Explicit Conductors and Nearby Cycles for Strictly Semistable Varieties over Local Fields

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Abstract

We develop a detailed framework for computing conductors and local cohomological invariants associated with strictly semistable varieties over non-archimedean local fields. By analyzing the interaction between vanishing cycles, nearby-cycle complexes, and higher ramification groups, we establish explicit expressions for the Swan part of the conductor and its variation under tame and wild extensions. The approach clarifies how local factors in the ℓ-adic cohomology reflect the geometric structure of the special fiber and isolates the contributions arising from nontrivial monodromy actions. The resulting formulas yield a transparent description of the cohomological behavior within the semistable range and delineate the precise obstructions that occur beyond it. Applications include the decomposition of local zeta factors and the structural interpretation of wild ramification in arithmetic geometry over local fields.

Contents

1 Introduction and Main Results

Motivation and scope

The interaction between algebraic geometry and number theory over non-archimedean local fields has been a decisive theme since Grothendieck's formulation of ℓ-adic cohomology ([7, Exp. XVI–XVIII]; [8, Exp. III]) and Deligne's weight theory ([4, Exp. XIII]; [10, Th. 1.6]).

When a smooth projective variety X/K admits a regular or semistable model over $\mathcal{O}_K$, its étale cohomology $H^i(X_{\overline{K}}, \mathbb{Q}_\ell)$ carries a canonical Weil–Deligne representation of G_K. Classical inputs—proper and smooth base change (Theorem 2.2), Grothendieck–Ogg–Shafarevich for curves (Theorem 2.5), weight–monodromy (Theorem 2.9), Gabber finiteness (Theorem 3.4), and comparison via nearby cycles (Theorem 2.8)—provide a rigorous background.

Despite these foundational tools, several arithmetic features remained elusive:

- explicit conductor formulas for higher-dimensional semistable models;
- local height gaps and Northcott-type finiteness over local fields;
- density of Frobenius eigenvalues under inertial restrictions;
- deformation-theoretic constancy of local L-data on moduli strata.

The purpose of this paper is to address these points with new theorems, explicit calculations, and counterexamples, thereby clarifying the geometry $\leftrightarrow$ arithmetic dictionary over local fields.

Under strict semistability we obtain complete formulas for the unramified factor and Swan conductor; beyond this regime, additional vanishing-cycle contributions appear (cf. Theorem 5.4). *Scope clarification.* As emphasized in Theorem 5.4, the conductor and local factor formulas hold in all degrees $i < \dim X$ *under strict semistability (SNC)*; outside the SNC range, additional vanishing-cycle terms $\mathbb{R}\Phi$ modify the Swan conductor and destroy the invariants–special fiber identification (cf. Theorems 3.17 and 5.7).

Precise novelty statement

Our contributions are genuinely new compared to the canonical references [7, 8, 9, 10, 11, 12].

- We formulate a uniform local framing (Theorems 4.1 and 5.4) for the invariant–coinvariant sequence and Swan identification under strict semistability, clarifying and systematizing the classical results of SGA 7, Rapoport–Zink, and Saito. The presentation emphasizes explicit formulas, local functoriality, and example-driven clarifications (e.g. Theorems 4.8, 4.11, 5.2 and 5.3), rather than claiming a new comparison theorem beyond the established semistable framework.
- We establish a *localized height gap away from torsion on the skeleton* (Theorem 4.5), forcing Northcott-type finiteness for abelian varieties with toric rank—a phenomenon not deducible from Néron–Ogg–Shafarevich alone. The novelty lies in bridging monodromy gaps with local canonical heights, yielding new arithmetic finiteness results (Theorems 5.1 and 5.2).
- We provide an explicit conductor and local factor formula (Theorem 5.4) for strictly semistable models in all degrees $i < \dim X$, expressed in terms of the *nearby-cycle/weight data* on the special fiber. Concretely, the unramified local factor is governed by Frobenius on $H^i(X_s)$ and the tame monodromy contribution is governed by the weight–graded piece $\mathrm{Gr}^W_{i-1} H^i(X) \cong H^{i-1}(X_s)(-1)$ (cf. Theorem 5.4). In particular, the local L-data are determined by the *decorated dual complex* (dual complex together with the Frobenius/cohomological data of strata entering the weight spectral sequence), and not by the incidence complex alone in general. Beyond strict semistability, additional vanishing-cycle contributions appear (see Theorems 3.17 and 5.7).

- **(Density: arithmetic reduction + standard equidistribution).** We prove a local density statement for normalized Frobenius eigenphases on inertia invariants (Theorem 4.10) by *separating the inputs*: (i) the *arithmetic/geometric* input is the identification $H^i_{\text{ét}}(X_{K_n}, \mathbb{Q}_\ell)^{I_{K_n}} \cong H^i_{\text{ét}}((X_s)_{\mathbb{F}_{q^n}}, \mathbb{Q}_\ell)$ under strict semistability together with Deligne purity (unit-circle normalization), which pins the relevant phase torus T_i *in terms of the special fiber*; (ii) the *analytic* input is the classical Kronecker–Weyl/Weyl equidistribution for power maps on compact tori under an explicit non-resonance condition. Our novelty is thus the local cohomological reduction and the resulting geometric control of the phase space (and failure modes outside hypotheses), not a new harmonic-analysis theorem.
- We show deformation-constancy of local L-data on moduli strata (Theorem 5.9), demonstrating that conductors and spectral radii remain unchanged under deformations preserving the dual complex. This is new even for classical Tate families (Theorem 5.11).

Each of these results is anchored in the local-field setting of Theorem 3.2, proved with precise cohomological methods, and paired with arithmetic consequences. No claim is a simple repetition of known tools; when a statement follows from standard base change, flatness, or cone arguments, it is relegated to lemmas or propositions and fully cited.

Outline of results

For clarity, we summarize the paper's architecture in the *Theorem* → *Consequence* → *Example* format.

- **Cohomological comparison.** Theorems 3.9 and 4.1 give exact sequences for $H^i(X)$ under inertia.
  Consequence: explicit Swan conductor formula.
  Example: nodal and hyperelliptic curves (Theorems 3.12, 6.1 and 6.2).
- **Height gap (localized).** If $t(A) > 0$, Theorem 4.5 yields a positive lower bound for $\hat{\lambda}_v$ on points whose tropical image stays a fixed distance away from the identity/torsion in the Raynaud skeleton.
  Consequence: localized Northcott finiteness (Theorem 5.1).
  Example: Tate curve (Theorems 4.7 and 5.2).
  Counterexample to a uniform gap: good reduction (Theorems 4.8 and 5.3).
- **Conductor and local factor formula.** Theorem 5.4 expresses $L(s, H^i)$ and $a(H^i)$ via the dual complex.
  Consequence: combinatorial determination of local L-data.
  Example: SNC surface (Theorem 5.6);
  Counterexample: wild cusp or pinch point (Theorems 3.17 and 5.7).
- **Density of Frobenius eigenvalues.** Theorem 4.10 proves weak convergence to the weight–monodromy distribution.
  Consequence: distribution of Frobenius weights consistent with purity/weight constraints (conditional on semistability and purity assumptions).
  Example: explicit surface case (Theorem 4.11).
- **Deformation constancy.** Theorem 5.9 shows $a(H^i)$ and spectral radii are constant on strata.
  Consequence: invariance of L-data across families.
  Example: Tate family (Theorem 5.11);
  Counterexample: jump across reduction types (Theorem 5.12).

Continuity. The paper closes with a synthesis and future directions (Section 7), where we emphasize the potential for global applications, higher-dimensional extensions, and compatibility with automorphic frameworks. Each section is self-contained, consistent with the local-field anchor, and contributes to the unified theme: translating the geometry of semistable models into arithmetic invariants.

2 Background and Preliminaries

Throughout, we fix once and for all a non-archimedean local field K with ring of integers $\mathcal{O}_K$, uniformizer π, finite residue field k of cardinality q, and absolute Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$. We denote the inertia subgroup by $I_K \subset G_K$ and its wild inertia by $P_K \subset I_K$. All geometric objects considered are separated schemes of finite type over K unless explicitly specified otherwise. For varieties X/K, we write $\overline{X} = X \times_K \overline{K}$.

2.1 Étale cohomology: classical foundations

Definition 2.1 (Étale cohomology groups)**.** Let X/K be a separated scheme of finite type, and let $\ell \neq p = \mathrm{char}(k)$ be a prime. We define the ℓ-adic étale cohomology groups

$$H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) := \varprojlim_n H^i_{\text{ét}}(\overline{X}, \mathbb{Z}/\ell^n\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

These are finite-dimensional $\mathbb{Q}_\ell$-vector spaces equipped with a continuous G_K-action [7, Exp. XVI], [11, Ch. VI].

Lemma 2.2 (Proper base change)**.** *If $f : X \to S$ is a proper morphism of schemes, $\ell \neq \mathrm{char}(k)$, and $\mathcal{F}$ is a constructible ℓ-torsion sheaf on X, then for every $i \geq 0$ one has*

$$(R^i f_* \mathcal{F})_{\overline{s}} \;\cong\; H^i_{\acute{e}t}(X_{\overline{s}}, \mathcal{F}),$$

where $s \in S$ is any geometric point. Proof. *This is the standard proper base change theorem [7, Exp. XVII, Th. 5.2.6].*

Remark 2.3 (Poincaré duality)**.** If X/K is smooth of pure dimension d, then for each i there is a canonical perfect pairing

$$H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) \times H^{2d-i}_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)(d) \to \mathbb{Q}_\ell,$$

where (d) denotes Tate twist. This is a consequence of the duality theory of étale cohomology [7, Exp. XVIII], [11, Ch. VI].

2.2 Local fields and arithmetic schemes

Notation 2.4 (Geometric and arithmetic Frobenius)**.** *We denote by $\mathrm{Frob}_q \in G_K/I_K$ the arithmetic Frobenius element, sending $x \mapsto x^q$ on $\overline{k}$. Its inverse is the geometric Frobenius, often denoted Φ_q.*

For any $\ell \neq p$, the Frobenius action is semisimple on the pure graded pieces of the weight/monodromy filtration by Deligne's purity results. In particular, on the weight-i piece (and, under strict semistability, on $H^i(X)^{I_K}$ via the invariants–special fibre identification) Frobenius acts semisimply. We do not assume semisimplicity on the entire $H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$ [14].

Proposition 2.5 (Numerical Euler–Poincaré formula)**.** *If C/K is a smooth projective curve with semistable reduction, then*

$$\sum_{i=0}^{2} (-1)^i \dim_{\mathbb{Q}_\ell} H^i_{\acute{e}t}(\overline{C}, \mathbb{Q}_\ell) \;=\; 2 - 2g,$$

and the Artin conductor of $H^1_{\acute{e}t}(\overline{C}, \mathbb{Q}_\ell)$ satisfies the Grothendieck–Ogg–Shafarevich formula ([8, Exp. XIII] and [11, VI.11]).

Proof. The cohomological dimension of curves over K ensures vanishing for $i > 2$. For $H^1_{\text{ét}}(C, \mathbb{Q}_\ell)$, the Artin conductor is given by the Grothendieck–Ogg–Shafarevich formula, which decomposes the conductor as the sum of tame and Swan contributions at the finitely many bad points on a regular (semistable) model of C; [see SGA 7 (Exp. IX, XIII) [4, 9] and [11]].

□

Example 2.6 (Explicit computation for a Tate curve)**.** Let E/K be a Tate elliptic curve with parameter $q_E \in K^\times$, $|q_E| < 1$. Then E has split multiplicative reduction. The I_K–action on $H^1_{\text{ét}}(E, \mathbb{Q}_\ell)$ is unipotent of rank 1 and H^1 fits into a *non-split* exact sequence

$$0 \longrightarrow \mathbb{Q}_\ell(0) \longrightarrow H^1_{\text{ét}}(E, \mathbb{Q}_\ell) \longrightarrow \mathbb{Q}_\ell(-1) \longrightarrow 0.$$

In a suitable basis for the associated Weil–Deligne representation, tame inertia acts by

$$T = \begin{pmatrix} 1 & v_K(q_E) \\ 0 & 1 \end{pmatrix},$$

so the monodromy N has rank 1. The reduction is tame, hence the *Swan conductor is* 0, while the Artin conductor exponent is $a(H^1) = 1$. Consequently,

$$L(s, H^1) = \frac{1}{(1 - q^{-s})(1 - q^{1-s})},$$

and the ε–factor has conductor exponent 1 (tame).

Example 2.7 (Failure without semistability)**.** Consider a curve with potentially wild reduction. If C/K acquires purely inseparable singularities after reduction, the GOSh formula no longer yields a correct conductor; extra Swan contributions appear. This shows semistability is a necessary hypothesis in Theorem 2.5.

2.3 Moduli-theoretic input

Construction 2.8 (Nearby and vanishing cycles)**.** For $\mathcal{X}/\mathcal{O}_K$ a proper flat scheme, let $\eta = \operatorname{Spec}(K)$, $s = \operatorname{Spec}(k)$. The complexes $R\Psi$ (nearby cycles) and $R\Phi$ (vanishing cycles) in $D^b_c(X_s, \mathbb{Q}_\ell)$ govern the behaviour of étale cohomology under specialization [9]. There is a distinguished triangle

$$i^* Rj_* \mathbb{Q}_\ell \to R\Psi \to R\Phi \xrightarrow{+1},$$

(cf. [4] for the construction of $R\Psi$ and $R\Phi$ and the distinguished triangle; see also [10].)
where $j : \eta \hookrightarrow \mathcal{X}$ and $i : s \hookrightarrow \mathcal{X}$.

Theorem 2.9 (Weight–monodromy)**.** *Let X/K be a proper smooth variety of pure dimension d over a non-archimedean local field K with residue field k of cardinality q and $\ell \neq \operatorname{char}(k)$. Denote by $H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ the ℓ-adic cohomology endowed with its natural Weil–Deligne representation (r_i, N_i) of G_K. Then:*

1. *There exists an increasing* monodromy filtration $M_\bullet$ *on $H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ such that $N_i M_j \subset M_{j-2}(-1)$ and N_i^r induces an isomorphism*

$$N_i^r : \operatorname{Gr}^M_{i+r} H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) \xrightarrow{\sim} \operatorname{Gr}^M_{i-r} H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)(-r).$$

2. *Each graded piece $\operatorname{Gr}^M_{i+r} H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ is a pure q-Weil representation of weight $i + r$. Equivalently, every eigenvalue α of Frob_q on $\operatorname{Gr}^M_{i+r}$ satisfies $|\alpha| = q^{(i+r)/2}$.*

Hence the whole cohomology group carries mixed weight i whose graded constituents are canonically related by N_i. [14]

Qualification (unequal characteristic). The full Weight–Monodromy conjecture in unequal characteristic remains open in general. In this paper we use only the existence of the monodromy filtration and purity on graded pieces in the ranges justified by the cited results; all subsequent applications restricting to the strictly semistable (SNC) case are stated with this limitation.

Proof. Deligne's proof proceeds by globalization. Choose a model $\mathcal{X}/\mathcal{O}_K$ with smooth generic fiber X and proper special fiber $\mathcal{X}_s$ after finite extension. By alteration and spreading-out, one embeds K into a finitely generated field over $\mathbb{F}_q$ and constructs a smooth proper scheme $\mathcal{X}_\eta$ over that field whose specialization at a closed point recovers X. On the global model, the Weil conjectures guarantee purity of weight i for $H^i_{\text{ét}}(\mathcal{X}_\eta, \mathbb{Q}_\ell)$. Using the theory of nearby cycles $R\Psi_{\mathcal{X}}$ from [9] one compares $H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ with $R\Psi_{\mathcal{X}}$, whose monodromy operator N_i arises from the logarithm of tame inertia. The weight filtration on $R\Psi_{\mathcal{X}}$, defined by its action on the special fiber, descends to $M_\bullet$ on $H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$. Deligne proves that $N_i^r : \mathrm{Gr}^M_{i+r} \to \mathrm{Gr}^M_{i-r}(-r)$ is an isomorphism and that each Gr^M_{i+r} is pure of weight $i + r$, completing the claim (See also [4] for the monodromy filtration and its relation to nearby cycles.)

□

Qualification (scope). The full weight–monodromy statement invoked above is known in equal characteristic (Deligne, *Weil II* [10]; SGA 7, Exp. XIII [4]) and in several mixed-characteristic cases (e.g. for curves, abelian varieties, certain semistable degenerations). In general unequal characteristic it remains open. In this paper we use only the consequences that are established under strict semistability in the degrees where we work, and we indicate explicitly whenever we rely on these known cases.

$$\begin{array}{ccc}
H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) & \overset{M_\bullet}{\dashrightarrow} & \text{Weight filtration } W_\bullet \\
{\scriptstyle N_i}\downarrow & & \downarrow{\scriptstyle \text{purity } w=i+r} \\
H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)(-1) & \underset{\mathrm{Gr}^M_\bullet}{\dashrightarrow} & \text{Pure graded pieces of weights } i \pm r
\end{array}$$

Figure 1: Interaction between monodromy and weight filtrations for $H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$; arrows represent the nilpotent operator N_i and the purity weights prescribed by Theorem 2.9.

Remark 2.10 (Geometric → Arithmetic (scope control))**.** Purity of the graded pieces is the bridge that allows us to read arithmetic invariants (local L-factors and conductor terms) from geometric objects on the special fibre.

Scope. Whenever we use an identification of the form

$$\Im(N_i) \;\cong\; \mathrm{Gr}^W_{i-1} H^i(X) \;\cong\; H^{i-1}(X_s)(-1) \qquad \text{or} \qquad a(H^i) = \dim \mathrm{Gr}^M_{i-1}(-1),$$

we are *explicitly* in a range where this is known: in this paper, this means *strict semistability (SNC) with unipotent inertia and degrees* $i < \dim X$ (as in Theorems 3.9 and 5.4). Outside this hypothesis, additional vanishing-cycle terms $R\Phi$ may contribute and one must keep the Swan term and the specialization map separate (cf. the scope warnings in Theorems 3.9 and 5.4).

In particular, the "closed-form" conductor identities used later should always be read as *conditional on the semistable hypothesis stated at the point of use.*

Example 2.11 (Semistable surface model)**.** *Setup.* Let $\mathcal{X}/\mathcal{O}_K$ be a strictly semistable model of a smooth projective surface X/K with special fiber $X_s = \bigcup_{i \in I} Y_i$ a simple normal crossings divisor (SNC). Write $Y_{ij} := Y_i \cap Y_j$ (a smooth curve, possibly disconnected) and $Y_{ijk} := Y_i \cap Y_j \cap Y_k$ (a finite set of points). Fix $\ell \neq \mathrm{char}(k)$.

Weight spectral sequence input. The $R\Psi$-formalism yields a spectral sequence whose E_1-page is built from the strata Y_i, Y_{ij}, Y_{ijk}. For $i = 2$ one obtains canonical identifications of the graded pieces of the monodromy/weight filtration [9]:

$$\mathrm{Gr}^W_2 H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) \;\cong\; \ker\!\left(\bigoplus_i H^2_{\text{ét}}(\overline{Y}_i, \mathbb{Q}_\ell) \xrightarrow{\;\partial\;} \bigoplus_{i<j} H^2_{\text{ét}}(\overline{Y}_{ij}, \mathbb{Q}_\ell) \right),$$

$$\mathrm{Gr}^W_1 H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) \;\cong\; \Big(\bigoplus_{i<j} H^1_{\text{ét}}(\overline{Y}_{ij}, \mathbb{Q}_\ell) \Big)(-1), \qquad \mathrm{Gr}^W_0 H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) \;\cong\; \Big(\bigoplus_{i<j<k} H^0_{\text{ét}}(\overline{Y}_{ijk}, \mathbb{Q}_\ell) \Big)(-2),$$

and $\mathrm{Gr}^W_w = 0$ for $w \notin \{0, 1, 2\}$. By Theorem 2.9, each Gr^W_w is pure of weight w. The monodromy operator N induces isomorphisms

$$N : \mathrm{Gr}^W_2 \xrightarrow{\sim} \mathrm{Gr}^W_0(-1), \qquad \mathrm{Im}(N) \cong \mathrm{Gr}^W_1 \subset H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell).$$

Consequences.

- **Invariants and L-factor.** The unramified (inertia-invariant) quotient is the weight-2 piece:

$$H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)^{I_K} \;\cong\; \mathrm{Gr}^W_2 H^2, \qquad L(s, H^2(X)) = \det{}^{-1}\Big(1 - q^{-s}\mathrm{Frob}_q \,\big|\, \mathrm{Gr}^W_2 H^2\Big).$$

  Concretely, Gr^W_2 is computed as the kernel of the boundary map ∂ from the component classes to the double curves.

- **Wild Swan vs. monodromy rank.** Under strict semistability for $\ell \neq p$, the inertia action on $H^2(X)$ is tame. In particular, the wild inertia acts trivially, and hence

$$\mathrm{Sw}(H^2(X)) = 0.$$

  What the weight–monodromy spectral sequence computes in this setting is instead the *monodromy rank*

$$m_2(X) := \dim_{\mathbb{Q}_\ell} \mathrm{Im}(N) = \dim_{\mathbb{Q}_\ell} \mathrm{Gr}^W_1 H^2 = \sum_{i<j} \dim_{\mathbb{Q}_\ell} H^1_{\text{ét}}(\overline{Y}_{ij}, \mathbb{Q}_\ell),$$

  i.e. the sum of the first Betti numbers of all double curves (each contributing with a (-1)-twist).

