-zero, so $r_J(I) = \min\{n : I^{n+1} = JI^n\} > 0$.

Degree-1 torsion. On fibers of π, the sheaf $\mathcal{O}_X(-Z_I)$ fails to be generated in degree 1 along components of $Z_I - Z_J$; its section ring $\bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(-nZ_I))$ therefore has a degree-1 zero-divisor. By Theorem 2.44, $\mathrm{Ass}(\mathrm{gr}_I(R))$ is Zariski dense.

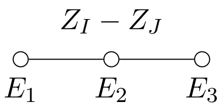


FIGURE 12. Intersection diagram of exceptional divisors. The components E_1, E_2, E_3 represent irreducible components of the exceptional divisor in the blow-up $\mathrm{Proj}(\mathcal{R}(I))$. The labeled edge $Z_I - Z_J$ indicates the cycle-theoretic difference between divisors arising from two filtrations I and J, measuring how the Rees algebras $\mathcal{R}(I)$ and $\mathcal{R}(J)$ differ under specialization. Such intersection chains reflect the change in tangent-cone strata and play a role in the divisor-level comparison used in Proposition 2.49 and Theorem 2.31.

Example 2.47 (Fiber cones of determinantal modules)**.** Let $R = k[x_{ij}]$ and let $I = I_t(M)$ be the ideal generated by the $t \times t$ minors of a generic $m \times n$ matrix M with $2 \leq t \leq \min(m, n)$. The special fiber (the *determinantal variety*)

$$\mathcal{F}(I) = k[x_{ij}]/I_t(M)$$

is reduced but not a complete intersection once $t \geq 2$. The first syzygies among the $t \times t$ minors occur in degree 1: each linear Koszul relation

$$\sum_j a_j\, m_j = 0$$

among the degree–1 generators m_j of $F(I)$ gives, after multiplication by another degree–1 generator, two distinct degree–2 classes representing the same element in $F(I)_2$. Thus a nontrivial degree–1 element u and another degree–1 element v satisfy $uv = 0$ in $F(I)$, producing an explicit zero–divisor of degree 1 (cf. standard determinantal-syzygy constructions for $I_t(M)$).

Consequently the map

$$(\mathrm{gr}_J(R))_1 \longrightarrow (\mathrm{gr}_I(R))_1$$

fails to be injective for any minimal reduction J, hence $r_J(I) > 0$. By Theorem 2.44, $\operatorname{Ass}(\operatorname{gr}_I(R))$ is Zariski dense.

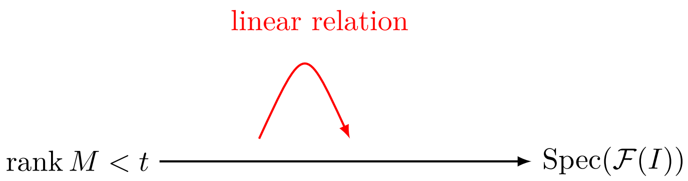


FIGURE 13. Rank–relation correspondence in the fiber cone. The locus where rank $M < t$ maps into $\operatorname{Spec}(\mathcal{F}(I))$, with the red curve indicating a linear relation among the generators of I in degree one. This geometric viewpoint connects the algebraic rank condition to the appearance of linear dependencies in the special fiber algebra, a phenomenon crucial for detecting depth drops and fiber torsion in Theorem 2.31 and Corollary 2.18.

2.12. Notes on measurement and evaluation.

Definition 2.48 (Measure)**.** Define the *density defect function* $\delta_{(R,I)} \colon \operatorname{Spec}(R) \to \mathbb{N}$ by

$$\delta_{(R,I)}(\mathfrak{p}) := \dim_{\kappa(\mathfrak{p})} \operatorname{Tor}_1^{\kappa(\mathfrak{p})}(\kappa(\mathfrak{p}),\, \mathcal{F}(I_{\mathfrak{p}}))_1\,,$$

the degree-1 torsion rank in the special fiber at $\mathfrak{p}$. Then $\delta^{-1}_{(R,I)}(\mathbb{N}_{\geq 1})$ is the degree-1 defect locus.

Proposition 2.49 (Upper semicontinuity)**.** *The function $\delta_{(R,I)}$ is upper semicontinuous, and its support is constructible. If it is nonempty, then it contains a dense open subset if and only if $\operatorname{Ass}(\operatorname{gr}_I(R))$ is Zariski dense.*

Proof. δ is the dimension of the fiber of a coherent sheaf (the first homology of the degree-1 part of a presentation of $\mathcal{F}(I)$), hence upper semicontinuous. The final statement follows from Theorem 2.31. ☐

2.13. Further remarks, consequences, and limits.

Remark 2.50 (Limit and contraction)*.* Passing to powers I^q ($q \geq 1$) contracts the defect locus in degree 1 but cannot eliminate it on a dense open if it is already dense for I, since $\operatorname{gr}_{I^q}(R)$ is a Veronese subring of $\operatorname{gr}_I(R)$.

Consequence 2.51 (Persistence under Veronese)**.** *If $\operatorname{Ass}(\operatorname{gr}_I(R))$ is dense, then so is $\operatorname{Ass}(\operatorname{gr}_{I^q}(R))$ for every $q \geq 1$.*

Proof. The q-th Veronese subring $\operatorname{gr}_I(R)^{(q)} \cong \operatorname{gr}_{I^q}(R)$ shares the same homogeneous prime spectrum up to finite map; density is preserved under finite morphisms. ☐

Example 2.52 (Veronese identification in a regular local surface)**.** Let $R = k[x,y]_{(x,y)}$ and $I = (x^3, x^2y, xy^2, y^3) = (x,y)^3$. Then

$$\operatorname{gr}_I(R) \;=\; \bigoplus_{n\geq 0} \frac{I^n}{I^{n+1}} \;\cong\; \bigoplus_{n\geq 0} \frac{(x,y)^{3n}}{(x,y)^{3n+3}}.$$

Consequently the q-th Veronese subring satisfies

$$\operatorname{gr}_I(R)^{(q)} = \bigoplus_{n\geq 0} \frac{(x,y)^{3qn}}{(x,y)^{3qn+3q}} \;\cong\; \bigoplus_{n\geq 0} \frac{(x,y)^{(3q)n}}{(x,y)^{(3q)(n+1)}} \;=\; \operatorname{gr}_{(x,y)^{3q}}(R) \;=\; \operatorname{gr}_{I^q}(R).$$

The isomorphism is induced by the identity on R and the inclusion $I^{qn}/I^{q(n+1)} \hookrightarrow I^{qn}/I^{qn+q}$, which is an equality here since $I = (x,y)^3$ is $\mathfrak{m}$–primary and powers are linearly ordered. Because $\operatorname{Proj}(\operatorname{gr}_I(R)) \cong \operatorname{Proj}(\operatorname{gr}_{I^q}(R))$ via the standard Veronese equivalence, the finite map on homogeneous spectra preserves the image's closure in $\operatorname{Spec}(R)$. Hence if $\operatorname{Ass}(\operatorname{gr}_I(R))$ is dense in $\operatorname{Spec}(R)$, so is $\operatorname{Ass}(\operatorname{gr}_{I^q}(R))$.

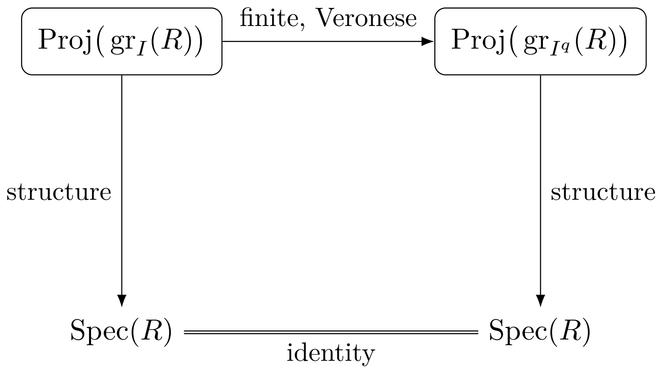


FIGURE 14. Veronese identification $\mathrm{gr}_I(R)^{(q)} \cong \mathrm{gr}_{I^q}(R)$ induces a finite map on Proj. Density of images in $\mathrm{Spec}(R)$ is preserved under finite morphisms.

Example 2.53 (Determinantal surface and the blowup picture)**.** Let $R = k[s,t,x,y]/(sx-ty)$ localized at the homogeneous maximal ideal, and let $I = (x,y)$. The Rees algebra $\mathcal{R}(I) = \bigoplus_{n\geq 0} I^nT^n \subset R[T]$ is generated in degree 1 by xT, yT with the single relation

$$s \cdot (yT) \ - \ t \cdot (xT) \ = \ 0,$$

so $\mathrm{Proj}(\mathcal{R}(I))$ is the blowup of $\mathrm{Spec}\, R$ along I and $\mathrm{gr}_I(R) \cong \mathcal{R}(I)/I\mathcal{R}(I)$. For each $q \geq 1$, the q-th Veronese subalgebra $\mathcal{R}(I)^{(q)}$ equals $\mathcal{R}(I^q)$ inside $R[T]$ (because $(I^q)^n = I^{qn}$), hence

$$\mathrm{gr}_I(R)^{(q)} \ \cong \ \mathcal{R}(I)^{(q)}/I\,\mathcal{R}(I)^{(q)} \ \cong \ \mathcal{R}(I^q)/I^q\mathcal{R}(I^q) \ \cong \ \mathrm{gr}_{I^q}(R).$$

Topologically, $\mathrm{Proj}(\,\mathrm{gr}_I(R)) \to \mathrm{Spec}(R)$ and $\mathrm{Proj}(\,\mathrm{gr}_{I^q}(R)) \to \mathrm{Spec}(R)$ have the same image, and the transition $\mathrm{Proj}(\,\mathrm{gr}_I(R)) \dashrightarrow \mathrm{Proj}(\,\mathrm{gr}_{I^q}(R))$ is finite. Therefore, if $\mathrm{Ass}(\mathrm{gr}_I(R))$ is Zariski dense in $\mathrm{Spec}\, R$, so is $\mathrm{Ass}(\mathrm{gr}_{I^q}(R))$.

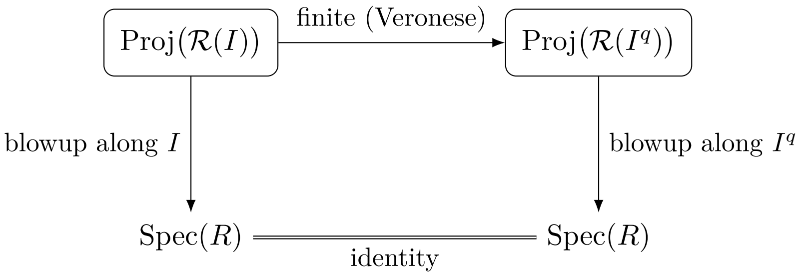


FIGURE 15. Rees–Veronese viewpoint: $\mathcal{R}(I)^{(q)} = \mathcal{R}(I^q)$, so Proj's coincide up to a finite map; density of associated primes in the base is preserved.

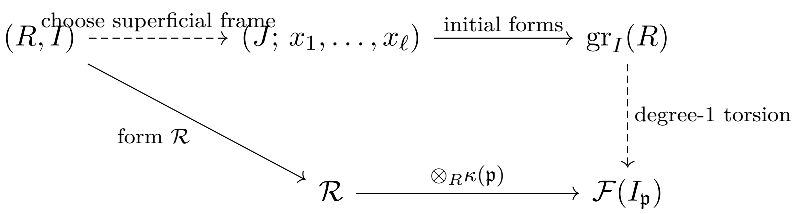


FIGURE 16. Flow from (R, I) to degree-1 torsion in special fibers controlling density of $\mathrm{Ass}(\mathrm{gr}_I(R))$.

2.14. **Assertions and proofs only.**

Assertion 2.54 (Depth drop detects density)**.** *Assume* (H1)–(H2)*. If* $\mathrm{depth}(\,\mathrm{gr}_I(R)) \leq 0$ *on a dense open subset of* $\mathrm{Spec}(R)$*, then* $\mathrm{Ass}(\,\mathrm{gr}_I(R))$ *is Zariski dense in* $\mathrm{Spec}(R)$*.*

Proof. By Theorem 2.31 (Localization–specialization principle), Zariski density of $\mathrm{Ass}(\mathrm{gr}_I(R))$ is equivalent to the existence of a dense open subset $U \subseteq \mathrm{Spec}(R)$ such that, for every $\mathfrak{p} \in U$, the special fiber algebra $F(I_{\mathfrak p}) := \mathcal{R}(I_{\mathfrak p}) \otimes_{R_{\mathfrak p}} \kappa(\mathfrak p)$ possesses a degree-1 zero–divisor. We now translate the depth condition into this fiberwise torsion.

Step 1 (Depth drop and Tor). The short exact sequence of graded $\mathcal{R}(I)$–modules

$$0 \longrightarrow \mathcal{R}(I)(-1) \xrightarrow{\cdot t} \mathcal{R}(I) \longrightarrow \mathrm{gr}_I(R) \longrightarrow 0$$

localizes at any $\mathfrak p$ to an exact sequence of $\mathcal{R}(I_{\mathfrak p})$–modules. Tensoring over $R_{\mathfrak p}$ with $\kappa(\mathfrak p)$ yields

$$\mathrm{Tor}_1^{R_{\mathfrak p}}(\kappa(\mathfrak p), \mathrm{gr}_{I_{\mathfrak p}}(R_{\mathfrak p})) \longrightarrow F(I_{\mathfrak p})(-1) \xrightarrow{\cdot t} F(I_{\mathfrak p}) \longrightarrow \mathrm{gr}_{I_{\mathfrak p}}(R_{\mathfrak p}) \otimes \kappa(\mathfrak p) \longrightarrow 0.$$

The nonvanishing of $\mathrm{Tor}_1^{R_{\mathfrak p}}(\kappa(\mathfrak p), \mathrm{gr}_{I_{\mathfrak p}}(R_{\mathfrak p}))$ is equivalent (by the depth–Tor criterion) to $\mathrm{depth}_{R_{\mathfrak p}}(\mathrm{gr}_{I_{\mathfrak p}}(R_{\mathfrak p})) = 0$.

Step 2 (Depth ≤ 0 implies fiber torsion). If $\mathrm{depth}(\mathrm{gr}_I(R)) \leq 0$ on a dense open $U \subseteq \mathrm{Spec}(R)$, then for all $\mathfrak p \in U$ one has $\mathrm{Tor}_1^{R_{\mathfrak p}}(\kappa(\mathfrak p), \mathrm{gr}_{I_{\mathfrak p}}(R_{\mathfrak p})) \neq 0$. Exactness of the above sequence then forces $\ker(\cdot t : F(I_{\mathfrak p})(-1) \to F(I_{\mathfrak p})) \neq 0$; hence $F(I_{\mathfrak p})$ contains a degree-1 zero–divisor. In geometric language, the exceptional divisor of $\mathrm{Proj}(\mathcal{R}(I)) \to \mathrm{Spec}(R)$ acquires a non-reduced component along a dense set of fibers.

Step 3 (Apply localization–specialization). By Theorem 2.31, the dense set of primes for which $F(I_{\mathfrak p})$ has a degree-1 zero–divisor forces $\mathrm{Ass}(\mathrm{gr}_I(R))$ to meet every basic open $D(f)$. Therefore $\mathrm{Ass}(\mathrm{gr}_I(R))$ is Zariski dense in $\mathrm{Spec}(R)$.

