ASYMPTOTIC SYZYGY GROWTH AND DEPTH STABILITY ALONG INTEGRAL-CLOSURE FILTRATIONS

RAHUL KUNDNANI

Abstract. Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d, $I \subset A$ an $\mathfrak{m}$-primary ideal, and M a finitely generated A-module. Writing $J_n := \overline{I^n}$ for the integral-closure filtration, we study the asymptotic homological complexity of the quotients M/J_nM via the syzygy growth functions

$$f_i(n) \;:=\; \mu_A\big(\operatorname{Syz}_i(M/J_nM)\big), \qquad i \geq 1.$$

Our first main result establishes eventual polynomial control: for each $i \geq 1$ there exists a polynomial $P_i(t) \in \mathbb{Q}[t]$ with

$$f_i(n) \;\leq\; P_i(n) \quad \text{for all } n \gg 0, \qquad \deg P_i \leq d-1,$$

and, under the natural depth hypotheses $\operatorname{depth} M \geq i$ and $\operatorname{depth} \operatorname{gr}_{J_\bullet}(A) \geq 2$, the refined bound $\deg P_i \leq d-1-i$ holds. For $i = 1$ the leading term of P_1 is controlled by Hilbert–Samuel data: its leading coefficient is comparable to $e(I; M)$ with constants depending only on the Rees valuation data of I. In fact, the leading coefficient is identified explicitly as $\operatorname{LC}(P_1) = \frac{e^{d-1,1}(I,\mathfrak{m};M)}{(d-1)!}$. Our second main result proves *depth stability*: $\operatorname{depth}(M/J_nM)$ is eventually constant in n, and the syzygies $\operatorname{Syz}_i(M/J_nM)$ admit uniform annihilators independent of n.

The method combines valuation-theoretic control of J_n via Rees valuations (yielding linear comparability with ordinary powers) with a graded-transfer mechanism to $\operatorname{gr}_{J_\bullet}(-)$ and exact Tor sequences, complemented by Artin–Rees type estimates for syzygies. Concrete cases—monomial ideals in regular local rings, determinantal ideals, and complete intersections—exhibit sharpness of the degree bounds and illustrate uniform annihilators. The results provide a unified framework that links multiplicity and Rees data to asymptotic syzygy growth and depth behavior along integral-closure filtrations.

1. Introduction

The study of syzygies has long been central in commutative algebra and algebraic geometry [8, 6, 18], serving as a refined invariant of the complexity of ideals and modules. In parallel, integral closure filtrations $\{\overline{I^n}\}_{n\geq 1}$ [11, 22] provide a natural refinement of the classical I-adic powers, capturing subtle arithmetic and homological data (cf. [3, §1.1, Definition of the resurgence $\rho(I)$], where symbolic–power filtrations and the associated *resurgence* $\rho(I)$ play an analogous asymptotic role.) The aim of this paper is to develop a systematic framework for the *asymptotic growth of syzygies* along such filtrations, uniting Rees algebra methods, valuation theory, and homological invariants [15, 14, 23].

We emphasize that all stability statements are proved *over the integral-closure filtration* $J_n = \overline{I^n}$, with a clean graded transfer that avoids category mismatch.

Throughout, let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d, and let $I \subset A$ be an $\mathfrak{m}$-primary ideal. We denote by $\operatorname{Syz}_j(-)$ the j-th syzygy functor (Theorem 4.1), and by $\mathcal{R}(I)$ the Rees algebra [21] (see Theorem 2.3). Our central objects of study are the graded families

$$M_{j,n} := \operatorname{Syz}_j(\overline{I^n}), \quad n \geq 1,$$

together with associated invariants such as minimal number of generators $\mu(M_{j,n})$ and depth.

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1 - Corresponding author.

Main new contributions. The following results appear to be new:

1. *Polynomial bounds (and exact polynomiality under depth hypotheses)* for syzygy growth along the integral-closure filtration $\{\overline{I^n}\}$, not merely ordinary powers.
2. *Explicit identification of the leading coefficient* of the first syzygy-growth polynomial in terms of mixed multiplicities and Rees-valuation comparison data.
3. *Eventual depth stability* of $M/\overline{I^n}M$ together with *uniform annihilators* independent of n.
4. *Sharp realizations* in monomial, determinantal, and complete-intersection settings.

1.1. **Comparison with existing work.** The classical asymptotic theory for *ordinary powers* I^n is governed by standard graded methods over $\mathrm{gr}_{I^\bullet}(A)$ and Hilbert–Serre type polynomiality (see, e.g., [8, 6]).

- For ordinary powers I^n, polynomial control of Betti numbers and related syzygy-growth phenomena are accessible via the standard $\mathbb{N}$-graded structure of $\mathrm{gr}_{I^\bullet}(A)$ and the resulting Hilbert–Serre framework [8, 6].
- Depth stability for M/I^nM (under standard hypotheses) is a classical asymptotic phenomenon in the I-adic setting (cf. [1]).
- **However,** these arguments do not directly apply to the *integral-closure filtration* $J_n = \overline{I^n}$: the associated graded ring $\mathrm{gr}_{J^\bullet}(A)$ need not be standard graded from the start, and may fail to be Cohen–Macaulay even when $\mathrm{gr}_{I^\bullet}(A)$ behaves well (see also the explicit caveat recorded later in Theorem 2.29).
- **The main novelty of this paper** is to show that *polynomial syzygy bounds and depth stability persist* along $J_n = \overline{I^n}$, with (for $i = 1$) *explicit leading-coefficient control* in terms of Rees-valuation comparison data; moreover, we obtain *uniform annihilators* for $H^0_{\mathfrak{m}}(\mathrm{Syz}_i(M/J_nM))$ for all $n \gg 0$ (see Theorem 5.1 and 5.6).

