Counting Exact Prime-Number Inclusion-Exclusion Method

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Abstract

Prime Number Theorem provide asymptotic estimates but do not yield exact results. This paper presents a complete, closed-form mathematical equation that exactly computes the prime-counting function π(N) for any integer N ≥ 2. Unlike existing methods which are either asymptotic approximations or recursive algorithms, our formulation is a single evaluable expression. The equation operates in two distinct modes: (1) using a sequence of known primes, or (2) using the simple sequence $J_1 = 2$, J_n = (n-1)-th odd integer ≥ 3 for n ≥ 2, with an intrinsic primality test $\mu_n = \lceil \prod_{x=1}^{n-1} (J_n/J_x - \lfloor J_n/J_x \rfloor) \rceil$ where $\mu_n = 1$ if and only if J_n is prime. The formula directly yields π(N) through elementary arithmetic operations without recursion, iteration, or algorithmic procedures.

The implications of this formula are explored in comparison to existing prime counting functions and its potential impact on the study of prime distribution, it is an explicit sieve-theoretic expression and a self-contained rewriting. This complements classical exact prime-counting methods (Meissel–Lehmer and descendants), which are vastly more efficient for computation

Introduction

The prime-counting function $\pi(x)$, defined as the number of prime numbers not exceeding x, has been a central object of study in number theory for centuries. While the celebrated Prime Number Theorem provides the asymptotic estimate $\pi(x) \sim \frac{x}{\log x}$, exact computation of $\pi(x)$ has led to recursive algorithms such as Legendre's formula, the Meissel-Lehmer method, and sieve techniques like the sieve of Eratosthenes.

In addition to these recursive and iterative methods, several "closed form" expressions for $\pi(x)$ or the n −th prime have been discovered, typically using Wilson's theorem or summation identities. Notable examples include Willans' formula (1964) and Mináč's formula, which express primality conditions through trigonometric functions and floor operations. While mathematically exact, these formulas are computationally impractical, serving primarily as theoretical curiosities or pedagogical tools.

In this expository note, we present an alternative explicit formulation that directly encodes the inclusion-exclusion principle underlying Legendre's formula. Our expression is self-contained in that it does not require a precomputed list of primes; instead, it operates on a simple sequence of odd numbers and includes a primality indicator constructed from elementary arithmetic functions. The formula is presented as a single mathematical expression that can be evaluated through direct substitution.

Our contribution is not the discovery of a new prime-counting method, but rather the presentation of a complete, self-contained formulation that explicitly demonstrates how the sieve of Eratosthenes can be written as a closed-form expression. This formulation may be particularly useful for pedagogical purposes, as it clearly illustrates the mechanics of inclusion-exclusion in prime counting.

Riemann explicit formula	Analytic (zeros)	Exact when expressed via ζ-zeros; used for analysis.
Meissel–Lehmer methods	Algorithmic (recursive)	Efficient computation of π(x) for large x.
Legendre (ϕ-formula)	Combinatorial (sieve)	π(x)=ϕ(x,a)+a−1 with a=π(⌊√x⌋).
This paper	Explicit inclusion–exclusion	Self-contained ; primarily expository.

Preliminaries: Legendre's Formula and Inclusion-Exclusion

Let $N \geq 2$ be an integer and let $p_1, p_2, \dots, p_a$ denote the primes not exceeding $\sqrt{N}$, where $a = \pi(\sqrt{N})$. Legendre's formula for $\pi(N)$ is

$$\pi(N) = \phi(N, a) + a - 1,$$

where $\phi(N, a)$ counts the integers $\leq N$ that are not divisible by any of the first a primes.

Applying the inclusion-exclusion principle to $\phi(N, a)$ yields the explicit expansion:

$$\phi(N, a) = N - \sum_{i=1}^{a} \left\lfloor \frac{N}{p_i} \right\rfloor + \sum_{1 \leq i < j \leq a} \left\lfloor \frac{N}{p_i p_j} \right\rfloor - \sum_{1 \leq i < j < k \leq a} \left\lfloor \frac{N}{p_i p_j p_k} \right\rfloor + \cdots + (-1)^a \left\lfloor \frac{N}{p_1 p_2 \cdots p_a} \right\rfloor.$$

The formula requires knowing the primes $p_1, \dots, p_a \leq \sqrt{N}$. In the next section, we show how to modify this expression so that it can operate without a precomputed prime list.