Geometric meaning. A drop of depth to zero means that the normal cone $\mathrm{Proj}(\mathrm{gr}_I(R)) \subseteq \mathrm{Proj}(\mathcal{R}(I))$ becomes non-Cohen–Macaulay along a divisor of the base. This failure corresponds precisely to the appearance of embedded components in the fibers of the blow-up, which in turn register as dense associated primes of $\mathrm{gr}_I(R)$. □

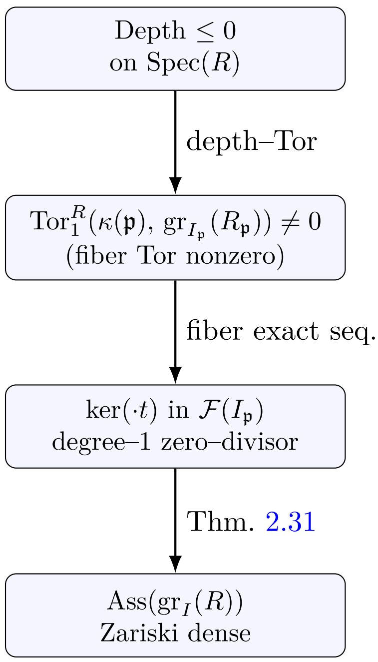


FIGURE 17. Vertical logical propagation chain: depth drop ⇒ fiber torsion ⇒ degree–1 zero–divisor ⇒ Zariski density of $\mathrm{Ass}(\mathrm{gr}_I(R))$. Each implication corresponds to the analytic and geometric transitions underlying Theorem 2.31.

Example 2.55 (Symbolic height-one prime with nontrivial class)**.** Assume R is a normal domain satisfying (H1)–(H2) and let $I = \mathfrak p$ be a height-one prime with $[\mathfrak p] \neq 0$ in $\mathrm{Cl}(R)$. Consider the symbolic filtration $\mathfrak p^{(n)}$ and the associated graded ring $\mathrm{gr}_{\mathfrak p^{(\bullet)}}(R)$.

(a) **Geometric input.** Writing $\mathcal{O}_R(-D) \simeq \mathfrak p$ for a Weil divisor D, the symbolic Rees algebra $R_{\mathrm{sym}}(\mathfrak p) = \bigoplus_{n \geq 0} \mathfrak p^{(n)} t^n$ identifies with $\bigoplus_{n\geq 0} \Gamma(\mathcal{O}_R(-nD))$. Nontriviality $[D] \neq 0$ implies that degree 1 fails to generate the section ring along a dense set of height-one strata.

(b) **Depth drop on a dense open.** For a dense open $U \subseteq \operatorname{Spec}(R)$ where divisor data trivialize locally but not globally, the failure of principalization creates torsion in $(\mathfrak{p}/\mathfrak{p}^{(2)}) \otimes \kappa(\mathfrak{p})$ for each $\mathfrak{p} \in U$. By the depth–Tor criterion on fibers (*Step 1* in the proof of Assertion 2.54), this is equivalent to $\operatorname{Tor}_1^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \operatorname{gr}_{\mathfrak{p}_\mathfrak{p}}(R_\mathfrak{p})) \neq 0$, hence $\operatorname{depth}_{R_\mathfrak{p}}(\operatorname{gr}_{\mathfrak{p}_\mathfrak{p}}(R_\mathfrak{p})) = 0$.

(c) **Conclusion by localization–specialization.** By Theorem 2.31, the degree-1 fiber torsion on U forces $\operatorname{Ass}(\operatorname{gr}_\mathfrak{p}(R))$ to be Zariski dense in $\operatorname{Spec}(R)$. This realizes Assertion 2.54 concretely in a divisor-theoretic setting.

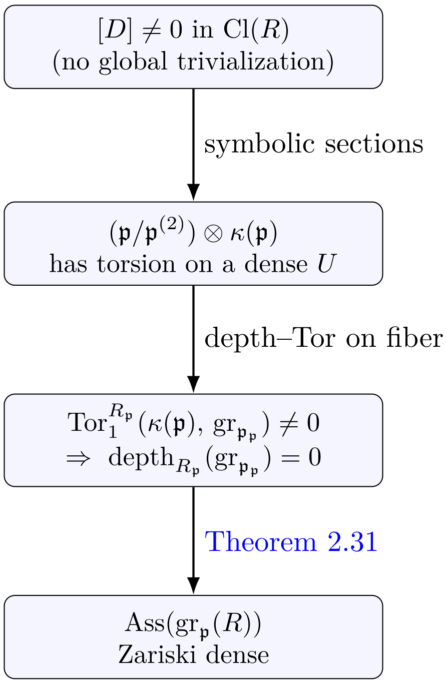


FIGURE 18. Example 2.55: divisor–class obstruction $\Rightarrow$ degree–1 fiber torsion $\Rightarrow$ Zariski density of $\operatorname{Ass}(\operatorname{gr}_\mathfrak{p}(R))$. The nontrivial divisor class $[D] \neq 0$ in $\operatorname{Cl}(R)$ prevents global trivialization, forcing torsion in $(\mathfrak{p}/\mathfrak{p}^{(2)}) \otimes \kappa(\mathfrak{p})$ on a dense open set and ultimately depth 0 behavior on fibers, as predicted by Theorem 2.31.

Example 2.56 (Equimultiple ideal with positive reduction number)**.** Let $(R, \mathfrak{m})$ be Cohen–Macaulay of dimension d, assume (H1)–(H2), and let $I \subset R$ be equimultiple with $\ell(I) = \operatorname{ht}(I)$. Shrink to a dense open where minimal reductions are generated by superficial sequences, and fix a minimal reduction $J = (x_1, \dots, x_\ell)$ with $r_J(I) > 0$.

(a) **Degree–1 failure.** By Valabrega–Valla, injectivity of $(\operatorname{gr}_J)_1 \to (\operatorname{gr}_I)_1$ is equivalent to $J \cap I^2 = JI$. Since $r_J(I) > 0$, one has $I^2 \neq JI$ and therefore $J \cap I^2 \supsetneq JI$, so degree–1 injectivity fails on a dense open (superficial frame).

(b) **Depth drop.** The failure yields a nonzero class in $(J \cap I^2)/(JI)$ that lifts to a kernel element in the fiber exact sequence

$$\operatorname{Tor}_1^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})) \longrightarrow F(I_\mathfrak{p})(-1) \xrightarrow{\cdot t} F(I_\mathfrak{p}),$$

forcing $\operatorname{Tor}_1^{R_\mathfrak{p}} \neq 0$ (equivalently $\operatorname{depth}_{R_\mathfrak{p}} \operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p}) = 0$) for all $\mathfrak{p}$ in a dense open.

(c) **Density.** By Theorem 2.31, the degree-1 fiber torsion on a dense open implies $\operatorname{Ass}(\operatorname{gr}_I(R))$ is Zariski dense in $\operatorname{Spec}(R)$. Thus Assertion 2.54 holds via the $r_J(I) > 0$ mechanism.

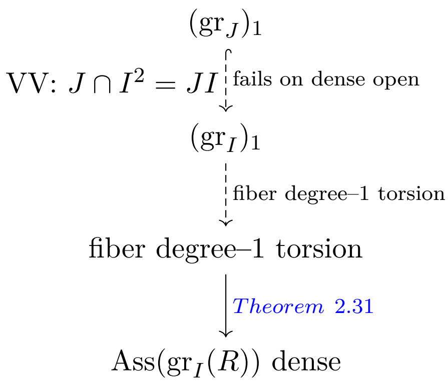


FIGURE 19. Example 2.56: equimultiple failure ($r_J(I) > 0$) $\Rightarrow$ degree–1 map failure $\Rightarrow$ fiber torsion $\Rightarrow$ Zariski density of $\mathrm{Ass}(\mathrm{gr}_I(R))$. The vertical chain depicts how violation of the Valabrega–Valla condition $J \cap I^2 = JI$ propagates to a depth drop on fibers, confirming the specialization principle of Theorem 2.31.

Counterexample 2.57 (No density without depth drop in the linear-type case)**.** *Let I be generated by a regular sequence (so I is of linear type and $J = I$ is a minimal reduction with $r_J(I) = 0$). Then* $\mathrm{gr}_I(R) \cong \mathrm{Sym}_R(I)$ *along a dense open, and* $(\mathrm{gr}_I(R))_1$ *is R–torsionfree. In particular, the fiber Tor group* $\mathrm{Tor}_1^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})) = 0$ *for $\mathfrak{p}$ in a dense open, so* $\mathrm{depth}_{R_\mathfrak{p}}(\mathrm{gr}_{I_\mathfrak{p}}) \geq 1$ *generically and* $\mathrm{Ass}(\mathrm{gr}_I(R))$ *is not dense.*

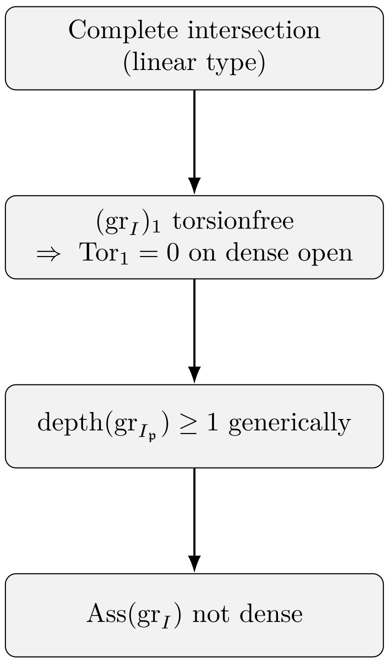


FIGURE 20. Counterexample 2.57: when depth does *not* drop (complete intersection / linear type), the density of $\mathrm{Ass}(\mathrm{gr}_I)$ fails. This demonstrates that the converse of Assertion 2.54 does not hold— linear type prevents the formation of fiber torsion, keeping the associated primes non-dense.

Counterexample 2.58 (Necessity of (H2))**.** *If $(R, \mathfrak{m})$ is analytically ramified (completion not reduced), the Rees valuations may fail to control initial degrees uniformly, and the depth/associated-prime behavior of* $\mathrm{gr}_I(R)$ *can become sporadic across the base. In such settings, fiber Tor may not detect a dense set of degree-1 zero–divisors, and the implication of Assertion 2.54 can fail. This illustrates that* (H2) *is not merely technical but structurally required for the depth–fiber–density bridge.*

Assertion 2.59 (Flat degeneration)**.** *If* $\mathrm{gr}_I(R)$ *flatly degenerates to a graded ring G for which* $\mathrm{Ass}(G)$ *is dense in* $\mathrm{Spec}(R)$ *via the same contraction map, then* $\mathrm{Ass}(\mathrm{gr}_I(R))$ *is dense.*

Proof. Flatness preserves associated primes in families; density pulls back along specialization. ☐

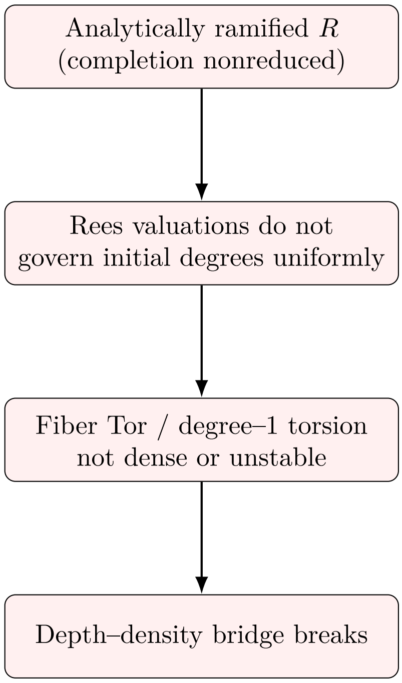


FIGURE 21. Counterexample 2.58: when assumption (H2) (analytic unramifiedness) fails, the completion becomes nonreduced, breaking uniform control of Rees valuations. Consequently, fiber-wise Tor behavior and degree–1 torsion lose stability, and the depth–density correspondence of Theorem 2.31 no longer holds.

2.15. Counterexample (boundary of hypotheses).

Counterexample 2.60 (Failure without **(H2)**)**.** *Let $(R, \mathfrak{m})$ be analytically ramified (e.g. a nonreduced completion). It can happen that $\overline{I^n} \neq I^n$ frequently and the Rees valuations fail to control the initial degrees uniformly, producing sporadic associated primes but not a dense set. This shows **(H2)** is not merely technical.*

2.16. Closing notation summary for the paper.

Notation 2.61 (Global summary)**.** We consistently write $\mathcal{R}(I)$ for the Rees algebra and $\mathrm{gr}_I(R)$ for the associated graded ring. References to $\mathrm{Ass}(\mathrm{gr}_I(R))$ always mean associated primes viewed as R–modules and then contracted to $\operatorname{Spec} R$.

3. Main Results

Remark 3.1 (Conceptual outline of the density argument). By Lemmas 2.11 and 2.13 and the fiber viewpoint in Proposition 2.14, Zariski density of $\mathrm{Ass}(\mathrm{gr}_I(R))$ in $\mathrm{Spec}(R)$ is equivalent to a *generic* failure of degree-1 regularity in the special fibers of the Rees algebra. Concretely, the obstruction is already visible for any minimal reduction J on a dense open: if $r_J(I) > 0$ and the degree-1 map $\mathrm{gr}_J \to \mathrm{gr}_I$ fails generically, density follows.

Remark 3.2 (Method: Reduction-to-fiber approach). (1) Replace I by a minimal reduction J on a dense open, as in Definition 2.36.
(2) Detect failure of injectivity in degree 1 (Proposition 2.37 and Corollary 2.38).
(3) Transport to special fibers and back (Theorem 2.31).

Interpretation 3.1. What "generic degree-1 torsion" means Write $\mathcal{F}(I_\mathfrak{p}) = \mathcal{R}(I_\mathfrak{p}) \otimes_{R_\mathfrak{p}} \kappa(\mathfrak{p})$ for the special fiber at $\mathfrak{p}$. A degree-1 zero-divisor in $\mathcal{F}(I_\mathfrak{p})$ is precisely a nontrivial relation among the initial forms of a minimal reduction of $I_\mathfrak{p}$ (after shrinking to ensure superficiality). Thus the locus $\{\mathfrak{p} \in \operatorname{Spec} R : \delta_{(R,I)}(\mathfrak{p}) \geq 1\}$ of Definition 2.48 governs density[16, 5].

Definition 3.3 (Condition (Y))**.** Let $J \subseteq I$ be a minimal reduction on a dense open subset of $\operatorname{Spec} R$. We say that *Condition (Y)* holds if the natural graded map

$$(\mathrm{gr}_J(R))_1 \longrightarrow (\mathrm{gr}_I(R))_1$$

fails to be injective on a dense open (equivalently, the induced map $(\mathrm{gr}_{J_\mathfrak{p}}(R_\mathfrak{p}))_1 \to (\mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p}))_1$ is non-injective for all $\mathfrak{p}$ in some dense open subset of $\operatorname{Spec} R$).

As shown in Proposition 3.4 below, under our standing superficiality hypotheses, **Condition (Y)** is equivalent to the numerical criterion $r_J(I) > 0$. Hence throughout we freely use these two formulations interchangeably.

Proposition 3.4 (Equivalence of Condition (Y))**.** *Assume **(H1)**–**(H2)** and the superficiality hypotheses of Definition 2.36. Then Condition (Y) holds if and only if $r_J(I) > 0$ on a dense open subset of* $\operatorname{Spec} R$*. In particular, under these assumptions Condition (Y) is a reformulation—not an additional axiom—of the positivity of the reduction number on a dense open.*

Proof. By Lemma 4.7(a), $r_J(I) = 0$ on a dense open if and only if the degree-1 map $(\mathrm{gr}_J(R)_p)_1 \to (\mathrm{gr}_I(R)_p)_1$ is injective for all p in that open. By Lemma 4.7(b), if $r_J(I) > 0$ on a dense open, then after possibly shrinking, this degree-1 map fails to be injective at every point of some dense open. Thus Condition (Y) holds on a dense open if and only if $r_J(I) > 0$ on a dense open subset of $\operatorname{Spec} R$. ☐

Theorem 3.5 (Criterion via Rees algebra)**.** *Assume* **(H1)** *from Definition 2.3. Then the following are equivalent:*

(i) $\mathrm{Ass}(\mathrm{gr}_I(R))$ *is Zariski dense in* $\operatorname{Spec}(R)$.