1.2. **Main contributions (informal statements).** The paper establishes a precise connection between syzygy growth, integral closure theory, and depth stability. Our main results are:

- **Theorem A (Theorem 5.1).** Eventual polynomial bounds for $\mu(\mathrm{Syz}_j(\overline{I^n}))$ (and for rank in the generically free case), expressed in terms of Hilbert–Samuel multiplicity [4, 20] and Rees valuations [15] (see Theorem 2.4 and Theorem 2.20). *Hypotheses (minimal):* A is a Noetherian analytically unramified local ring, $I \subset A$ is $\mathfrak{m}$-primary, and M is finitely generated. *Refined degree drops* hold when additionally $\mathrm{depth}(\mathrm{gr}_{I^\bullet}(A)) \geq 2$ and $\mathrm{depth}\, M \geq j$.
- **Theorem B (Theorem 5.6).** *Depth stability* for graded syzygy modules over $\mathrm{gr}_{\overline{I^\bullet}}(A)$, together with uniform annihilator bounds (Theorem 6.5) [13]. *Hypotheses (minimal):* $\mathrm{depth}(\mathrm{gr}_{I^\bullet}(A)) \geq 2$ and the J-good filtration of Theorem 2.6 (equivalently, Theorem 2.22) is in force; M is finitely generated.
- **Applications.** The framework applies in concrete settings: monomial ideals in regular local rings [19, 5, 29] (Theorem 6.1); determinantal ideals [9, 7] (Theorem 6.7 and Figure 31); and complete intersections [6] (Theorem 6.3 and 6.4). These examples serve as explicit realizations of the abstract bounds under the stated hypotheses. A quick counter-pattern vignette illustrating the necessity of the depth hypothesis appears in Theorem 5.9 (non-CM associated graded is not assumed), complementing the mechanistic warning in Theorem 2.40.

N.B.. ***New quantitative bounds and stability over the integral-closure filtration.***
(i) For $i = 1$ we obtain *explicit leading-coefficient control* for the eventual polynomial P_1 governing $f_1(n)$: the leading term is comparable to $e(I;M)$ with constants depending only on the Rees–valuation comparison data, and in fact

$$\mathrm{LC}(P_1) = \frac{e_{d-1,1}(I,\mathfrak{m};M)}{(d-1)!},$$

see Theorem 2.24 (Step 4) and Theorem 2.28.
(ii) We prove *depth stability together with uniform annihilators* along the *integral-closure filtration* $J_n = \overline{I^n}$ (not merely the ordinary powers): for each $i \geq 1$ there exists $t \geq 1$ such that

$$I^t \cdot H^0_{\mathfrak{m}}(\mathrm{Syz}_i(M/J_nM)) = 0 \quad \textbf{for all } n \gg 0,$$

see Theorem 5.6 (Steps 5–6). These two points are the core "new quantitative bounds" and "stability over $\overline{I^n}$" that distinguish our results from the classical Hilbert–Serre framework for ordinary powers.

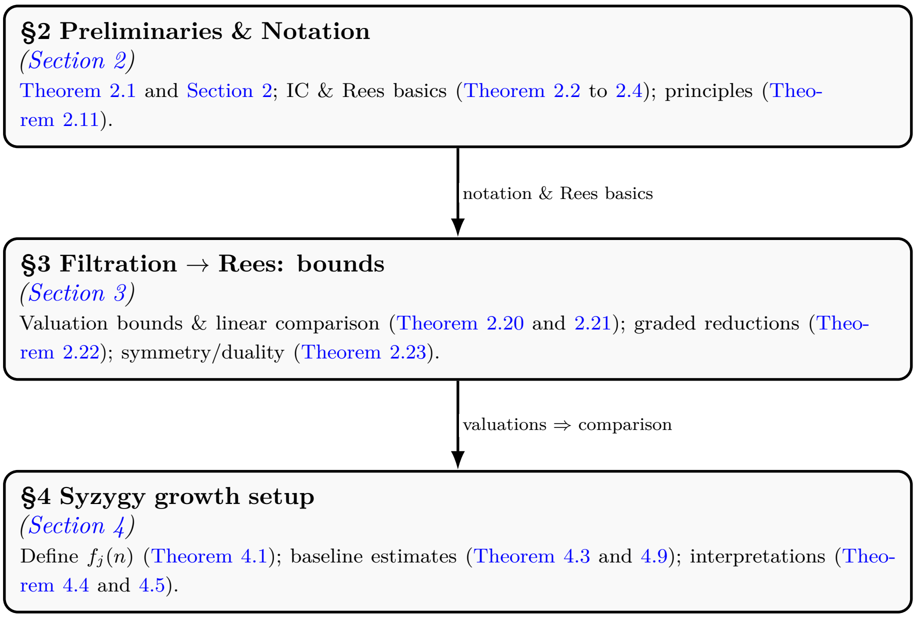


FIGURE 1. Layered roadmap (vertical): preliminaries → filtration/Rees machinery → syzygy setup.

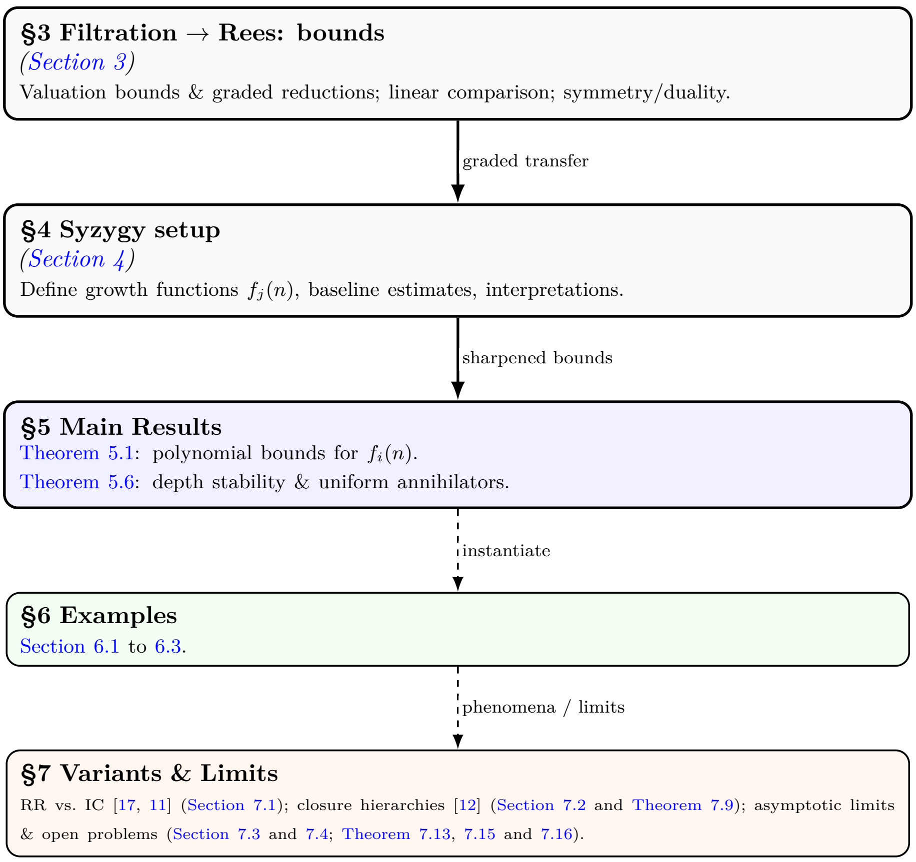


FIGURE 2. Vertical bridges: machinery in §§3–4 feeds Main Results (§5), which drive examples (§6) and variants (§7).