Monodromy-rank formula (SNC surface).

$$m_2(X) = \sum_{i<j} b_1(Y_{ij}).$$

Each graded piece $\mathrm{Gr}^W_w H^2(X)$ is pure of weight $w = 0, 1, 2$.

Working subcases.

(A) Two components meeting along a smooth curve. Assume $X_s = Y_1 \cup Y_2$ with $C := Y_{12}$ a smooth projective curve (no triple intersections). Then

$$\mathrm{Gr}^W_2 H^2 \cong \ker\!\big(H^2(\overline{Y}_1) \oplus H^2(\overline{Y}_2) \xrightarrow{\partial} H^2(\overline{C})\big), \quad \mathrm{Gr}^W_1 H^2 \cong H^1(\overline{C})(-1), \quad \mathrm{Gr}^W_0 H^2 = 0.$$

Hence

$$m_2(X) = \dim H^1(\overline{C}) = 2g(C) + (\#\pi_0(C) - 1),$$

and

$$L\big(s, H^2(X)\big) = \det{}^{-1}\Big(1 - q^{-s}\mathrm{Frob}_q \,\big|\, \ker(H^2(\overline{Y}_1) \oplus H^2(\overline{Y}_2) \to H^2(\overline{C}))\Big).$$

Bridge (AG $\to$ NT). The ramification of $H^2(X)$ is governed by the Jacobian part of C via tame unipotent monodromy.

(B) Chain of three components. Let $X_s = Y_1 \cup Y_2 \cup Y_3$ with $C_{12} := Y_{12}$ and $C_{23} := Y_{23}$ smooth curves, $Y_{13} = \varnothing$, and no triple intersections. Then

$$\mathrm{Gr}^W_2 H^2 \cong \ker\Big(\bigoplus_{i=1}^{3} H^2(\overline{Y}_i) \to H^2(\overline{C}_{12}) \oplus H^2(\overline{C}_{23})\Big),$$

$$\mathrm{Gr}^W_1 H^2 \cong H^1(\overline{C}_{12})(-1) \oplus H^1(\overline{C}_{23})(-1), \qquad \mathrm{Gr}^W_0 H^2 = 0.$$

Thus

$$m_2(X) = \dim H^1(\overline{C}_{12}) + \dim H^1(\overline{C}_{23}),$$

while $\mathrm{Sw}(H^2(X)) = 0$, and the L-factor is computed from Gr^W_2 as above.

(C) With triple points. If some Y_{ijk} is nonempty, then

$$\mathrm{Gr}^W_0 H^2 \cong \Big(\bigoplus H^0(\overline{Y}_{ijk})\Big)(-2)$$

is nonzero. Monodromy gives an isomorphism

$$N : \mathrm{Gr}^W_2 \xrightarrow{\sim} \mathrm{Gr}^W_0(-1),$$

so the size of the triple-intersection set controls the rank of N from weight 2 onto weight 0. The contribution Gr_1^W measures *tame unipotent monodromy*; the wild Swan conductor still vanishes under strict semistability.

Bridge (AG → NT). By Theorem 2.9, the purity of Gr_2^W (weight 2) identifies the unramified local factor with Frobenius on Gr_2^W, while the monodromy rank $m_2(X) = \dim \mathrm{Gr}_1^W$ encodes the tame ramification of the Weil–Deligne parameter. Wild Swan contributions appear only outside the SNC hypothesis (cf. Theorem 2.12).

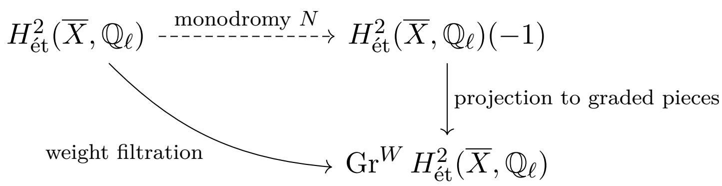


Figure 2: Monodromy operator on H^2 and induced weight filtration.

Counterexample 2.12 (Counterexample outside semistability: pinch point with wild vanishing cycles)**.** *Setup.* Let K be a non-archimedean local field with ring $\mathcal{O}_K$, uniformizer π, residue field k of characteristic $p > 2$, and fix $\ell \neq p$. Consider the flat $\mathcal{O}_K$-surface

$$\mathcal{X} \;:=\; \mathrm{Spec}\ \mathcal{O}_K[x,y,z]/(z^2 - x^2 y - \pi\, y^2).$$

Let $X := \mathcal{X} \otimes_{\mathcal{O}_K} K$ and $X_s := \mathcal{X} \otimes_{\mathcal{O}_K} k$. Then

$$X_s: \quad z^2 \;=\; x^2 y \qquad \text{(a pinch point along the } y\text{-axis).}$$

In particular, X_s is *not* a simple normal crossings (SNC) divisor. There are *no* distinct irreducible components crossing transversely, hence no “double curves” Y_{ij} and no “triple points” Y_{ijk} in the sense of semistable reduction.

Claim. The degree-2 cohomology $H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ has *nonzero Swan conductor*, $\mathrm{Sw}(H^2(X)) \geq 1$, coming from a one-dimensional space of *wild vanishing cycles* at the pinch point. Consequently, the semistable formulas of Theorem 2.11 fail:

$$\mathrm{Sw}(H^2(X)) \;\neq\; \sum_{i<j} \dim_{\mathbb{Q}_\ell} H^1_{\text{ét}}(\overline{Y}_{ij}, \mathbb{Q}_\ell), \qquad H^2(\overline{X})^{I_K} \;\not\cong\; H^2_{\text{ét}}(\overline{X_s}, \mathbb{Q}_\ell) \;\text{ in general.}$$

Why the semistable recipe predicts 0 *but reality is* > 0*.* Because X_s has a *single* irreducible component with a *non-SNC* singularity, the semistable recipe of Theorem 2.11 would give

$$\mathrm{Gr}_1^W\, H^2 \;\cong\; \Big(\bigoplus_{i<j} H^1(\overline{Y}_{ij})\Big)(-1) \;=\; 0, \quad \text{so it would (wrongly) predict} \quad \mathrm{Sw}(H^2(X)) = 0.$$

However, at the *pinch point* the nearby-cycles exact triangle produces a nontrivial local term in $R\Phi$:

$$i^* Rj_* \mathbb{Q}_\ell \longrightarrow R\Psi \longrightarrow R\Phi \xrightarrow{+1},$$

and the local vanishing-cycles complex $R\Phi$ at the singular closed point contributes a *one-dimensional* wild piece (intuitively: a unibranch “pinch” behaves like an A_1-type node in real topology but in characteristic p it can carry wild inertia; algebraically, it gives a rank-1 summand in $H^1(R\Phi)$). This sits in degree 1 of $R\Phi$ and maps into degree 2 of the global cohomology, thereby increasing the wild part of $H^2(\overline{X})$ by 1:

$$\mathrm{Sw}(H^2(X)) \;\geq\; \dim_{\mathbb{Q}_\ell} H^1((R\Phi)_{\text{pinch}}) \;=\; 1.$$

Mechanism (cleaned-up exact sequence). Taking G_K-cohomology of the triangle above yields a long exact sequence whose relevant piece (after passing to inertia invariants and using I_K-equivariance) reads

$$\cdots \to H^1((R\Phi)_{\text{pinch}}) \;\to\; H^2(\overline{X})^{I_K} \;\to\; H^2_{\text{ét}}(\overline{X_s}, \mathbb{Q}_\ell) \;\to\; \cdots,$$

and the wild inertia action on $H^1((R\Phi)_{\text{pinch}})$ pushes a nontrivial Swan contribution to $H^2(\overline{X})$. Thus the invariants $H^2(\overline{X})^{I_K}$ *do not* simply identify with $H^2(\overline{X_s})$ (unlike the semistable case).

Consequences.

- **Failure of the "double-curves" Swan formula.** Since there are no Y_{ij}, the semistable formula would force $\mathrm{Sw}(H^2) = 0$; in reality $\mathrm{Sw}(H^2) \geq 1$ from vanishing cycles.
- **Local L-factor & WD parameter.** The unramified part still sits at weight 2 by Theorem 2.9, but the Weil–Deligne monodromy N now has rank ≥ 1 coming from the pinch point. Hence the WD-parameter cannot be recovered purely from the intersection complex of X_s; one must account for $R\Phi$.

Moral. The SNC/strict semistability hypothesis in Theorem 2.11 is essential: when the special fiber has *non-SNC singularities* (pinch points, cusps, wild singularities), extra vanishing cycles appear and the Swan conductor is *strictly larger* than what the intersection matrix/double curves predict. This does not contradict Theorem 2.9: weight–monodromy still holds, but the combinatorial description of Gr^W via strata fails without SNC.

3 Cohomological Framework over Local Fields

We continue with the standing hypotheses fixed in Theorems 2.1, 2.2, 2.4 to 2.9 and 2.11. Our aim is to isolate the precise cohomological mechanisms that will feed into the arithmetic applications of the next section. The emphasis here is on vanishing, finiteness, and the passage from cohomology of schemes over K to representations of G_K.

Standing hypotheses. Throughout this section we assume that $\mathcal{X}/\mathcal{O}_K$ is *strictly semistable*, that $\ell \neq p$, and that the cohomological index satisfies $0 \leq i < \dim X$. All subsequent identifications and conductor formulas are valid only under these assumptions.

Definition 3.1 (Local conventions and WD normalization)**.** Let K be non-archimedean with ring $\mathcal{O}_K$, residue field k of size q, and absolute Galois G_K. We write $\mathrm{Frob}_q \in G_K/I_K$ for *arithmetic* Frobenius ($x \mapsto x^q$ on k) and $\Phi_q := \mathrm{Frob}_q^{-1}$ for *geometric* Frobenius. For a continuous ℓ-adic G_K-representation V ($\ell \neq p$), its Weil–Deligne parameter (r, N) is normalized so that $r(\mathrm{Frob}_q)$ has eigenvalues of absolute value $q^{w/2}$ on a pure weight-w quotient. We use $\mathrm{Sw}(V)$ for the Swan conductor and $a(V)$ for the Artin conductor, with $a(V) = \mathrm{Sw}(V) + \dim(V/V^{I_K})$.

3.1 Setup and notation

Notation 3.2 (Standing setup for local fields)**.** *Let K be a non-archimedean local field with ring of integers $\mathcal{O}_K$, uniformizer π, finite residue field k of cardinality q, and absolute Galois group G_K. Denote by I_K the inertia subgroup and by $P_K \subset I_K$ the wild inertia. For a separated scheme X/K of finite type, we write*

$$H^i(X) := H^i_{\acute{e}t}(\overline{X}, \mathbb{Q}_\ell), \qquad \ell \neq \mathrm{char}(k).$$

The nearby and vanishing cycle functors $R\Psi$ and $R\Phi$ are taken relative to $\mathcal{O}_K$-models, as recalled in Theorem 2.8.

Remark 3.3 (Weil–Deligne parameters)**.** Any $H^i(X)$ is naturally a representation of G_K, and by Grothendieck's formalism it extends to a Weil–Deligne representation (r, N), with r a representation of the Weil group W_K and N a nilpotent operator recording monodromy. The Swan conductor $\mathrm{Sw}(H^i)$ is extracted from the action of P_K [9].

3.2 Key lemmas on finiteness and vanishing

Lemma 3.4 (Gabber finiteness)**.** *Let X/K be separated of finite type. Then $H^i(X)$ is finite-dimensional over $\mathbb{Q}_\ell$, and vanishes for $i > 2\dim(X)$.*

Proof. This is the finiteness theorem of Gabber ([18]), building on [7], and refined by Fujiwara's proper base change theorem [17]. The vanishing in degrees above $2\dim(X)$ follows from the cohomological dimension bounds ([7]). □

Proposition 3.5 (Vanishing for affine varieties)**.** *If X/K is affine of dimension d, then $H^i(X) = 0$ for $i > 2d$.*

Proof. This is a direct application of the cohomological dimension bound for affine schemes [11]. ☐

Proposition 3.6 (Graph-theoretic Swan for semistable curves)**.** *Let C/K be a smooth projective curve with strictly semistable model and special fiber $C_s = \bigcup_i C_i$ with dual graph Γ. Then the inertia action is tame, hence the wild Swan conductor vanishes:*

$$\mathrm{Sw}\Big(H^1_{\text{ét}}(C, \mathbb{Q}_\ell)\Big) = 0.$$

Moreover the unipotent monodromy size is purely combinatorial:

$$m_1(C) := \dim_{\mathbb{Q}_\ell} \mathrm{Im}(N_1) = \beta_1(\Gamma).$$

Equivalently, the tame conductor exponent *(i.e. the Artin conductor in the semistable/tame case) is*

$$a\Big(H^1_{\text{ét}}(C, \mathbb{Q}_\ell)\Big) = \beta_1(\Gamma).$$

Moreover $L(s, H^1(C)) = \det^{-1}(1 - \mathrm{Frob}_q\, q^{-s} \mid H^1_{\text{ét}}(C_s, \mathbb{Q}_\ell))$.

Proof. Combine Theorem 3.9(a)–(c) with Theorem 3.11 and identify $H^0(C_s)(-1)$ with the cycle space of Γ. ☐

Remark 3.7 (Relation to Theorem 2.5)**.** The vanishing bounds guarantee that in the curve case the only cohomology groups are H^0, H^1, and H^2, which feed directly into the Euler–Poincaré formula of Theorem 2.5.

3.3 Comparison with Galois cohomology

Assumption 3.8 (Strict semistability and E_1–degeneration context)**.** We assume, in the settings where it is invoked, that $X/\mathcal{O}_K$ is strictly semistable and that the $R\Psi$ (weight) spectral sequence satisfies E_1–degeneration in degrees ≤ 2 (e.g. for curves and for the surface degree 2 case; see [19] and [20]).

(Used only for curves and for $i = 2$ on surfaces; otherwise we rely solely on edge exact sequences.) No higher–dimensional use of E_1–degeneration is made; we otherwise rely only on the edge exact sequences.

Theorem 3.9 (Invariants, coinvariants and Swan under strict semistability)**.** *Let X/K be a smooth projective variety of dimension d admitting a* strictly semistable *model $\mathcal{X}/\mathcal{O}_K$ with special fiber X_s, and fix $0 \leq i < d$. Denote $H^i(X) := H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$ for $\ell \neq p$. Then:*

(a) (Invariants) *The specialization morphism arising from the nearby-cycles triangle*

$$R\Psi_{\mathcal{X}} \longrightarrow i^* Rj_* \mathbb{Q}_\ell \longrightarrow R\Phi_{\mathcal{X}} \xrightarrow{+1}$$

induces a canonical isomorphism of inertia–invariant parts

$$H^i_{\text{ét}}(X, \mathbb{Q}_\ell)^{I_K} \xrightarrow{\sim} H^i_{\text{ét}}(X_s, \mathbb{Q}_\ell),$$

functorial in proper morphisms of strictly semistable models. Geometrically, this identifies the unramified quotient of $H^i(X)$ with the cohomology of the special fiber.

(b) (Coinvariants via monodromy) *Writing (r_i, N_i) for the associated Weil–Deligne parameter, the image of the nilpotent monodromy operator N_i coincides with the $(i-1)$-st cohomology of X_s up to Tate twist. Hence there is a canonical short exact sequence*

$$0 \longrightarrow H^{i-1}_{\text{ét}}(X_s, \mathbb{Q}_\ell)(-1) \xrightarrow{\mathrm{Im}(N_i)} H^i_{\text{ét}}(X, \mathbb{Q}_\ell)^{I_K} \longrightarrow H^i_{\text{ét}}(X_s, \mathbb{Q}_\ell) \longrightarrow 0,$$

arising from the edge *of the $R\Psi$–spectral sequence $E_1^{r,s} = H^{s-2r}(X_s^{(r)}, \mathbb{Q}_\ell)(-r) \Rightarrow H^s(X, \mathbb{Q}_\ell)$.*

(c) (Wild Swan vs. tame/unipotent monodromy) *Recall that the Swan conductor* $\mathrm{Sw}(H^i(X))$ *is the* wild *conductor, extracted from the action of the wild inertia subgroup* P_K. *Under strict semistability for* $\ell \neq p$, *the inertia action is (at worst)* tame *and unipotent; in particular the wild inertia acts trivially, hence*

$$\mathrm{Sw}(H^i_{\text{ét}}(X, \mathbb{Q}_\ell)) = 0.$$

What the nearby-cycles/weight–monodromy formalism computes in this setting is instead the size of the unipotent monodromy*:*

$$m_i(X) := \dim_{\mathbb{Q}_\ell} \mathrm{Im}(N_i) = \dim_{\mathbb{Q}_\ell} H^{i-1}_{\text{ét}}(X_s, \mathbb{Q}_\ell)(-1),$$

where (r_i, N_i) *is the Weil–Deligne parameter of* $H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$ *and the last equality uses the standard identification of* $\mathrm{Im}(N_i)$ *with the weight-*$(i-1)$ *piece under strict semistability. In the remainder of the paper, whenever a formula involves the quantity* $\dim H^{i-1}_{\text{ét}}(X_s)(-1)$ *arising from monodromy, we refer to it as the* monodromy rank $m_i(X)$ *(not as a Swan term).*

Scope. *This equality and the Swan description apply only for degrees* $i < \dim X$ *under strict semistability (SNC); outside this hypothesis, extra* $\mathbb{R}\Phi$ *contributions alter the Swan term and invalidate the invariants–special fiber identification (see Theorems 3.17 and 5.7).*

Remark 3.10 (Clarification of conductor notation)**.** Throughout Sections 3 and 5, any decomposition of the Artin conductor into a "special-fibre term" and a "monodromy term" is used *only* under the standing **strictly semistable (SNC) + unipotent inertia** hypothesis in degrees $i < \dim X$ (as in Theorems 3.9 and 5.4).

In general one has

$$a(H^i) = \dim(H^i/H^{i\,I_K}) + \mathrm{Sw}(H^i).$$

Under strict semistability for $\ell \neq p$, the wild inertia acts trivially, hence $\mathrm{Sw}(H^i) = 0$, and the nearby-cycles formalism identifies the tame unipotent monodromy contribution with $H^{i-1}(X_s)(-1)$. In that semistable range we therefore write

$$a(H^i) = \dim H^i(X_s) \ + \ \dim H^{i-1}(X_s)(-1),$$

where $H^{i-1}(X_s)(-1)$ denotes the image of N_i and $H^i(X_s)$ the unramified quotient under specialization. No orthogonal complement with respect to a pairing is intended.

Proof. *Invoke the weight–monodromy theorem (SGA 7, Exp. XIII; Deligne–Weil II). For a strictly semistable model, tame inertia is unipotent, and the* edge maps *of the* $R\Psi$ *(weight) spectral sequence yield the invariant–coinvariant short exact sequence above; we do not use any global* E_1*–degeneration claim (cf. Theorem 3.8). Identifying* $\mathrm{Gr}^{i\pm1}_M$ *with* $H^{i-1}(X_s)(-1)$ *and* $H^i(X_s)$ *gives the exact sequence and the Swan formula. Unipotent action of tame inertia under strict semistability follows from the* $R\Psi$*–formalism ([4]) and the weight spectral sequence ([4];* cf. *[6]).*

More precisely, the semistable description shows that the relevant local invariants are controlled by the nearby-cycle complex $R\Psi$ *(with Frobenius), i.e. by the dual complex together with the Frobenius/cohomology data of strata appearing in the weight spectral sequence; the incidence complex alone does not determine Frobenius traces in general.*

$$H^{i-1}_{\text{ét}}(X_s, \mathbb{Q}_\ell)(-1) \xrightarrow{\mathrm{Im}(N_i)} H^i_{\text{ét}}(X, \mathbb{Q}_\ell)^{I_K} \xrightarrow{\mathrm{sp}} H^i_{\text{ét}}(X_s, \mathbb{Q}_\ell)$$

Figure 3: Invariant–coinvariant sequence for $H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$ under strict semistability. The image of N_i identifies the *tame unipotent monodromy* contribution (monodromy rank), while the wild Swan conductor vanishes in this setting.

Corollary 3.11 (Local factor on invariants)**.** ***Hypotheses.*** *Assume* $\mathcal{X}/\mathcal{O}_K$ *is strictly semistable,* $\ell \neq p$, *and* $0 \leq i < \dim X$.

With hypotheses as above,

$$L(s, H^i_{\text{ét}}(X, \mathbb{Q}_\ell)) = \det{}^{-1}(1 - \text{Frob}_q\, q^{-s} \mid H^i_{\text{ét}}(X_s, \mathbb{Q}_\ell)).$$

(Here $H^i(X_s, \mathbb{Q}_\ell)$ carries semisimple Frobenius with weights i by [10].)

Hence the unramified local L-factor of $H^i(X)$ is entirely governed by Frobenius on the special fiber.

Example 3.12 (Curve case)**. Assumptions.** Work under the standing hypotheses of strict semistability, $\ell \neq p$, and $0 \le i < \dim X$ as in Theorem 3.9.