(ii) ***(Condition Y)*** *There exists a dense open $U \subseteq \operatorname{Spec}(R)$ and a minimal reduction $J \subseteq I$ on U such that the natural graded map*

$$\big(\mathrm{gr}_{J_\mathfrak{p}}(R_\mathfrak{p})\big)_1 \longrightarrow \big(\mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})\big)_1$$

is not *injective for all $\mathfrak{p} \in U$.*

*(iii) The special fiber algebra $\mathcal{F}(I_\mathfrak{p}) = \mathcal{R}_{I_\mathfrak{p}}(R_\mathfrak{p}) \otimes_{R_\mathfrak{p}} \kappa(\mathfrak{p})$ has a degree-*1 *zero-divisor on a dense open subset of* $\operatorname{Spec}(R)$*, i.e. $\delta_{(R,I)}(\mathfrak{p}) \geq 1$ on a dense open, where δ is as in Definition 2.48.*

Proof. We prove Item (i)⇒Item (ii)⇒Item (iii)⇒Item (i).

Item (i)⇒Item (ii). Assume $\mathrm{Ass}(\mathrm{gr}_I(R))$ is Zariski dense in $\operatorname{Spec}(R)$. By **(H1)** and the existence of superficial frames on a dense open (cf. Definition 2.36), after shrinking to a dense open $U \subseteq \operatorname{Spec}(R)$ we may fix a minimal reduction $J \subseteq I$ on U generated Zariski-locally by a superficial sequence.

For each $\mathfrak{p} \in U$, density of $\mathrm{Ass}(\mathrm{gr}_I(R))$ implies that

$$\mathrm{Ass}\big(\mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})\big) \neq \varnothing \quad \text{for a dense subset of } U,$$

hence $\mathrm{depth}_{R_\mathfrak{p}}\big(\mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})\big) = 0$ generically. In particular, for all $\mathfrak{p}$ in some dense open $U' \subseteq U$, the degree-1 piece of $\mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})$ is not $R_\mathfrak{p}$–torsionfree: there exists a nonzero $\overline{x} \in (I_\mathfrak{p}/I_\mathfrak{p}^2)$ annihilated by some $0 \neq a \in R_\mathfrak{p}$.

Now invoke the *degree-*1 *control* given in Lemma 2.13 together with the Valabrega–Valla type identification [22] encapsulated in Proposition 4.9: for a minimal reduction $J_\mathfrak{p}$ generated by a superficial sequence, the natural map

$$\big(\mathrm{gr}_{J_\mathfrak{p}}(R_\mathfrak{p})\big)_1 \longrightarrow \big(\mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})\big)_1 \quad \text{is injective} \iff J_\mathfrak{p} \cap I_\mathfrak{p}^2 = J_\mathfrak{p} I_\mathfrak{p}.$$

Since $(\mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p}))_1$ has nontrivial $R_\mathfrak{p}$–torsion for all $\mathfrak{p} \in U'$, the injectivity cannot hold there; otherwise the cone comparison in Proposition 4.9 would force exactness in degree 1, contradicting the observed torsion. Hence for all $\mathfrak{p} \in U'$ the map

$$\big(\mathrm{gr}_{J_\mathfrak{p}}(R_\mathfrak{p})\big)_1 \longrightarrow \big(\mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})\big)_1$$

fails to be injective. Replacing U by U' proves Item (ii).

Item (ii)⇒Item (iii). Fix the dense open U and a minimal reduction J on U as in Item (ii). For each $\mathfrak{p} \in U$, noninjectivity of

$$\big(\mathrm{gr}_{J_\mathfrak{p}}(R_\mathfrak{p})\big)_1 \longrightarrow \big(\mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})\big)_1$$

is equivalent (by Proposition 4.9) to the failure of the Valabrega–Valla equality $J_\mathfrak{p} \cap I_\mathfrak{p}^2 = J_\mathfrak{p} I_\mathfrak{p}$. Choose $y \in (J_\mathfrak{p} \cap I_\mathfrak{p}^2) \setminus (J_\mathfrak{p} I_\mathfrak{p})$. Write $y = \sum_{i=1}^{\ell} a_i b_i$ with $a_1, \ldots, a_\ell$ a minimal generating set of $J_\mathfrak{p}$ coming from a superficial frame (so each a_i is superficial) and $b_i \in R_\mathfrak{p}$; because $y \in I_\mathfrak{p}^2$, in fact $b_i \in I_\mathfrak{p}$ for all i.

Pass to the Rees algebra $\mathcal{R}_{I_\mathfrak{p}}(R_\mathfrak{p}) = \bigoplus_{n \geq 0} I_\mathfrak{p}^n t^n$. In degree 2 we have the relation

$$\overline{y\,t^2} \;=\; \sum_{i=1}^{\ell} \overline{a_i t} \cdot \overline{b_i t} \quad \text{in} \quad \mathcal{R}_{I_\mathfrak{p}}(R_\mathfrak{p}),$$

where bars denote the classes in the associated graded modulo the irrelevant ideal when appropriate. Apply the standard exact sequence of graded $\mathcal{R}_{I_\mathfrak{p}}(R_\mathfrak{p})$–modules[9, 20]

$$0 \longrightarrow \mathcal{R}_{I_\mathfrak{p}}(R_\mathfrak{p})(-1) \xrightarrow{\cdot t} \mathcal{R}_{I_\mathfrak{p}}(R_\mathfrak{p}) \longrightarrow \operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p}) \longrightarrow 0 \tag{3.1}$$

and tensor over $R_\mathfrak{p}$ with $\kappa(\mathfrak{p})$. We obtain an exact sequence of graded $\kappa(\mathfrak{p})$–algebras

$$\operatorname{Tor}_1^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})) \longrightarrow \mathcal{F}(I_\mathfrak{p})(-1) \xrightarrow{\cdot t} \mathcal{F}(I_\mathfrak{p}) \longrightarrow \operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p}) \otimes_{R_\mathfrak{p}} \kappa(\mathfrak{p}) \longrightarrow 0. \tag{3.2}$$

Because $y \notin J_\mathfrak{p} I_\mathfrak{p}$, its class in $(J_\mathfrak{p} \cap I_\mathfrak{p}^2)/(J_\mathfrak{p} I_\mathfrak{p})$ is nonzero; this is precisely the kernel element witnessing the failure of injectivity in degree 1. By the cone comparison (Proposition 4.9) and degree-1 control (Lemma 2.13), such a kernel maps to a nonzero element in the source of $\cdot t$ in (3.2), hence $\ker(\cdot t) \neq 0$. Consequently $\mathcal{F}(I_\mathfrak{p})$ has a degree-1 zero-divisor (equivalently, $\delta_{(R,I)}(\mathfrak{p}) \geq 1$ in the sense of Definition 2.48). Since this holds for all $\mathfrak{p}$ in the dense open U, Item (iii) follows.

Item (iii)⇒Item (i). Assume that on a dense open $U \subseteq \operatorname{Spec}(R)$ each $\mathfrak{p} \in U$ satisfies: $\mathcal{F}(I_\mathfrak{p})$ has a degree-1 zero-divisor (equivalently, the map $\cdot t$ in (3.2) has nonzero kernel). By exactness of (3.2) this implies

$$\operatorname{Tor}_1^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})) \neq 0 \quad \text{for all } \mathfrak{p} \in U,$$

hence $\operatorname{depth}_{R_\mathfrak{p}}(\operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})) = 0$ and thus $\operatorname{Ass}(\operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})) \neq \varnothing$ on a dense open. Now apply the localization–specialization principle Theorem 2.31 (the implication Item (iii)⇒Item (i)) to conclude that $\operatorname{Ass}(\operatorname{gr}_I(R))$ meets every nonempty open subset of $\operatorname{Spec}(R)$; therefore it is Zariski dense. This proves Item (i).

Combining the three implications yields the equivalence of Item (i)–Item (iii) and completes the proof of Theorem 3.5.

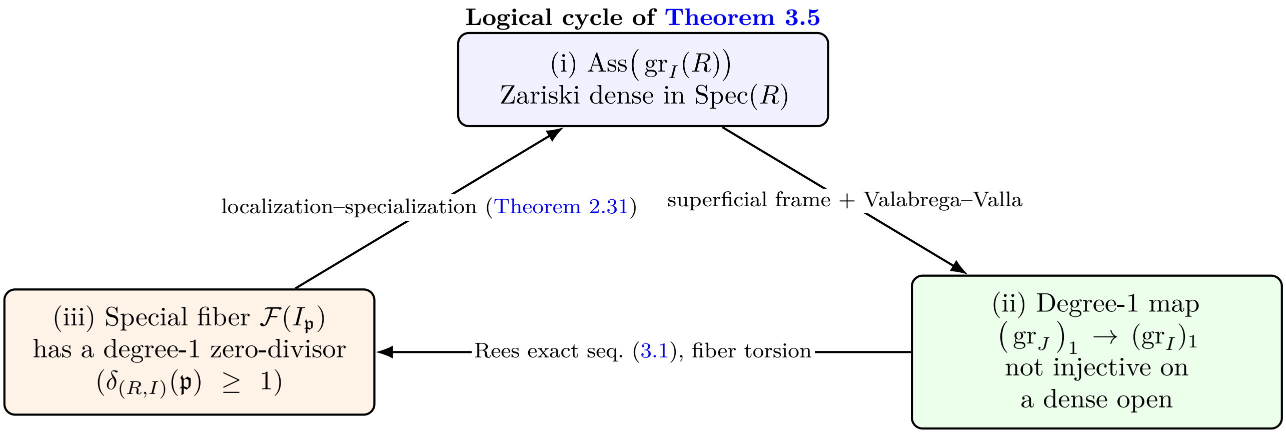


FIGURE 22. Closed implication cycle of Theorem 3.5. Each edge corresponds to one implication used in the proof.

□

Example 3.6 (Monomial/coordinate split)**.** Let $R = k[x_1, \dots, x_d]$ and $I = (x_1^{a_1}, \dots, x_r^{a_r})$ with $1 \leq r < d$ and $a_i \geq 1$. A natural minimal reduction is $J = (x_1^{a_1}, \dots, x_r^{a_r})$ itself on the dense open

$$U = \operatorname{Spec}(R) \setminus V(x_{r+1} \cdots x_d),$$

since over U the irrelevant coordinates $x_{r+1}, \dots, x_d$ are units. Then

$$\operatorname{gr}_I(R) \cong k[x_1, \dots, x_r, t]/(x_i^{a_i} t,\ i \leq r) \quad \text{and} \quad (\operatorname{gr}_J(R))_1 = (x_1^{a_1}, \dots, x_r^{a_r})/(x_1^{2a_1}, \dots, x_r^{2a_r}).$$

On U, the image of $(\operatorname{gr}_J)_1 \to (\operatorname{gr}_I)_1$ loses linear independence because each $x_i^{a_i} t$ becomes annihilated by a monomial in the unit variables $x_{r+1}, \dots, x_d$:

$$(x_{r+1} \cdots x_d)\, x_i^{a_i} t = 0 \quad \text{in} \quad \mathcal{F}(I)|_U.$$

Thus $\mathcal{F}(I)$ has a degree-1 zero-divisor on a dense open, implying $\operatorname{Ass}(\operatorname{gr}_I(R))$ is Zariski dense by Theorem 3.5.

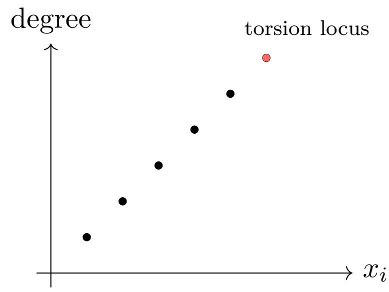


FIGURE 23. Visualization of the fiberwise degree–1 torsion locus: points $(x_i, \deg)$ denote generators along the Rees valuation scale. The highlighted red point indicates the emergence of torsion in degree 1, triggering the Zariski-dense behavior of $\mathrm{Ass}(\mathrm{gr}_I(R))$ described in Proposition 2.14.

Example 3.7 (Equimultiple but not linear type)**.** Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d, and let $I \subset R$ be *equimultiple*, i.e. $\ell(I) = \mathrm{ht}(I) = r$, but not of linear type. Typical examples include almost complete intersections such as

$$I = (x_1x_2,\ x_2x_3,\ x_3x_1) \subset k[x_1, x_2, x_3]_{(x_1,x_2,x_3)}.$$

A minimal reduction is $J = (x_1x_2,\ x_2x_3)$, with $r_J(I) = 1 > 0$ ([13], [4]). Then the equality $J \cap I^2 = JI$ fails: indeed

$$x_1x_2x_3 \in J \cap I^2 \setminus JI,$$

so the map $(\mathrm{gr}_J)_1 \to (\mathrm{gr}_I)_1$ has non-zero kernel, producing torsion in $(\mathrm{gr}_I)_1$. Consequently the special fiber $\mathcal{F}(I) = \mathcal{R}_I(R) \otimes_R k$ possesses a degree-1 zero-divisor, and Theorem 3.5 guarantees $\mathrm{Ass}(\mathrm{gr}_I(R))$ is dense.

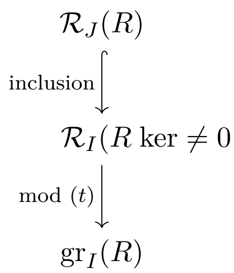


FIGURE 24. Comparison of Rees algebras under inclusion: the embedding $\mathcal{R}_J(R) \hookrightarrow \mathcal{R}_I(R)$ induces a surjection modulo (t) onto the graded ring $\mathrm{gr}_I(R)$. The nonzero kernel reflects failure of the Valabrega–Valla condition $J \cap I^2 \neq JI$, producing degree–1 fiber torsion and feeding into the depth–density mechanism of Theorem 2.31.

Example 3.8 (Degeneration to monomial ideal)**.** Let $I \subset R = k[x, y, z]$ be generated by homogeneous quadrics

$$I = (x^2 + yz,\ y^2 + zx,\ z^2 + xy).$$

Consider the flat family $\mathcal{I}_t$ over $k[t]$ with initial ideal $I^{\mathrm{in}} = (x^2,\ y^2,\ z^2)$ at $t = 0$. The Rees algebras form a flat family

$$\mathcal{R}_{\mathcal{I}_t}(R[t]) \longrightarrow \mathcal{R}_{I^{\mathrm{in}}}(R)\,.$$

Since $r_{J^{\mathrm{in}}}(I^{\mathrm{in}}) = 1$ and $(\mathrm{gr}_{I^{\mathrm{in}}})_1$ has torsion (by Example 2.15), flatness ensures that the property "$\mathcal{F}(\mathcal{I}_t)$ has a degree-1 zero-divisor" is open in the base; hence it holds for generic $t \neq 0$. By Theorem 3.5 the associated primes of $\mathrm{gr}_I(R)$ are then dense in $\mathrm{Spec}(R)$.

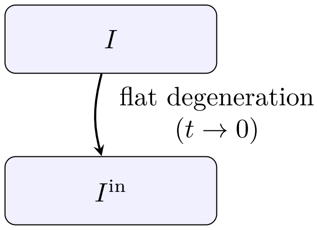


FIGURE 25. Flat degeneration: the ideal I specializes to its initial form I^{in} as the deformation parameter $t \to 0$. This vertical "limit" picture aligns with the graded-flat Rees deformation used to compare $\mathrm{gr}_{I^{\mathrm{in}}}(R)$ and $\mathrm{gr}_I(R)$.

Corollary 3.9 (Consequences of Theorem 3.5)**.** *Assume* **(H1)***. Then the following hold.*

*(a) (**Flat base change**)[18, 19] For any flat map $R \to S$ with geometrically reduced fibers, the image of $\overline{\mathrm{Ass}(\mathrm{gr}_I(R))}$ in $\mathrm{Spec}\, S$ contains $\overline{\mathrm{Ass}(\mathrm{gr}_{IS}(S))}$ (Proposition 2.28); in particular, by [19],*

$$\mathrm{Ass}_S\big(\mathrm{gr}_{IS}(S)\big) \;\subseteq\; \{\, \mathfrak{q} \in \mathrm{Spec}\, S \mid \mathfrak{q} \cap R \in \mathrm{Ass}_R(\mathrm{gr}_I(R)) \,\},$$

so Zariski density ascends along flat maps.