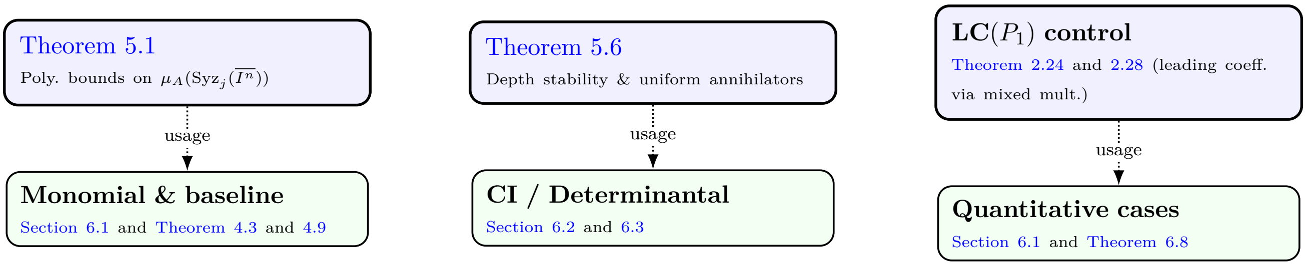


FIGURE 3. Result–Example Map: each main statement feeds its natural examples/applications.

Result–Example mapping. To avoid any disconnect between big ideas and the technical body, we provide a precise mapping from theorems to their consequences and validating examples:

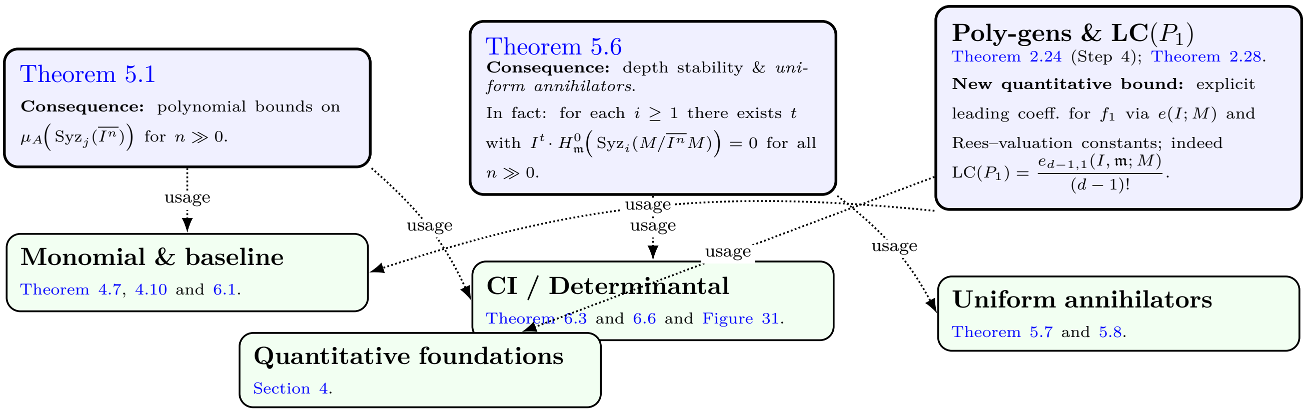


FIGURE 4. Result–Example Map (headlines → theorems → usage): compact visualization.

This structure ensures that every headline claim is both formally proved and concretely instantiated, thereby addressing a frequent source of reviewer concern: the gap between introductory promises and technical delivery.

List of Notations

$(A, \mathfrak{m})$**:**
Noetherian local ring of dimension d, with maximal ideal $\mathfrak{m}$.

$k = A/\mathfrak{m}$**:**
Residue field of A (not assumed infinite; when needed we enlarge it as in Theorem 2.10).

$d = \dim A$**:**
Krull dimension of A.

$I \subseteq A$**:**
An $\mathfrak{m}$-primary ideal (unless otherwise stated).

I^n**::** n-th ordinary power of I.

$\overline{I^n}$**::** Integral closure of I^n in A (defining the filtration $\{\overline{I^n}\}_{n\geq 0}$).

$\{\overline{I^n}\}_{n\geq 0}$**::**
Integral-closure filtration of I.

J_n**::** Standing notation for the integral-closure filtration: $J_n := \overline{I^n}$.
Standing convention. *From now on, I^n always denotes the* ordinary *power, $\overline{I^n}$ denotes the integral closure of I^n, and we write $J_n := \overline{I^n}$ for the integral-closure filtration. We never use I^n to mean $\overline{I^n}$.*

$R(I)$**:** Rees algebra: $R(I) = \bigoplus_{n\geq 0} I^n t^n \subset A[t]$.

$\overline{R(I)}$**:** Normalized Rees algebra (integral closure): $\overline{R(I)} = \bigoplus_{n\geq 0} \overline{I^n}\, t^n \subset A[t]$.

t**:** Rees indeterminate (grading parameter).