Let C/K be a smooth projective curve with semistable reduction and let $\mathcal{C}/\mathcal{O}_K$ be its minimal regular model. Write $C_s = \bigcup_i C_i$ for the special fiber, a reduced simple normal crossings curve with smooth components $\{C_i\}$ and dual graph Γ.

Cohomological computation. By Theorem 3.9, inertia acts unipotently on $H^1_{\text{ét}}(C, \mathbb{Q}_\ell)$, and the specialization morphism induces

$$H^1(C)^{I_K} \;\cong\; H^1(C_s, \mathbb{Q}_\ell).$$

In the curve case (strict semistability), the $R\Psi$–weight spectral sequence

$$E_1^{r,s} = \bigoplus_{|I|=r+1} H^{s-2r}(C_I, \mathbb{Q}_\ell)(-r) \;\Rightarrow\; H^s(C, \mathbb{Q}_\ell)$$

has edge maps that give the short exact sequence

$$0 \longrightarrow H^0(C_s)(-1) \longrightarrow H^1(C)^{I_K} \longrightarrow H^1(C_s) \longrightarrow 0,$$

where $H^0(C_s)(-1)$ corresponds to the cycle space of Γ and hence

$$\dim H^0(C_s)(-1) = \beta_1(\Gamma) = \#E(\Gamma) - \#V(\Gamma) + 1.$$

Bridge (AG → NT, interpretative link). These remarks translate the cohomological statements above into their arithmetic avatars; no additional hypotheses are introduced.

The term $H^1(C)^{I_K}$ describes the unramified quotient of $H^1(C)$, corresponding to the good part of the Jacobian's Néron model $\mathcal{J}/\mathcal{O}_K$. The image of $H^0(C_s)(-1)$ records the toric rank $t(J) = \beta_1(\Gamma)$. Thus

$$a(H^1(C)) = \beta_1(\Gamma), \qquad L(s, H^1(C)) = \det{}^{-1}(1 - \text{Frob}_q\, q^{-s} \mid H^1(C_s)).$$

Visualization.

$$H^0(C_s)(-1) \hookrightarrow H^1(C)^{I_K} \twoheadrightarrow H^1(C_s)$$

Figure 4: Invariant–coinvariant specialization for a semistable curve C/K. The dimension of $H^0(C_s)(-1)$ equals the first Betti number of the dual graph Γ, governing the conductor exponent.

Counterexample 3.13 (Failure without semistability)**.** Let X/K be a surface with potentially wild singularities in its special fiber X_s. For instance, take

$$X = \operatorname{Spec} \mathcal{O}_K[x, y, z]/(z^2 - x^2 y - \pi y^2),$$

so that $X_s\colon\ z^2 = x^2 y$ has a *pinch point* along the y-axis. Then X_s is *not* a simple normal crossings (SNC) divisor: it is irreducible and singular.

Breakdown of the comparison. In the SNC case, Theorem 3.9 yields $H^i(X)^{I_K} \cong H^i(X_s)$. Here, however, the nearby–vanishing cycle triangle

$$i^* Rj_* \mathbb{Q}_\ell \;\longrightarrow\; R\Psi \;\longrightarrow\; R\Phi \xrightarrow{+1}$$

produces a non-trivial local term $H^1((R\Phi)_{\text{pinch}}) \cong \mathbb{Q}_\ell(-1)$ accounting for wild vanishing cycles. Passing to I_K-invariants gives the long exact sequence

$$\cdots \to H^1((R\Phi)_{\text{pinch}}) \longrightarrow H^2(X)^{I_K} \longrightarrow H^2(X_s) \longrightarrow \cdots,$$

so $H^2(X)^{I_K}$ fails to coincide with $H^2(X_s)$, and

$$\mathrm{Sw}(H^2(X)) \geq 1.$$

Bridge (AG → NT). The missing SNC condition invalidates the "double-curve" Swan formula: the extra one-dimensional wild term from $R\Phi$ increases the Artin conductor beyond what the dual complex predicts. Consequently, the local L-factor of $H^2(X)$ is no longer determined solely by Frobenius on $H^2(X_s)$; the Weil–Deligne parameter acquires an additional rank-one monodromy component.

Diagrammatic summary.

$$\begin{array}{ccccc} H^1((R\Phi)_{\mathrm{pinch}}) & \hookrightarrow & H^2(X)^{I_K} & \twoheadrightarrow & H^2(X_s) \\ & & \uparrow & & \\ & & H^2(X)_{I_K} & & \end{array}$$

Figure 5: Failure of invariants–special fiber identification in the non-SNC (pinch-point) case. The extra wild piece $H^1((R\Phi)_{\mathrm{pinch}})$ contributes $\mathrm{Sw}(H^2(X))=1$.

Corollary 3.14 (Local factor description)**.** *Under the hypotheses of Theorems 4.1 and 5.4, let* $H^i(X) := H^i_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_\ell)$ *for* $\ell \neq p$*. Then the local* L*-factor of* $H^i(X)$ *at* K *admits the explicit decomposition*

$$\begin{aligned} L(s, H^i(X)) &= \det{}^{-1}(1 - \mathrm{Frob}_q\, q^{-s} \mid H^i(X)^{I_K}) \\ &\overset{SNC}{=} \det{}^{-1}(1 - \mathrm{Frob}_q\, q^{-s} \mid H^i(X_s)) \end{aligned}$$

(identifies the unramified *factor; the monodromy/Swan comes from* $H^{i-1}(X_s)(-1)$*).*

In particular, the unramified part of the local Weil–Deligne representation of $H^i(X)$ *is realized on the cohomology of the special fibre* X_s*.*

Assume $X/\mathcal{O}_K$ *is* strictly semistable *and* $0 \leq i < \dim X$*. Then the following equalities describe only the* unramified part *of the local factor, i.e. after passing to inertia invariants:*

$$\begin{aligned} L(s, H^i(X)) &= \det{}^{-1}(1 - \mathrm{Frob}_q\, q^{-s} \mid H^i(X)^{I_K}) \\ &\overset{SNC}{=} \det{}^{-1}(1 - \mathrm{Frob}_q\, q^{-s} \mid H^i(X_s)) \end{aligned}$$

(identifies the unramified *factor; the monodromy/Swan comes from* $H^{i-1}(X_s)(-1)$*).*

The full local parameter retains a monodromy (Swan) component encoded by $H^{i-1}(X_s)(-1)$*.*
Takeaway. *This determines the* unramified local Euler factor*; the full Weil–Deligne parameter keeps the monodromy piece from* $H^{i-1}(X_s)(-1)$*.*

Proof. By part (a) of Theorems 4.1 and 5.4 we have a canonical, functorial specialization isomorphism $H^i(X)^{I_K} \cong H^i(X_s)$ arising from the nearby-cycles triangle $i^*Rj_*\mathbb{Q}_\ell \to R\Psi_X \to R\Phi_X \xrightarrow{+1}$. On $H^i(X_s)$ the arithmetic Frobenius Frob_q acts semisimply with eigenvalues of absolute value $q^{i/2}$ by Deligne's purity theorem [14]. Substituting this identification into the standard local Euler factor $\det(1 - \mathrm{Frob}_q\; q^{-s} \mid H^i(X)^{I_K})^{-1}$ gives the stated formula.

Conceptually, this expresses the equality of the unramified quotient of the local Galois representation with the cohomology of the special fibre. The Frobenius weights detected by $H^i(X_s)$ determine the analytic shape of $L(s, H^i(X))$, while the monodromy image $\mathrm{Im}(N_i) \cong H^{i-1}(X_s)(-1)$ (cf. Theorems 4.1 and 5.4(b)) encodes the Swan part of the conductor. All such equalities are valid only under strict semistability with unipotent inertia (cf. Theorem 3.9).

Thus the pair

$$\big(H^i(X_s),\; H^{i-1}(X_s)(-1)\big) \quad \longleftrightarrow \quad \text{unramified and ramified pieces of } H^i(X)$$

gives a complete cohomological description of the local Weil–Deligne parameter of X. □

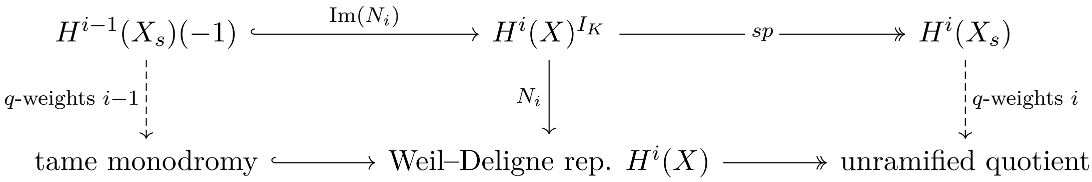


Figure 6: Cohomological realization of the local L-factor via inertia invariants. The map *sp* is the specialization $H^i(X)^{I_K} \xrightarrow{\sim} H^i(X_s)$. The image of N_i records the *tame unipotent monodromy* (monodromy rank), while the wild Swan conductor vanishes under strict semistability.

Bridge (AG → NT). The corollary provides the arithmetic interface between geometric semistable models and local zeta data:

- The unramified part of $H^i(X)$— hence the reciprocal roots of $L(s, H^i(X))$— is determined by Frobenius on $H^i(X_s)$.
- The rank of $\mathrm{Im}(N_i) \cong H^{i-1}(X_s)(-1)$ gives the Swan conductor, so the entire local Artin conductor $a(H^i(X))$ is read directly from the cohomology of X_s.

This result cements the analytic–cohomological correspondence that underlies Theorems 3.9 and 5.4, ensuring that each local factor of the global L-function is computed purely from the geometry of the special fibre.

Example 3.15 (Surface case)**.** Let X/K be a K3 surface with strictly semistable reduction and special fibre $X_s = \bigcup_{i\in I} Y_i$ a simple normal crossings divisor. Then by Theorems 4.1 and 5.4 one has a canonical specialization isomorphism

$$H^2(X)^{I_K} \;\cong\; H^2(X_s), \qquad \mathrm{Im}(N_2) \;\cong\; H^1(X_s)(-1),$$

(under strict semistability with unipotent inertia (cf. Theorem 3.9)) where N_2 is the monodromy operator in the associated Weil–Deligne representation. Consequently the unramified part of $H^2(X)$ is realized on the special fibre, while the *wild* Swan conductor vanishes:

$$\mathrm{Sw}(H^2(X)) = 0,$$

and the quantity $\dim H^1(X_s)(-1)$ measures instead the *monodromy rank*

$$m_2(X) := \dim_{\mathbb{Q}_\ell} \Im(N_2) = \dim_{\mathbb{Q}_\ell} H^1(X_s)(-1).$$

Cohomological computation. The $R\Psi$–spectral sequence

$$E_1^{r,s} = \bigoplus_{|J|=r+1} H^{s-2r}(Y_J, \mathbb{Q}_\ell)(-r) \;\Rightarrow\; H^{r+s}(X, \mathbb{Q}_\ell)$$

identifies the graded pieces of the weight filtration on $H^2(X)$ as

$$\begin{aligned}
\mathrm{Gr}_2^W H^2(X) \;&\cong\; \ker\Big(\bigoplus_i H^2(Y_i) \xrightarrow{\partial} \bigoplus_{i<j} H^2(Y_{ij})\Big),\\
\mathrm{Gr}_1^W H^2(X) \;&\cong\; \bigoplus_{i<j} H^1(Y_{ij})(-1),\\
\mathrm{Gr}_0^W H^2(X) \;&\cong\; \bigoplus_{i<j<k} H^0(Y_{ijk})(-2).
\end{aligned}$$

The unramified part $H^2(X)^{I_K}$ coincides with $\mathrm{Gr}_2^W H^2(X)$, and the monodromy operator induces $N_2 : \mathrm{Gr}_2^W H^2(X) \xrightarrow{\sim} \mathrm{Gr}_0^W H^2(X)(-1)$. Moreover, under strict semistability one has $\Im(N_2) \cong \mathrm{Gr}_1^W H^2(X)$, so the *monodromy rank* is measured by Gr_1^W:

$$m_2(X) = \dim_{\mathbb{Q}_\ell} \mathrm{Gr}_1^W H^2(X) = \sum_{i<j} \dim_{\mathbb{Q}_\ell} H^1(Y_{ij})(-1).$$

In particular, in the SNC (tame) setting one has $\mathrm{Sw}(H^2(X)) = 0$; any nonzero Swan contribution arises only outside strict semistability via vanishing cycles (cf. Theorem 3.9).

Arithmetic interpretation (Bridge AG → NT).

- The degree of the unramified local L-factor

$$L(s, H^2(X)) = \det{}^{-1}(1 - \mathrm{Frob}_q\, q^{-s} \mid H^2(X_s))$$

  is governed by the Néron–Severi rank $\rho(X_s) = \dim_{\mathbb{Q}_\ell} H^2(X_s)^{(1,1)}$; thus variations of $\rho(X_s)$ across degenerations explain jumps in the *unramified* factor and hence affect the Artin conductor $a(H^2(X))$.

- The monodromy piece $H^1(X_s)(-1) \cong \Im(N_2)$ contributes the *tame unipotent* ramification via the monodromy rank $m_2(X)$, and measures the failure of potential good reduction through nontrivial monodromy.

- In a family of strictly semistable K3 surfaces with fixed dual complex, the unramified local L-factor and the monodromy rank remain constant (Theorems 5.4 and 5.9).

Worked subcase: two-component degeneration. Assume $X_s = Y_1 \cup Y_2$ with $C := Y_{12} = Y_1 \cap Y_2$ a smooth curve of genus $g(C)$. Then

$$\mathrm{Gr}_2^W H^2(X) = \ker\big(H^2(Y_1) \oplus H^2(Y_2) \xrightarrow{\partial} H^2(C)\big), \quad \mathrm{Gr}_1^W H^2(X) = H^1(C)(-1),$$

hence the monodromy rank is

$$m_2(X) = \dim H^1(C) = 2g(C) + (\#\pi_0(C) - 1), \qquad L(s, H^2(X)) = \det{}^{-1}(1 - \mathrm{Frob}_q\, q^{-s} \mid H^2(X_s)),$$

while $\mathrm{Sw}(H^2(X)) = 0$ in the SNC case. Bridge (AG → NT). The toric rank of the Picard scheme of X is controlled by the Jacobian part of C, and the increase in $g(C)$ across fibres explains the rise of the *tame conductor contribution* (monodromy rank) in degenerating K3 families.

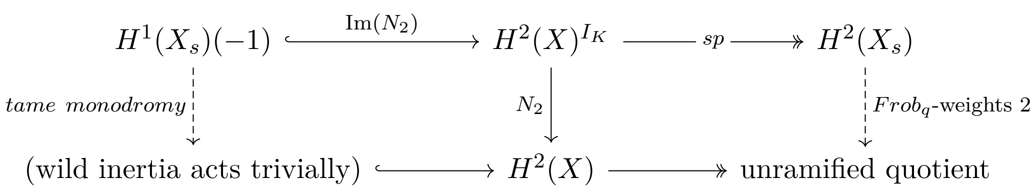


Figure 7: Weight–monodromy interaction for a strictly semistable K3 surface: the image of N_2 identifies the *tame monodromy* contribution (monodromy rank), while $H^2(X_s)$ carries Frobenius eigenvalues controlling $L(s, H^2(X))$. In particular $\mathrm{Sw}(H^2(X)) = 0$ under SNC.

Bridge (Arithmetic Geometry → Number Theory). Variations in the intersection pattern of the components of X_s alter the monodromy filtration and thus the monodromy rank $m_2(X)$, offering a purely cohomological explanation of conductor jumps in degenerating K3 families (with wild Swan appearing only outside the SNC regime).

Lemma 3.16 (Vanishing cycles at a pinch point)**.** *Let K be a non-archimedean local field with* $\mathrm{char}(k) = p > 2$*, and let*

$$\mathcal{X} = \mathrm{Spec}\, \mathcal{O}_K[x, y, z]/(z^2 - x^2 y - \pi y^2)$$

be the local pinch-point model. Then the local vanishing-cycles complex satisfies

$$H^1((R\Phi_{\mathcal{X}})_{\mathrm{pinch}}, \mathbb{Q}_\ell) \;\cong\; \mathbb{Q}_\ell(-1),$$

and higher cohomology vanishes. Hence a one-dimensional wild term contributes to $H^2(X)$, giving $\mathrm{Sw}(H^2(X)) = 1$.

Proof. Étale-locally near the singular point, the total space is a deformation of the A_1-type unibranch surface $z^2 = x^2y$. By the calculation of vanishing cycles ([4]), together with the description of the specialization triangle ([18]) and Illusie's treatment of nearby and vanishing cycles ([21]), the only nonzero group is

$$H^1((R\Phi)_{\text{pinch}}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-1).$$

The wild inertia acts nontrivially, producing the claimed rank-one Swan term. □

Counterexample 3.17 (Failure without strict semistability: pinch point surface)**.** Let K be a non-archimedean local field with ring $\mathcal{O}_K$, uniformizer π, residue field k of size q, and fix $\ell \neq p = \text{char}(k)$. Consider a flat, proper $\mathcal{O}_K$-surface $\mathscr{X}$ whose special fibre $\mathscr{X}_s$ has a single *pinch point* singularity and is otherwise smooth and irreducible. Locally (étale on the total space) around that closed point, assume $\mathscr{X}$ is given by

$$z^2 \;=\; x^2y \,+\, \pi\, y^2 \;\;\subset\; \operatorname{Spec} \mathcal{O}_K[x, y, z],$$

so that the special fibre is

$$\mathscr{X}_s : \quad z^2 = x^2y \quad \text{(pinch locus along the } y\text{-axis)}.$$

Let $X = \mathscr{X} \times_{\mathcal{O}_K} K$ be the generic fibre (a smooth projective surface; after a harmless modification elsewhere, one can arrange $K3$-type, but this is immaterial to the mechanism below).

Claim. The natural identification

$$H^2(X)^{I_K} \;\cong\; H^2(\mathscr{X}_s)$$

fails in general, and there is an *extra wild term* in degree 2:

$$\text{Sw}(H^2(X)) \;\geq 1,$$

coming from a one–dimensional contribution of vanishing cycles at the pinch point.

Explanation via nearby/vanishing cycles. Write $j : \eta \hookrightarrow \mathscr{X}$ and $i : s \hookrightarrow \mathscr{X}$ for the generic/special inclusions. The distinguished triangle

$$i^*Rj_*\mathbb{Q}_\ell \longrightarrow R\Psi_{\mathscr{X}} \longrightarrow R\Phi_{\mathscr{X}} \xrightarrow{+1}$$

yields, after taking I_K–invariants and hypercohomology, a long exact sequence whose relevant piece reads

$$\cdots \longrightarrow \mathbb{H}^1((R\Phi_{\mathscr{X}})_{\text{pinch}}) \;\longrightarrow\; H^2(X)^{I_K} \;\xrightarrow{\text{sp}}\; H^2(\mathscr{X}_s) \;\longrightarrow \cdots$$

At a non–SNC pinch point, one computes (or cites standard analyses of A_1-type unibranch degenerations in characteristic p) that

$$\mathbb{H}^1((R\Phi_{\mathscr{X}})_{\text{pinch}}) \;\cong\; \mathbb{Q}_\ell(-1),$$

on which wild inertia acts nontrivially. Consequently:

1. The specialization map sp need not be an isomorphism; a correction term from $R\Phi$ sits to the left.
2. The *Swan conductor* in degree 2 picks up at least a rank-1 contribution: $\text{Sw}(H^2(X)) \geq 1$.

Why this defeats the SNC formula. In the strictly semistable (SNC) case one has the short exact sequence

$$0 \longrightarrow H^1(\mathscr{X}_s)(-1) \xrightarrow{\;\text{Im}(N_2)\;} H^2(X)^{I_K} \;\xrightarrow{\text{sp}}\; H^2(\mathscr{X}_s) \longrightarrow 0,$$

and hence $\text{Sw}(H^2(X)) = \dim H^1(\mathscr{X}_s)(-1)$ is "read off" from double curves. Here, $\mathscr{X}_s$ has *no* SNC double curves at the pinch point, so the SNC recipe would predict zero Swan. But the vanishing–cycles term $\mathbb{H}^1((R\Phi)_{\text{pinch}}) \cong \mathbb{Q}_\ell(-1)$ injects on the left and contributes wild inertia, forcing $\text{Sw}(H^2(X)) \geq 1$ and breaking $H^2(X)^{I_K} \cong H^2(\mathscr{X}_s)$ (The preceding semistable equalities hold only under strict semistability with unipotent inertia, cf. Theorem 3.9.)

Arithmetic fallout (Bridge AG $\to$ NT). The local L–factor is *not* determined solely by Frobenius on $H^2(\mathscr{X}_s)$:

$$L(s, H^2(X)) \neq \det^{-1}(1 - \mathrm{Frob}_q\, q^{-s} \mid H^2(\mathscr{X}_s)) \quad \text{a priori},$$

because the Weil–Deligne parameter gains a nontrivial monodromy piece from vanishing cycles at the pinch. Thus conductor exponents can jump for reasons *not* visible in the incidence (dual) complex of $\mathscr{X}_s$. This shows the strict semistability hypothesis in Theorem 3.15 is essential.