Self-Contained Formulation

The Sequence J_n and Primality Indicator μ_n

We define a sequence J_n for $n \geq 1$ as follows:

$J_1 = 2$
For $n \geq 2$:
$J_n =$ the $(n-1)$-th odd integer ≥ 3

$$\mu_n = \left\lceil \prod_{x=1}^{n-1} \left(\frac{J_n}{J_x} - \left\lfloor \frac{J_n}{J_x} \right\rfloor \right) \right\rceil$$

Thus, $J_2 = 3, J_3 = 5, J_4 = 7, J_5 = 9, J_6 = 11, \ldots$.

For $n \geq 2$, we define a primality indicator μ_n by:

$$\mu_n = \left\lceil \prod_{x=1}^{n-1} \left(\frac{J_n}{J_x} - \left\lfloor \frac{J_n}{J_x} \right\rfloor \right) \right\rceil.$$

We also set $\mu_1 = 1$, since $J_1 = 2$ is prime.

Proposition 1 (Primality Indicator Theorem). For $n \geq 2$, $\mu_n = 1$ if and only if J_n is prime.

Proof. If J_n is composite, then it has a divisor J_x with $1 \leq x < n$. For that x, we have $J_x \mid J_n$, so $\frac{J_n}{J_x}$ is an integer, and thus

$$\frac{J_n}{J_x} - \left\lfloor \frac{J_n}{J_x} \right\rfloor = 0.$$

Hence the product in (2) is zero, and $\mu_n = \lceil 0 \rceil = 0$.

If J_n is prime, then for every $x < n$, J_x does not divide J_n, so $\frac{J_n}{J_x}$ is not an integer. Therefore,

$$0 < \frac{J_n}{J_x} - \left\lfloor \frac{J_n}{J_x} \right\rfloor < 1.$$

The product of finitely many numbers in $(0, 1)$ also lies in $(0, 1)$, so the ceiling of the product is 1.

Remark. The product in (2) runs over all $x < n$, but to determine primality it suffices to check divisibility by primes up to $\sqrt{J_n}$. However, the given definition is simpler and does not affect the correctness of the formula.

Inclusion-Exclusion Method

For prime numbers in range (N):

$$\pi(N) = N + \alpha - 1 - \left\lfloor \frac{N}{J_1} \right\rfloor$$

$$- \sum_{n=2}^{\alpha} \left(\left\lfloor \frac{N}{J_n} \right\rfloor \right)$$

$$+ \sum_{n=2}^{\alpha} \sum_{x=1, x<n}^{j} \left\lfloor \frac{N}{J_x \times J_n} \right\rfloor$$

$$- \sum_{x_3=3}^{\alpha} \sum_{x_2=2}^{x_3-1} \sum_{x_1=1}^{x_2-1} \left\lfloor \frac{N}{J_{x_1} \times J_{x_2} \times J_{x_3}} \right\rfloor$$

$$- \sum_{x_4=4}^{\alpha} \sum_{x_3=3}^{x_4-1} \sum_{x_2=2}^{x_3-1} \sum_{x_1=1}^{x_2-1} \left\lfloor \frac{N}{J_{x_1} \times J_{x_2} \times J_{x_3} \times J_{x_4}} \right\rfloor$$

...

$$- \sum_{x_n=\alpha}^{\alpha} \sum_{x_{n-1}=n-1}^{x_n-1} \dots . \sum_{x_2=2}^{x_3-1} \sum_{x_1=1}^{x_2-1} \left\lfloor \frac{N}{J_{x_1} \times J_{x_2} \times \dots \times J_{x_{n-1}} \times J_{x_n}} \right\rfloor$$

where: $N_T(N)$ = Prime numbers in range N, $N \in \mathbb{N} = \{\text{Natural Numbers}\}$, $\sqrt{N}$: principal square root of N, $\alpha = \max\{n : J_n \leq \sqrt{N}\}$, $J_j \leq \left\lfloor \frac{N}{J_n} \right\rfloor$, $\boldsymbol{J} \in \{\textbf{prime numbers}\}$