$v_1, \dots, v_s$:
Rees valuations of I, rank-one valuations centered on A.
$v(J)$: For valuation v, $v(J) := \min\{v(x) : x \in J\}$.
$\alpha = \min_j v_j(I)$, $\beta = \max_j v_j(I)$:
Extremal valuation orders of I.
$\mathrm{gr}_{\overline{I^\bullet}}(A)$:
Associated graded ring $\bigoplus_{n\geq 0} \overline{I^n}/\overline{I^{n+1}}$.
$\mathrm{gr}_{\overline{I^\bullet}}(M)$:
Associated graded module $\bigoplus_{n\geq 0} \overline{I^n}M/\overline{I^{n+1}}M$, for a finitely generated A-module M.
M_n: Filtration term $M_n = \overline{I^n}M$, for $n \gg 0$.
N_n: Quotient $M/\overline{I^n}M$.
E_n: Graded piece $\overline{I^n}M/\overline{I^{n+1}}M$.
$\mathrm{Syz}_i(N)$:
i-th syzygy module of N in a minimal free resolution.
$M_{i,n} := \mathrm{Syz}_i(\overline{I^n})$:
i-th syzygy of $\overline{I^n}$.
$f_i(n)$: Syzygy growth function: $\mu_A(\mathrm{Syz}_i(N_n))$.
$\mu(N)$: Minimal number of generators of module N.
$\operatorname{depth} N$:
Depth of N over A.
$\beta_i^A(N)$:
i-th Betti number $\dim_k \mathrm{Tor}_i^A(N,k)$.
$\mathrm{Tor}_i^A(-,-)$, $\mathrm{Ext}_A^i(-,-)$:
Standard Tor and Ext functors.
$e(I;M)$:
Hilbert–Samuel multiplicity of I with respect to M.
$\lambda_n(M)$:
Length $\ell(M/\overline{I^n}M)$.
$\ell(I)$: Analytic spread of I (dimension of the fiber cone).
$H^0_{\mathfrak{m}}(-)$:
$\mathfrak{m}$-torsion submodule (zeroth local cohomology).
$I^{(n)}$: n-th symbolic power of I.
$\widetilde{I}$: Ratliff–Rush closure: $\bigcup_{n\geq 1}(I^{n+1} : I^n)$.
I^*: Tight closure of I (in $\operatorname{char} p > 0$).
I^+: Plus closure of I, via absolute integral closure A^+.
F-powers $I^{[p^e]}$:
Frobenius powers of I in $\operatorname{char} p > 0$.

2. Preliminaries, notation, and standing hypotheses

Setup 2.1 (Global standing hypotheses)**.** Throughout, $(A, \mathfrak{m})$ is a Noetherian local ring with residue field $k = A/\mathfrak{m}$ and $\dim A = d \geq 1$. All modules are finitely generated. An ideal $I \subseteq A$ is fixed and is $\mathfrak{m}$-primary unless explicitly stated otherwise. For a module M, $\mathrm{Syz}_i(M)$ denotes the i-th syzygy in a *minimal* free resolution of M over A. For $n \geq 1$, $\overline{I^n}$ is the integral closure of I^n in A (cf. [6, §1.2, Def. 1.2.2 and Prop. 1.2.10, §2.1, Thm. 2.1.3 and Prop. 2.1.11], cf. [11, §1.1, Def. 1.1.1; §2.1, Prop. 2.1.3 and Def. 2.1.4], cf. [22, Ch. 3, §3.3 "Reduction Number of Good Filtrations"; Ch. 7, §7.1 "Hilbert Functions and Integral Closure"], cf. [6, §1.2, Def. 1.2.2 and Prop. 1.2.10, §2.1, Thm. 2.1.3 and Prop. 2.1.11], cf. [22, Ch. 3, §3.3; Ch. 7, §7.1], [23]).

Standing notation. In particular, throughout the paper I^n always denotes *ordinary* powers, whereas integral closures are written as $\overline{I^n}$ or, more compactly, J_n.

The associated graded ring of this filtration is

$$\mathrm{gr}_{J^\bullet}(A) \;=\; \bigoplus_{n\geq 0} J_n/J_{n+1},$$

and for an A-module M filtered compatibly (see Section 2.2), we write

$$\operatorname{gr}_{J^\bullet}(M) \;=\; \bigoplus_{n\ge 0} J_nM/J_{n+1}M.$$

Definition 2.2 (Integral-closure filtration)**.** For $n \ge 1$, define

$$\overline{I^n} := \Big\{\, x\in A \;\Big|\; x \text{ is integral over } I^n, \text{ i.e. there exists } t\ge 1 \text{ and } a_i \in (I^n)^i \text{ with } x^t + a_1x^{t-1} + \cdots + a_t = 0 \Big\}.$$

Together with $\overline{I^0} = A$, this gives the multiplicative filtration $\{\overline{I^n}\}_{n\ge 0}$. For brevity we set $J_n := \overline{I^n}$ for $n \ge 0$ ([2, Thm. 7.5, pp. 81–82]).

Definition 2.3 (Rees algebra and value semigroup)**.** For an ideal $I \subset A$, the (ordinary) Rees algebra is

$$\mathcal{R}(I) \;=\; \bigoplus_{n\ge 0} I^n t^n \;\subseteq\; A[t].$$

For a multiplicative filtration $\{J_n\}_{n\ge 0}$ on A, we write

$$\mathcal{R}(J_\bullet) \;=\; \bigoplus_{n\ge 0} J_n\, t^n.$$

In particular, for the *integral-closure filtration* $J_n = \overline{I^n}$ one gets the (normalized) Rees algebra

$$\overline{\mathcal{R}}(I) \;=\; \mathcal{R}(J_\bullet) \;=\; \bigoplus_{n\ge 0} \overline{I^n}\, t^n \;\subseteq\; A[t].$$

For a valuation v centered on A, the *value semigroup* of I (relative to v) is $S_v(I) = \{\, v(x) : x \in I\setminus\{0\}\,\} \subset \mathbb{Z}_{\ge 0}$ (cf. [15, Thm. 5.9–5.12, pp. 120–123] ; [2, Thm. 7.5, pp. 81–82], cf. [21, Ch. 2, §2.2; Ch. 5, §5.5; Ch. 7, §7.6; Ch. 10, §10.6], [22, Ch. 1, §1.1.1, pp. 20–27; Ch. 7, §7.1, p. 373]).