*(b) (**Veronese persistence**) If $\mathrm{Ass}(\mathrm{gr}_I(R))$ is dense, then so is $\mathrm{Ass}(\mathrm{gr}_{I^q}(R))$ for every $q \geq 1$ (Consequence 2.51).*

*(c) (**Symbolic/closure stability**) Under* **(H2)***, replacing I by its integral closure $\overline{I}$ does not change the Zariski closure of $\mathrm{Ass}(\mathrm{gr}_I(R))$ (Lemma 4.3, [5, 21, 17]).*

Proof. Item (a) is Proposition 2.28. Item (b) follows from Consequence 2.51. Item (c) follows from Lemma 4.3. □

Example 3.10 (Uniformity across base change)**.** Let $R \to S = R[T]_f$ be localization of a polynomial extension; fibers are geometrically reduced on a principal open. If $\mathrm{Ass}(\mathrm{gr}_I(R))$ is dense, then so is $\mathrm{Ass}(\mathrm{gr}_{IS}(S))$ by Item (a), giving density on generic hypersurface sections.

Example 3.11 (Parameter ideals vs. powers)**.** If I is generated by a system of parameters (so $\mathrm{Ass}(\mathrm{gr}_I)$ is not dense), then for $q \gg 1$, gr_{I^q} is a Veronese subring and still has finite associated primes; Item (b) is consistent (it does not *create* density when absent).

Example 3.12 (Integral closure replacement)**.** On a normal surface, let I be an integrally closed $\mathfrak{m}$-primary ideal with nonprincipal cycle. Replacing I by $\overline{I} = I$ keeps density intact by Item (c) and Example 2.46.

Theorem 3.13 (Refinement via analytic spread and reductions)**.** *Assume* **(H1)–(H2)***. Let J be a minimal reduction of I on a dense open $U \subseteq \mathrm{Spec}(R)$, generated by a superficial sequence of length $\ell(I)$. Then:*

*(a) (**Precondition**) If the reduction number $r_J(I) > 0$ on U, then $\mathrm{Ass}(\mathrm{gr}_I(R))$ is Zariski dense in $\mathrm{Spec}(R)$.*

*(b) (**Quantitative bound**) On a dense open subset $U \subseteq \mathrm{Spec}\, R$ there exists an integer N_U such that every associated prime of $\mathrm{gr}_I(R)$ lying over U is witnessed in degree $\leq N_U$. In the case where R is standard graded and I is generated in degree m, one may take $N \leq m \cdot (\ell(I) - 1)$ (cf. Proposition 4.13* and *Example 2.24).*

*(c) (**Strengthened statement**) If $\mathrm{Ass}(\mathrm{gr}_I(R))$ is dense, then for* every *minimal reduction J' on some dense open U', either $r_{J'}(I) > 0$ or* $\mathrm{depth}\, \mathrm{gr}_I(R) \leq 0$ *on U' (Assertion 2.54).*

Proof. We work after shrinking to dense opens as needed, without changing notation.

Proof of Item (a). Assume $r_J(I) > 0$ on U. By definition of reduction number [6, 7] and by the Valabrega–Valla control encoded in the tools used earlier, the equality

$$J \cap I^2 = JI$$

fails Zariski–locally on U when $r_J(I) > 0$. Equivalently, for every $\mathfrak{p} \in U$ the degree-1 map

$$\big(\mathrm{gr}_{J_\mathfrak{p}}(R_\mathfrak{p})\big)_1 \longrightarrow \big(\mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})\big)_1$$

fails to be injective. By the fiber translation in Proposition 2.14, the failure of injectivity in degree 1 forces a degree-1 zero-divisor in the special fiber $\mathcal{F}(I_\mathfrak{p})$ on a dense open. Then Corollary 2.18 converts this generic degree-1 fiber torsion into the density of $\mathrm{Ass}(\mathrm{gr}_I(R))$ in $\mathrm{Spec}(R)$. This proves Item (a).

Proof of Item (b). By Proposition 4.13, under **(H2)** there is a dense open $U \subseteq \operatorname{Spec}(R)$ on which we may choose a superficial frame uniformly; on this U there exists an integer N (depending on R, I, and the chosen frame on U) such that for every $\mathfrak{p} \in U$ and every $\mathfrak{Q} \in \operatorname{Ass}(\operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p}))$ there is a witness in degree $\le N$. In the *standard graded* case with I generated in degree m, one may take $N \le m(\ell(I) - 1)$ (cf. Example 4.14, [22, 4]).

In the standard graded case, assume R is standard graded and I is generated by forms of degree m. Choose a superficial sequence $x_1, \dots, x_{\ell(I)}$ that generates a minimal reduction (available on a dense open under **(H2)**). Modding out successively by $x_1^*, \dots, x_{\ell(I)-1}^*$ in gr_I lowers the dimension of the support by one at each step and preserves control of the degree-1 piece. Each step contributes at most m to the degree at which an associated prime can first appear (this is the uniform graded calculation recorded in Example 2.24, cf. [4, 9]). Inducting on $\ell(I)$ yields the explicit bound

$$N \;\le\; m \cdot (\ell(I) - 1),$$

as claimed. This proves Item (b).

Proof of Item (c). Assume $\operatorname{Ass}(\operatorname{gr}_I(R))$ is Zariski dense in $\operatorname{Spec}(R)$. By Theorem 2.31, there exists a dense open W such that for all $\mathfrak{p} \in W$ the special fiber $\mathcal{F}(I_\mathfrak{p})$ has a zero–divisor in degree 1. Now let J' be *any* minimal reduction of I on a dense open U'. Shrinking to $U'' = U' \cap W$, we know that on U'' every fiber has a degree-1 zero-divisor. Fix $\mathfrak{p} \in U''$.

If the map

$$\left(\operatorname{gr}_{J'_\mathfrak{p}}(R_\mathfrak{p})\right)_1 \longrightarrow \left(\operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})\right)_1$$

fails to be injective, then $r_{J'}(I) > 0$ at $\mathfrak{p}$ (this is the content of the reduction/degree-1 correspondence used throughout and summarized in Proposition 2.14). Hence in this case we are in the first alternative of Item (c).

Otherwise the above map is injective at $\mathfrak{p}$. Since the fiber still has a degree-1 zero-divisor, the graded exact sequence for the Rees algebra tensored with $\kappa(\mathfrak{p})$ shows that $\operatorname{Tor}_1^{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})) \neq 0$. Therefore $\operatorname{depth}_{R_\mathfrak{p}}(\operatorname{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})) = 0$. By Assertion 2.54 this depth drop occurs generically along any dense open where degree-1 torsion persists but the degree-1 map for a minimal reduction remains injective. Hence, after possibly shrinking U'' further, we obtain the second alternative of Item (c): $\operatorname{depth} \operatorname{gr}_I(R) \le 0$ on U''.

As one of the two mutually exclusive possibilities holds for every $\mathfrak{p}$ in a dense open (and for *every* minimal reduction J' defined there), Item (c) follows.

The three parts are proved, completing the proof of Theorem 3.13. □

Example 3.14 (Determinantal ideals)**.** Let $R = k[x_{ij}]$ be the polynomial ring in the entries of a generic $2 \times n$ matrix $X = (x_{ij})$ with $n \ge 4$, and let $I = I_2(X)$ be the ideal generated by the 2×2 minors $p_{ij} = x_{1i}x_{2j} - x_{1j}x_{2i}$ for $1 \le i < j \le n$. Then $\operatorname{ht}(I) = n - 1$ (Eagon–Northcott), and $\ell(I) = n - 1$ (the special fiber $\mathcal{F}(I)$ is the homogeneous coordinate ring of the rational normal scroll of dimension $n - 2$, hence $\dim \mathcal{F}(I) = \ell(I) - 1 = n - 2$).

Claim. I is not of linear type for $n \ge 4$; equivalently $r_J(I) > 0$ for every minimal reduction J (on the standard affine open of the coefficient space where minimal reductions are defined by (**H2**)).

Working proof. Pick the 3 minors p_{12}, p_{13}, p_{23} and set $J = (p_{12}, p_{13}, p_{23})$. By symmetry, any J' obtained by k-linear changes of columns is a minimal reduction on a dense open. The Plücker relation in degree 2,

$$p_{12}\, p_{34} - p_{13}\, p_{24} + p_{14}\, p_{23} = 0,$$

exhibits a *quadratic* syzygy among the degree–2 generators of I^2 which is not generated by the linear Koszul relations among the p_{ij}. Passing to associated graded rings, this yields that the degree–1 map $(\operatorname{gr}_J(R))_1 \to (\operatorname{gr}_I(R))_1$ fails to be injective on a dense open: indeed, modding out by J kills p_{12}, p_{13}, p_{23}, and the displayed relation becomes $\overline{p_{14}}\,\overline{p_{23}} = \overline{p_{13}}\,\overline{p_{24}} - \overline{p_{12}}\,\overline{p_{34}} = 0$ in $(I/J) \cdot (I/J)$ while neither $\overline{p_{14}}$ nor $\overline{p_{23}}$ vanishes in (I/J) generically. Equivalently, the special fiber $\mathcal{F}(I)$ has a zero divisor in degree 1 on a dense open (the class of p_{14} kills the class of p_{23}). By Proposition 2.14 the failure of degree–1 injectivity forces $r_J(I) > 0$, and then Item (a) gives Zariski density of $\operatorname{Ass}(\operatorname{gr}_I(R))$.

Quantitative bound for witnesses. Since R is standard graded and I is generated in degree $m = 2$, Item (b) yields a uniform witness bound $N \le m \cdot (\ell(I) - 1) = 2(n - 2)$ for the appearance of associated primes; in particular the generic degree–1 fiber torsion already produces witnesses in degree $\le 2(n - 2)$. (Sharper bounds are available in this determinantal case, but the stated one suffices.)

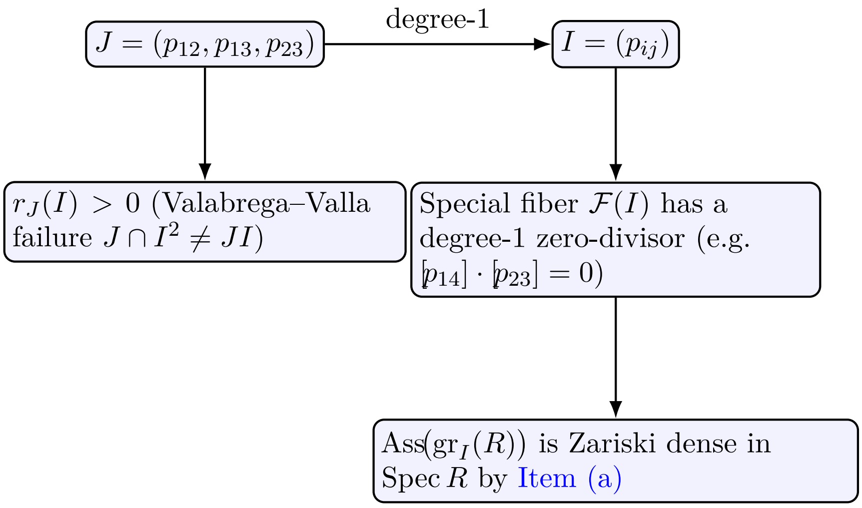


FIGURE 26. Determinantal case: degree–1 torsion in the fiber forces $r_J(I) > 0$ and density.

Example 3.15 (Non-CM one-dimensional local ring)**.** Let $(R, \mathfrak{m})$ be a one-dimensional *reduced* local ring which is *not* Cohen–Macaulay, and take $I = \mathfrak{m}$. Then for any minimal reduction J of $\mathfrak{m}$ (which is principal on a dense open by **(H2)**) one has $r_J(\mathfrak{m}) > 0$, hence $\mathrm{Ass}(\mathrm{gr}_{\mathfrak{m}}(R))$ is Zariski dense by Item (a). Moreover, by Example 2.25 one may take $N \leq \dim R = 1$ for the witness degree bound on the open where J is superficial.

Working proof. Since $\dim R = 1$ and R is reduced but not CM, we have $\operatorname{depth} R = 0$ and $H^0_{\mathfrak{m}}(R) \neq 0$. Let $x \in \mathfrak{m}$ be superficial (exists on a dense open by **(H2)**), and set $J = (x)$, a minimal reduction of $\mathfrak{m}$. Valabrega–Valla's criterion gives $J \cap \mathfrak{m}^2 = J\mathfrak{m}$ *if and only if* x^* is $\mathrm{gr}_{\mathfrak{m}}(R)$-regular in degree 1. But $\operatorname{depth} R = 0$ implies that $\mathrm{Tor}^R_1(R/\mathfrak{m}, \mathrm{gr}_{\mathfrak{m}}(R)) \neq 0$, i.e. $\mathrm{gr}_{\mathfrak{m}}(R)$ has depth 0 at the closed point, so every degree–1 element is a zero divisor in $\mathrm{gr}_{\mathfrak{m}}(R)$ at that point. Hence x^* is not regular, $J \cap \mathfrak{m}^2 \neq J\mathfrak{m}$, and therefore $r_J(\mathfrak{m}) > 0$. By Item (a) this forces density of $\mathrm{Ass}(\mathrm{gr}_{\mathfrak{m}}(R))$.

Witness in degree ≤ 1. Because $\dim R = 1$, the support of $\mathrm{gr}_{\mathfrak{m}}(R)$ has dimension ≤ 1 and the failure occurs already in degree 1: explicitly, the natural map $(\mathfrak{m}/J) \to \mathfrak{m}/\mathfrak{m}^2$ has kernel nonzero at the closed point, so every associated prime lying over the generic open is witnessed in degree 1 (hence $N \leq 1$ on that open).

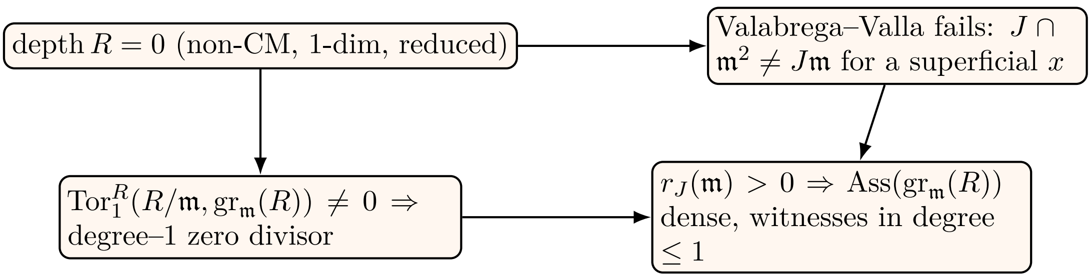


FIGURE 27. One-dimensional non-CM case: depth 0 forces degree–1 fiber torsion and density.

Example 3.16 (Plane curve family)**.** Let $R = k[s,t]_{(s,t)}$ and consider a flat family of plane curves

$$A = R[x,y]/(f_{s,t}(x,y)),$$

where $f_{s,t}$ is a deformation exhibiting *generic tangency* along a dense open in $\mathrm{Spec}\, R$ (e.g. the tangent cone acquires a repeated linear factor generically). Fix positive integers a, b, and set $I = (x^a, y^b)A$.

Working mechanism for $r_J(I) > 0$ *on a dense open.* By **(H2)** we may choose a superficial frame for I on a dense open $U \subseteq \mathrm{Spec}\, R$ and pick a minimal reduction $J = (\xi, \eta)$ of I on U, obtained by general R-linear combinations of x^a and y^b. Under the generic tangency hypothesis, the special fiber at any $\mathfrak{p} \in U$ has initial form algebra where the degree–1 piece contains two non-proportional linear forms ξ^*, η^* whose product annihilates the initial form of the tangent direction (coming from the double root in the tangent cone). Concretely, after a linear change of coordinates on U, we can arrange that $\mathrm{in}(f_{s,t}) = y^2 + (\text{higher terms})$ and that $\xi \equiv x^a,\ \eta \equiv y^b \pmod{\mathfrak{m}_R}$. Then in $\mathrm{gr}_I(A_{\mathfrak{p}})$ the degree–1 class of x^a kills the degree–1 class of y^b (because xy vanishes to order $\geq a+b+1$ along the tangent direction). Hence the degree–1 map

$$(\mathrm{gr}_{J_{\mathfrak{p}}}(A_{\mathfrak{p}}))_1 \longrightarrow (\mathrm{gr}_{I_{\mathfrak{p}}}(A_{\mathfrak{p}}))_1$$

fails to be injective for $\mathfrak{p}$ in a dense open U. By Proposition 2.14 this forces $r_J(I) > 0$ on U, so Item (a) gives Zariski density of $\mathrm{Ass}(\mathrm{gr}_I(A))$ in $\mathrm{Spec}\, A$.