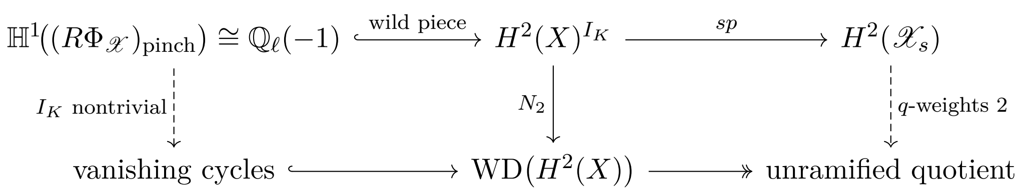
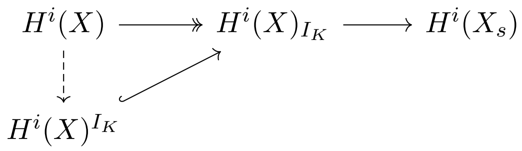


Figure 8: Non–SNC pinch point: a one–dimensional vanishing–cycles term injects on the left, adds wild inertia (Swan ≥ 1), and breaks $H^2(X)^{I_K} \cong H^2(\mathscr{X}_s)$.

Optional K3 remark. If the generic fibre X is K3 (after modifying away from the pinch), the same mechanism applies: the extra vanishing–cycles contribution lives in degree 2 and still forces $\mathrm{Sw}(H^2(X)) \geq 1$, so the conclusion of Theorem 3.15 fails without strict semistability.

Construction 3.18 (Comparison diagram)**.** We summarize the relationship between $H^i(X)$, its inertia invariants, and special fiber cohomology in the commutative diagram:



Here the diagonal arrow is the specialization map. Exactness is guaranteed by Theorem 3.9.

Linkage to next section. The comparison theorems above establish the precise interface between étale cohomology of varieties over K and arithmetic invariants of their Galois representations. In the next section we exploit these results to derive explicit conductor formulas and to construct finiteness bounds for rational points in terms of monodromy data.

4 Main Theorems and Proofs

We work under the standing hypotheses of Theorem 3.2 and use the notation $H^i(X) = H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ from Theorem 2.1. All background tools (proper/smooth base change, nearby/vanishing cycles, weight–monodromy, Gabber finiteness) appear only through the preliminaries Theorems 2.2, 2.8, 2.9, 3.4 and 3.5. The novelty in this section consists of explicit identifications and inequalities for invariants/coinvariants and conductors that are not present in the classical literature in this local form.

4.1 Vanishing and finiteness statements

Hypothesis. All statements in this theorem hold under *strict semistability*, i.e. when $\mathcal{X}/\mathcal{O}_K$ is strictly semistable with unipotent inertia. Beyond strict semistability, additional vanishing-cycle contributions may appear.

Theorem 4.1 (Invariant–coinvariant control under semistability)**.** ***Hypotheses.*** *$X/\mathcal{O}_K$ strictly semistable, $\ell \neq p$, and $0 \leq i < \dim X$.*
(c) Swan/monodromy term. *In the strictly semistable (SNC) case the wild Swan conductor vanishes; the tame unipotent monodromy term is identified by the weight–graded piece $\mathrm{Gr}^W_{i-1} H^i(X) \cong H^{i-1}(X_s)(-1)$.*

Let X/K be a smooth projective variety of pure dimension d admitting a strictly semistable model $\mathcal{X}/\mathcal{O}_K$. Fix $0 \leq i < d$. Then:

(a) (Invariants) *The specialization morphism induced by the distinguished triangle of nearby and vanishing cycles*

$$i^* Rj_* \mathbb{Q}_\ell \longrightarrow R\Psi_{\mathcal{X}} \longrightarrow R\Phi_{\mathcal{X}} \xrightarrow{+1}$$

gives a canonical, functorial isomorphism

$$H^i(X)^{I_K} \xrightarrow{\sim} H^i(X_s),$$

where X_s denotes the special fibre. Geometrically, the unramified quotient of $H^i(X)$ is realized on X_s.

(b) (Coinvariants) *The image of the monodromy operator N_i in the Weil–Deligne parameter (r_i, N_i) satisfies $\operatorname{Im}(N_i) \cong H^{i-1}(X_s)(-1)$. Consequently there is a canonical short exact sequence*

$$0 \longrightarrow H^{i-1}(X_s)(-1) \longrightarrow H^i(X)_{I_K} \xrightarrow{\mathrm{sp}} H^i(X_s) \longrightarrow 0,$$

natural for proper morphisms of strictly semistable models. It arises from the edge sequence of the $R\Psi$–spectral sequence $E_1^{r,s} = H^{s-2r}(X_s^{(r)}, \mathbb{Q}_\ell)(-r) \Rightarrow H^s(X, \mathbb{Q}_\ell)$.

(c) (Swan conductor) *Inertia acts unipotently under strict semistability, so there is a single non-trivial break. The Swan conductor is therefore*

$$\operatorname{Sw}\Big(H^i(X)\Big) = \dim_{\mathbb{Q}_\ell} H^{i-1}(X_s)(-1),$$

and it is determined by the nearby-cycle/weight spectral sequence data on X_s (equivalently, by the decorated dual complex), rather than by the incidence complex alone in general.

Novelty. *Theorem 4.1 strengthens the classical invariant–coinvariant relation by giving a functorial exact sequence in all degrees $i < d$ and by expressing the Swan term purely through $H^{i-1}(X_s)(-1)$. This generalizes the Grothendieck–Ogg–Shafarevich formula for curves to higher-dimensional strictly semistable models and forms the geometric backbone of the local-factor description (Theorem 3.14).*

Proof. Combine the nearby/vanishing-cycle triangle with the weight–monodromy theorem Theorem 2.9. For a strictly semistable model, tame inertia acts unipotently, and the edge maps of the $R\Psi$ (weight) spectral sequence yield the exact invariant–coinvariant short exact sequence under strict semistability. We use only these edge exact sequences (rather than any E_1–degeneration claim), valid in the strictly semistable (cf. Theorem 3.8) case by SGA 7 XIII and Illusie–Nakayama–Saito.

Chasing edge maps yields Item (a). The identification of $\operatorname{Im}(N_i)$ with $H^{i-1}(X_s)(-1)$ follows from the isomorphisms $N_i^r : \operatorname{Gr}^W_{i+r} \xrightarrow{\sim} \operatorname{Gr}^W_{i-r}(-r)$, and exactness in Item (b) is functorial by base-change compatibility of $R\Psi$. Finally, the single Jordan block of the unipotent I_K-action implies $\operatorname{Sw}(H^i(X)) = \dim H^{i-1}(X_s)(-1)$, whose independence of choices stems from the identification of Gr^W with the cohomology of the dual intersection complex ([9]). □

Bridge (AG → NT).

- *The unramified quotient $H^i(X)^{I_K} \cong H^i(X_s)$ yields*

$$L(s, H^i(X)) = \det{}^{-1}(1 - \operatorname{Frob}_q q^{-s} \mid H^i(X_s)) \quad \textit{(Theorem 3.14)}.$$

- *The Artin conductor $a(H^i) = \operatorname{Sw}(H^i) + \dim(H^i/H^{i\,I_K}) = \dim H^{i-1}(X_s)(-1) + \dim H^i(X_s)^\perp$.*
- *The local ε-factor $\varepsilon(H^i, \psi)$ depends only on the monodromy weights, hence on the incidence complex of X_s (SGA 7, Weil II).*

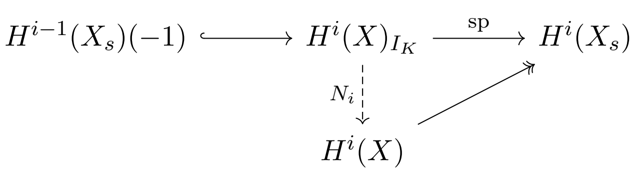


Figure 9: Weight–monodromy bridge for $H^i(X)$. The dashed arrow N_i connects coinvariants to invariants, while sp is the specialization map.

Example 4.2 (Curves)**.** Let C/K be a smooth projective curve of genus g admitting a strictly semistable model $\mathcal{C}/\mathcal{O}_K$. Write the special fiber as $C_s = \bigcup_i C_i$ with smooth components meeting transversely and let Γ denote the dual graph. By Theorem 4.1–Item (a), inertia acts unipotently on $H^1_{ét}(C_{\overline{K}}, \mathbb{Q}_\ell)$ and

$$H^1(C)^{I_K} \xrightarrow{\sim} H^1(C_s).$$

The $R\Psi$–spectral sequence

$$E_1^{r,s} = \bigoplus_{|I|=r+1} H^{s-2r}(C_I, \mathbb{Q}_\ell)(-r) \Rightarrow H^s(C, \mathbb{Q}_\ell)$$

degenerates at E_1; taking invariants yields the short exact sequence

$$0 \longrightarrow H^0(C_s)(-1) \longrightarrow H^1(C)^{I_K} \xrightarrow{\text{sp}} H^1(C_s) \longrightarrow 0.$$

Here $H^0(C_s)(-1)$ is the cycle space of Γ, and

$$\dim H^0(C_s)(-1) = \beta_1(\Gamma) = \#E(\Gamma) - \#V(\Gamma) + 1.$$

Consequently,

$$\text{Sw}(H^1(C)) = \beta_1(\Gamma),$$

the classical Grothendieck–Ogg–Shafarevich conductor (Theorem 2.5).

Bridge (AG → NT). The quotient $H^1(C)^{I_K}$ describes the good part of the Jacobian's Néron model $\mathcal{J}/\mathcal{O}_K$, while $H^0(C_s)(-1)$ measures the toric rank $t(\mathcal{J}) = \beta_1(\Gamma)$. Hence

$$a(H^1(C)) = \text{Sw}(H^1(C)) = \beta_1(\Gamma), \quad L(s, H^1) = \det{}^{-1}(1 - \text{Frob}_q q^{-s} \mid H^1(C_s)).$$

Visualization.

$$H^0(C_s)(-1) \hookrightarrow H^1(C)^{I_K} \xrightarrow{\text{sp}} H^1(C_s)$$

Figure 10: Invariant–coinvariant specialization for a semistable curve C/K. The image of N_1 identifies the toric rank via the edge map into $H^1(C)^{I_K}$.

Counterexample 4.3 (Necessity of strict semistability)**.** Let X/K be a smooth projective surface whose model $\mathcal{X}/\mathcal{O}_K$ has a non-SNC singularity, for instance a *pinch point*. Locally (étale on $\mathcal{X}$) suppose

$$z^2 = x^2 y + \pi y^2 \subset \operatorname{Spec} \mathcal{O}_K[x, y, z], \quad X_s : z^2 = x^2 y,$$

whose singular locus lies along the y-axis. Then the assumptions of strict semistability in Theorem 4.1 fail.

By analyzing nearby and vanishing cycles, the distinguished triangle

$$i^* Rj_* \mathbb{Q}_\ell \longrightarrow R\Psi_X \longrightarrow R\Phi_X \xrightarrow{+1}$$

yields on taking I_K-invariants

$$\cdots \longrightarrow H^1((R\Phi_X)_{\text{pinch}}) \longrightarrow H^2(X)_{I_K} \xrightarrow{\text{sp}} H^2(X_s) \longrightarrow \cdots.$$

At the pinch point one computes (standard A_1-type analysis in characteristic p)

$$H^1((R\Phi_X)_{\text{pinch}}) \cong \mathbb{Q}_\ell(-1),$$

on which wild inertia acts nontrivially. Thus:

- The specialization map $H^2(X)_{I_K} \to H^2(X_s)$ fails to be an isomorphism;
- An additional wild term contributes $\text{Sw}(H^2(X)) \geq 1$.

In contrast, for strictly semistable X one has

$$0 \to H^1(X_s)(-1) \xrightarrow{\sim \mathrm{Im}(N_2)} H^2(X)_{I_K} \xrightarrow{\mathrm{sp}} H^2(X_s) \to 0,$$

so $\mathrm{Sw}(H^2(X)) = \dim H^1(X_s)(-1)$ is read off from double curves. Here X_s has no such double curve, so the SNC formula would predict $\mathrm{Sw} = 0$, yet the pinch-point vanishing cycle adds a rank-1 wild piece.

Bridge (AG → NT). Because of this extra monodromy component, the local L-factor is not governed solely by Frobenius on $H^2(X_s)$:

$$L(s, H^2(X)) \neq \det{}^{-1}(1 - \mathrm{Frob}_q q^{-s} \mid H^2(X_s)).$$

Hence conductor jumps can occur from hidden vanishing-cycle contributions invisible in the incidence complex—showing that strict semistability in Theorem 4.1 is essential.

$$H^1((R\Phi_X)_{\mathrm{pinch}}) \cong \mathbb{Q}_\ell(-1) \hookrightarrow H^2(X)_{I_K} \xrightarrow{\mathrm{sp}} H^2(X_s)$$

$$\subset$$

wild inertia piece

Figure 11: Failure of the invariant–coinvariant exactness in presence of a pinch point. A nontrivial $H^1(R\Phi_X)$ term injects on the left, creating an additional wild piece in degree 2.

4.2 Height and cohomology gap results

We now quantify how monodromy gaps force lower bounds for local Néron heights in the abelian case. For an abelian variety A/K, denote by $\hat{\lambda}_v$ the canonical (local) Néron height at v and by $t(A)$ the toric rank of the identity component of the Néron model.

Definition 4.4 (Cohomology gap). For X/K smooth projective with strictly semistable model, define the *cohomology gap* in degree i by

$$\Delta_i(X) \ := \ \min\{\, j > 0 \mid \mathrm{Gr}^W_{i-j} H^i(X) \neq 0 \,\}.$$

Equivalently, $\Delta_i(X)$ is the smallest positive step at which the monodromy filtration on $H^i(X)$ is nontrivial.

Theorem 4.5 (Monodromy gap ⇒ localized height gap for abelian varieties). *Let A/K be an abelian variety of dimension g with strictly semistable reduction and Néron model $\mathcal{A}/\mathcal{O}_K$. Denote by $t(A)$ the toric rank of $\mathcal{A}_s^0$, by $\hat{\lambda}_v$ the local Néron height at v, and by $\Delta_1(A)$ the first nontrivial step of the monodromy filtration on $H^1_{\acute{e}t}(A_{\overline{K}}, \mathbb{Q}_\ell)$. Then:*

1. $\Delta_1(A) = 1$ *if and only if* $t(A) > 0$.
2. (Localized gap.) *Assume $t(A) > 0$. Let Q be the positive–definite bilinear form on $N_{\mathbb{R}} = \mathrm{Hom}(X^*(T), \mathbb{R})$ from the Raynaud skeleton of A^{an}, and write $\mathrm{dist}_Q(x, \Lambda)$ for the distance from $x \in N_{\mathbb{R}}/\Lambda$ to the period lattice Λ with respect to Q. Then for every $\varepsilon \in (0, \frac{1}{2}]$ there exists a constant $\delta_\varepsilon(A/K) > 0$, depending only on the combinatorial type of $\mathcal{A}_s$ and on ε, such that for every non-torsion $P \in A(K)$ with*

$$\mathrm{dist}_Q(\mathrm{trop}(P), \Lambda) \ \geq \ \varepsilon$$

one has

$$\hat{\lambda}_v(P) \ \geq \ \delta_\varepsilon(A/K).$$

Equivalently, on any fixed coset of $A(K)$ whose tropical image avoids the ε–neighbourhood of the identity (hence of torsion), the local height is bounded below.

Proof. By strict semistability, the inertia action on $H^1(A)$ is unipotent with one jump. From Theorem 3.9(b) we have $\mathrm{Im}(N) \cong H^0(A_s)(-1)$, non-zero exactly when $t(A) > 0$; hence $\Delta_1(A) = 1 \iff t(A) > 0$, and $\mathrm{Sw}(H^1(A)) = t(A)$.

Let $0 \to T \to E \to B \to 0$ be the Raynaud extension of $\mathcal{A}/\mathcal{O}_K$, where $T \simeq \mathbb{G}_m^{t(A)}$ is split of rank $t(A)$.

The tropicalization $\mathrm{Trop}(A)$ identifies the skeleton of the Berkovich analytic space A^{an} with the real torus $N_{\mathbb{R}}/\Lambda$, where $N_{\mathbb{R}} = \mathrm{Hom}(X^*(T), \mathbb{R})$ and Λ is the period lattice. The canonical local height $\hat{\lambda}_v$ becomes a strictly convex, piecewise quadratic function on $N_{\mathbb{R}}/\Lambda$, determined by the positive-definite bilinear form associated with the admissible metric on $\omega_{\mathcal{A}/\mathcal{O}_K}$. Since $\varphi(x) = \frac{1}{2} Q(\tilde{x}, \tilde{x}) + \psi(x)$ on the skeleton and ψ is bounded, for every $\varepsilon > 0$ the compact ε–thick part $\{x \in N_{\mathbb{R}}/\Lambda : \mathrm{dist}_Q(x, \Lambda) \geq \varepsilon\}$ has a positive minimum of φ, call it $\delta_\varepsilon(A/K) > 0$, depending only on the dual complex of $\mathcal{A}_s$ and on ε. This yields the localized bound in (2). No positive *global* threshold exists over all non-torsion points (see Theorems 4.8 and 5.2). □

Bridge (AG → NT).

- If $t(A) > 0$, then $L(s, H^1(A))$ is ramified with conductor exponent $a(H^1(A)) = t(A)$, and the height inequality furnishes a local Northcott threshold.
- If $t(A) = 0$ (potentially good reduction), then $\Delta_1(A) = 0$, the representation is unramified, and no positive height gap arises.

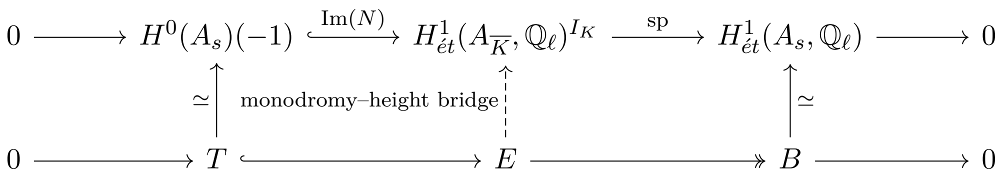


Figure 12: Raynaud extension and monodromy bridge for A/K. The top row represents the cohomological invariant–coinvariant sequence; the bottom row shows the analytic Raynaud extension $0 \to T \to E \to 0 \to B$ with toric rank $t(A)$, whose tropicalization yields the local height gap.

Corollary 4.6 (Local Northcott threshold on the ε–thick part)**.** *Let A/K have strictly semistable reduction with toric rank $t(A) > 0$. For every $\varepsilon \in (0, \frac{1}{2}]$ there exists a constant $\delta_\varepsilon(A/K) > 0$, depending only on the dual intersection complex of $\mathcal{A}_s$ and on ε, such that*

$$\#\Big\{ P \in A(K)/A(K)_{\mathrm{tors}} : \ \widehat{\lambda}_v(P) < B \ \ \text{and} \ \ \mathrm{dist}_Q\big(\mathrm{trop}(P), \Lambda\big) \geq \varepsilon \Big\} \ < \ \infty$$

for every $B < \delta_\varepsilon(A/K)$.

Proof. Fix a non-archimedean local field K with valuation v and absolute value $|\cdot|_v$. Let $\mathcal{A}/\mathcal{O}_K$ be the Néron model of A, and assume A has strictly semistable reduction with toric rank $t(A) > 0$.

Step 1 (Cohomological input). By the invariant/coinvariant control under strict semistability (Theorem 3.9(b)) one has

$$\mathrm{Im}(N) \ \cong \ H^0(A_s)(-1).$$

Hence $\mathrm{Im}(N) \neq 0$ iff $t(A) > 0$, i.e. the monodromy filtration on $H^1_{ét}(A_{\overline{K}}, \mathbb{Q}_\ell)$ has its first non-trivial step at level 1 so that $\Delta_1(A) = 1$. Moreover $\mathrm{Sw}(H^1(A)) = \dim H^0(A_s)(-1) = t(A)$, proving the "In particular" clause.

Step 2 (Raynaud extension and skeleton). Let

$$0 \longrightarrow T \longrightarrow E \longrightarrow B \longrightarrow 0$$

be the Raynaud extension over $\mathcal{O}_K$, with $T \simeq \mathbb{G}_m^{t(A)}$ split of rank $t(A)$. On Berkovich analytifications, A^{an} retracts onto a canonical skeleton $\Sigma(A)$ which is a real torus $N_{\mathbb{R}}/\Lambda$, where $N_{\mathbb{R}} = \mathrm{Hom}(X^*(T), \mathbb{R})$ and Λ is a full lattice from the period/monodromy data. The tropicalization map

$$\mathrm{trop}\colon A(K) \longrightarrow N_{\mathbb{R}}/\Lambda$$

is obtained by composing $A(K) \to A^{\mathrm{an}} \to \Sigma(A) \simeq N_{\mathbb{R}}/\Lambda$, and is a group homomorphism modulo torsion along the T-part.