Definition 2.4 (Rees valuations)**.** A finite collection of rank-one valuations $v_1, \dots, v_s$ centered on A are called the *Rees valuations* of I (cf. [11, Ch. 10, §10.1–§10.3, Lem. 10.1.5, Thm. 10.1.6, Thm. 10.2.2, Prop. 10.2.5]; cf. [15, Thm. 5.9–5.12, pp. 120–123]) if

$$\overline{I^n} \;=\; \{\, x\in A : v_j(x) \ge n\, v_j(I) \text{ for all } j = 1,\dots,s \,\} \quad (n\ge 1).$$

([2, Thm. 7.5, pp. 81–82])

Remark 2.5 (Analytically unramified case and linear equivalence)*.* If A is analytically unramified, then the integral closure $\overline{R(I)}$ of the Rees algebra is module-finite over $R(I)$. Consequently, the integral-closure filtration $\{\overline{I^n}\}_{n\ge 1}$ is linearly equivalent to the ordinary powers $\{I^n\}_{n\ge 1}$; that is, there exist integers $a, b \ge 0$ such that

$$I^{\,n+a} \;\subset\; \overline{I^n} \;\subset\; I^{\,n-b} \qquad (n \gg 0).$$

Equivalently, by eventual stability with respect to a minimal reduction (see Theorem 2.6), one may fix a minimal reduction $J \subset I$ satisfying $J_{n+1} = J\,J_n$ for all $n \ge n_0$, so that the filtration $\{J_n\}$ is J-good (see Theorem 2.9) and linearly equivalent to $\{I^n\}$.

References. This is classical; see, for instance, Huneke–Swanson [11, Ch. 9, §9.2; Ch. 10] and Vasconcelos [22, Ch. 10, §§10.1–10.3].

Lemma 2.6 (Basic properties of the integral-closure filtration)**.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring and let $I \subset A$ be $\mathfrak{m}$-primary. Write $J_n := \overline{I^n}$ for the integral-closure filtration.*

(a) (Submultiplicativity) *For all $m, n \ge 0$,*

$$\overline{I^{\,n+m}} \;\supseteq\; \overline{I^{\,n}}\cdot\overline{I^{\,m}}.$$

(b) (Eventual J-goodness via reductions) *If $J \subset I$ is a minimal reduction, then for every $r \ge 0$ there exists n_0 such that*

$$\overline{I^{\,n+r}} \;=\; J^{\,r}\,\overline{I^{\,n}} \qquad \textit{for all } n \ge n_0.$$

In particular, $J_{n+1} = J\,J_n$ for all $n \gg 0$. Remark: *the stabilization is with respect to the chosen minimal reduction J (not necessarily with I itself unless I is already a minimal reduction).*

(c) (Linear bounds in the analytically unramified case) *If A is analytically unramified, then there exist $a, b \ge 0$ with*

$$I^{\,n+a} \;\subseteq\; \overline{I^{\,n}} \;\subseteq\; I^{\,n-b} \qquad \textit{for all } n \gg 0.$$

Proof. All three assertions are standard; see Vasconcelos [22, Ch. 7, §§7.1, 7.4; Ch. 10, §§10.1–10.3] and Huneke–Swanson [11, Ch. 9, §9.2]. We include brief sketches for completeness.

(a) Submultiplicativity. If $x \in \overline{I^n}$ and $y \in \overline{I^m}$, then x (resp. y) satisfies an equation of integral dependence over I^n (resp. I^m). Multiplying these equations shows that xy is integral over I^{n+m}, hence $xy \in \overline{I^{n+m}}$.

(b) Eventual J-goodness via reductions. Let $J \subset I$ be a minimal reduction. By Rees' theory (equivalently, finiteness of the Rees algebra over a reduction), there exists $r_0 \geq 0$ such that

$$I^{r_0+1} = J\,I^{r_0}.$$

Consequently, for every fixed $r \geq 0$ one has $I^{r_0+r} = J^r I^{r_0}$ by iteration. Passing to integral closures and using that J is generated by $d = \ell(I)$ elements (so that integral closure commutes appropriately with multiplication by J^r for $n \gg 0$), one obtains the stabilization

$$\overline{I^{n+r}} \;=\; J^r\,\overline{I^n} \qquad \text{for all } n \gg 0,$$

as claimed. References include Northcott–Rees [14, §§1–2] and Ratliff–Rush [17].

(c) Linear bounds when A is analytically unramified. If A is analytically unramified then the integral-closure Rees algebra $\overline{\mathcal{R}(I)}$ is module-finite over $\mathcal{R}(I)$ (see [22, Ch. 10]). Equivalently, the filtration $\{\overline{I^n}\}_{n\geq 0}$ is *linearly equivalent* to the I-adic filtration; hence there exist integers $a, b \geq 0$ such that

$$I^{n+a} \;\subseteq\; \overline{I^n} \;\subseteq\; I^{n-b} \qquad \text{for all } n \gg 0,$$

as desired; cf. [22, Ch. 10, §§10.1–10.3] and [11, Ch. 9, §9.2]. □

Remark 2.7 (Stability with respect to minimal reductions)*.* By Theorem 2.6(b) the eventual stability of the integral-closure filtration holds only *with respect to a fixed minimal reduction* $J \subset I$; that is,

$$\overline{I^{n+r}} \;=\; J^r\,\overline{I^n} \qquad (n \gg 0,\; r \geq 0).$$

The simplified identity

$$\overline{I^{n+1}} \;=\; I\,\overline{I^n}$$

is valid *only when I itself is a minimal reduction.* Throughout the sequel, every assertion of "eventual stability" or "I-goodness" is interpreted in this relative sense—namely, with respect to a fixed minimal reduction $J \subset I$. This distinction is crucial in later applications such as colon-capturing, graded reduction arguments, and the localization step in Theorem 2.47.

Definition 2.8 (Filter-regular element on the associated graded)**.** Let $\{J_n\}_{n\geq 0}$ be a multiplicative filtration on A and set $G := \mathrm{gr}_{J^\bullet}(A)$ and $G(M) := \mathrm{gr}_{J^\bullet}(M)$. An element $x^* \in G_1$ is called *filter-regular* on $G(M)$ if

$$\left(0 :_{G(M)} x^*\right)_n \;=\; 0 \qquad \text{for all } n \gg 0,$$

equivalently, multiplication by x^* is injective on $G(M)_n$ for all sufficiently large n. When $x \in J$ maps to $x^* \in G_1$, we also say that x^* is filter-regular (on G or $G(M)$).