For Odd numbers in range (N):

$$\pi(N) = N + \alpha - 1 - \left\lfloor \frac{N}{J_1} \right\rfloor$$

$$- \sum_{n=2}^{\alpha} \left(\left\lceil \prod_{x=1}^{n-1} \left(\frac{J_n}{J_x} - \left\lfloor \frac{J_n}{J_x} \right\rfloor \right) \right\rceil \right) \left(\left\lfloor \frac{N}{J_n} \right\rfloor \right)$$

$$+ \sum_{n=2}^{\alpha} \left(\left\lceil \prod_{x=1}^{n-1} \left(\frac{J_n}{J_x} - \left\lfloor \frac{J_n}{J_x} \right\rfloor \right) \right\rceil \right) \left(\sum_{x=1, x<n}^{j} \left\lfloor \frac{N}{J_x \times J_n} \right\rfloor \right)$$

$$- \sum_{x_3=3}^{\alpha} \left(\left\lceil \prod_{x=1}^{n-1} \left(\frac{J_n}{J_x} - \left\lfloor \frac{J_n}{J_x} \right\rfloor \right) \right\rceil \right) \left(\sum_{x_2=2}^{x_3-1} \sum_{x_1=1}^{x_2-1} \left\lfloor \frac{N}{J_{x_1} \times J_{x_2} \times J_{x_3}} \right\rfloor \right)$$

$$- \sum_{x_4=4}^{\alpha} \left(\left\lceil \prod_{x=1}^{n-1} \left(\frac{J_n}{J_x} - \left\lfloor \frac{J_n}{J_x} \right\rfloor \right) \right\rceil \right) \left(\sum_{x_3=3}^{x_4-1} \sum_{x_2=2}^{x_3-1} \sum_{x_1=1}^{x_2-1} \left\lfloor \frac{N}{J_{x_1} \times J_{x_2} \times J_{x_3} \times J_{x_4}} \right\rfloor \right)$$

...

$$- \sum_{x_n=\alpha}^{\alpha} \left(\left\lceil \prod_{x=1}^{n-1} \left(\frac{J_n}{J_x} - \left\lfloor \frac{J_n}{J_x} \right\rfloor \right) \right\rceil \right) \left(\sum_{x_{n-1}=n-1}^{x_n-1} \dots . \sum_{x_2=2}^{x_3-1} \sum_{x_1=1}^{x_2-1} \left\lfloor \frac{N}{J_{x_1} \times J_{x_2} \times \dots \times J_{x_{n-1}} \times J_{x_n}} \right\rfloor \right)$$

Where: $N_T(N)$ = Prime numbers in range N, $N \in \mathbb{N} = \{\text{Natural Numbers}\}$, $\sqrt{N}$: principal square root of N, $\alpha = \max\{n: J_n \leq \sqrt{N}\}$, $J_j \leq \left\lfloor \frac{N}{J_n} \right\rfloor$, $\boldsymbol{J} \in \{\boldsymbol{odd\ numbers}\}$

Note that α is the number of terms in the sequence J_n that are $\leq \sqrt{N}$. By construction, the set $\{J_n: \mu_n = 1, 1 \leq n \leq \alpha\}$ is exactly the set of primes $\leq \sqrt{N}$, so $\alpha \geq \pi(\sqrt{N})$ (with equality if and only if $\sqrt{N}$ is prime or composite but not divisible by any J_n that is prime).

Our self-contained formula for $\pi(N)$ is:

$$\pi(N) = N + \alpha - 1 - \sum_{n=1}^{\alpha} \mu_n \left\lfloor \frac{N}{J_n} \right\rfloor + \sum_{1 \leq i < j \leq \alpha} \mu_i \mu_j \left\lfloor \frac{N}{J_i J_j} \right\rfloor - \sum_{1 \leq i < j < k \leq \alpha} \mu_i \mu_j \mu_k \left\lfloor \frac{N}{J_i J_j J_k} \right\rfloor + \cdots + (-1)^{\alpha} \left\lfloor \frac{N}{J_1 J_2 \cdots J_\alpha} \right\rfloor$$

Theorem 1. Formula correctly computes $\pi(N)$ for any integer $N \geq 2$.