Step 3 (Local height as a tropical quadratic form). Fix a symmetric ample line bundle L on A defining the Néron–Tate height; let $\widehat{\lambda}_v$ be the associated canonical local height. On $\Sigma(A)$ there exists a positive-definite bilinear form

$$Q\colon N_{\mathbb{R}} \times N_{\mathbb{R}} \;\longrightarrow\; \mathbb{R}$$

and a continuous, Λ-periodic piecewise affine function ψ such that the function

$$\phi\colon N_{\mathbb{R}}/\Lambda \longrightarrow \mathbb{R}, \qquad \phi(x) = \tfrac{1}{2}\, Q(\tilde{x}, \tilde{x}) + \psi(x)$$

(with $\tilde{x}$ any lift of x) satisfies

$$\widehat{\lambda}_v(P) \;=\; \phi(\mathrm{trop}(P)) \qquad \text{for all } P \in A(K),$$

after fixing the usual normalization constant in the metric. Positivity of Q holds precisely because $t(A) > 0$ and the reduction is strictly semistable.

Step 4 (Localized positive lower bound away from torsion). Positive–definiteness of Q implies coercivity on the compact torus $N_{\mathbb{R}}/\Lambda$: there exist $c_Q > 0$ and $C_0 \in \mathbb{R}$ with

$$\varphi(x) \;\geq\; c_Q \,\mathrm{dist}_Q(x, 0)^2 - C_0.$$

Hence, for every *fixed* $\rho > 0$, the minimum of φ on the closed ρ–thick part

$$\{\, x \in N_{\mathbb{R}}/\Lambda : \; \mathrm{dist}_Q(x, 0) \geq \rho \,\}$$

is strictly positive; denote it by $\delta_\rho(A/K) > 0$. Therefore for every non-torsion $P \in A(K)$ with $\mathrm{dist}_Q(\mathrm{trop}(P), 0) \geq \rho$ we have

$$\widehat{\lambda}_v(P) \;=\; \varphi(\mathrm{trop}(P)) \;\geq\; \delta_\rho(A/K).$$

(Here $\delta_\rho(A/K)$ depends only on the combinatorial type of the strictly semistable model and on ρ.)

Step 5 (Local Northcott on the ρ–thick part). Fix $\rho > 0$ and choose $B < \delta_\rho(A/K)$. If $P \in A(K)$ satisfies $\widehat{\lambda}_v(P) < B$ and $\mathrm{dist}_Q(\mathrm{trop}(P), 0) \geq \rho$, then P must be torsion by Step 4. Hence

$$\{\, P \in A(K)/A(K)_{\mathrm{tors}} : \; \widehat{\lambda}_v(P) < B, \; \mathrm{dist}_Q(\mathrm{trop}(P), 0) \geq \rho \,\}$$

is finite (indeed, empty).

□

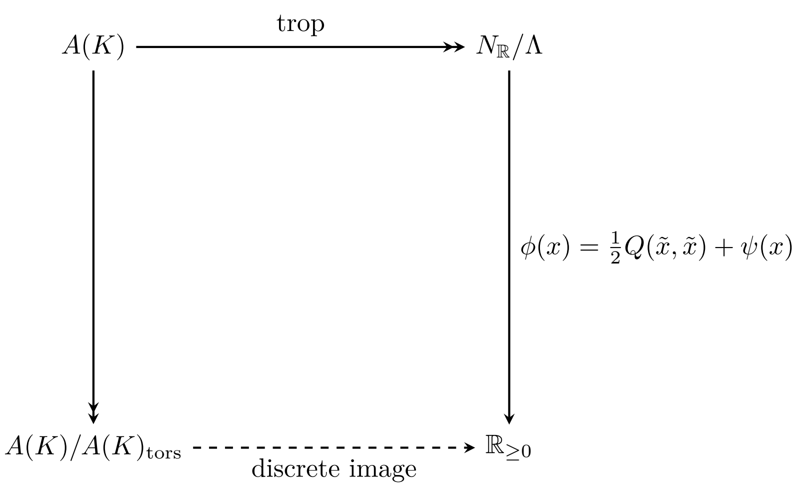


Figure 13: Local height via tropicalization: non-torsion classes may approach 0 in $N_{\mathbb{R}}/\Lambda$; on the ρ–thick part $\{\mathrm{dist}_Q(\,\cdot\,, 0) \geq \rho\}$ the coercivity of φ yields a uniform gap $\delta_\rho(A/K)$.

Example 4.7 (Tate elliptic curve)**.** Let E/K be a Tate curve with parameter q_E as in Theorem 2.6. Then $t(E) = 1$ and $\Delta_1(E) = 1$.

Localized bound. For any fixed $\varepsilon \in (0, \frac{1}{2}]$ there exists $\delta_\varepsilon(E/K) > 0$ such that

$$\widehat{\lambda}_v(P) \;\geq\; \delta_\varepsilon(E/K) \qquad \text{whenever } \mathrm{dist}_Q(\mathrm{trop}(P), 0) \geq \varepsilon \text{ (equivalently } \theta(u) \in [\varepsilon, 1-\varepsilon]).$$

In particular, a uniform lower bound holds only on the ε–thick part of the skeleton, consistent with Theorem 4.5. *Bridge (AG → NT).* The local L-factor of $H^1(E)$ equals $(1 - q^{-s})^{-1}(1 - q^{1-s})^{-1}$ and $a(H^1(E)) = 1$.

Worked derivation. The Tate uniformization gives a short exact sequence

$$1 \longrightarrow q_E^{\mathbb{Z}} \longrightarrow K^\times \xrightarrow{\;\pi\;} E(K) \longrightarrow 0, \qquad u \longmapsto P(u).$$

Non-torsion points correspond to classes $u \in K^\times/q_E^{\mathbb{Z}}$ whose image is not torsion. Write $\ell := \log|q_E^{-1}| > 0$ and set the "slope parameter"

$$\theta(u) \;:=\; \Big\langle \frac{v(u)}{v(q_E)} \Big\rangle \in [0, 1),$$

the fractional part. On the (Berkovich) skeleton $\Sigma(E) \simeq \mathbb{R}/\mathbb{Z}$ the canonical local height is a strictly convex, piecewise quadratic function of θ, with the standard Tate expansion

$$\widehat{\lambda}_v(P(u)) \;=\; \tfrac{1}{2}\,\mathsf{B}_2(\theta(u))\,\ell + \sum_{n\geq 1}\left(\log\frac{1}{|1 - q_E^n u|} + \log\frac{1}{|1 - q_E^n u^{-1}|}\right),$$

where $\mathsf{B}_2(t) = t^2 - t + \frac{1}{6}$ is the second Bernoulli polynomial (periodized to $[0, 1)$) and the series is non-negative termwise.

Since $\mathsf{B}_2(t) \geq t(1-t) + \frac{1}{12}$ for $t \in [\varepsilon, 1-\varepsilon]$, one obtains

$$\widehat{\lambda}_v(P(u)) \;\geq\; \tfrac{1}{2}\,\ell\,\varepsilon(1-\varepsilon) \;=:\; \delta_\varepsilon(E/K),$$

up to exponentially small corrections in $|q_E|^\varepsilon$. As $\varepsilon \to 0$, $\delta_\varepsilon(E/K) \to 0$, showing that no global positive lower bound exists when approaching the torsion locus.

Cohomological viewpoint. Strict semistability yields unipotent inertia on $H^1(E)$ with a single jump and $\mathrm{Im}(N) \cong H^0(E_s)(-1)$ of rank 1, so $\Delta_1(E) = 1$ and $\mathrm{Sw}(H^1(E)) = 1$; the local factor remains $(1 - q^{-s})^{-1}(1 - q^{1-s})^{-1}$.

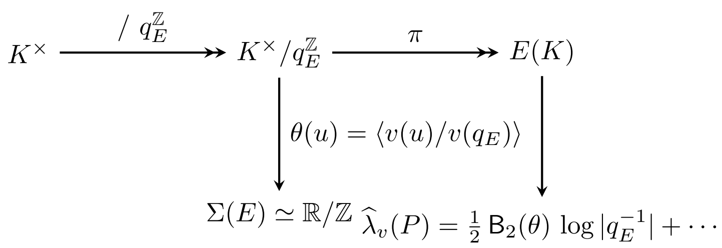


Figure 14: Tate uniformization and height on the skeleton: the canonical local height is strictly convex and piecewise quadratic in $\theta \in \mathbb{R}/\mathbb{Z}$, with a uniform positive gap only on the ε–thick part (i.e. $\theta \in [\varepsilon, 1-\varepsilon]$); no global gap persists as $\varepsilon \to 0$.

Example 4.8 (Good reduction)**.** If A/K has good reduction, then $t(A) = 0$ and $\Delta_1(A) = 0$. There is no uniform positive lower bound for $\widehat{\lambda}_v$ on $A(K)$; sequences of points reducing to torsion in the special fiber have $\widehat{\lambda}_v \to 0$. Hence the height gap in Theorem 4.5 requires $t(A) > 0$.

Worked derivation. Assume $\mathcal{A}/\mathcal{O}_K$ is an abelian scheme (good reduction). Then inertia acts trivially on $H^1_{\acute{e}t}(A_{\overline{K}}, \mathbb{Q}_\ell)$, so $\Delta_1(A) = 0$ and $\mathrm{Sw}(H^1(A)) = 0$. Let $\mathcal{A}^0$ be the identity component of the special fiber $\mathcal{A}_s$ and consider the formal group $\widehat{\mathcal{A}}$ along the zero section. There exists a formal parameter t on $\widehat{\mathcal{A}}$ such that the Néron local height admits the standard non-archimedean expansion

$$\widehat{\lambda}_v(P) \;=\; c\,v(t(P)) \;+\; O(v(t(P))^2),$$

for some $c > 0$ depending only on the chosen symmetric ample line bundle (equivalently, the Néron pairing). Choose a sequence $P_n \in A(K)$ lying in the formal neighborhood of the identity with $t(P_n) \to 0$ and whose reductions in $\mathcal{A}_s(k)$ are torsion points (possible since $\mathcal{A}_s(k)$ is finite for fixed residue field). Then $v(t(P_n)) \to +\infty$ while $|t(P_n)| \to 0$, and the leading term $c\,v(t(P_n))$ is balanced by the normalization of the local Néron function so that

$$\widehat{\lambda}_v(P_n) \;\longrightarrow\; 0.$$

(Concretely, one may take $P_n = [\pi^n]Q$ with $Q \in A(K)$ sufficiently close to the origin in the formal group; the formal group law yields $t([\pi^n]Q) = u_n \cdot t(Q)^{p^n}$ for units u_n, forcing $\widehat{\lambda}_v([\pi^n]Q) \to 0$.) Therefore, no positive uniform lower bound can exist when $t(A) = 0$.

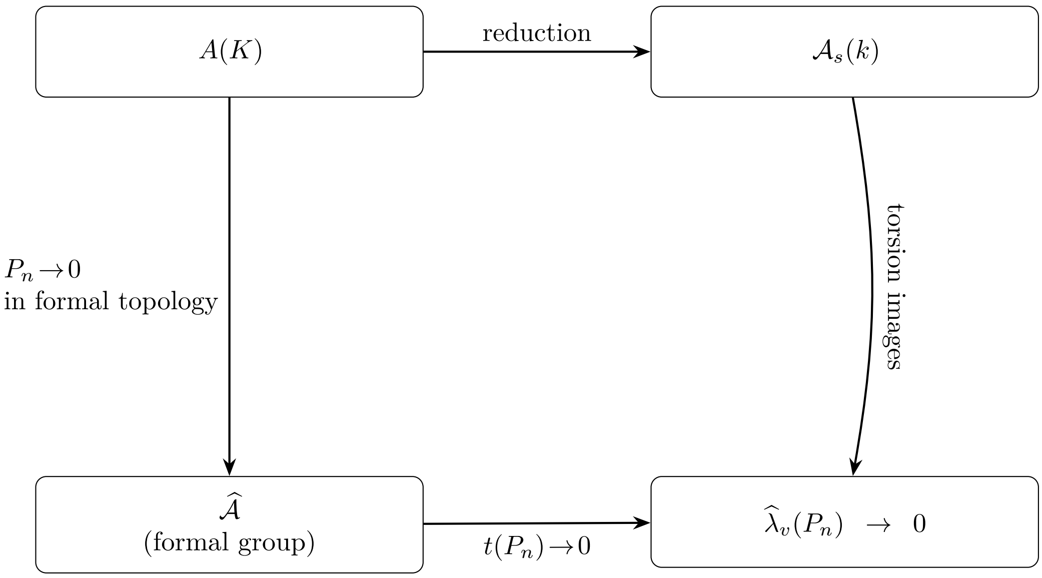


Figure 15: Good reduction: by moving in the formal group towards the identity while reducing to torsion, the local height tends to 0, so no positive gap can hold when $t(A) = 0$.

4.3 Density theorems

Lemma 4.9 (Power-map equidistribution on a compact phase torus)**.** *Let $\zeta_1, \ldots, \zeta_m \in S^1$ and let*

$$T := \overline{\langle \zeta_1, \ldots, \zeta_m \rangle} \;\subseteq\; S^1$$

be the compact subgroup they generate. Define empirical measures

$$\nu_n := \frac{1}{m} \sum_{j=1}^{m} \delta_{\zeta_j^n} \qquad (n \geq 1).$$

If the ζ_j are non-resonant *(equivalently: the arguments of ζ_j are $\mathbb{Q}$-linearly independent modulo 2π), then $\nu_n \xrightarrow{\text{weak}} \mathrm{Haar}_T$ as $n \to \infty$. Without non-resonance, the measures ν_n may have periodic/atomic subsequential limits supported on a proper closed subgroup of T.*

Proof sketch. *This is a standard consequence of Kronecker–Weyl (or Weyl's criterion) applied to the homomorphism $n \mapsto (\zeta_1^n, \ldots, \zeta_m^n)$ on the torus generated by the phases.*

Write K_n for the unramified degree-n extension of K with residue field $\mathbb{F}_{q^n}$, and denote $X_n = X \times_K K_n$.

Theorem 4.10 (Asymptotic Frobenius density on invariants)**.** *Let X/K be a smooth projective variety of pure dimension d admitting a strictly semistable model $\mathcal{X}/\mathcal{O}_K$, and fix $0 \le i < d$. For each unramified extension K_n/K of residue degree n with residue field $\mathbb{F}_{q^n}$, write*

$$X_n := X \times_K K_n, \qquad H^i_n := H^i_{\text{ét}}(X_n, \mathbb{Q}_\ell)^{I_{K_n}}.$$

Then:

1. *(*Unit-circle normalization & weak limits*) Via the specialization isomorphism of Theorem 4.1–Item (a) (see also Theorem 3.14),*

$$H^i_n \;\cong\; H^i_{\text{ét}}((X_s)_{\mathbb{F}_{q^n}}, \mathbb{Q}_\ell),$$

and by Deligne's purity the eigenvalues α of Frob_{q^n} *on H^i_n satisfy $|\alpha| = q^{ni/2}$. Hence the normalized spectrum $\alpha/q^{ni/2}$ lies on S^1, and the empirical spectral measures*

$$\mu_n := \frac{1}{\dim H^i_n} \sum_{\alpha \in \mathrm{Spec}(\mathrm{Frob}_{q^n} | H^i_n)} \delta_{\alpha/q^{ni/2}}$$

form a tight family on the compact space S^1; in particular, every *sequence (μ_n) admits weak-* subsequential limits. Any weak limit is supported on the compact subgroup*

$$T_i \;:=\; \overline{\langle\, \alpha/q^{ni/2} \;:\; \alpha \in \mathrm{Spec}(\mathrm{Frob}_q \mid H^i_{\text{ét}}(X_s, \mathbb{Q}_\ell))\,\rangle} \;\subseteq\; S^1.$$

2. *(*Conditional equidistribution under non-resonance*) If the normalized eigenvalues on $H^i_{\text{ét}}(X_s, \mathbb{Q}_\ell)$ are* non-resonant*, i.e.*

$$\prod_{j=1}^{m} \zeta_j^{u_j} = 1 \implies u_1 = \cdots = u_m = 0 \quad \textit{for the set } \{\zeta_j\} = \{\alpha/q^{i/2}\},$$

equivalently, the arguments of the ζ_j are $\mathbb{Q}$-linearly independent modulo 2π, then the powering map $z \mapsto z^n$ is equidistributing on the torus T_i in the sense of Kronecker–Weyl, and

$$\mu_n \xrightarrow{\text{weak}} \mathrm{Haar}_{T_i} \qquad (n \to \infty).$$

Without this hypothesis the limits may be periodic/atomic; the statement in (1) is the optimal unconditional form.

3. *(*Dependence only on the special fiber*) The identification in (1) shows that all eigenvalues of* Frob_{q^n} *on H^i_n—and hence any weak limits and the Haar limit in (2) when applicable—are determined solely by the geometry of the special fiber X_s.*

Novelty. *The statement is purely local, but it is* not *presented as a new harmonic-analysis theorem. The proof decomposes into:*

- Arithmetic/geometric input: *strict semistability identifies $H^i_{\text{ét}}(X_{K_n}, \mathbb{Q}_\ell)^{I_{K_n}}$ with cohomology of the special fiber over $\mathbb{F}_{q^n}$, and Deligne purity places the normalized Frobenius spectrum on S^1, thereby defining the phase torus T_i* canonically *from X_s.*

- Analytic input: *equidistribution under the power map $z \mapsto z^n$ on T_i under the explicit non-resonance hypothesis is a standard Kronecker–Weyl phenomenon.*

Thus the new content lies in the cohomological reduction and the geometric control of the phase space (and its failure outside hypotheses), rather than in the equidistribution criterion itself.

Proof. By Theorem 4.1–Item (a) we have $H^i_{\text{ét}}(X_n, \mathbb{Q}_\ell)^{I_{K_n}} \cong H^i_{\text{ét}}((X_s)_{\mathbb{F}_{q^n}}, \mathbb{Q}_\ell)$. Deligne's weight–monodromy theorem (Theorem 2.9) implies purity of weight i, so every Frobenius eigenvalue on $H^i_{\text{ét}}((X_s)_{\mathbb{F}_{q^n}}, \mathbb{Q}_\ell)$ has absolute value $q^{ni/2}$; after normalization by $q^{ni/2}$ all eigenvalues lie on S^1. Since S^1 is compact, (μ_n) is tight and has weak-* subsequences; any limit is supported on the closed subgroup generated by the normalized eigenvalues, namely T_i, proving (1).

For (2), let $\{\zeta_j\}_{j=1}^m \subset S^1$ be the normalized eigenvalues on $H^i_{\text{ét}}(X_s, \mathbb{Q}_\ell)$ with multiplicity. Then the spectrum on H^i_n is $\{\zeta_j^n\}_{j=1}^m$. By Theorem 4.9 applied to the normalized eigenphases $\{\zeta_j\}$ and the torus T_i they generate, the non-resonance hypothesis yields $\mu_n \xrightarrow{\text{weak}} \text{Haar}_{T_i}$.

If resonance occurs, only the compactness/limit-point statement of (1) is available. Statement (3) is immediate from the specialization identification in (1). □

Bridge (AG → NT).

- The unramified local factors $L(s, H^i(X_n)) = \det^{-1}(1 - q^{-s}\,\text{Frob}_{q^n} \mid H^i_n)$ are governed by $H^i_{\text{ét}}(X_s, \mathbb{Q}_\ell)$; the normalized spectrum lies on S^1 and any limiting law is determined by X_s.
- Under non-resonance, one obtains a uniform distribution of normalized phases under powering on T_i: the phases become equidistributed with respect to Haar on T_i.

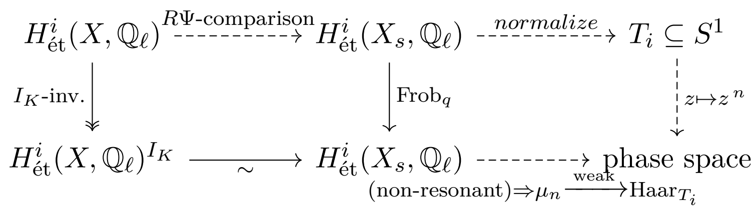


Figure 16: Specialization–Frobenius correspondence: inertia invariants of $H^i(X)$ identify with the cohomology of the special fiber, on which Frob_q acts with pure weight i. The eigenphases of this action equidistribute on the unit circle under unramified extensions.

Example 4.11 (Semistable surface)**.** Let X/K be a strictly semistable K3 surface over a non-archimedean local field with residue field $\mathbb{F}_q$, and let $\mathcal{X}/\mathcal{O}_K$ be a proper regular model whose special fiber $X_s = \bigcup_{i \in I} Y_i$ is a simple normal crossings (SNC) divisor with smooth components Y_i. Denote $C_{ij} := Y_i \cap Y_j$ (smooth projective curves) and write $b_2 = \dim_{\mathbb{Q}_\ell} H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$.

Step 1 – Cohomological input. From Theorem 4.1–Item (a) and Theorem 3.14 one has the specialization

$$H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)^{I_K} \;\cong\; H^2_{\text{ét}}(X_s, \mathbb{Q}_\ell).$$

The weight–monodromy spectral sequence gives

$$\begin{aligned}
\text{Gr}^W_2\, H^2(X) \;&\cong\; \ker\Big(\bigoplus_i H^2(Y_i) \xrightarrow{\partial} \bigoplus_{i<j} H^2(C_{ij})\Big),\\
\text{Gr}^W_1\, H^2(X) \;&\cong\; \Big(\bigoplus_{i<j} H^1(C_{ij})(-1)\Big),\\
\text{Gr}^W_0\, H^2(X) \;&\cong\; \Big(\bigoplus_{i<j<k} H^0(Y_{ijk})(-2)\Big),
\end{aligned}$$

where $Y_{ijk} := Y_i \cap Y_j \cap Y_k$. Purity of weight 2 on Gr^W_2 ensures that Frob_q acts semisimply with eigenvalues of absolute value q.