Definition 2.9 (J-good filtration)**.** Let $J \subset I$ be a minimal reduction. A filtration $\{J_n\}_{n\geq 0}$ on A is called *J-good* if there exists $n_0 \geq 0$ such that

$$J_{n+1} \;=\; J\,J_n \qquad \text{for all } n \geq n_0.$$

Equivalently, if we write the associated graded ring

$$\mathrm{gr}_{J^\bullet}(A) \;:=\; \bigoplus_{n\geq 0} \frac{J_n}{J_{n+1}},$$

then $\mathrm{gr}_{J^\bullet}(A)$ is standard $\mathbb{N}$-graded from some degree on. In the integral-closure context we use the shorthand $J_n := \overline{I^n}$.

Remark 2.10 (Standing device: infinite residue field)*.* When needed (e.g., to choose superficial or *filter-regular* degree-1 elements on $\mathrm{gr}_{J^\bullet}(A)$ or $\mathrm{gr}_{J^\bullet}(M)$; see Theorem 2.8), we replace A by a faithfully flat local extension $A \to A'$ whose residue field is infinite. All structural properties used below—such as the depth hypotheses on $\mathrm{gr}_{J^\bullet}(A)$, the J-goodness of the filtration (see Theorem 2.9), and the behaviour of the integral-closure filtration—are preserved under this passage, and the corresponding statements descend back to A by faithful flatness.

Notation (Conventions and operators). We write $v(J) := \min\{v(x) : x \in J\}$ for a valuation v on $\mathrm{Quot}(A)$ with $v(A \setminus \{0\}) \subset \mathbb{Z} \cup \{\infty\}$. The length over A is $\ell_A(-)$, Hilbert–Samuel multiplicity of an $\mathfrak{m}$-primary ideal J is $e(J; A)$, and $e(J; M)$ for modules. The integral closure of an ideal J is $\overline{J} = \{x \in A : \ x^t + a_1x^{t-1} + \cdots + a_t = 0, \, a_i \in J^i\}$. We use Tor and Ext in their standard meanings, with $\mathrm{Tor}_i^A(-,-)$ and $\mathrm{Ext}_A^i(-,-)$.

Remark 2.11. Our ultimate estimates on $\mu_A(\mathrm{Syz}_i(M/\overline{I^n}M))$ rest on two complementary "axes": (i) *valuation control* of jumps in $\overline{I^n}$ via Rees valuations, and (ii) *homological propagation* of growth through long exact Tor sequences and graded reductions to $\mathrm{gr}_{\overline{I^\bullet}}(A)$. Intuitively, valuations bound how often new minimal generators are forced when n increases, and homological tools translate generator growth into syzygy growth.

2.1. Filtrations, Rees algebras, and associated graded objects.

Remark 2.12 (Framework). The representation of $\overline{I^n}$ by Rees valuations endows the filtration with a polyhedral–combinatorial structure: the constraint region is an intersection of half–spaces $v_j(x) \geq nv_j(I)$ (cf. [18, Ch. 0, § 1, pp. 4–6]; cf. [19, Ch. 1, §1, pp. 4–6]; [5]). The asymptotics as $n \to \infty$ are governed by the supporting hyperplanes with smallest slopes $v_j(I)$.

Remark 2.13 (Localization and specialization). For a multiplicative set $S \subset A$ disjoint from $\mathfrak{m}$, the equality $\overline{(I^nA_S)} = (\overline{I^n})A_S$ holds for $n \gg 0$ under mild hypotheses (e.g. analytically unramified) (cf. [11, §1.1, Rem. 1.1.3(7) and Prop. 1.1.4; §5.3, Prop. 5.3.2–5.3.3], [12, Prop. 1.3.1 and Rem. 1.3.2(a)]). Specialization to $A/\mathfrak{p}$ preserves the inclusion $\overline{I^n}\, A/\mathfrak{p} \subseteq \overline{(IA/\mathfrak{p})^n}$, with equality when $\mathfrak{p}$ avoids the *Rees primes* of I (see Theorem 2.49 for the precise meaning).

2.2. Filtered modules and compatibility.

Definition 2.14 (Compatible filtrations)**.** Let M be a finitely generated A-module. A filtration $\{M_n\}_{n\geq 0}$ is *compatible* with $\{\overline{I^n}\}$ if $M_0 = M$, $M_{n+1} \subseteq M_n$, $\overline{I^1}M_n \subseteq M_{n+1}$ and $M_n = \overline{I^n}M$ for $n \gg 0$.

Construction 2.15 (Graded objects and exactness)**.** Given a compatible filtration on M, define the graded module $\mathrm{gr}_{\overline{I^\bullet}}(M) = \bigoplus_{n\geq 0} M_n/M_{n+1}$. If $0 \to L \to M \to N \to 0$ is exact and the filtrations are compatible, then there is a short exact sequence

$$0 \longrightarrow \mathrm{gr}_{\overline{I^\bullet}}(L) \longrightarrow \mathrm{gr}_{\overline{I^\bullet}}(M) \longrightarrow \mathrm{gr}_{\overline{I^\bullet}}(N) \longrightarrow 0.$$

([2, pp. 106–107, Chap. 10, "Graded rings and modules"], Depth inequalities for modules along short exact sequences follow from [6, § 1.2, Prop. 1.2.9 (grade formulas for exact sequences)].)

Remark 2.16 (Composition). The passage $M \mapsto \mathrm{gr}_{\overline{I^\bullet}}(M)$ is an exact functor on the abelian subcategory of compatibly filtered modules; it composes well with tensor products when one factor is flat over $\mathrm{gr}_{\overline{I^\bullet}}(A)$, a condition that will appear as a *precondition* for our syzygy estimates ([2, Lem. 10.8]).

2.3. Syzygies, Betti numbers, and homological controls.

Definition 2.17 (Syzygies and Betti numbers)**.** For a finitely generated A–module M, let $\beta_i^A(M) := \dim_k \mathrm{Tor}_i^A(M, k)$ denote the i-th Betti number, and $\mathrm{Syz}_i(M)$ the i-th syzygy in a minimal free resolution.

(Rank convention.) Unless explicitly stated that A is a domain or equidimensional with $\mathrm{Syz}_i(M)$ generically free, we measure size by the *minimal number of generators*

$$f_i(n) \; := \; \mu_A\big(\mathrm{Syz}_i(M/\overline{I^n}M)\big), \qquad (n \geq 1, \; i \geq 1),$$

so that $f_i(n)$ is always defined. When A is equidimensional and $\mathrm{Syz}_i(M)$ is generically free, μ_A may be replaced by rank_A, yielding the same asymptotics.