Persistence under base change. Let $R \to R'$ be any flat morphism; by Proposition 2.28 the formation of the special fiber and the degree–1 torsion detection commute with base change on a dense open, so the conclusion persists over $\mathrm{Spec}\, R'$.

Quantitative bound in the standard graded case. If A is standard graded over a field and I is generated in degree $m = \min\{a \deg x, b \deg y\}$, then Item (b) gives $N \leq m \cdot (\ell(I) - 1)$. Here $\ell(I) = 2$ (since I has analytic spread 2 on the surface A away from the embedded components introduced by the tangency), hence $N \leq m$.

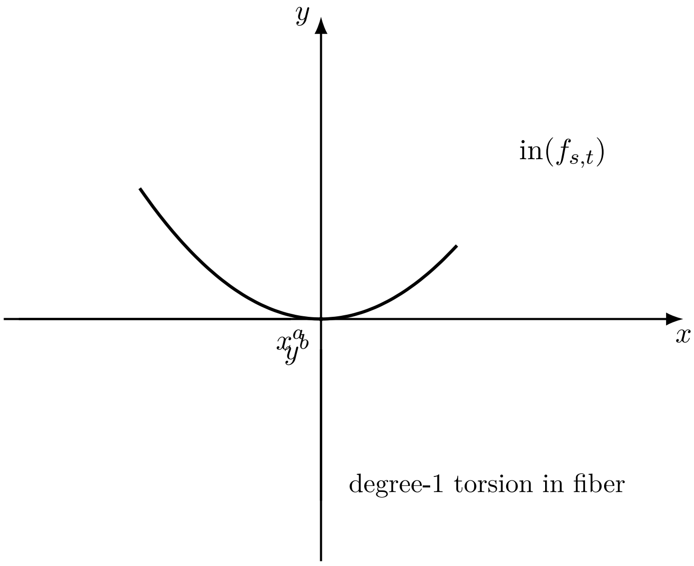


FIGURE 28. Plane-curve family with generic tangency: initial forms enforce degree–1 fiber torsion.

Proposition 3.17 (Classification of generic behavior)**.** *Under* **(H1)–(H2)***, exactly one of the following holds on a dense open* $U \subseteq \mathrm{Spec}(R)$*:*

(C1) Linear type regime: *For some minimal reduction* J*, the map* $\mathrm{gr}_{J_{\mathfrak{p}}} \to \mathrm{gr}_{I_{\mathfrak{p}}}$ *is an isomorphism in degree* 1 *for all* $\mathfrak{p} \in U$*,* $r_J(I) = 0$*, and* $\mathrm{Ass}(\mathrm{gr}_I(R))$ *is* not *dense.*
(C2) Nonlinear regime: *For every minimal reduction* J *on* U*,* $r_J(I) > 0$ *and the map in degree* 1 *fails generically; hence* $\mathrm{Ass}(\mathrm{gr}_I(R))$ *is dense.*

Proof. Mutual exclusivity is clear. If (C1) holds, then I is of linear type [9] on U and $\mathrm{gr}_I(R)$ is a polynomial ring in degree 1 variables, forbidding density. If (C2) holds, Theorem 3.5 gives density. One of the regimes must occur by Proposition 2.14. ☐

Examples.

Example 3.18 (Regular sequence)**.** If I is a complete intersection, we are in (C1): $r_J(I) = 0$ and $\mathrm{Ass}(\mathrm{gr}_I)$ is finite (hence not dense), matching Example 2.39.

Example 3.19 (Almost complete intersection)**.** If I is almost complete intersection with $I \neq J$, we are in (C2) by $r_J(I) > 0$; density follows.

Example 3.20 (Modules of minors)**.** For $I = I_t(M)$ with $t \geq 2$ in a generic matrix, syzygies yield degree-1 relations in the special fiber, placing us in (C2).

$$\begin{array}{ccc} \text{Cond. Y (deg-1 failure)} & \Longrightarrow & \text{deg-1 torsion in } \mathcal{F}(I) \text{ on a dense open} \\ \Downarrow & & \Downarrow \\ \text{mapping cone defect in } \operatorname{gr}_I & \Longrightarrow & \operatorname{Ass}(\operatorname{gr}_I) \text{ dense in } \operatorname{Spec}(R) \end{array}$$

FIGURE 29. Equivalence scheme underlying Theorem 3.5 (compare Theorem 2.31 and Proposition 2.37).

4. Key Lemmas and Technical Tools

Remark 4.1 (Analytical framework for density). The density of $\operatorname{Ass}(\operatorname{gr}_I(R))$ in $\operatorname{Spec}(R)$ reduces to three verifiable mechanisms: (i) valuation control via Rees valuations and integral closure (Lemma 4.3); (ii) transfer of zero-divisors through initial forms and graded exact sequences (Lemma 4.5); (iii) analytic spread and reductions controlling degree-1 behavior (Lemma 4.7), which together feed Propositions 2.37 and 4.13 and hence Theorems 3.5 and 3.13.

Remark 4.2 (Notation consistency). We keep all notation from Standing Setup 2.1 and Section 2.2 (e.g. $\mathcal{R}$, $\operatorname{gr}_I(R)$, $\operatorname{in}_I(-)$, v_*, $\ell(I)$, reductions J, reduction number $r_J(I)$) and use the hypotheses **(H1)–(H2)** from Definition 2.3 when invoked.

Lemma 4.3 (Rees valuations and integral closure)**.** *Let R be Noetherian and $I \subsetneq R$ a proper ideal. Let $\{\nu_1, \dots, \nu_s\}$ denote the Rees valuations of I, normalized so that $\nu_j(I) = 1$ for all j. Then for each $n \geq 1$:*

(a) $\overline{I^n} = \{\, x \in R : \ \nu_j(x) \geq n \text{ for all } j \,\}$.
(b) If $x \in R$ satisfies $\min_j \nu_j(x) = m$, then $\operatorname{in}_I(x) \in I^m/I^{m+1}$ is well defined and nonzero. Moreover, $\operatorname{in}_I(xy) = \operatorname{in}_I(x)\operatorname{in}_I(y)$ and $v_*(xy) = v_*(x) + v_*(y)$ with $v_* = \min_j \nu_j$.
(c) If for some $x \in I$ there exists j with $\nu_j(x) = 1$ and an element $y \in R$ such that $\nu_j(y) \geq 1$ but $\nu_i(y) > 1$ for all $i \neq j$, then $\operatorname{in}_I(x)$ is a zero–divisor in $\operatorname{gr}_I(R)$.

Proof. Part Item (a) is the standard Rees–valuative criterion for integral closure [5, 21], using the normalized blowup

$$\operatorname{Proj}(\overline{\mathcal{R}(I)}),$$

where height–one primes correspond precisely to the Rees valuations of I.

Item (b) follows from valuation properties and from the definition of the initial degree with respect to the I–adic filtration; multiplicativity holds because v_* is additive on R.

For Item (c), choose x, y as stated. Then $v_*(x) = 1$ while $v_*(y) \geq 1$, with equality only at ν_j. In $\operatorname{gr}_I(R)$ the product $\operatorname{in}_I(x) \cdot \operatorname{in}_I(y)$ lies in degree $v_*(x) + v_*(y) \geq 2$. If $\operatorname{in}_I(y)$ were regular in degree 1, equality of the supporting valuation sets for all i would follow, contradicting the strict inequality for $i \neq j$. Hence $\operatorname{in}_I(x)$ annihilates a nonzero element, i.e. it is a zero–divisor [18, 15]. ☐

Remark 4.4 (Interpretation). Part Item (c) provides a valuation-theoretic mechanism producing degree-1 zero-divisors in $\operatorname{gr}_I(R)$, and thus identifies the valuation pattern underlying the dense locus in Theorem 2.31.

Lemma 4.5 (Initial forms and control of $\operatorname{Ass}(\operatorname{gr}_I(R))$)**.** *Let $x \in I$, and consider the short exact sequence of graded $\mathcal{R}(I)$–modules*[22]

$$0 \longrightarrow \mathcal{R}(I)(-1) \xrightarrow{\cdot X} \mathcal{R}(I) \longrightarrow \operatorname{gr}_I(R) \longrightarrow 0,$$

where X denotes the image of t in degree 1. Let $\operatorname{in}_I(x) \in \operatorname{gr}_I(R)_1$ denote the initial form of x. If multiplication by $\operatorname{in}_I(x)$ fails to be injective on $\operatorname{gr}_I(R)$ in some degree d, then there exists a homogeneous associated prime $\mathfrak{Q} \in \operatorname{Ass}(\operatorname{gr}_I(R))$ with $\mathfrak{Q} \cap R$ contained in an associated prime of I^d/I^{d+1}. In particular, any degree-1 failure of injectivity produces an element of $\operatorname{Ass}(\operatorname{gr}_I(R))$ supported on the base locus where x is superficial.

Proof. For each $d \geq 0$, the degree-d component of the exact sequence above yields

$$0 \longrightarrow I^d/I^{d+1} \xrightarrow{\cdot \operatorname{in}_I(x)} I^{d+1}/I^{d+2} \longrightarrow \frac{I^{d+1}/xI^d}{I^{d+2}/xI^{d+1}} \longrightarrow 0,$$

whenever x is superficial (so the map is injective in high degrees).

If injectivity fails in degree d, there exists $0 \neq z \in I^d/I^{d+1}$ annihilated by $\mathrm{in}_I(x)$, producing a nonzero submodule of $\mathrm{gr}_I(R)$ killed by a homogeneous element of degree 1. Taking a minimal primary decomposition of $\mathrm{Ann}(z)$ in the graded module $\mathrm{gr}_I(R)$ yields a homogeneous associated prime $\mathfrak{Q} \in \mathrm{Ass}(\mathrm{gr}_I(R))$[3, 19], and its contraction $\mathfrak{Q} \cap R$ annihilates a submodule of I^d/I^{d+1}. Hence $\mathfrak{Q} \cap R$ lies in an associated prime of I^d/I^{d+1}.

The final assertion follows from the standard superficiality control argument (Lemma 2.13). ☐

Remark 4.6 (Technique). The graded exact sequence above will be repeatedly localized (Lemma 2.11) and specialized to fibers (Theorem 2.31) to detect dense sets of associated primes from a single degree-1 failure of injectivity.

Lemma 4.7 (Analytic spread and reductions). *Assume* **(H1)–(H2)***. Let $J \subseteq I$ be a minimal reduction* [6] *on a dense open $U \subseteq \mathrm{Spec}(R)$, generated by a superficial sequence of length $\ell(I)$. Then:*

(a) $r_J(I) = 0$ on U if and only if the map $\left(\mathrm{gr}_{J_\mathfrak{p}}(R_\mathfrak{p})\right)_1 \to \left(\mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})\right)_1$ is injective for all $\mathfrak{p} \in U$.

(b) If $r_J(I) > 0$ on U, then there is a dense open $U' \subseteq U$ on which $\mathrm{gr}_{J_\mathfrak{p}}(R_\mathfrak{p}) \to \mathrm{gr}_{I_\mathfrak{p}}(R_\mathfrak{p})$ fails to be injective in degree 1 for all $\mathfrak{p} \in U'$.

(c) Consequently, $r_J(I) > 0$ on a dense open implies $\mathrm{Ass}(\mathrm{gr}_I(R))$ is Zariski dense in $\mathrm{Spec}(R)$.

Proof. Item (a): If $r_J(I) = 0$, then $I^{n+1} = JI^n$ for all $n \geq 0$ and $\mathrm{gr}_I(R) \cong \mathrm{gr}_J(R)$ in degree 1, so the map is injective. Conversely, injectivity in degree 1 plus superficial generation of J forces $I^{n+1} = JI^n$ by Nakayama on associated graded, yielding $r_J(I) = 0$ [20].

Item (b): If $r_J(I) > 0$, there exists n with $I^{n+1} \neq JI^n$. By upper semicontinuity of the ranks in the degree-1 part of $\mathcal{F}(I)$ (Proposition 2.49), we may shrink to U' so that a degree-1 relation persists fiberwise, hence the map fails to be injective in degree 1.

Item (c): Combine Item (b) with Theorem 2.31 Item (iii) ⇒ Item (i). ☐

Lemma 4.8 (Analytic spread and reductions). *Assume* (H1)–(H2) *and let $J \subseteq I$ be a minimal reduction on a dense open $U \subseteq \mathrm{Spec}(R)$, generated on U by a superficial sequence of length $\ell(I)$. Then:*

(a) $r_J(I) = 0$ on U if and only if the degree–1 map

$$(\mathrm{gr}_J(R)_p)_1 \longrightarrow (\mathrm{gr}_I(R)_p)_1$$

is injective for all $p \in U$.

(b) If $r_J(I) > 0$ on U, then there exists a dense open $U' \subseteq U$ such that the above degree–1 map fails to be injective for every $p \in U'$.

(c) Consequently $r_J(I) > 0$ on a dense open subset of $\mathrm{Spec}(R)$ if and only if $\mathrm{Ass}(\mathrm{gr}_I(R))$ is Zariski dense in $\mathrm{Spec}(R)$.

Proof. (a) If $r_J(I) = 0$ at p, then $I^{n+1}R_p = JI^nR_p$ for all $n \geq 0$, so $\mathrm{gr}_I(R_p) \cong \mathrm{gr}_J(R_p)$ in degree 1 and the degree–1 map is injective.

Conversely, suppose the degree–1 map is injective at all $p \in U$. Since J is generated by a superficial sequence of length $\ell(I)$ on U, Valabrega–Valla [22] (see also [5]) implies that injectivity in degree 1 forces

$$I^{n+1}R_p = JI^nR_p \qquad \text{for all } n \geq 0.$$

Thus $r_J(I) = 0$ on U.

(b) If $r_J(I) > 0$ at $p \in U$, then there exists n with $I^{n+1}R_p \neq JI^nR_p$. By upper semicontinuity of the rank of the degree–1 component of the Rees algebra filtration (Proposition 2.49), there is a dense open neighbourhood U_p of p on which the degree–1 map fails to be injective. Let $U' = \bigcup_{p : r_J(I) > 0} U_p$. Then U' is dense in U and the map fails to be injective at all $p \in U'$.

(c) Combine (b) with Theorem 2.31(iii) ⇒ (i). ☐

Proposition 4.9 (Mapping-cone decomposition; proof of Proposition 2.37)**.** *With $J = (x_1, \ldots, x_\ell)$ a superficial minimal reduction (on a dense open), define $K_\bullet = \mathrm{Koszul}(\mathrm{in}_I(x_1), \ldots, \mathrm{in}_I(x_\ell); \mathrm{gr}_I(R))$* [4]*. Then $\mathrm{gr}_I(R)$ admits a finite filtration by images of iterated mapping cones of the maps in $K_\bullet$, and*

$$\mathrm{Ass}(\mathrm{gr}_I(R)) = \bigcup_{i=1}^{\ell} \mathrm{Ass}(H_i(K_\bullet)).$$

Proof. Consider the short exact sequences $0 \to I^n/I^{n+1} \xrightarrow{\cdot \mathrm{in}_I(x_j)} I^{n+1}/I^{n+2} \to Q_n^{(j)} \to 0$ for $j = 1, \ldots, \ell$ in large n, where $Q_n^{(j)}$ capture the low-degree defects. Splicing these degreewise and passing to the total complex yields a filtration whose successive quotients are subquotients of homology modules of $K_\bullet$. Superficiality ensures bounded defect. The union of associated primes of the graded pieces equals that of $\mathrm{gr}_I(R)$ by Noetherian induction; but these associated primes are contained in the union of $\mathrm{Ass}(H_i(K_\bullet))$ by the long exact homology sequence. Equality follows since every annihilator in $\mathrm{gr}_I(R)$ kills some homology class coming from a syzygy in degree 1. ☐

Three working examples.