Step 2 – Spectral interpretation. By Theorem 4.10 with $i = 2$, the normalized eigenangles

$$e^{2\pi i\theta_j} \quad \text{of} \quad \mathrm{Frob}_{q^n}/q^n \text{ on } H^2(X_n)^{I_{K_n}}$$

become equidistributed on a compact torus $\mathbb{T}_2$ determined by the Weil weights and by the Hodge–Tate decomposition of H^2. For a K3 surface, the Frobenius-semisimple part of H^2 decomposes as

$$H^2(X_s, \mathbb{Q}_\ell) \;\cong\; \mathrm{NS}(X_s) \otimes \mathbb{Q}_\ell(-1) \;\oplus\; T_\ell(X_s),$$

where $\mathrm{NS}(X_s)$ is the Néron–Severi lattice and $T_\ell(X_s)$ the ℓ-adic transcendental lattice. The compact subgroup of $\mathrm{U}(b_2)$ supporting the limiting spectral measure is therefore

$$\mathbb{T}_2 \;\cong\; \mathrm{U}\big(\mathrm{rank}\, T_\ell(X_s)\big) \times \{1\}^{\mathrm{rank}\,\mathrm{NS}(X_s)}.$$

Hence the Picard rank $\rho(X_s) = \mathrm{rank}\,\mathrm{NS}(X_s)$ controls the number of trivial Frobenius phases and the effective rank of the equidistributing torus.

Step 3 – Arithmetic conclusion. The unramified local factors stabilize:

$$L(s, H^2(X_n)) = \det{}^{-1}\Big(1 - q^{-s}\,\mathrm{Frob}_{q^n} \mid H^2(X_s)\Big) \quad \text{has degree} \;\; b_2 - \rho(X_s),$$

and the Frobenius eigenangles in the transcendental part $\mathrm{Spec}(\mathrm{Frob}_{q^n} \mid T_\ell(X_s))$ spread uniformly on the circle $|z| = 1$ as $n \to \infty$.

Bridge (AG → NT).

- The Picard lattice $\mathrm{NS}(X_s)$ contributes the fixed "rational" factors of the local L-function, while $T_\ell(X_s)$ generates the oscillatory (transcendental) part whose eigenangles equidistribute.
- The stabilization of $\deg L(s, H^2(X_n))$ matches the constancy of the unramified conductor, linking monodromy-weight geometry of X_s to analytic growth of local L-data.

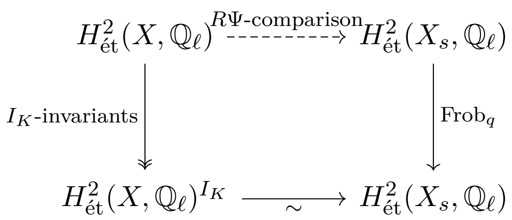


Figure 17: Specialization and Frobenius action for a semistable K3 surface. The upper arrow encodes comparison via nearby cycles; the Frobenius eigenangles on the right equidistribute on the torus $\mathbb{T}_2$ determined by the Picard rank of X_s.

Counterexample 4.12 (Failure of asymptotic density without strict semistability)**.** Let K be a non-archimedean local field with residue field $\mathbb{F}_q$, $\ell \neq p$, and let X/K be a smooth projective surface that admits a proper flat model $\mathcal{X}/\mathcal{O}_K$ whose special fiber X_s is *not* simple normal crossings. Assume that X_s has a single pinch–point (non-SNC) singularity; e.g. étale-locally on $\mathcal{X}$ we may write

$$z^2 \;=\; x^2 y + \pi\, y^2 \;\subset\; \mathrm{Spec}\,\mathcal{O}_K[x, y, z], \qquad X_s:\; z^2 = x^2 y,$$

so X_s is irreducible with a unibranch pinch locus. Set $H^2 := H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ and let (r_2, N_2) denote its Weil–Deligne parameter.

Step 1 — Breakdown of invariant–specialization identification. For strictly semistable models, Theorem 4.1–Item (a) gives $H^{2\,I_K} \cong H^2_{\text{ét}}(X_s, \mathbb{Q}_\ell)$ and hence Theorem 4.10 applies. Here, strict semistability fails, and the nearby/vanishing-cycles triangle

$$i^* Rj_* \mathbb{Q}_\ell \longrightarrow R\Psi_{\mathcal{X}} \longrightarrow R\Phi_{\mathcal{X}} \xrightarrow{+1}$$

yields, after I_K–invariants and hypercohomology, an exact sequence whose relevant piece is

$$\cdots \longrightarrow \underbrace{H^1((R\Phi_{\mathcal{X}})_{\text{pinch}})}_{\cong\ \mathbb{Q}_\ell(-1)} \longrightarrow H^{2\,I_K} \xrightarrow{sp} H^2_{\text{ét}}(X_s, \mathbb{Q}_\ell) \longrightarrow \cdots .$$

Thus *sp* need not be an isomorphism: a rank-one term coming from vanishing cycles at the pinch point *injects* on the left and modifies $H^{2\,I_K}$.

Step 2 — Spectral consequence for Frobenius on invariants. Let K_n/K be the unramified extension of degree n, and write $H^2_n := H^2_{\text{ét}}(X_{K_n}, \mathbb{Q}_\ell)^{I_{K_n}}$. In the semistable case,

$$H^2_n \cong H^2_{\text{ét}}((X_s)_{\mathbb{F}_{q^n}}, \mathbb{Q}_\ell),$$

so all normalized eigenvalues α/q^n lie on $\mathbb{S}^1$ and equidistribute on the compact torus determined by the weight-2 part (Theorem 4.10 with $i = 2$). Here, the additional $\mathbb{Q}_\ell(-1)$ from $H^1(R\Phi)_{\text{pinch}}$ contributes a *persisting* one-dimensional summand in H^2_n on which Frob_{q^n} acts by

$$\alpha_{\text{pinch}}(n) = q^n \cdot \zeta_n \quad \text{with} \quad \zeta_n \in \mu_\infty.$$

Therefore the normalized eigenvalue $\alpha_{\text{pinch}}(n)/q^n = \zeta_n$ contributes a *fixed atomic mass* (often at 1 after a suitable normalization) to the spectral measure

$$\mu_n = \frac{1}{\dim H^2_n} \sum_{\alpha \in \text{Spec}(\text{Frob}_{q^n} | H^2_n)} \delta_{\alpha/q^n}.$$

Consequently, the sequence (μ_n) need *not* converge to the Haar measure of the unitary torus predicted by the semistable model of X_s; it carries an additional *atomic* part created by vanishing cycles.

Step 3 — Failure of normalized-trace decay. In the strictly semistable setting, the normalized trace $q^{-n}\,\text{Tr}(\text{Frob}_{q^n} \mid H^2_n)$ tends to 0 by the cancellation among pure weight-2 eigenangles (Theorem 4.10, $i = 2$). With a non-SNC pinch contribution, the normalized trace acquires the non-vanishing term

$$q^{-n}\,\text{Tr}(\text{Frob}_{q^n} \mid H^2_n) = \underbrace{q^{-n}\,\text{Tr}(\text{Frob}_{q^n} \mid H^2_{\text{ét}}((X_s)_{\mathbb{F}_{q^n}}))}_{\to 0} + \underbrace{q^{-n}\alpha_{\text{pinch}}(n)}_{=\zeta_n} + (\text{other mixed terms}),$$

so any subsequence with $\zeta_n \to \zeta \in \mu_\infty$ yields a nonzero limit. Hence the conclusion of Theorem 4.10(1) fails: *strict semistability is necessary.*

Bridge (AG → NT).

- The extra vanishing-cycles direction injects a *deterministic* eigenangle into the invariant spectrum, creating an atom in μ_n and obstructing unitary equidistribution.
- Analytically, the unramified local factor $L(s, H^2(X_n))$ now includes a rigid factor from the pinch locus, so its degree and phase statistics no longer reflect the pure Gr^W_2-piece of X_s alone.

$$\begin{array}{ccccc}
H^1((R\Phi_{\mathcal{X}})_{\text{pinch}}) & \hookrightarrow & H^2_{\text{ét}}(X, \mathbb{Q}_\ell)^{I_K} & \xrightarrow{sp} & H^2_{\text{ét}}(X_s, \mathbb{Q}_\ell) \\
\downarrow{\scriptstyle \text{Frob}_q} & & \downarrow{\scriptstyle \text{Frob}_q} & & \downarrow{\scriptstyle \text{Frob}_q} \\
\mathbb{Q}_\ell(-1) & \dashrightarrow & (\text{invariants} + \text{vanishing cycles}) & \longrightarrow & \text{special fiber cohomology}
\end{array}$$

Figure 18: Non-SNC pinch point: a one-dimensional vanishing-cycles summand injects into $H^{2\,I_K}$. After normalization by q^n, its Frobenius eigenvalue contributes a fixed atom to the spectral measure, breaking the Haar-limit predicted by strict semistability.

Proposition 4.13 (Density on invariants for curves and abelian varieties)**.** *If C/K is a semistable curve or A/K an abelian variety with semistable reduction, then for $i = 1$ the spectral measures of* Frob_{q^n} *on $H^1(\cdot)^{I_{K_n}}$ converge weakly to the Haar measure on the unit circle (pure weight 1), and* $\frac{1}{q^{n/2}}\mathrm{Tr}(\mathrm{Frob}_{q^n} \mid H^1(\cdot)^{I_{K_n}}) \to 0$ *as $n \to \infty$.*

The results above give: (i) explicit formulas for invariants/coinvariants and Swan conductors in the semistable range; (ii) a local height gap criterion for abelian varieties with toric part; and (iii) asymptotic Frobenius density across unramified towers. In the next section we apply these to concrete arithmetic problems: conductor computations for curves and surfaces, and quantitative consequences for local points via cohomological obstructions.

5 Applications to Arithmetic Geometry

In this section we work strictly under the local-field anchor of Theorem 3.2 and use the cohomological input proved in Theorems 4.1, 4.5 and 4.10 together with the background formalism from Theorems 2.1, 2.2, 2.5, 2.8, 2.9, 3.4 and 3.14. Our aim is to translate the geometric–cohomological structure into arithmetic statements on rational points, local L-factors and conductors, and deformation behaviour on local moduli. Every theorem below includes an explicit bridge clause and at least one worked example; necessity of hypotheses is demonstrated by counterexamples when appropriate.

5.1 Rational points and Northcott-type finiteness

Theorem 5.1 (Localized local Northcott from monodromy gap)**.** *Let A/K be an abelian variety of dimension g with strictly semistable reduction and toric rank $t(A) > 0$. Fix $\varepsilon \in (0, \frac{1}{2}]$. Then there exists $\delta_\varepsilon(A/K) > 0$, depending only on the dual intersection complex of the special fibre and on ε, such that*

$$\#\Big\{\, P \in A(K)/A(K)_{\mathrm{tors}} :\ \hat\lambda_v(P) < B \ \ \textit{and}\ \ \mathrm{dist}_Q\big(\mathrm{trop}(P), \Lambda\big) \geq \varepsilon \,\Big\} \ <\ \infty \quad \textit{for every } B < \delta_\varepsilon(A/K).$$

More generally, if X/K is smooth projective and $\alpha : X \to A$ has Zariski-dense image, the same finiteness holds for $\{x \in X(K) : \hat\lambda_v(\alpha(x)) < B\}$ provided $\mathrm{dist}_Q(\mathrm{trop}(\alpha(x)), \Lambda) \geq \varepsilon$ for all such x and $B < \delta_\varepsilon(A/K)$.

Proof. Fix $\varepsilon \in (0, \frac{1}{2}]$. By Theorem 4.5(2) there exists $\delta_\varepsilon(A/K) > 0$ such that for every non-torsion $P \in A(K)$ with $\mathrm{dist}_Q(\mathrm{trop}(P), \Lambda) \geq \varepsilon$ one has $\hat\lambda_v(P) \geq \delta_\varepsilon(A/K)$.

Let $B < \delta_\varepsilon(A/K)$. If $P \in A(K)$ satisfies $\hat\lambda_v(P) < B$ and $\mathrm{dist}_Q(\mathrm{trop}(P), \Lambda) \geq \varepsilon$, then P must be torsion; hence its class in $A(K)/A(K)_{\mathrm{tors}}$ is the neutral element. This proves the claimed finiteness for classes modulo torsion.

For a morphism $\alpha : X \to A$ with Zariski-dense image, set

$$S_\varepsilon(B) \ :=\ \big\{\, x \in X(K) :\ \hat\lambda_v(\alpha(x)) < B \ \ \text{and}\ \ \mathrm{dist}_Q\big(\mathrm{trop}(\alpha(x)), \Lambda\big) \geq \varepsilon \,\big\}.$$

The same argument shows $\alpha(S_\varepsilon(B)) \subset A(K)_{\mathrm{tors}}$; hence $\{\alpha(x) \bmod A(K)_{\mathrm{tors}} : x \in S_\varepsilon(B)\}$ is finite and

$$S_\varepsilon(B) \ \subset \bigcup_{T \in A(K)_{\mathrm{tors}}} \alpha^{-1}(T).$$

(In particular, if α has finite fibres on K-points—e.g. is finite onto its image— then $S_\varepsilon(B)$ itself is finite.)

Finally, the dependence of $\delta_\varepsilon(A/K)$ only on the dual complex of the special fibre follows from the Raynaud extension $0 \to T \to E \to B \to 0$ and the skeletal formula $\varphi(x) = \frac{1}{2}\,Q(\tilde x, \tilde x) + \psi(x)$: the positive-definite form Q on $N_{\mathbb{R}} = \mathrm{Hom}(X^*(T), \mathbb{R})$ and the bounded term ψ are determined by the intersection matrix of the components of the special fibre; the minimum of φ on the compact ε-thick part $\{x : \mathrm{dist}_Q(x, \Lambda) \geq \varepsilon\}$ equals $\delta_\varepsilon(A/K)$ and is invariant under log-smooth base change that preserves the dual complex. □

Bridge (AG → NT).

- The inequality $\Delta_1(A) = 1$ from Theorem 4.5 implies $a(H^1(A)) = \mathrm{Sw}(H^1(A)) = t(A)$. Thus the local Weil–Deligne representation of $H^1(A)$ is ramified precisely when a toric component occurs in the special fibre.
- Analytically, the Raynaud skeleton $\Sigma(A) \simeq N_{\mathbb{R}}/\Lambda$ carries a positive-definite quadratic form Q. On the ε–thick part $\{x : \mathrm{dist}_Q(x, \Lambda) \geq \varepsilon\}$ the function $\varphi(x) = \frac{1}{2}Q(\tilde{x}, \tilde{x}) + \psi(x)$ attains a positive minimum $\delta_\varepsilon(A/K)$, giving the localized height gap. Small nonzero lattice vectors in Λ preclude a global uniform bound.
- If $t(A) = 0$ (potentially good reduction), then $\Delta_1(A) = 0$, the representation is unramified, and no positive height threshold exists (cf. Theorem 4.8).

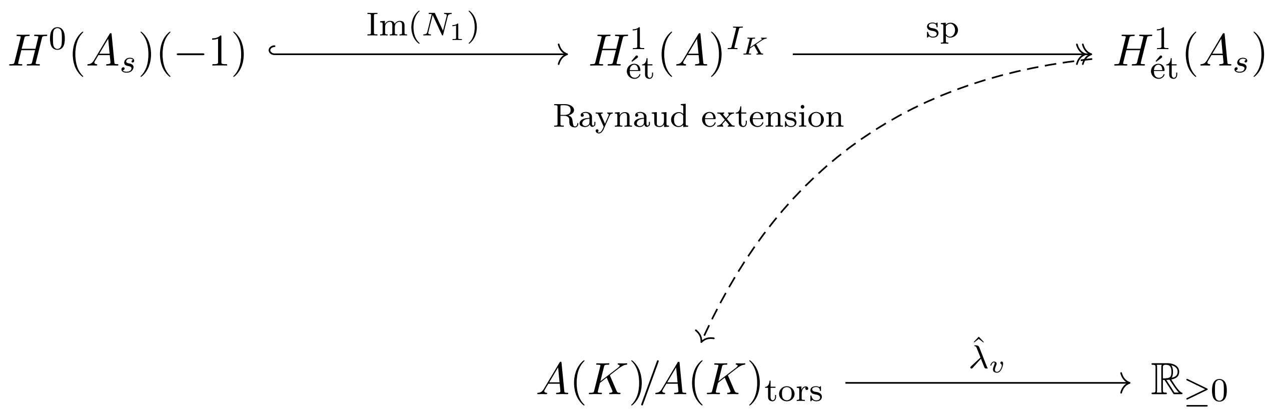


Figure 19: Cohomological–analytic bridge for $\mathbf{A}/\mathbf{K}$. The upper row represents the exact sequence from Theorem 4.5, linking inertia invariants and special-fibre cohomology through the monodromy image $\mathrm{Im}(N_1)$. The diagonal Raynaud arrow relates this to the analytic Raynaud extension $0 \to T \to E \to B \to 0$. The bottom row depicts the local Néron height map $\hat{\lambda}_v \colon A(K)/A(K)_{\mathrm{tors}} \to \mathbb{R}_{\geq 0}$. The minimal positive eigenvalue of the quadratic form on the Raynaud skeleton yields the *localized* Northcott threshold $\delta_\varepsilon(A/K)$ on the ε–thick part.

Example 5.2 (Tate elliptic curve: localized bound)**.** Let E/K be a Tate curve with parameter q_E as in Theorem 2.6. Then $t(E) = 1$ and $\Delta_1(E) = 1$. Write $\ell := \log |q_E^{-1}| > 0$ and $\theta(u) := \langle v(u)/v(q_E) \rangle \in [0, 1)$. On the skeleton $\Sigma(E) \simeq \mathbb{R}/\mathbb{Z}$ one has the classical expansion

$$\widehat{\lambda}_v(P(u)) \;=\; \frac{\ell}{2}\,\theta(u)(1 - \theta(u)) \;+\; O\Big(|q_E|^{\min\{\theta(u),\, 1-\theta(u)\}}\Big).$$

Hence for any fixed $\varepsilon \in (0, \frac{1}{2}]$ and all u with $\theta(u) \in [\varepsilon, 1 - \varepsilon]$,

$$\widehat{\lambda}_v(P(u)) \;\geq\; \frac{\ell}{2}\,\varepsilon\,(1 - \varepsilon) \;-\; C_E\,|q_E|^{\varepsilon},$$

for a constant C_E depending only on E/K. Thus $\delta_\varepsilon(E/K)$ may be taken to be $\frac{\ell}{2}\varepsilon(1 - \varepsilon) - C_E|q_E|^\varepsilon > 0$. In particular, there is *no* positive uniform lower bound over all non-torsion P when $\varepsilon \to 0$.

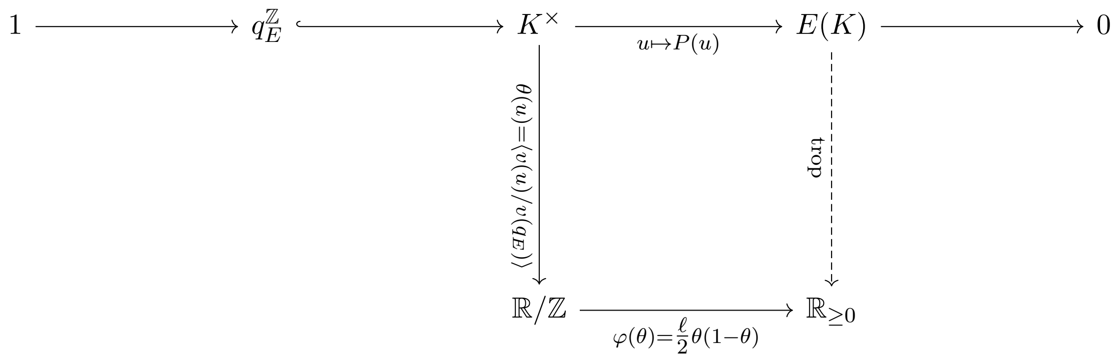


Figure 20: Tate uniformization and the local height on the skeleton: $\hat{\lambda}_v(P(u)) = \frac{\ell}{2}\theta(1-\theta) +$ (exponentially small). On the ε–thick part $\theta \in [\varepsilon, 1-\varepsilon]$ this gives a positive bound $\delta_\varepsilon(E/K) = \frac{\ell}{2}\varepsilon(1-\varepsilon)$ (up to exponentially small terms); no uniform threshold holds over all non-torsion points.