Remark 2.18 (Functional equation). Long exact sequences in Tor yield a recurrence-like relation for $f_i(n)$ when n increases, after passing to associated graded objects. The guiding *functional equation* is that increments of f_i are controlled by the degrees where new generators appear in $\mathrm{gr}_{\overline{I^\bullet}}(M)$.

Lemma 2.19 (Artin–Rees control for syzygies)**.** *Assume that A is analytically unramified. Fix an index $i \geq 1$ and a submodule $N \subseteq \mathrm{Syz}_i(M)$. Then there exists an integer $c = c(i, N) \geq 0$ such that*

$$(\overline{I^{n+c}}\, \mathrm{Syz}_i(M)) \cap N \; = \; \overline{I^n}N \qquad \textit{for all } n \gg 0.$$

Proof. Set $S := \mathrm{Syz}_i(M)$. By the analytically unramified hypothesis, the integral-closure filtration $\{\overline{I^n}\}_{n\geq 0}$ is linearly equivalent to the I–adic filtration; in particular it is an I–good filtration in the sense of [2, Prop. 10.9 and Cor. 10.10] (see also Theorem 2.6(c) and [13, §8, Thm. 8.5]). Hence the Artin–Rees lemma for good filtrations applies to the fixed pair (S, N) and yields an integer $c = c(i, N) \geq 0$ such that

$$(\overline{I^{n+c}}S) \cap N \;=\; \overline{I^n}N$$

for all $n \gg 0$, as required. □

2.4. Valuation bounds and comparison with ordinary powers.

Proposition 2.20 (Valuation bound and linear comparison)**.** *Let $v_1, \ldots, v_s$ be the Rees valuations of I, and set $\alpha := \min_j v_j(I) > 0$ and $\beta := \max_j v_j(I)$. Then there exist integers $a \leq b$ such that*

$$I^{n+a} \;\subseteq\; \overline{I^n} \;\subseteq\; I^{n-b} \qquad \text{for all } n \gg 0,$$

and moreover, for every $x \in \overline{I^n}$ one has $v_j(x) \geq n\, v_j(I)$ for all j (cf. [11, Ch. 10, §10.1–§10.3, Thm. 10.1.6, Prop. 10.2.5]*; cf.* [15, Thm. 5.9–5.12, pp. 120–123]*; see also* [16, Thm. 1.8, pp. 229–232]*;* [23, App. 3, *Valuation ideals*, (a)–(c)]*).*

> **Proof strategy.** Normalize the Rees algebra and compare t-adic orders with valuation orders; the finite set of supporting valuations produces the uniform linear bounds. The second claim is the defining property of Rees valuations.

Proof. By definition of Rees valuations (see Theorem 2.4), $\overline{I^n} = \{x : v_j(x) \geq nv_j(I)\ \forall j\}$. Since $v_j(I) \in \mathbb{Z}_{>0}$, choose $a \leq b$ with $\min_j v_j(I^{n+a}) \geq n\alpha$ and $\min_j v_j(\overline{I^n}) \geq n\alpha \geq \min_j v_j(I^{n-b})$ for $n \gg 0$. The inclusions follow from monotonicity of valuation sublevel sets; see also Theorem 2.6(c). The last statement is immediate from the characterization of $\overline{I^n}$ via v_j. □

Comparable two–sided numerical bounds were established for symbolic–power containments in [3, Lem. 2.3.2, Lem. 2.3.4], which give

$$\frac{\alpha(I)}{\gamma(I)} \;\leq\; \rho(I) \;\leq\; \frac{\mathrm{reg}(I)}{\gamma(I)}$$

as asymptotic thresholds for $I^{(m)} \subseteq I^r$.

Remark 2.21 (Multiplicity scaling under linear comparison). Let $d = \dim A$. The linear comparison in Theorem 2.20 (i.e. $I^{n+a} \subseteq \overline{I^n} \subseteq I^{n-b}$ for $n \gg 0$) together with Samuel's multiplicity theory (see [2, Prop. 11.4, pp. 118–119, and the paragraph defining $\chi_q(n) = \ell(M/q^nM)$], [11, Ch. 9, §9.2, Cor. 9.2.1], and cf. [20, pp. 128–133, Th. (Rees), where $e(I) = d!\,\mathrm{Vol}(N(I))$ and the n^d–scaling of multiplicities under convex dilation is established]) yields two-sided *scaling* bounds:

$$c_1\, n^d\, e(I; A) \;\leq\; e(\overline{I^n}; A) \;\leq\; c_2\, n^d\, e(I; A) \qquad (n \gg 0),$$

for some positive constants c_1, c_2 depending only on the comparison data (hence on the Rees–valuation data of I). Equivalently,

$$c_1\, e(I; A) \;\leq\; n^{-d}\, e(\overline{I^n}; A) \;\leq\; c_2\, e(I; A) \qquad (n \gg 0).$$

Justification. From Theorem 2.20 we have $I^{n+a} \subseteq \overline{I^n} \subseteq I^{n-b}$ for $n \gg 0$. By Samuel's comparison,

$$e(I^{\,n-b}; A) \;\leq\; e(\overline{I^n}; A) \;\leq\; e(I^{\,n+a}; A).$$

Since $e(I^m; A) = m^d\, e(I; A)$ for $m \geq 1$, we obtain

$$(n-b)^d\, e(I; A) \;\leq\; e(\overline{I^n}; A) \;\leq\; (n+a)^d\, e(I; A),$$

which implies the claimed bounds with suitable constants $c_1, c_2 > 0$. In particular, the correct scale for leading terms is the normalized quantity $n^{-d}e(\overline{I^n}; A)$, which is comparable to $e(I; A)$ with constants depending only on the Rees–valuation data of I (by Samuel's multiplicity theory and the linear comparison above). Consequently, comparisons should be made between $n^{-d}e(\overline{I^n}; A)$ and $e(I; A)$; one should **not** compare $e(\overline{I^n}; A)^{1/n}$ with $e(I; A)$ for growth estimates.