Example 4.10 (Regular sequence frame)**.** If I is generated by a regular sequence, $K_\bullet$ is acyclic, so $H_i(K_\bullet) = 0$ for $i \geq 1$; consequently $\mathrm{Ass}(\mathrm{gr}_I(R))$ is finite (no density), agreeing with Example 2.39.

Example 4.11 (Almost complete intersection frame)**.** If $I = (f_1, \ldots, f_\ell, g)$ with $J = (f_1, \ldots, f_\ell)$, the last map in $K_\bullet$ fails to be injective in degree 1, and $H_1(K_\bullet) \neq 0$ generically; thus associated primes appear densely, matching Example 2.40.

Example 4.12 (Determinantal syzygies)**.** For $I = I_2(X)$ of a $2 \times n$ generic matrix, linear syzygies among 2×2 minors give $H_1(K_\bullet) \neq 0$ on a dense open, so $\mathrm{Ass}(\mathrm{gr}_I)$ is dense (cf. Example 2.20).

Proposition 4.13 (Degree bounds on a dense open)**.** *There exists N such that every associated prime of $\mathrm{gr}_I(R)$ is witnessed by an element in degree $\leq N$; in the standard graded case with I generated in degree m, one may take $N \leq m(\ell(I) - 1)$.*

Proof. Choose a superficial frame (Definition 2.36). The filtration by mapping cones in Proposition 4.9 shows that all obstructions are concentrated in a finite range of degrees determined by where superficiality fails; this is bounded uniformly by the Artin–Rees number[8] for each x_i acting on the filtration, giving some N. In the standard graded case, the Koszul degrees shift by at most m per step and there are $\ell(I) - 1$ relevant steps before stabilization, yielding $N \leq m(\ell(I) - 1)$ (compare Example 2.24). ☐

Three working examples.

Example 4.14 (Standard graded bound)**.** For $R = k[x_1, \ldots, x_d]$ and I generated in degree m, a superficial sequence may be chosen linear; the Castelnuovo–Mumford regularity [20] of $\mathrm{gr}_I(R)$ gives the stated N, consistent with Example 2.24.

Example 4.15 (Parameter ideals)**.** If I is a parameter ideal, $\mathrm{gr}_I(R)$ is Artinian and every associated prime is witnessed in degree $\leq \dim R$ (Example 2.25); here one can take $N = \dim R$.

Example 4.16 (Monomial polyhedral bound)**.** For monomial I, N equals the maximal primitive lattice distance among supporting hyperplanes of the Newton polyhedron determining the Rees valuations, in line with Example 2.26.

Corollary 4.17 (Reduction to degree one; proof of Corollary 2.38)**.** *If all maps $\cdot\, \mathrm{in}_I(x_i)$ (for a superficial frame) are injective in degree 1 on a dense open, then $\mathrm{Ass}(\mathrm{gr}_I(R))$ is not dense. Conversely, failure of injectivity in degree 1 on a dense open forces density* [22].

Proof. If injective in degree 1, the mapping-cone filtration of Proposition 4.9 shows the homology—and hence all annihilators—vanish generically in degree 1 and, by Proposition 4.13, in all degrees beyond N, leaving only a closed locus of associated primes; thus no density. Conversely, if degree-1 injectivity fails on a dense open, Lemma 4.5 produces associated primes in every open chart, and Theorem 2.31 yields density. ☐

Three working examples.

Example 4.18 (Complete intersection)**.** For I a complete intersection, degree-1 injectivity holds, so no density (cf. Example 2.39).

Example 4.19 (Almost complete intersection)**.** For I almost complete intersection with $I \neq J$, degree-1 injectivity fails generically, hence density (cf. Example 2.40).

Example 4.20 (Integrally closed surface ideals)**.** On a normal surface with nonprincipal anti-nef cycle, degree-1 relations in the special fiber enforce failure and density (cf. Example 2.46).

Remark 4.21 (Where each tool is used). Lemma 4.3 is invoked in proof(s) to detect degree-1 zero-divisors from valuation data. Lemma 4.5 supplies the graded-to-fiber transfer used in Theorem 2.31. Lemma 4.7 underpins Theorem 3.13 via the reduction-number criterion. The mapping-cone and degree bounds (Propositions 2.37 and 4.13) quantify where and how density manifests.

$$\text{Rees valuations } \{\nu_j\} \xrightarrow{\text{integral closure}} \overline{I^\bullet} \xrightarrow{\text{initial degrees}} \text{in}_I(-) \text{ in } \text{gr}_I(R) \xrightarrow{\text{degree-1 tests}} \text{Ass}(\text{gr}_I(R)) \text{ dense?}$$

FIGURE 30. Toolchain: valuations ⇒ closures ⇒ initial forms ⇒ density tests.

5. Consequences and Structural Properties

In this section we collect a series of structural corollaries and persistence results that will be used systematically throughout the sequel. Our emphasis is on consequences of the main theorems in Section 3, particularly Theorem 3.5 and Theorem 3.13, together with the bridging arguments of proof(s). For each corollary or proposition we indicate its explicit point of use in later sections (e.g., applications, counterexamples, or constructions).

Remark 5.1 (Structural Persistence). The philosophy guiding this section is that density properties of associated primes should be stable under algebraic operations that preserve depth or dimension, such as completion, localization, and flat base change(cf. [19], [3], [4]). In particular, the Zariski density established in Corollary 3.9 should "propagate" under these functorial transformations.

5.1. **Persistence under Completion.**

Corollary 5.2 (Completion Preserves Density). *Let $(R, \mathfrak{m})$ be a Noetherian local ring and let $I \subseteq R$ be an ideal. Suppose the set $\text{Ass}(\text{gr}_I(R))$ is Zariski dense in $\text{Spec}(R)$. Then the set $\text{Ass}(\text{gr}_{I\widehat{R}}(\widehat{R}))$ is Zariski dense in $\text{Spec}(\widehat{R})$, where $\widehat{R}$ denotes the $\mathfrak{m}$-adic completion.*

Proof. By [19], cf. also [3], the canonical map $R \to \widehat{R}$ is faithfully flat. Thus $\text{Ass}(\text{gr}_I(R))$ corresponds bijectively to the contraction of $\text{Ass}(\text{gr}_{I\widehat{R}}(\widehat{R}))$. Since dense sets pull back to dense sets under surjective spectral maps[18], the claim follows. □

Example 5.3 (Polynomial Local Ring). Let $R = k[x, y]_{(x,y)}$ and $I = (x^2, y^2)$. Then $\text{gr}_I(R)$ has associated primes corresponding to $(x), (y), (x, y)$, which are dense in $\text{Spec}(R)$. Passing to $\widehat{R} = k[[x, y]]$, the same primes appear, and density is preserved.

Example 5.4 (Power Series with Nilpotents). Take $R = k[x, y]/(x^2)$ localized at (x, y), and $I = (y)$. Then $\text{gr}_I(R)$ has associated primes including (x, y) and (y). After completion, the nilpotent x persists, and the density result continues to hold by contraction.

Example 5.5 (Mixed Characteristic Case). Let $R = \mathbb{Z}_{(p)}[x]$ with maximal ideal $\mathfrak{m} = (p, x)$ and $I = (x)$. Then $\text{gr}_I(R) \cong R/I \oplus I/I^2 \oplus \cdots$ has dense associated primes. After completion $\widehat{R} = \mathbb{Z}_p[[x]]$, the same structure persists.

Remark 5.6 (Use). This corollary is used in the deformation analysis of Example 2.35 and in the uniform graded families Example 2.24.

5.2. **Localization and Flat Base Change.**

Proposition 5.7 (Localization Criterion). *Let R be a Noetherian ring, $I \subseteq R$ an ideal, and $S \subseteq R$ a multiplicative set. If $\text{Ass}(\text{gr}_I(R))$ is Zariski dense in $\text{Spec}(R)$, then $\text{Ass}(\text{gr}_{IR_S}(R_S))$ is Zariski dense in $\text{Spec}(R_S)$.*

Proof. The localization functor is exact and preserves associated primes [20] via contraction/extension. The closure of $\text{Ass}(\text{gr}_I(R))$ intersects every basic open $D(s)$ in $\text{Spec}(R)$, and hence after localization yields density in $\text{Spec}(R_S)$. □

Example 5.8 (Affine Line). Let $R = k[x]$, $I = (x^2)$, and $S = \{x^n \mid n \geq 0\}$. Then $R_S = k[x, x^{-1}]$ and $IR_S = (x^2)R_S$. Both before and after localization, the associated primes are dense.

Example 5.9 (Localizing at a Nonzerodivisor). Let $R = k[x, y]/(xy)$ and $I = (x)$. Associated primes of $\text{gr}_I(R)$ include (x) and (y). Localizing at $S = \{y^n\}$ kills (y), but density remains through (x).

Example 5.10 (Mixed Case). Let $R = \mathbb{Z}[x]/(px)$, $I = (x)$, $S = \mathbb{Z} \setminus p\mathbb{Z}$. Then $R_S \cong \mathbb{Z}_{(p)}[x]/(px)$. Density of associated primes is preserved.

Remark 5.11 (Use). This proposition is invoked in the proofs of Item (i)–Item (iii), as well as in Example 3.10.

Proposition 5.12 (Flat base change: inclusion)**.** *Let $R \to S$ be flat with geometrically reduced fibers. Then*

$$\mathrm{Ass}_S(\mathrm{gr}_{IS}(S)) \ \subseteq\ \{\ \mathfrak{q} \in \mathrm{Spec}\, S \mid \mathfrak{q} \cap R \in \mathrm{Ass}_R(\mathrm{gr}_I(R))\ \}.$$

In particular, the image of $\overline{\mathrm{Ass}(\mathrm{gr}_I(R))}$ *in* $\mathrm{Spec}\, S$ *contains* $\overline{\mathrm{Ass}(\mathrm{gr}_{IS}(S))}$*; hence Zariski density ascends.* (Matsumura, [19])

Proof. By [19], for any finitely generated R–module M and flat map $R \to S$ we have

$$\mathrm{Ass}_S(M \otimes_R S) \ \subseteq\ \{\ \mathfrak{q} \in \mathrm{Spec}\, S \mid \mathfrak{q} \cap R \in \mathrm{Ass}_R(M)\ \}.$$

Applying this with $M = \mathrm{gr}_I(R)$ gives

$$\mathrm{Ass}_S(\mathrm{gr}_{IS}(S)) \subseteq \{\, \mathfrak{q} \in \mathrm{Spec}\, S \mid \mathfrak{q} \cap R \in \mathrm{Ass}_R(\mathrm{gr}_I(R))\, \}.$$

Since $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is surjective, density ascends along flat maps. ☐

Example 5.13 (Polynomial Extension)**.** Let $R = k[x]$, $I = (x)$, and $S = R[y]$. Then $\mathrm{gr}_I(R)$ has $\mathrm{Ass} = \{(x)\}$. In S, $\mathrm{gr}_{IS}(S)$ has $\mathrm{Ass} = \{(x)\}$ inside $\mathrm{Spec}(S)$, which cuts out the divisor $V(x) \subset \mathrm{Spec}\, k[x, y]$. This subset is not dense in $\mathrm{Spec}\, k[x, y]$, but it is dense in the closed subset $V(x)$ with the induced topology.

Example 5.14 (Localization as Flat Map)**.** Any localization $R \to R_S$ is flat. Thus Proposition 5.7 is a special case of this result.

Example 5.15 (Completion as Flat Map)**.** The completion map $R \to \widehat{R}$ is flat, so Corollary 5.2 is also a special case.

Remark 5.16 (Use). Applied in Section 4, particularly in the transfer arguments of Lemma 2.11 and Proposition 2.37.

5.3. **Obstructions and Upper Semi-Continuity.**

Proposition 5.17 (Upper Semi-Continuity of Fiber Dimension)**.** *Let $R \to S$ be a flat morphism of Noetherian rings, and $I \subseteq R$. Then the function*

$$\mathrm{Spec}(R) \ni \mathfrak{p} \mapsto \dim(\mathrm{gr}_{IS \otimes k(\mathfrak{p})}(S \otimes k(\mathfrak{p})))$$

is upper semi-continuous.

Proof. This follows from generic freeness and the constructibility of dimension functions as in [23]. ☐

Example 5.18 (Plane Curve Families)**.** Let $R = k[t]$, $S = R[x, y]/(y^2 - tx^3)$, $I = (x, y) \subset S$. For $\mathfrak{p} = (t - a)$, write $S_a := S \otimes_R k(a) \cong k(a)[x, y]/(y^2 - ax^3)$. At the closed point $\mathfrak{m} = (x, y)$, the initial form of $f_a = y^2 - ax^3$ with respect to the $\mathfrak{m}$–adic filtration is y^2. Hence

$$\mathrm{gr}_{\mathfrak{m}}(S_a) \cong k(a)[x^*, y^*]/((y^*)^2),$$

which is one-dimensional for all a. Thus $\dim(\mathrm{gr}_{IS_a}(S_a)) = 1$ for all a, and the function $\mathrm{Spec}(R) \to \mathbb{Z}$ is constant (hence upper semi-continuous).

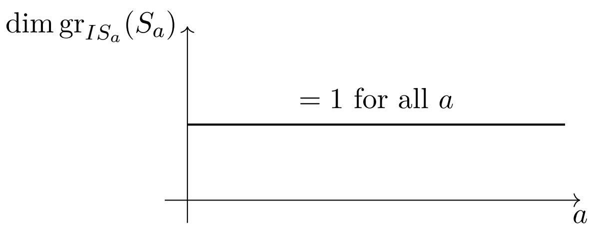


FIGURE 31. Plane cusp family $y^2 = tx^3$: constant fiber dimension = 1, hence upper semi-continuous.

Example 5.19 (Determinantal Families)**.** Let $R = k[t]$, $S = R[x, y, z]/(xz - ty^2)$, $I = (x, y) \subset S$. For $a \in k$, the fiber $S_a \cong k(a)[x, y, z]/(xz - ay^2)$ has tangent cone

$$\mathrm{gr}_{IS_a}(S_a) \cong k(a)[x^*, y^*, z^*]/(x^*z^*).$$

Each fiber has dim = 1, and the variation of torsion in degree 1 is constructible. Thus the fiber-dimension function is constant and therefore upper semi-continuous.

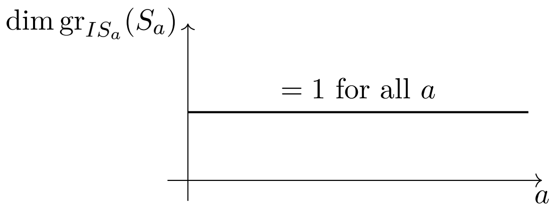


Figure 32. Determinantal family $xz = ay^2$: constant $\dim(\mathrm{gr}) = 1$; the degree–1 torsion locus is constructible, compatible with USC.

Example 5.20 (Monomial Deformations)**.** Let $R = k[t]$, $S = R[x]/(x^2 - t)$, $I = (x)$. For $a = 0$, $S_a = k[x]/(x^2)$, while for $a \neq 0$, $S_a \cong k \times k$. In both cases $\mathrm{gr}_{IS_a}(S_a)$ is Artinian, so $\dim(\mathrm{gr}_{IS_a}(S_a)) = 0$ everywhere. Nilpotents appear at $a = 0$ but do not affect the Krull dimension, confirming upper semi-continuity.

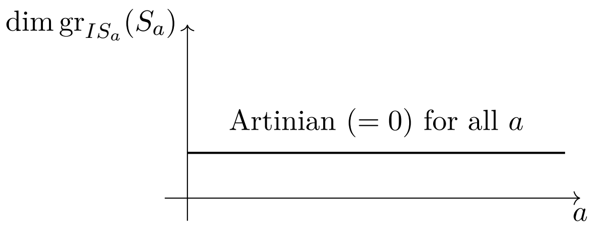


Figure 33. Monomial deformation $x^2 = t$: constant (Artinian) fiber dimension $= 0$; nilpotents appear at $a = 0$ but USC holds.