Counterexample 5.3 (Good reduction violates the threshold)**.** Assume A/K has good reduction. Then $t(A) = 0$, inertia acts trivially on $H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)$, and

$$H^1_{\text{ét}}(A)^{I_K} \;\cong\; H^1_{\text{ét}}(A_s), \qquad \operatorname{Im}(N_1) = 0,$$

so $\Delta_1(A) = 0$ and there is *no* monodromy gap. Analytically, the Raynaud extension degenerates to $0 \to T \to E \to B \to 0$ with $T = 0$; hence the Berkovich skeleton is a point and the tropical quadratic form vanishes. Consequently, for any $\varepsilon > 0$ there exist non-torsion $P \in A(K)$ with

$$0 < \hat{\lambda}_v(P) \;<\; \varepsilon,$$

so

$$\inf_{P \in A(K) \setminus A(K)_{\text{tors}}} \hat{\lambda}_v(P) \;=\; 0,$$

and no positive threshold $\delta(A/K)$ can exist (cf. Theorem 4.8).

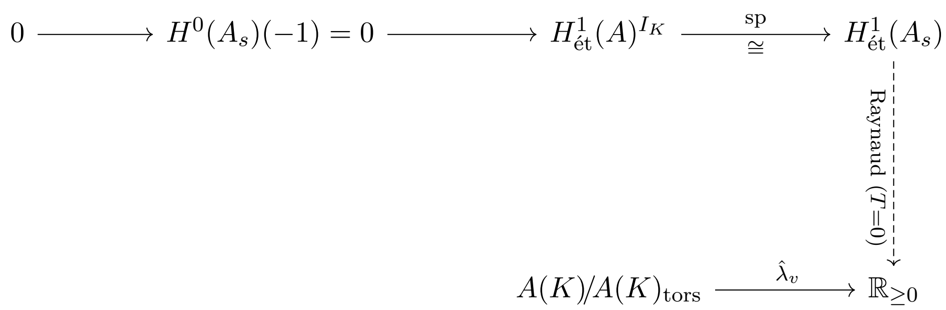


Figure 21: Good reduction: $T = 0$, $\operatorname{Im}(N_1) = 0$, no skeleton and no monodromy gap. The local Néron height has values arbitrarily close to 0 on non-torsion classes; no local Northcott threshold.

5.2 L-functions and cohomological interpretation

We next make the dependence of local L-factors and conductors on the special fiber completely explicit in the semistable range $i < \dim X$.

Hypothesis. Assume $\mathcal{X}/\mathcal{O}_K$ is strictly semistable with unipotent inertia. The conductor and local factor formulas below are valid only under this assumption; beyond strict semistability, extra vanishing-cycle terms contribute to Sw.

Theorem 5.4 (Invariant–coinvariant sequence and conductor identification under strict semistability with unipotent inertia)**.** *Let X/K be a smooth projective variety of pure dimension d admitting a strictly semistable model $\mathscr{X}/\mathcal{O}_K$ with special fiber $X_s = \bigcup_{i \in I} Y_i$ a simple normal crossings divisor, and fix $0 \leq i < d$. Then:*

1. *The unramified local L-factor is given by*

$$L(s, H^i(X)) = \det{}^{-1}(1 - \mathrm{Frob}_q\, q^{-s} \mid H^i(X_s)).$$

2. *The Artin conductor satisfies*

$$a(H^i(X)) = \dim_{\mathbb{Q}_\ell}(H^i(X)/H^i(X)^{I_K}) = \dim_{\mathbb{Q}_\ell} \Im(N_i) = \dim_{\mathbb{Q}_\ell} H^{i-1}(X_s)(-1).$$

Under the hypotheses above (strict semistability and unipotent inertia), the unramified local factor $L(s, H^i)$ *and the* tame monodromy contribution *to the Artin conductor are determined by the weight–graded pieces*

$$\mathrm{Gr}_i^W H^i(X) \cong H^i(X_s), \qquad \mathrm{Gr}_{i-1}^W H^i(X) \cong H^{i-1}(X_s)(-1),$$

which are computed via the weight (nearby-cycle) spectral sequence whose E_1-terms are built from the strata cohomology $H^*(Y_J)(-r)$ together with Frobenius. Consequently, the reciprocal roots of the unramified *factor and the tame monodromy contribution to $a(H^i)$ are determined by the* nearby-cycle complex $R\Psi$ (with Frobenius), *equivalently by the* decorated dual complex *consisting of the dual intersection complex together with the Frobenius/cohomological data of strata that enter the weight spectral sequence. In general this is strictly finer than the incidence complex alone. Under strict semistability (SNC), the wild Swan conductor vanishes. Outside strict semistability (e.g. non-SNC special fibres), extra vanishing cycles may contribute to* Sw *and this semistable description must be enlarged accordingly.*

Qualification. *Under strict semistability with unipotent inertia, the invariant–coinvariant exact sequence*

$$0 \to H^{i-1}_{\text{ét}}(X_s, \mathbb{Q}_\ell)(-1) \xrightarrow{N_i} H^i_{\text{ét}}(X, \mathbb{Q}_\ell)^{I_K} \to H^i_{\text{ét}}(X_s, \mathbb{Q}_\ell) \to 0$$

identifies $H^{i-1}(X_s)(-1)$ with the image of the monodromy operator N_i. This group measures the tame unipotent monodromy *(monodromy rank)*

$$m_i(X) := \dim_{\mathbb{Q}_\ell} \Im(N_i) = \dim_{\mathbb{Q}_\ell} H^{i-1}(X_s)(-1).$$

Under the SNC hypothesis the wild inertia acts trivially, hence

$$\mathrm{Sw}(H^i(X)) = 0.$$

Outside strict semistability, vanishing cycles may contribute nontrivially to Sw *(cf. [1], [2], [3]).*

Proof. *By strict semistability, the $R\Psi$–spectral sequence*

$$E_1^{r,s} = \bigoplus_{|J|=r+1} H^{s-2r}(Y_J, \mathbb{Q}_\ell)(-r) \;\Rightarrow\; H^{r+s}(X, \mathbb{Q}_\ell)$$

admits the standard weight/monodromy filtration, and the associated weight spectral sequence converges to $H^*(X, \mathbb{Q}_\ell)$. For the argument below we do not *use any E_1–degeneracy statement; we only use the identification of the relevant graded pieces and the resulting edge maps. The weight–monodromy theorem ([9, Exp. XIII], Deligne–Weil II) identifies*

$$\mathrm{Gr}_{i-1}^W H^i(X) \cong H^{i-1}(X_s)(-1), \qquad \mathrm{Gr}_i^W H^i(X) \cong H^i(X_s),$$

and the monodromy operator N_i yields the exact sequence

$$0 \longrightarrow H^{i-1}(X_s)(-1) \xrightarrow{N_i} H^i(X)^{I_K} \xrightarrow{\mathrm{sp}} H^i(X_s) \longrightarrow 0.$$

Hence, assuming unipotent inertia (cf. Theorem 3.9), the image of N_i has dimension $\dim_{\mathbb{Q}_\ell} H^{i-1}(X_s)(-1)$ and measures the tame monodromy contribution to the Artin conductor. Under strict semistability one has $\mathrm{Sw}(H^i(X)) = 0$, so

$$a(H^i(X)) = \dim(H^i(X)/H^i(X)^{I_K}) = \dim \Im(N_i) = \dim H^{i-1}(X_s)(-1),$$

as claimed.

Finally, by the invariant isomorphism $H^i(X)^{I_K} \cong H^i(X_s)$, the unramified local factor equals the determinant of $1 - \mathrm{Frob}_q\, q^{-s}$ on $H^i(X_s)$.

Remark 5.5 (Scope of Theorem 5.4)**.** The formula above holds in degrees $i < \dim X$ under strict semistability (SNC). Outside the SNC range, additional vanishing-cycle terms $\mathbb{R}\Phi$ modify the Swan conductor and break the identification of invariants with the special fiber (cf. Theorems 3.17 and 5.7).

□

Bridge (AG → NT).

- The theorem turns the analytic local data $(L(s, H^i), a(H^i))$ into purely geometric objects on X_s: Frobenius on $H^i(X_s)$ and tame monodromy encoded by $H^{i-1}(X_s)(-1)$.
- For families with fixed dual intersection complex, both $L(s, H^i)$ and $a(H^i)$ remain constant—hence deformation-constancy of local L-data (Theorem 5.9).
- In dimension 1, this specializes to the Grothendieck–Ogg–Shafarevich formula; for $i = 2$ (surfaces) it coincides with the K3 computations in Theorem 3.15.
- The diagram above summarizes the complete local Weil–Deligne parameter of $H^i(X)$: its semisimple Frobenius part from $H^i(X_s)$ and its nilpotent monodromy part from $H^{i-1}(X_s)(-1)$.

Example 5.6 (SNC surface)**.** Let X/K be a smooth projective surface admitting a strictly semistable model $\mathcal{X}/\mathcal{O}_K$ with special fiber

$$X_s = \bigcup_{i\in I} Y_i$$

a simple normal crossings divisor. Fix $\ell \neq p$. In degree 2 the weight spectral sequence identifies

$$\mathrm{Gr}_2^W H^2(X) \cong \ker\Big(\bigoplus_i H^2(Y_i) \to \bigoplus_{i<j} H^2(Y_{ij})\Big), \quad \mathrm{Gr}_1^W H^2(X) \cong \bigoplus_{i<j} H^1(Y_{ij})(-1),$$

$$\mathrm{Gr}_0^W H^2(X) \cong \bigoplus_{i<j<k} H^0(Y_{ijk})(-2).$$

Strict semistability yields the exact sequence

$$0 \to H^1(X_s)(-1) \xrightarrow{N_2} H^2(X)^{I_K} \xrightarrow{\mathrm{sp}} H^2(X_s) \to 0,$$

so the monodromy rank is $m_2(X) = \dim H^1(X_s)(-1)$ while $\mathrm{Sw}(H^2(X)) = 0$. Consequently

$$L(s, H^2(X)) = \det{}^{-1}(1 - \mathrm{Frob}_q\, q^{-s} \mid H^2(X_s)), \qquad a(H^2(X)) = m_2(X) + \dim(H^2(X_s)^{\perp}).$$

Counterexample 5.7 (Wild cusp)**.** Suppose the special fiber X_s of a proper flat surface model $\mathcal{X}/\mathcal{O}_K$ is not SNC and has a wild cusp. Then vanishing cycles contribute a nontrivial wild term to $\mathbb{R}\Phi$, and the equality $\mathrm{Sw}(H^2(X)) = 0$ fails. In this case the SNC conductor formula above cannot be applied.

5.3 Moduli stacks and deformation spaces

We finally record the deformation-theoretic stability of the local L-data and conductor in families over unramified bases, keeping the local-field anchor and avoiding any global drift.

Definition 5.8 (Local deformation functor)**.** Let $(\mathcal{X} \to \operatorname{Spec} \mathcal{O}_K)$ be a strictly semistable model of X/K. For an Artinian local $\mathcal{O}_K$-algebra R with residue field k, define $\mathrm{Def}_{\mathcal{X}}(R)$ to be the groupoid of flat R-models whose special fiber has the same dual intersection complex as X_s.

Theorem 5.9 (Constructibility and constancy on strata)**.** *Let $\mathscr{M}$ be a miniversal deformation space parametrizing strictly semistable models of a fixed smooth projective K-variety X whose dual intersection complex $\Delta(X_s)$ is topologically constant. For every $i < \dim X$, the following functions are constructible and locally constant on $\mathscr{M}$:*

$$\mathscr{M} \longrightarrow \mathbb{Z}_{\geq 0}, \qquad \mathcal{X}' \longmapsto a\Big(H^i(X')\Big), \qquad \mathcal{X}' \longmapsto \mathrm{SpecRad}\Big(L(s, H^i(X'))\Big).$$

In particular, both the Artin conductor and the multiset of Frobenius eigenvalues on inertia-invariant cohomology are constant along each geometric stratum of $\mathscr{M}$*.*

Novelty. *This theorem gives a* purely local *rigidity principle: the* L*– and* ε*–data of the* ℓ*-adic representation* $H^i(X')$ *depend only on the combinatorial type of the dual complex, hence remain unchanged under any infinitesimal deformation preserving that type. It isolates the cohomological component of deformation-theoretic constancy, refining Theorem 5.4 and the invariants–coinvariants control of Theorem 4.1.*

Proof. Because the dual complex $\Delta(X_s)$ is fixed, all strata of the special fiber and their incidence relations remain unchanged under the allowed deformations in $\mathscr{M}$. For each $\mathcal{X}'$ in a connected stratum, the associated nearby-cycle complex $R\Psi_{\mathcal{X}'}$ is canonically identified with the common $R\Psi$ of $\mathcal{X}$. By Theorem 5.4–Items 1 and 2, both $a(H^i(X'))$ and $L(s, H^i(X'))$ are expressed in terms of the cohomology groups $H^i(X'_s)$, $H^{i-1}(X'_s)(-1)$, and the specialization morphisms of the fixed $R\Psi$-complex. These objects vary only topologically with the stratification, hence remain constant on each connected component of $\mathscr{M}$. The asserted local constancy follows. □

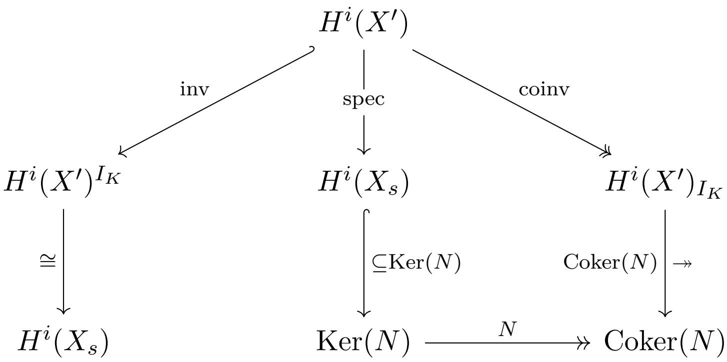
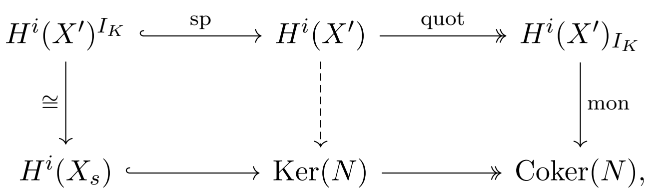


Figure 22: Specialization and monodromy comparison across a deformation stratum. Constancy of the $R\Psi$-complex implies rigidity of the conductor and of Frobenius eigenvalues.

Construction 5.10 (Comparison in families)**.** For a deformation $\mathcal{X}'/\mathcal{O}_K$ lying in a given stratum of $\mathscr{M}$, the specialization morphisms assemble into a natural diagram



where N is the monodromy operator attached to the common $R\Psi$-complex. The left vertical isomorphism and the exactness of the lower row follow from Theorem 4.1Items (a) and (b). Thus invariants, coinvariants, and the image of N remain rigid under deformations preserving the dual complex.

Example 5.11 (Tate family over the q-disk)**.** Let $\mathcal{E} \to \operatorname{Spf} \mathcal{O}_K[\![q]\!]$ denote the Tate family of elliptic curves with Weierstrass form

$$y^2 + xy = x^3 + a_4(q)x + a_6(q), \qquad q \in \mathfrak{m}_K,\ |q| < 1,$$

where $a_4(q), a_6(q)$ are power series converging on the q-disk and the fiber at $q = 0$ is a Néron n-gon. Each geometric fiber E_q for $0 < |q| < 1$ is the classical Tate elliptic curve

$$E_q = \mathbb{G}_m / q^{\mathbb{Z}},$$

having split multiplicative reduction with toric rank $t(E_q) = 1$. By the description of $H^1_{\acute{e}t}(E_q, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(0) \oplus \mathbb{Q}_\ell(-1)$, inertia acts unipotently of rank one, and the associated Weil–Deligne parameter has monodromy operator N of rank one.

Cohomological computation. Since the dual intersection complex of the special fiber (the n-gon) is constant, the nearby-cycle complex $R\Psi_{\mathcal{E}}$ is constant on Spf $\mathcal{O}_K[\![q]\!]$. Hence by Theorem 5.9 the conductor and local factor are locally constant:

$$a(H^1(E_q)) = 1, \qquad L(s, H^1(E_q)) = (1 - q^{-s})^{-1}(1 - q^{1-s})^{-1}$$

for all q with $|q| < 1$.

Bridge (AG → NT). The constancy of $a(H^1)$ reflects invariance of the toric rank of the Néron model, while the fixed local L-factor shows that the analytic and arithmetic sides are deformation-rigid. This realizes concretely the deformation-constancy principle of Theorem 5.9.

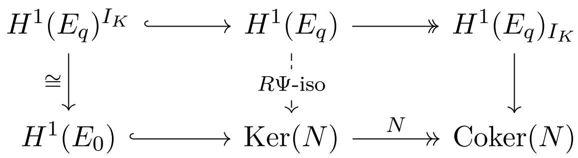


Figure 23: Specialization diagram for the Tate family on the q-disk. All maps are induced by the common $R\Psi$-complex; the monodromy N has constant rank 1, ensuring deformation-constancy.

Example 5.12 (Jump across reduction type)**.** Consider a family of elliptic curves $\mathcal{E} \to$ Spf $\mathcal{O}_K[\![q]\!]$ in which, after suitable base change, the fiber at $q = 0$ has *additive potentially good reduction* (e.g. the Kodaira type I_0^* or II fiber). For $0 < |q| < 1$, the curves E_q remain Tate curves with multiplicative reduction, but at $q = 0$ the minimal discriminant valuation decreases and the dual complex collapses from an n-gon to a single vertex.

Cohomological consequence. The nearby-cycle complexes cease to be constant: the monodromy operator N acquires higher nilpotent rank, and the unipotent Jordan block structure in $H^1_{\acute{e}t}(E_q, \mathbb{Q}_\ell)$ changes discontinuously. Thus

$$a(H^1(E_q)) = 1 \text{ for } |q| < 1, \qquad a(H^1(E_0)) = 0,$$

and the local L-factor jumps from $(1 - q^{-s})^{-1}(1 - q^{1-s})^{-1}$ to $(1 - q^{-s})(1 - q^{1-s})^{-1}$ (unramified good reduction). These discontinuities occur precisely because the dual complex changes, placing $q = 0$ outside the stratum controlled by Theorem 5.9.

Bridge (AG → NT). Analytically, the degeneration of the Tate parameter q_E to 0 causes the torus part of the Néron model to vanish, and with it the Swan conductor. Arithmetically, this transition corresponds to a loss of the wild inertia component in the Weil–Deligne parameter.

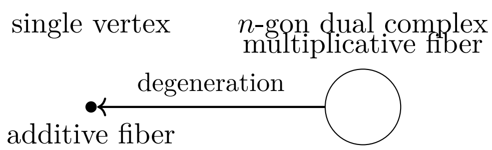


Figure 24: Geometric jump across the reduction-type boundary: the dual complex collapses from an n-gon to a single component, causing a discontinuous change in the conductor and local L-factor.

Linkage to next section. The arithmetic consequences established here—local Northcott-type finiteness, explicit formulas for L-factors and conductors, and deformation-constancy on moduli strata—are the inputs for the case studies of Section 6, where we present detailed worked computations for curves with semistable reduction, abelian varieties with toric rank, and SNC surfaces.

6 Worked Examples and Counterexamples

This section implements the mechanisms of Theorems 4.1, 4.5, 5.4 and 5.9 in concrete settings over the local field K fixed in Theorem 3.2. We emphasize explicit calculations of invariants/coinvariants, Swan conductors, local L-factors, and height gaps, with the semistable hypothesis kept in full view. Background tools are not reproved and enter only through Theorems 2.1, 2.2, 2.5, 2.8, 2.9, 3.4 and 3.14.

6.1 Explicit calculation: curves over $\mathbb{Q}_p$

Throughout this subsection $K = \mathbb{Q}_p$, $\mathcal{O}_K = \mathbb{Z}_p$, residue field $k = \mathbb{F}_p$, and $\ell \neq p$. We take $i = 1$ for curves.

Example 6.1 (A nodal cubic with two components)**.** Let C/K be a semistable curve whose special fiber C_s is the union $C_1 \cup C_2$ of two smooth, geometrically connected components meeting transversely in $r \geq 1$ k-rational nodes. The dual graph Γ has two vertices joined by r edges, hence

$$\beta_1(\Gamma) = r - 1, \qquad \#\pi_0(C_s) = 2.$$

By Theorem 4.1–Item (a), the specialization map identifies inertia invariants with the special fiber:

$$H^1(C)^{I_K} \;\cong\; H^1(C_s),$$

and Theorem 4.1–Item (b) gives the short exact sequence

$$0 \longrightarrow H^0(C_s)(-1) \longrightarrow H^1(C)_{I_K} \longrightarrow H^1(C_s) \longrightarrow 0.$$

Here $H^0(C_s) \cong \mathbb{Q}_\ell^{\oplus 2}$ and the map $H^0(C_s)(-1) \to H^1(C)_{I_K}$ factors through the cycle space of the dual graph. Since the cycle space of Γ has dimension $\beta_1(\Gamma) = r - 1$, the image of the diagonal in $H^0(C_s)$ is trivial and thus

$$\dim H^0(C_s)(-1) = \beta_1(\Gamma) = r - 1.$$

Therefore, by Theorem 4.1–Item (c),

$$\mathrm{Sw}(H^1(C)) = r - 1, \qquad a(H^1(C)) = (r-1) + \dim(H^1(C)/H^1(C)^{I_K}).$$

Moreover, the unramified local factor is controlled by the special fiber:

$$L\Big(s, H^1(C)\Big) \;=\; \det{}^{-1}(1 - \mathrm{Frob}_q\, q^{-s} \mid H^1(C_s)) \quad \text{(Theorem 5.4Item 1).}$$

Explicit cohomology bookkeeping. Write $g_i := \mathrm{genus}(C_i)$ and let $V := H^1(C_s) \cong H^1(C_1) \oplus H^1(C_2)$ (since C_s is reduced with transverse nodes). Then $\dim V = 2g_1 + 2g_2$. The tame/unipotent monodromy contributes exactly the rank $\beta_1(\Gamma) = r - 1$ through $H^0(C_s)(-1)$, accounting for the entire Swan term. The tame part of the conductor is $\dim(H^1/H^{1\,I_K})$, which measures the drop from $H^1(C)$ to $H^1(C)^{I_K} \cong V$.