Proof. By Proposition 1, $\mu_n = 1$ if and only if J_n is prime. Therefore, the product $\mu_{i_1} \mu_{i_2} \cdots \mu_{i_m}$ equals 1 precisely when all of $J_{i_1}, J_{i_2}, \ldots, J_{i_m}$ are prime, and 0 otherwise. Consequently, the only nonzero terms in (4) are those corresponding to products of distinct primes among the J_n with $n \leq \alpha$. Since every prime $\leq \sqrt{N}$ appears in the sequence J_n (and only primes yield $\mu_n = 1$), the set of primes $\leq \sqrt{N}$ is exactly $\{J_n: \mu_n = 1, 1 \leq n \leq \alpha\}$. Thus, formula (4) is equivalent to Legendre's formula (1) with $a = \pi(\sqrt{N})$, and hence computes $\pi(N)$ correctly.

Remark. Formula can be written more compactly as:

$$\pi(N) = N + \alpha - 1 + \sum_{\substack{I \subseteq \{1,\ldots,\alpha\} \\ I \neq \emptyset}} (-1)^{|I|} \left(\prod_{i \in I} \mu_i \right) \left\lfloor \frac{N}{\prod_{i \in I} J_i} \right\rfloor.$$

Computational Complexity

Formula contains 2^{α} terms, where $\alpha \approx \pi(\sqrt{N}) \sim \frac{2\sqrt{N}}{\log N}$ by the Prime Number Theorem. This exponential growth makes direct evaluation impractical for large N. Moreover, computing each μ_n requires $n - 1$ divisibility checks (via the product in (2)), leading to an overall complexity that is far worse than that of the sieve of Eratosthenes or the Meissel-Lehmer algorithm.

Thus, this formula *is not intended as a practical computational tool*. Its value lies instead in its mathematical form and pedagogical utility.

Pedagogical Value

The formula serves as an excellent teaching device for illustrating several concepts:

- **Inclusion-Exclusion Principle:** Formula shows exactly how inclusion-exclusion is applied to count numbers not divisible by a set of primes.

- **Legendre's Formula:** It provides a fully expanded, explicit version of Legendre's formula that is often presented only recursively.
- **Primality Testing:** The indicator μ_n demonstrates how primality can be determined using only elementary arithmetic operations, without recourse to Wilson's theorem or other advanced results.
- **Sieve Methods:** The formula explicitly encodes the sieve of Eratosthenes as a single mathematical expression, revealing the underlying structure of sieve methods.
- **Mathematical Writing:** It exemplifies how logical conditions (like divisibility) can be encoded using floor and ceiling functions, a technique useful in various areas of discrete mathematics.

Relation to Other Closed-Form Formulas

Several "closed form" expressions for $\pi(x)$ or the n −th prime exist in the literature. Willans (1964) gave formulas using Wilson's theorem.

Mináč's formula is similar in spirit. These formulas, like ours, are mathematically exact but computationally impractical. Our formula differs in that it is derived directly from the inclusion-exclusion principle and uses a primality indicator based on divisibility rather than Wilson's theorem.

It should be noted that formulas involving summations and floor functions are sometimes called "closed-form" in a discrete sense, though they are not closed-form in the analytic sense (like expressions involving elementary functions only). Our formula falls into this discrete category.

Conclusion

We have presented a self-contained, explicit formula for the prime-counting function $\pi(N)$ that expands Legendre's formula using inclusion-exclusion and includes a built-in primality test. The formula is a single mathematical expression that can be evaluated by direct substitution, requiring no precomputed list of primes and no recursive or iterative procedures.

While computationally inefficient, the formula offers pedagogical clarity, demonstrating how sieve methods can be written in closed form. It may be useful for teaching number theory and discrete mathematics, and it provides a concrete example of how logical conditions can be encoded using elementary arithmetic functions.

Future work could explore variations of the primality indicator or ways to compress the formula to reduce the number of terms, though any such improvements would likely remain of theoretical interest rather than practical computational value.

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AI assistance. The author used AI to assist with drafting and editing prose, improving clarity, and formatting. The author retains full responsibility for the correctness, originality, and integrity of all results and citations.

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