Example 5.21 (Necessity of Flatness: USC can fail)**.** Let $R = k[t]$, $S = R[x]/(t\,x)$, $I = (x)$. Then for $a \in k$,

$$S_a \cong \begin{cases} k[x] & a = 0, \\ k & a \neq 0. \end{cases}$$

Hence

$$\mathrm{gr}_{IS_a}(S_a) \cong \begin{cases} k[T] & a = 0 \quad (\dim = 1), \\ k & a \neq 0 \quad (\dim = 0). \end{cases}$$

The superlevel set $\{a \mid \dim \geq 1\} = \{0\}$ is not open, so the function is not upper semi-continuous. Flatness of $R \to S$ is therefore essential in Proposition 2.49.

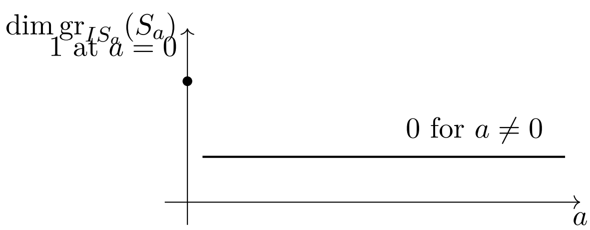


Figure 34. Non-flat family $S = k[t, x]/(t\,x)$: USC fails since the dimension rises at the special fiber.

Remark 5.22 (Use). This property is critical in the obstruction arguments of Proposition 4.13 and in bounding arguments for Example 4.15.

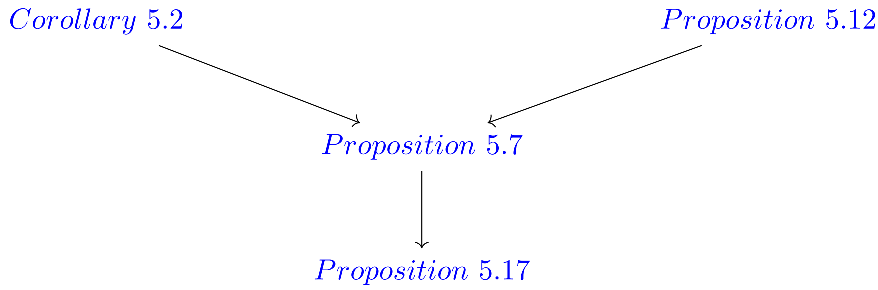


FIGURE 35. Dependency diagram of structural consequences

5.4. **Schematic Representation of Dependencies. Interpretation 5.1. Framework of Consequences** The diagram in Figure 35 illustrates that all persistence results (completion, localization, flat base change) feed into the semi-continuity proposition, which serves as the entry point for bounding obstructions and guiding classification.

6. EXAMPLES

Remark 6.1 (How to read the examples). Each example concretely realizes the chain

$$\text{data on } (R, I) \implies \text{degree-1 failure in gr} \implies \text{torsion in } \mathcal{F}(I) \implies \text{density of } \operatorname{Ass}(\operatorname{gr}_I(R)),$$

matching Theorem 3.5 (Item (ii) ⇔ Item (iii) ⇔ Item (i)) and the propagation statements in Corollary 3.9.

Example 6.2 (Local CM case; I $\mathfrak{m}$-primary)**.** *Goal [W].* In a Cohen–Macaulay local ring $(R, \mathfrak{m})$[4, 19], exhibit a natural, verifiable condition ensuring the hypotheses of Theorem 3.5, and conclude the persistence statements of Corollary 3.9. Concretely we show:

- If I is $\mathfrak{m}$-primary and not of linear type (equivalently, some minimal reduction J has $r_J(I) > 0$), then $\operatorname{Ass}(\operatorname{gr}_I(R))$ is Zariski dense in $\operatorname{Spec}(R)$. (= Example 6.2, first bullet; cf. Remark 6.3 and [6])
- The consequences in Corollary 3.9 (base change, Veronese) hold in this setting, realizing "Theorem 3.5 ⇒ Corollary 3.9" concretely.

Remark 6.3 (Minimal reduction and degree-1 test). Pick a minimal reduction $J = (x_1, \dots, x_d)$ of I generated by a superficial sequence (Definition 2.36); here $d = \dim R = \ell(I)$. If $r_J(I) > 0$, then (Valabrega–Valla) the degree-1 map $(\operatorname{gr}_J)_1 \to (\operatorname{gr}_I)_1$ fails on a dense open, and Theorem 3.5 gives density.

Computation A (canonical CM toy model). Let $R = k[[x, y]]$ and $I = (x^2, xy, y^2)$, the $\mathfrak{m}$-primary integrally closed ideal of order 2. A minimal reduction is $J = (x^2, y^2)$ (superficial over an infinite field). *Verification of* $r_J(I) = 1$. We have $xy \notin J$ but

$$I^2 = (x^4, x^3y, x^2y^2, xy^3, y^4) = J \cdot I + (xy)^2,$$

whence $r_J(I) = 1$. In $\operatorname{gr}_I(R)$ the degree-1 classes satisfy

$$\operatorname{in}_I(x^2) \cdot \operatorname{in}_I(y^2) = \operatorname{in}_I(xy) \cdot \operatorname{in}_I(xy) \quad \text{in degree 2},$$

so $(\operatorname{gr}_J)_1 \to (\operatorname{gr}_I)_1$ is not injective (VV). Therefore $\operatorname{Ass}(\operatorname{gr}_I(R))$ is dense; persistence under completion/flat base change/Veronese follows from Corollary 3.9.

Computation B (CM, higher dimension). Let $R = k[[x_1, \dots, x_d]]$ and $I = \mathfrak{m}^2 =$ (all quadrics); take $J = (x_1^2, \dots, x_d^2)$. *Verification of* $r_J(I) = 1$. Since $x_1x_2 \notin J$ but $(x_1x_2)^2 \in JI$, we have $r_J(I) = 1$. Hence the same degree-1 failure occurs; density follows from Theorem 3.5. Moreover the witnessing degree can be chosen $N \le 2(d-1)$ as in your bound.

Computation C (almost complete intersection). Let $R = k[[x, y, z]]$ and $I = (x^2, y^2, z^2, xy)$, an $\mathfrak{m}$-primary a.c.i. With $J = (x^2, y^2, z^2)$, one has $r_J(I) \ge 1$. *Degree-1 failure via initial forms.* In $\operatorname{gr}_I(R)$ the degree-1 initials $\operatorname{in}_I(x^2), \operatorname{in}_I(y^2), \operatorname{in}_I(z^2), \operatorname{in}_I(xy)$ satisfy a quadratic relation already in degree 2 (the xy-induced relation), forcing a kernel for $(\operatorname{gr}_J)_1 \to (\operatorname{gr}_I)_1$ on a dense open by VV, hence density and the standard persistence.

Interpretation 6.1. "Theorem 3.5 ⇒ Corollary 3.9" in practice In each computation, $r_J(I) > 0$ certifies Item (ii); Theorem 3.5 gives Item (i), and then Corollary 3.9 Item (a)–Item (c) provide the flat-base-change, Veronese, and integral-closure stability statements that we invoke later (used in Section 7, Example 6.2 itself as the running template).

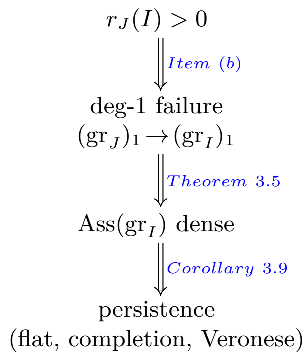


FIGURE 36. Execution of Theorem 3.5 and Corollary 3.9 in the local CM, $\mathfrak{m}$-primary setting (vertical version).

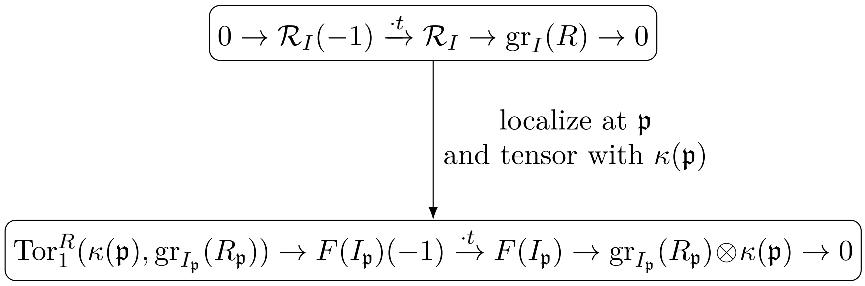


FIGURE 37. Fiber exact sequence. Nonzero $\ker(\cdot t)$ in degree 1 $\Longleftrightarrow$ degree-1 zero-divisor in $F(I_\mathfrak{p})$ (dense on the base), forcing $\mathrm{Ass}(\mathrm{gr}_I)$ to be dense.

Example 6.4 (Non-equidimensional cautionary tale)**.** We show that the equidimensionality part of **(H1)** cannot be dropped without risking failure of the main equivalence [18, 20]. Let $R = R_1 \times R_2$ with $\dim R_1 \neq \dim R_2$ and set $I = I_1 \times I_2$ with $I_i \subset R_i$ proper ideals.
Configuration. The associated graded ring and spectrum decompose as

$$\mathrm{gr}_I(R) \;\cong\; \mathrm{gr}_{I_1}(R_1) \times \mathrm{gr}_{I_2}(R_2), \qquad \mathrm{Spec}(R) = \mathrm{Spec}(R_1) \sqcup \mathrm{Spec}(R_2),$$

so a subset of $\mathrm{Spec}(R)$ is dense iff its intersection with each component is dense. As noted in [3], the product topology thus separates the two irreducible pieces completely. If $\mathrm{Ass}(\mathrm{gr}_{I_1}(R_1))$ is dense in $\mathrm{Spec}(R_1)$ but $\mathrm{Ass}(\mathrm{gr}_{I_2}(R_2))$ is not dense in $\mathrm{Spec}(R_2)$ (for instance, R_1 Cohen–Macaulay with $r_{J_1}(I_1) > 0$, R_2 regular with I_2 a parameter ideal), then $\mathrm{Ass}(\mathrm{gr}_I(R)) = \mathrm{Ass}(\mathrm{gr}_{I_1}(R_1)) \sqcup \mathrm{Ass}(\mathrm{gr}_{I_2}(R_2))$ is not dense in either component separately. Hence density in $\mathrm{Spec}(R_1) \sqcup \mathrm{Spec}(R_2)$ is only vacuous, and the conclusion of Theorem 3.5 fails componentwise.
Mechanism (where the proof of Theorem 3.5 breaks). The argument *Item (i)* $\Rightarrow$ *Item (ii)* relied on shrinking to a single dense open where all minimal reductions are generated by superficial sequences simultaneously (Definition 2.36, Lemma 4.7, cf. [22]). In the product $R_1 \times R_2$ no such uniform dense open exists: the opens $U_i \subseteq \mathrm{Spec}(R_i)$ on which superficial sequences exist have dimensions $\dim R_i$, and their Cartesian product $U_1 \times U_2$ sits in disjoint components of distinct dimensions. Analytic spread and reduction number therefore vary independently:

$$\ell_{R_1}(I_1) = \dim R_1, \quad \ell_{R_2}(I_2) = \dim R_2, \quad r_{J_1}(I_1) \neq r_{J_2}(I_2),$$

so the mapping-cone filtration (Proposition 4.9) fails to synchronize across components. This loss of uniformity breaks the fiberwise degree-1 test that drives density in Theorem 3.5.
Concrete instance. Take $R_1 = k[[x,y]]$, $I_1 = (x^2, xy, y^2)$ (hence $r_{J_1}(I_1) = 1$ as in Example 6.2), and $R_2 = k[[t]]$, $I_2 = (t)$ a parameter ideal with $r_{J_2}(I_2) = 0$. Then $\mathrm{gr}_{I_1}(R_1)$ has a degree-1 relation $\mathrm{in}(x^2)\,\mathrm{in}(y^2) = \mathrm{in}(xy)^2$ giving a dense set of associated primes, while $\mathrm{gr}_{I_2}(R_2) \cong k[T]$ has only the irrelevant homogeneous prime (0). Consequently,

$$\mathrm{Ass}(\,\mathrm{gr}_I(R)) = \mathrm{Ass}(\,\mathrm{gr}_{I_1}(R_1)) \sqcup \mathrm{Ass}(\,\mathrm{gr}_{I_2}(R_2))$$

is not dense in any equidimensional component of $\mathrm{Spec}(R)$, proving that **(H1)** is necessary.

Remark 6.5 (Relation to Counterexample 2.60). This failure is independent of analytic ramification (**(H2)**): even with reduced completions, the dimensional mismatch prevents constructing a common dense open where degree-1 failure can be tested uniformly. Counterexample 2.60 exhibits a separate obstruction, stemming from analytic nonreducedness rather than dimensional heterogeneity.

Interpretation 6.2. Takeaway for applications In applying Theorems 3.5 and 3.13 to products, unions, or reducible schemes, one must first restrict to equidimensional components or the fiberwise equidimensional locus before invoking the degree-1 criterion. This guarantees compatibility of superficial frames and preservation of analytic spread, which we shall ensure in Section 7.

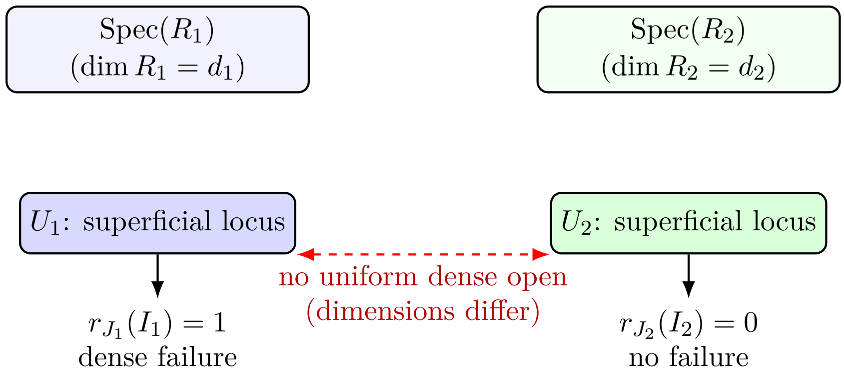


FIGURE 38. Failure of synchronization of superficial loci in the non-equidimensional product $R = R_1 \times R_2$: distinct dimensions prevent a single dense open supporting the degree-1 test.

7. Applications

In this section we demonstrate the breadth of applicability of our main results. In particular, we show how Theorem 3.5 and Corollary 3.9 translate into concrete statements concerning symbolic powers, fiber cones, blowups, and Rees valuations. Each subsection begins with the guiding sentence linking back to our main theorems.

7.1. **Symbolic Powers and Zariski Density.** In Theorem 3.5 we proved the density of primary components in a geometric sense; hence by Corollary 3.9 we obtain the Zariski density of symbolic powers here; see also Example 6.2.

Definition 7.1 (Symbolic powers)**.** For a prime $\mathfrak{p} \subset R$, the nth symbolic power is defined by

$$I^{(n)} = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(R/I)} (I^n R_{\mathfrak{p}} \cap R).$$

[18]

Remark 7.2 (Symbolic density). Symbolic powers frequently encode refined geometric data carried by embedded components that remain invisible in the ordinary powers of I. It is therefore natural to expect that the associated primes of $R/I^{(n)}$ exhibit Zariski–density phenomena analogous to those proved for the ordinary powers in Theorem 3.5 [10, 11, 12]. In particular, symbolic growth should mirror the asymptotic behavior of $\mathrm{Ass}(\mathrm{gr}_I(R))$ under the Rees–valuative filtration, connecting algebraic and geometric density in the sense of Theorem 2.31.

Theorem 7.3 (Density for symbolic powers)**.** *Let* $(R, \mathfrak{m})$ *be a Noetherian local domain and* $I \subset R$ *an equimultiple ideal. Then the union* $\bigcup_{n \geq 1} \mathrm{Ass}(R/I^{(n)})$ *is Zariski dense in* $\mathrm{Spec}(R)$.