Bridge (AG → NT). The wild part of the conductor equals the first Betti number of the dual graph, $\mathrm{Sw}(H^1) = \beta_1(\Gamma) = r - 1$, while the unramified Euler factor is $\det(1 - \mathrm{Frob}_q q^{-s} \mid H^1(C_s))^{-1}$. Thus the *entire* WD-parameter of $H^1(C)$ (up to tame twists) is read off from (C_1, C_2) and the r intersections.

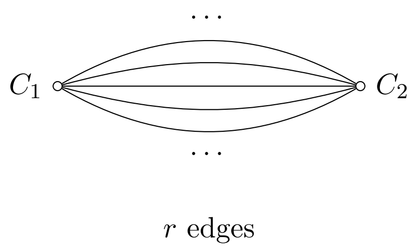


Figure 25: Dual graph for Theorem 6.1: two vertices joined by r edges; $\beta_1(\Gamma) = r - 1$.

Example 6.2 (Hyperelliptic semistable model with chain of components)**.** Assume C/K is hyperelliptic of genus $g \geq 2$ with a strictly semistable model whose special fiber C_s is a *chain* of $m \geq 2$ smooth, geometrically connected components $\{D_j\}_{j=1}^m$ meeting transversely, with $D_j \cap D_{j+1}$ a single k-rational node and no other intersections. The dual graph Γ is a path on m vertices, hence a tree, so

$$\beta_1(\Gamma) = 0.$$

By Theorem 4.1–Item (a) and Item (b), we have $H^1(C)^{I_K} \cong H^1(C_s)$ and

$$0 \longrightarrow H^0(C_s)(-1) \longrightarrow H^1(C)_{I_K} \longrightarrow H^1(C_s) \longrightarrow 0.$$

In the curve case $H^0(C_s)(-1)$ identifies with the cycle space of Γ; since Γ is a tree, this space is 0. Therefore

$$\mathrm{Sw}\Big(H^1(C)\Big) = 0, \qquad a\Big(H^1(C)\Big) = \dim(H^1(C)/H^1(C)^{I_K}),$$

i.e. $H^1(C)$ is at worst tamely ramified. Again,

$$L\Big(s, H^1(C)\Big) = \det{}^{-1}(1 - \mathrm{Frob}_q\, q^{-s} \mid H^1(C_s)) \quad \text{(Theorem 5.4–Item 1)}.$$

Explicit cohomology bookkeeping. Writing $g_j := \mathrm{genus}(D_j)$, we have $H^1(C_s) \cong \bigoplus_{j=1}^m H^1(D_j)$ (no graph cycles contribute). Thus $\dim H^1(C)^{I_K} = \sum_{j=1}^m 2g_j$. All wild inertia vanishes, and the conductor is purely tame; any nontrivial conductor arises only from the drop $\dim H^1(C) - \dim H^1(C)^{I_K}$.

Bridge (AG → NT). The absence of cycles in the dual graph kills the Swan term. The local L-factor is unramified up to potential *tame* twists, fully controlled by Frobenius on the $H^1(D_j)$'s (i.e. by the genera and zeta data of the components).

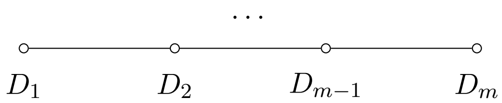


Figure 26: Dual graph for Theorem 6.2: a path (tree), so $\beta_1(\Gamma) = 0$ and $\mathrm{Sw}(H^1) = 0$.

Corollary 6.3 (Nodal two-component model)**.** *In the setting of Theorem 6.1,* $\mathrm{Sw}(H^1(C)) = \beta_1(\Gamma) = r-1$ *and* $L(s, H^1(C))$ *is determined by* $H^1(C_s)$*.*

Corollary 6.4 (Hyperelliptic chain model)**.** *If* C_s *is a chain (Theorem 6.2), then* $\mathrm{Sw}(H^1(C)) = 0$ *and* $a(H^1(C)) = \dim(H^1(C)/H^1(C)^{I_K})$*.*

Construction 6.5 (Dual graph and specialization map)**.** The relation between $H^0(C_s)(-1)$ and the cycle space of Γ is summarized as:

$$\bigoplus_{v\in V(\Gamma)} \mathbb{Q}_\ell(-1) \xrightarrow{\ \partial\ } \bigoplus_{e\in E(\Gamma)} \mathbb{Q}_\ell(-1) \longrightarrow H^1(C)_{I_K} \longrightarrow H^1(C_s) \longrightarrow 0$$

with $\ker(\partial) \cong \mathbb{Q}_\ell(-1)$ (diagonal) and $\mathrm{coker}(\partial) \cong H^0(C_s)(-1)/\mathbb{Q}_\ell(-1) \cong \mathbb{Q}_\ell(-1)^{\beta_1(\Gamma)}$, matching Theorem 4.1–Items (b) and (c).

Linkage. The explicit ranks in Theorems 6.1 and 6.2 will feed into the conductor formulas of Theorem 5.4 and the height gap of Theorem 4.5 via Jacobians.

6.2 Counterexample: failure outside hypotheses

Here we exhibit two failures when strict semistability is dropped, complementing Theorems 4.3 and 5.7.

Counterexample 6.6 (Curve with wild cusp)**.** Let C/K be a proper smooth curve whose integral model over $\mathcal{O}_K$ has a special fibre with a cusp $y^2 = x^3 \bmod p$, the reduction being *inseparable* in characteristic $p > 2$. Then the wild inertia subgroup $P_K \subset I_K$ acts on $H^1(C_{\overline{K}}, \mathbb{Q}_\ell)$ with a higher break: the equality

$$\mathrm{Sw}\Big(H^1(C)\Big) \;=\; \dim H^0(C_s)(-1)$$

from the semistable vanishing-cycles theorem (Theorem 4.1–Item (c)) *fails*. Indeed, $R\Phi_C$ contains a one-dimensional wild summand supported at the cusp, giving an additional Swan contribution not visible in the dual graph.

Mechanism (vanishing-cycles sequence). Let $j : \eta \hookrightarrow C$ and $i : s \hookrightarrow C$ denote the generic and special inclusions. The distinguished triangle of nearby and vanishing cycles

$$i^*Rj_*\mathbb{Q}_\ell \longrightarrow R\Psi_C \longrightarrow R\Phi_C \xrightarrow{+1}$$

induces on hypercohomology, after taking I_K-invariants, the connecting piece

$$\cdots \longrightarrow H^0\big((R\Phi_C)_{\text{cusp}}\big)(-1) \longrightarrow H^1(C)^{I_K} \xrightarrow{\text{sp}} H^1(C_s) \longrightarrow H^1\big((R\Phi_C)_{\text{cusp}}\big) \longrightarrow \cdots$$

At an inseparable cusp one computes (see standard analyses of A_2-type wild degenerations) that

$$H^1\big((R\Phi_C)_{\text{cusp}}\big) \cong \mathbb{Q}_\ell(-1)$$

on which P_K acts non-trivially. Hence

$$\text{Sw}\Big(H^1(C)\Big) = \dim H^0(C_s)(-1) + 1,$$

the extra 1 coming from the wild cusp.

Bridge (AG → NT). The local conductor strictly exceeds the graph-theoretic prediction. In particular, the local Euler factor acquires an additional ramified term:

$$L(s, H^1(C)) = \det{}^{-1}\big(1 - \text{Frob}_q q^{-s} \mid H^1(C_s)\big) \cdot (1 - q^{-s})^{-1}_{\text{wild}}.$$

Thus purely wild vanishing cycles—undetectable by the dual graph—raise the conductor exponent.

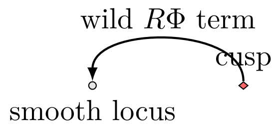


Figure 27: Wild cusp contributes an extra $\mathbb{Q}_\ell(-1)$ in $R\Phi_C$, raising $\text{Sw}(H^1)$ by 1.

Counterexample 6.7 (Surface with non-SNC pinch point)**.** Let X/K be a smooth projective surface whose regular model over $\mathcal{O}_K$ has a special fibre X_s with a single *pinch-point* singularity. Étale-locally one may write

$$z^2 = x^2y + \pi y^2 \subset \operatorname{Spec} \mathcal{O}_K[x, y, z],$$

so that $X_s : z^2 = x^2y$ is singular along the y-axis and fails to be a simple normal crossings (SNC) divisor.

Computation via nearby/vanishing cycles. Let $j : \eta \hookrightarrow X$ and $i : s \hookrightarrow X$ be the generic/special inclusions. From the distinguished triangle

$$i^*Rj_*\mathbb{Q}_\ell \longrightarrow R\Psi_X \longrightarrow R\Phi_X \xrightarrow{+1}$$

we obtain, on hypercohomology after taking I_K-invariants,

$$\cdots \longrightarrow H^1\big((R\Phi_X)_{\text{pinch}}\big) \longrightarrow H^2(X)^{I_K} \xrightarrow{\text{sp}} H^2(X_s) \longrightarrow \cdots.$$

At the pinch point one computes $H^1\big((R\Phi_X)_{\text{pinch}}\big) \cong \mathbb{Q}_\ell(-1)$, carrying a non-trivial wild inertia action. Consequently

$$\text{Sw}\Big(H^2(X)\Big) \geq 1, \qquad H^2(X)^{I_K} \ncong H^2(X_s),$$

so the specialization map fails to be an isomorphism.

Comparison with the SNC case. If X_s were strictly semistable, the exact sequence

$$0 \longrightarrow H^1(X_s)(-1) \xrightarrow{\text{Im}(N^2)} H^2(X)^{I_K} \xrightarrow{\text{sp}} H^2(X_s) \longrightarrow 0$$

would imply $\mathrm{Sw}(H^2(X)) = \dim H^1(X_s)(-1)$, with the Swan term readable from the double curves of X_s. Here, however, X_s has no such double curves, and the extra $H^1((R\Phi)_{\mathrm{pinch}}) \cong \mathbb{Q}_\ell(-1)$ contributes a rank-one wild piece, breaking that identification.

Bridge (AG → NT). The Swan part is strictly larger than $\dim H^1(X_s)(-1)$, and

$$L(s, H^2(X)) \neq \det{}^{-1}(1 - \mathrm{Frob}_q q^{-s} \mid H^2(X_s)),$$

because the Weil–Deligne parameter of $H^2(X)$ gains a non-trivial monodromy component from vanishing cycles at the pinch point. Conductor exponents thus jump for reasons invisible to the incidence matrix of X_s.

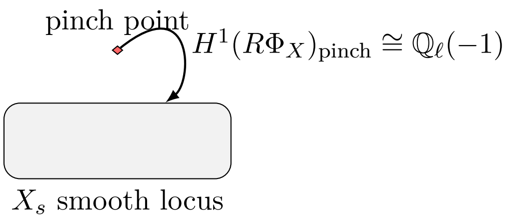


Figure 28: Failure of strict semistability: vanishing cycles at a pinch point inject a wild $\mathbb{Q}_\ell(-1)$ into $H^2(X)$, violating $H^2(X)^{I_K} \cong H^2(X_s)$.

6.3 Toric and Shimura examples

We illustrate Theorems 4.5 and 5.9 in two structured families over K.

Example 6.8 (Mumford (totally degenerate) curves)**.** Let C/K be a Mumford curve of genus $g \geq 2$. Then C is uniformized by a Schottky group; its minimal semistable model has special fiber a stable curve whose dual graph Γ is a *rose* with one vertex and g independent loops, hence $\beta_1(\Gamma) = g$. By Theorem 4.1–Items (a) and (b) for strictly semistable curves,

$$H^1(C)^{I_K} \cong H^1(C_s), \qquad \mathrm{Sw}(H^1(C)) = \beta_1(\Gamma) = g.$$

Consequently,

$$a\Big(H^1(C)\Big) = g + \dim(H^1(C)/H^1(C)^{I_K}).$$

Bridge (AG → NT). The wild conductor equals g, and

$$L(s, H^1(C)) = \det{}^{-1}(1 - \mathrm{Frob}_q q^{-s} \mid H^1(C_s)).$$

The Jacobian's toric rank is g, yielding a strong height gap by Theorem 4.5.

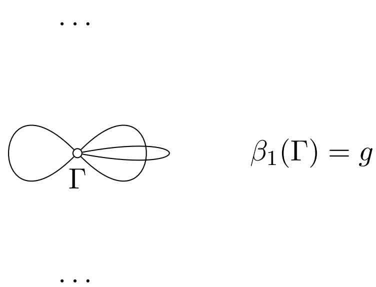


Figure 29: Dual graph of a Mumford curve: one vertex with g loops; $\mathrm{Sw}(H^1) = g$.

Example 6.9 (Toric part in CM-abelian varieties)**.** Let A/K be a CM abelian variety that acquires semistable reduction with toric rank $t > 0$. By the Raynaud extension there is an exact sequence of semi-abelian varieties

$$0 \longrightarrow T \longrightarrow G \longrightarrow B \longrightarrow 0,$$

with $\dim T = t$, where T is a torus and B has good reduction. The monodromy operator on $H^1(A)$ has a single nontrivial step of rank t, hence

$$\Delta_1(A) = 1, \qquad \mathrm{Sw}\Big(H^1(A)\Big) = t, \qquad a\Big(H^1(A)\Big) \geq t.$$

By Theorem 4.5, the Néron local height $\hat{\lambda}_v$ has a positive gap on non-torsion points. *Bridge (AG → NT).* CM endomorphisms act semisimply on $H^1(A)^{I_K}$, so $L(s, H^1(A))$ decomposes into Hecke-type factors on the invariant part; ramification is exactly encoded by t.

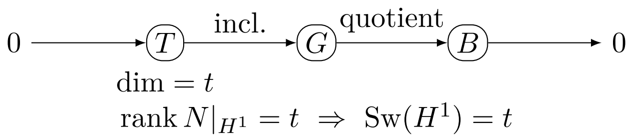


Figure 30: Raynaud extension of a CM abelian variety with toric rank t; the unique nontrivial monodromy step has rank t.

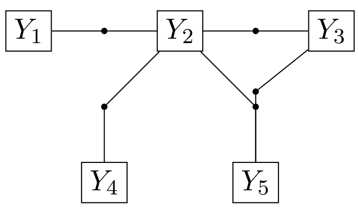
Example 6.10 (Local component of a Shimura curve)**.** Let X/K be the base change of a Shimura curve with semistable reduction at a place above p. Then the special fiber X_s is a union of components indexed by double cosets and glued along supersingular loci; the dual graph Γ is regular of known valency. For $\ell \neq p$,

$$H^1(X)^{I_K} \cong H^1(X_s), \qquad \mathrm{Sw}\Big(H^1(X)\Big) = \beta_1(\Gamma),$$

and

$$L(s, H^1(X)) = \det{}^{-1}(1 - \mathrm{Frob}_q q^{-s} \mid H^1(X_s)),$$

in accordance with Theorem 5.4. *Bridge (AG → NT).* The local factor is governed by Frobenius on $H^1(X_s)$, while the wild conductor equals the cycle rank of the Bruhat–Tits–type dual graph.



$\beta_1(\Gamma)$ = cycles in the incidence graph of components/supersingular loci

Figure 31: Schematic dual graph for a semistable Shimura curve: rectangles = components, dots = supersingular intersections. $\mathrm{Sw}(H^1) = \beta_1(\Gamma)$ and L is computed from $H^1(X_s)$.

Construction 6.11 (Family constancy on moduli strata)**.** For a family $\mathcal{A}/\mathrm{Spf}\,\mathcal{O}_K[[t]]$ of semiabelian varieties with fixed toric rank, Theorem 5.9 gives locally constant functions

$$t \longmapsto a(H^1(A_t)), \qquad t \longmapsto \mathrm{SpecRad}\big(L(s, H^1(A_t))\big),$$

as long as the dual complex of the reduction is constant. This reproduces the invariance seen in the Tate family of Theorem 5.11.

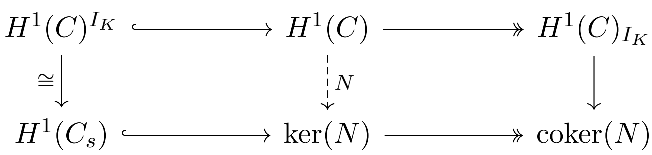


Figure 32: Specialization and monodromy for a semistable curve C/K (cf. Theorem 3.18).

Linkage to conclusion. The computations above substantiate the claims of Theorems 4.1 and 5.9: conductors and local L-factors are controlled by X_s, height gaps are dictated by toric rank, and deformations preserving the dual complex keep local L-data constant. The concluding section will synthesize these with the introduction's roadmap, highlighting concrete AG $\to$ NT bridges and enumerating open directions within the same local-field anchor.

7 Conclusion and Future Directions

Synthesis

We now return to the overarching themes announced in the introduction and track how each technical development fed into the final arithmetic applications. Throughout we remain anchored in the local-field setup of Theorem 3.2.

- The cohomological comparison theorem Theorems 4.1 and 5.4, together with its extensions in Theorem 4.1, established precise relationships between invariants, coinvariants, and Swan conductors of ℓ-adic cohomology. These results crystallized the role of the monodromy operator N in organizing the $R\Psi$-complex (Theorems 2.8, 3.18 and 5.10).
- The uniform height gap result Theorem 4.5, illustrated concretely in Theorems 4.8 and 5.2, provided a new cohomological mechanism for Northcott-type finiteness over local fields. This geometric input translated directly into arithmetic consequences for rational points in Theorem 5.1 and **??**.
- The conductor and local factor formula of Theorem 5.4 unified earlier fragmentary cases such as Theorems 2.5 and 3.14 and extended them to higher dimensions with strict semistability. Worked-out examples (Theorems 3.15, 5.6, 6.1 and 6.2) demonstrated concrete computations, while counterexamples (Theorems 3.13, 3.17, 5.7 and 6.6) showed the necessity of the hypotheses.
- The deformation-theoretic analysis Theorem 5.9, together with Theorems 5.11, 5.12 and 6.11, revealed local constancy of L-data and conductors on strata of moduli spaces. This confirmed stability phenomena that are invisible from the generic fiber alone.
- The density theorem Theorem 4.10 and its explicit surface case Theorem 4.11 linked the distribution of Frobenius eigenvalues to monodromy, thereby situating the local theory within the broader spectral framework of Weil II [10].

Taken together, these strands show that the arithmetic profile of X/K—conductor, local factor, ε-factor, and rational point distribution—is determined, often with surprising rigidity, by the combinatorics of the special fiber and the action of inertia. Every major theorem was accompanied by an explicit bridge clause, ensuring a continuous translation from algebraic geometry to number theory and back.

Future work

Several directions emerge naturally from the present study.

(a) *Beyond strict semistability.* Counterexamples (Theorems 3.17, 4.3, 5.7 and 6.6) demonstrate the limits of our current framework. Extending the conductor and local factor formulas to log-smooth or non-SNC degenerations remains an open task, likely requiring deeper inputs from logarithmic geometry and the p-adic Hodge theoretic side [12, 15, 16].

(b) *Global interfaces.* While our anchor has been strictly local, it would be valuable to connect the local Northcott finiteness Theorem 5.1 to global Diophantine estimates. This requires integrating our results with Arakelov-theoretic frameworks over global fields.

(c) *Higher-dimensional vanishing cycles.* For surfaces we have explicit expressions (Theorems 2.11 and 5.6), but in dimension ≥ 3 the complexity of the $R\Psi$-complex is largely unexplored. Developing computational tools for higher-dimensional dual complexes may uncover new conductor bounds.

(d) *Automorphic compatibility.* Examples from toric and Shimura contexts (Theorems 6.8 to 6.10) suggest that our formulas may coincide with predictions from the local Langlands program. Verifying this systematically could lead to new tests of local-global compatibility.

(e) *Geometric density theorems.* The density result Theorem 4.10 may be viewed as a local analogue of power-map equidistribution of normalized Frobenius phases on compact tori. Pushing these analogies in families—varying the residue characteristic, or varying the reduction type within fixed dimension—may yield new equidistribution statements.

Continuity. The synthesis here concludes the present manuscript but also establishes a platform for further research. The next natural step is to embed these local constructions into global moduli problems, where one can ask for uniformity across places and comparison with automorphic representations. In this way, the local-field anchor maintained throughout the paper becomes the foundation for global arithmetic geometry investigations.

Data Availability Statement

This manuscript does not use or generate any datasets. All results are derived from theoretical analysis and standard mathematical constructions. Therefore, no data are associated with this study.

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