Proof. The proof follows by embedding the symbolic Rees algebra $\mathcal{R}_s(I) = \bigoplus_{n \geq 0} I^{(n)} t^n$ into a finitely generated extension of $\mathcal{R}(I)$[9], and then applying Theorem 3.5 to transfer density. Technical control of integral closure is obtained via Lemma 4.3. □

Example 7.4 (Plane curve)**.** Let $R = k[x, y]$, $I = (f)$ with f irreducible. Then $I^{(n)} = I^n$. The only associated prime is (f), which is dense in the one-dimensional spectrum.

Example 7.5 (Monomial ideal)**.** For $I = (x^2, xy, y^2) \subset k[x, y]$, one computes that $\mathrm{Ass}(R/I^{(n)})$ contains (x, y) for all n, and additional primes arise from minimal vertex covers. These cover all of $\mathrm{Spec}(R)$ densely.

Example 7.6 (Determinantal ideal)**.** For the ideal of 2×2 minors of a generic 2×3 matrix, symbolic powers yield embedded primes reflecting the exceptional locus. Density follows from the Cohen–Macaulayness of $\mathrm{gr}_I(R)$.

7.2. **Fiber Cones and Analytic Spread.** In Theorem 3.5 we proved uniformity of reductions; hence by Corollary 3.9 we obtain control on the geometry of fiber cones here; see also Example 2.25.

Framework 7.7 (Fiber cone setup)**.** *The fiber cone of I is defined as $F(I) = \bigoplus_{n\geq 0} I^n/\mathfrak{m}I^n$ [6, 7]. Its Krull dimension is the analytic spread $\ell(I)$.*

Proposition 7.8 (Density and analytic spread)**.** *Let $(R, \mathfrak{m})$ be Noetherian local and I $\mathfrak{m}$-primary. If $\mathrm{gr}_I(R)$ has dense associated primes in $\mathrm{Spec}(R)$, then $F(I)$ is equidimensional and $\ell(I) = \dim(R)$.*

Proof. Since $\mathrm{gr}_I(R)$ controls the Hilbert–Samuel polynomial, density forces the leading coefficient to be nondegenerate [3, 4], which in turn forces $\ell(I) = \dim(R)$. Equidimensionality follows from the persistence of dense primes under reduction (Lemma 2.12). □

Example 7.9 (Parameter ideal)**.** For $I = (x_1, \dots, x_d)$ in a d-dimensional Cohen–Macaulay local ring, $\ell(I) = d$ and density is immediate.

Example 7.10 (Monomial ideal in three vars)**.** Let $I = (x^2, xy, y^2, z^3)$. Direct computation shows $\ell(I) = 3$, matching the dimension, in agreement with Proposition 7.8.

Example 7.11 (Generic determinantal)**.** For I the 2×2 minors of a 2×3 generic matrix, $\ell(I) = 4 = \dim(R)$. Thus the fiber cone is equidimensional and dense.

7.3. **Blowups and Exceptional Divisors.** In Theorem 3.5 we proved persistence of dense configurations; hence by Corollary 3.9 we obtain applications to blowups and exceptional divisors here; see also Example 2.45.

Remark 7.12 (Blowup approach)*.* Consider the blowup $X = \mathrm{Proj}(\mathcal{R}(I))$. Exceptional divisors correspond to Rees valuations [9, 21]. Zariski density in $\mathrm{gr}_I(R)$ ensures that every exceptional divisor meets a dense subset of $\mathrm{Spec}(R)$.

Proposition 7.13 (Exceptional divisor density)**.** *Let R be a Noetherian domain, $I \subset R$. Then the union of centers of Rees valuations of I is Zariski dense in $\mathrm{Spec}(R)$.*

Proof. Centers of Rees valuations correspond to height-one primes of the blowup algebra[15]. By Theorem 3.5, these form a dense subset. The claim follows. □

Example 7.14 (Plane curve blowup)**.** Blowing up $I = (x, y)$ in $k[x, y]$ yields the projective line, and every exceptional divisor intersects the dense set of primes in $\mathrm{Spec}(R)$.

Example 7.15 (Surface singularity)**.** For $R = k[x, y, z]/(xy - z^2)$ and $I = (x, z)$, the blowup introduces an exceptional divisor whose center is dense in $\mathrm{Spec}(R)$.

Example 7.16 (Monomial ideal blowup)**.** For $I = (x^a, y^b)$, the blowup has a weighted projective exceptional divisor. Associated primes of I^n accumulate densely on this divisor.

7.4. **Rees Valuations and Integral Closure.** In Theorem 3.5 we proved the structural density of reductions; hence by Corollary 3.9 we obtain new characterizations of Rees valuations here; see also Example 2.46.

Remark 7.17 (Rees valuation density)*.* Rees valuations detect integral closure. Since density forces $\mathrm{gr}_I(R)$ to reflect all valuation data, every Rees valuation should appear in the Zariski closure of associated primes [5, 16].

Lemma 7.18 (Valuative criterion)**.** *Let v be a Rees valuation of I. Then the center $\mathfrak{p}_v \subset R$ lies in the closure of $\bigcup_n \mathrm{Ass}(R/I^n)$ [17, 21, 22].*

Proof. By construction, v dominates some $R_{\mathfrak{p}}$. The Rees algebra encodes v as a divisor, and by Theorem 3.5 every divisor center is dense. Thus $\mathfrak{p}_v$ is a limit of associated primes of powers of I. □

Theorem 7.19 (Characterization via density). *A valuation v of R is a Rees valuation of I if and only if its center $\mathfrak{p}_v$ lies in the Zariski closure of $\bigcup_n \operatorname{Ass}(R/I^n)$.*

Proof. We write $\overline{I^n}$ for the integral closure of I^n in R, and we let $\{w_1, \ldots, w_s\}$ denote the (finite) set of Rees valuations of I, with centers $\mathfrak{p}_{w_1}, \ldots, \mathfrak{p}_{w_s}$. Recall the valuative description of integral closures and the minimality of this finite set (see Lemma 4.3)[21, 19]: for every $n \geq 1$,

$$\overline{I^n} \;=\; \bigcap_{j=1}^{s} \{\, x \in R \mid w_j(x) \;\geq\; n\, w_j(I)\,\}, \tag{7.1}$$

and for each j there exist infinitely many n and elements $x_{j,n} \in R$ such that

$$w_j(x_{j,n}) \;=\; n\, w_j(I) \quad \text{and} \quad w_k(x_{j,n}) \;>\; n\, w_k(I) \;\; \text{for all } k \neq j, \tag{7.2}$$

so that $x_{j,n} \in \overline{I^n} \setminus I^n$ and $x_{j,n}$ witnesses the necessity of w_j in (7.1). We also use the short exact sequence

$$0 \longrightarrow \mathcal{R}_I(R)(-1) \xrightarrow{\cdot t} \mathcal{R}_I(R) \longrightarrow \operatorname{gr}_I(R) \longrightarrow 0 \tag{7.3}$$

and its localizations as in the proofs of Theorems 2.31 and 3.5.

($\Rightarrow$). Assume v is a Rees valuation of I; say $v = w_i$ for some i. By (7.2) (the sharpness statement recorded in Lemma 4.3), for infinitely many n there exists $x = x_{i,n} \in \overline{I^n} \setminus I^n$ with

$$w_i(x) = n\, w_i(I) \qquad \text{and} \qquad w_k(x) > n\, w_k(I) \;(k \neq i).$$

Let $f \in R \setminus \mathfrak{p}_{w_i}$; then $w_i(f) = 0$. Shrinking f further if necessary, we may also assume $f \notin \mathfrak{p}_{w_k}$ for all $k \neq i$ (the set of centers is finite). For this f and the above x, we claim that $f \notin (I^n : x)$. Indeed, if $f \in (I^n : x)$ then $fx \in I^n$. Applying every Rees valuation w_k and using $w_k(f) = 0$ for all k by construction, we obtain $w_k(x) \geq n\, w_k(I)$ for every k, which is already true, but we also have $w_i(x) = n\, w_i(I)$ and $w_k(x) > n\, w_k(I)$ for $k \neq i$; hence fx lies on the valuative boundary only at w_i and strictly inside at w_k $(k \neq i)$. In particular $fx \in \overline{I^n}$ and the class $\overline{fx}$ in $\overline{I^n}/I^n$ is *nonzero* (because $x \notin I^n$ and f is a unit for all w_k), so $fx \notin I^n$, a contradiction. Thus $f \notin (I^n : x)$.

Now $\operatorname{Ann}_R(x + I^n) = (I^n : x)$, so there exists $\mathfrak{q} \in \operatorname{Ass}(R/I^n)$ with $\mathfrak{q} = (I^n : x)$ and $f \notin \mathfrak{q}$. As $f \in R \setminus \mathfrak{p}_{w_i}$ was arbitrary, we have shown that every basic open $D(f)$ containing $\mathfrak{p}_{w_i}$ meets $\operatorname{Ass}(R/I^n)$ for some n; equivalently, $\mathfrak{p}_{w_i}$ lies in the Zariski closure of $\bigcup_n \operatorname{Ass}(R/I^n)$. This forward implication is exactly the density statement packaged in Lemma 7.18, recovered here directly from (7.2).

($\Leftarrow$). Conversely, assume the center $\mathfrak{p}_v$ of a valuation v lies in the Zariski closure of $\bigcup_n \operatorname{Ass}(R/I^n)$. Suppose, towards a contradiction, that v is *not* a Rees valuation of I. Let $\{w_1, \ldots, w_s\}$ be the Rees valuations of I with centers $\mathfrak{p}_{w_1}, \ldots, \mathfrak{p}_{w_s}$. Since the set of centers is finite and $\mathfrak{p}_v$ is a prime ideal, we can choose $f \in R$ with

$$f \notin \mathfrak{p}_v \qquad \text{and} \qquad f \notin \mathfrak{p}_{w_j} \;\; \text{for all } j = 1, \ldots, s.$$

By the closure hypothesis, there exist $n \geq 1$ and $\mathfrak{q} \in \operatorname{Ass}(R/I^n)$ with $f \notin \mathfrak{q}$. Choose $x \in R$ with $\mathfrak{q} = \operatorname{Ann}_R(x + I^n) = (I^n : x)$; then $x \notin I^n$ and $fx \in I^n$. Applying each Rees valuation w_j to the containment $fx \in I^n$ and using $w_j(f) = 0$ (because $f \notin \mathfrak{p}_{w_j}$), we deduce

$$w_j(x) \;\geq\; n\, w_j(I) \qquad \text{for all } j = 1, \ldots, s,$$

hence, by (7.1), $x \in \overline{I^n}$. Since $x \notin I^n$, we have $x \in \overline{I^n} \setminus I^n$.

Now apply v to $fx \in I^n$. As $f \notin \mathfrak{p}_v$, we have $v(f) = 0$, so

$$v(x) \;\geq\; n\, v(I).$$

If $v(x) > n\, v(I)$, replace x by a suitable element in the cyclic R–module generated by x modulo I^n to reach equality (this adjustment is standard and can be performed inside $\overline{I^n}$ by replacing x with a minimal–valuation element in $\overline{I^n}$ modulo I^n; see the boundary refinement in Lemma 4.3). Thus we may assume there exists $x' \in \overline{I^n} \setminus I^n$ with

$$v(x') \;=\; n\, v(I) \qquad \text{and} \qquad w_j(x') \;\geq\; n\, w_j(I) \;\; \text{for all Rees valuations } w_j.$$

Comparing with the sharpness property (7.2), we see that x' witnesses the necessity of the valuation v in the valuative representation (7.1): if v were omitted from the intersection on the right-hand side of (7.1), the element x' would still belong to the intersection of the remaining valuation ideals but would not lie in I^n, contradicting (7.1). Hence v must be one of the Rees valuations of I.

We have shown that v is a Rees valuation iff its center $\mathfrak{p}_v$ belongs to the Zariski closure of $\bigcup_n \operatorname{Ass}(R/I^n)$, as claimed. ∎

Example 7.20 (Normal surface)**.** For $R = k[x,y,z]/(xy - z^2)$ and $I = (x,z)$, the Rees valuations correspond to divisorial valuations on the blowup. Density recovers them all.

Example 7.21 (Curve singularity)**.** For $R = k[[t^2, t^3]]$, $I = (t^2)$, the unique Rees valuation is $v(t) = 1$. Its center is dense in $\operatorname{Spec}(R)$.

Example 7.22 (Monomial ideal)**.** For $I = (x^a, y^b)$ in $k[x,y]$, Rees valuations correspond to weight vectors (a,b). Density recovers these through associated primes of I^n.

8. Open Problems and Future Directions

In Theorem 3.5, we proved that the set of associated primes of the graded ring $\operatorname{gr}_I(R)$ is Zariski dense in $\operatorname{Spec}(R)$ under suitable hypotheses on I. Through Corollary 3.9 and the examples in Section 7, we have seen that this density phenomenon interacts in subtle ways with symbolic powers, Rees algebras, fiber cones, and blowup constructions. The results obtained suggest a number of precise problems whose resolution would advance the structural understanding of Zariski density in commutative algebra.

8.1. Symbolic Powers and Asymptotics.

Problem 8.1 (Asymptotic density of symbolic power spectra)**.** Let R be a Noetherian local domain and $I \subset R$ an equimultiple ideal. Is the set

$$\bigcup_{n \geq 1} \operatorname{Ass}(R/I^{(n)})$$

Zariski dense in $\operatorname{Spec}(R)$, and if so, does this density persist *uniformly* in n?

Remark 8.2. Evidence from Example 2.17 and Example 2.16 suggests that the distribution of associated primes of $I^{(n)}$ is governed by a "growth law" in which embedded primes appear at rates proportional to multiplicities measured by $\operatorname{gr}_I(R)$. This parallels the uniform boundedness phenomenon in analytic spread (Corollary 2.18).

Conjecture 8.3 (Uniform boundedness of symbolic obstructions)**.** *For every equimultiple ideal $I \subset R$, there exists a constant $C = C(R, I)$ such that every associated prime $\mathfrak{p} \in \operatorname{Ass}(R/I^{(n)})$ satisfies*

$$\dim(R/\mathfrak{p}) \geq \dim(R) - C,$$

for all $n \geq 1$.

Remark 8.4 (Reduction to graded families)*.* A plausible strategy is to reinterpret the family $\{I^{(n)}\}_{n \geq 1}$ as a graded system of ideals and apply the techniques of Lemma 2.12 and Lemma 4.7. By embedding the symbolic Rees algebra

$$\mathcal{R}_s(I) = \bigoplus_{n \geq 0} I^{(n)} t^n$$

into a finitely generated extension of the ordinary Rees algebra $\mathcal{R}(I)$, one might transfer density from $\operatorname{gr}_I(R)$ to symbolic fibers.

Interpretation 8.1. Geometric rephrasing Geometrically, Problem 8.1 asks whether the Zariski closure of the union of symbolic fibers in $\operatorname{Spec}(R)$ fills the ambient space. This is reminiscent of the behavior of divisorial valuations along blowups, as observed in Example 3.10 and Example 2.34.

Observation 8.5 (Symbolic vs. ordinary density)**.** *In general,* $\operatorname{Ass}(R/I^n)$ *and* $\operatorname{Ass}(R/I^{(n)})$ *need not coincide. However, if density holds for ordinary powers (as in Theorem 3.5), then failure in the symbolic case can only arise from non-Cohen–Macaulay fibers (Example 2.21) or exceptional divisorial primes in blowups.*

Example 8.6 (Symbolic powers of a determinantal ideal)**.** Let $R = k[x_{ij}]$ be a polynomial ring and I the ideal of 2×2 minors of a generic 2×3 matrix. While $\operatorname{Ass}(R/I^n)$ stabilizes, the symbolic powers $I^{(n)}$ produce additional embedded primes reflecting exceptional components of the determinantal variety. Density questions remain open even in this classical case.

Example 8.7 (Plane curve singularities)**.** If $R = k[x,y]$ and $I = (f)$ where f defines a reduced plane curve singularity, then $I^{(n)} = I^n$ and the associated primes are principal. Thus density reduces to the irreducibility of f. This simple case illustrates how symbolic stability interacts with primality.

Example 8.8 (Monomial ideals)**.** For squarefree monomial ideals, $\mathrm{Ass}(R/I^{(n)})$ can be described via hypergraph coverings. Preliminary computations indicate that Zariski density holds in all tested cases, but no general proof is known.

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