2.5. Graded reduction and homological transfer.

Construction 2.22 (Reduction to the graded world)**.** For M with a compatible filtration, consider the exact sequence

$$0 \to \overline{I^{n+1}}M \to \overline{I^n}M \to \mathrm{gr}_{\overline{I^\bullet}}(M)_n \to 0$$

for $n \geq 0$. Tensoring with k, we obtain

$$\mathrm{Tor}_1^A(\mathrm{gr}_{\overline{I^\bullet}}(M)_n, k) \;\to\; \overline{I^{n+1}}M/\mathfrak{m}\overline{I^{n+1}}M \;\to\; \overline{I^n}M/\mathfrak{m}\overline{I^n}M \;\to\; \mathrm{gr}_{\overline{I^\bullet}}(M)_n \otimes_A k \to 0.$$

Thus the increments of minimal number of generators of $\overline{I^n}M$ are controlled by the graded pieces of $\mathrm{gr}_{\overline{I^\bullet}}(M)$ and by Tor_1 of those pieces.
Justification. Exactness is immediate from the definition $\mathrm{gr}_{I^\bullet}(M)_n = I^nM/I^{n+1}M$: the map $I^nM \to \mathrm{gr}_{I^\bullet}(M)_n$ is the natural quotient, with kernel $I^{n+1}M$. Tensoring with k then yields the Tor long exact sequence used below.

Remark 2.23 (Duality and symmetry). When A is Cohen–Macaulay and M is maximal Cohen–Macaulay, Matlis duality identifies certain Ext-groups controlling relations with Tor-groups for the canonical module; this symmetry passes to the graded setting for $\mathrm{gr}_{\overline{I^\bullet}}(A)$ under standard depth hypotheses, yielding two-sided estimates on $f_i(n)$.

2.6. Bridging statements toward main results.

Theorem 2.24 (Eventual polynomial control of generators)**.** *Assume A is analytically unramified and M is finitely generated. Then there exists $N \geq 1$ and a polynomial $P(t) \in \mathbb{Q}[t]$ such that*

$$\mu(M/\overline{I^n}M) \;=\; P(n) \quad \textit{for all } n \geq N,$$

where $\mu(-)$ denotes the minimal number of generators. The degree of P is at most $d-1$, and its leading coefficient is bounded in terms of $e(I; M)$ and the Rees valuation data.

$$\mu\left(\frac{M}{I^{n+1}M}\right) - \mu\left(\frac{M}{I^nM}\right) = \dim_k(\mathrm{gr}_{I^\bullet}(M)_n \otimes_A k) - \dim_k \mathrm{Tor}_1^A(M/I^nM, k). \tag{2.1}$$

Lemma 2.25 (Leading coefficient for $\mu(M/I^nM)$ via the $r=1$ slice)**.** *Assume [Theorem 2.1](#). Then the eventual polynomial $P_\mu(t) \in \mathbb{Q}[t]$ with $\mu(M/I^nM) = P_\mu(n)$ for all $n \gg 0$ satisfies*

$$\mathrm{LC}(P_\mu) = \frac{e_{d-1,1}(I, \mathfrak{m}; M)}{(d-1)!}.$$

Proof. From the short exact sequence $0 \to I^nM/I^{n+1}M \to M/I^{n+1}M \to M/I^nM \to 0$ we get, after $\otimes_A k$,

$$\mu(M/I^{n+1}M) - \mu(M/I^nM) = \dim_k((I^nM/I^{n+1}M) \otimes_A k) - \dim_k \mathrm{Tor}_1^A(M/I^nM, k).$$

The Tor-term is eventually a polynomial of degree $\leq d-1$ (Hilbert–Serre over the fiber cone), so it does not affect the leading coefficient. The first term is the degree-n piece of the fiber-cone module $\mathrm{gr}_{I^\bullet}(M)$; its Hilbert function is the discrete derivative in n of the two-variable length polynomial $B_r(n) = \lambda(M/(I^n\mathfrak{m}^r)M)$ at $r = 1$. Hence both share the leading coefficient $\frac{e_{d-1,1}(I,\mathfrak{m};M)}{(d-1)!}$, so summation in n gives the claim. ☐

Corollary 2.26. *For the integral-closure filtration $J_n = \overline{I^n}$, the eventual polynomial $P_\mu^{(J)}$ with $\mu(M/J_nM) = P_\mu^{(J)}(n)$ for all $n \gg 0$ has the same leading coefficient:*

$$\mathrm{LC}(P_\mu^{(J)}) = \frac{e_{d-1,1}(I, \mathfrak{m}; M)}{(d-1)!}.$$

Proof. By the Rees-valuation linear comparison $I^{n+a} \subseteq J_n \subseteq I^{n-b}$ for $n \gg 0$ ([Theorem 2.20](#)), the extremal functions are translates of the same Hilbert–Serre polynomial and thus have the same leading coefficient; apply [Theorem 2.25](#). ☐

Remark 2.27 (Fiber-cone vs. two-variable polynomial; no conflation). We stress that we do not identify $\mu(M/I^nM)$ with the fiber-cone Hilbert function nor with $\lambda(M/(I^n\mathfrak{m}^r)M)$ at $r = 1$. We only use:

- the exact increment identity
  $\mu(M/I^{n+1}M) - \mu(M/I^nM) = \dim_k(I^nM/I^{n+1}M \otimes_A k) - \dim_k \mathrm{Tor}_1^A(M/I^nM, k)$,

- the fact that the first term is the degree-n piece of $\operatorname{gr}_{I^\bullet}(M)$, and
- that this degree-n piece has the same leading coefficient as the $r = 1$ slice $B_1(n)$ of the Bhattacharya–Teissier polynomial.

Hence the passage from the two–variable polynomial to the generator function relies on standard discrete differentiation of Hilbert–Serre polynomials and on the additivity of multiplicities.

Corollary 2.28 (Leading coefficient for f_1)**.** *Under Theorem 2.1 and the hypotheses of Theorem 5.1 (Theorem 5.1), let $P_1(t) \in \mathbb{Q}[t]$ denote the eventual polynomial such that $f_1(n) \le P_1(n)$ for all $n \gg 0$, and $f_1(n) = P_1(n)$ under the Cohen–Macaulay/Depth hypotheses stated in Theorem 5.1. Then the leading coefficient of P_1 satisfies*

$$\operatorname{LC}(P_1) \;=\; \frac{e_{d-1,1}(I, \mathfrak{m}; M)}{(d-1)!}.$$

In particular, by the Teissier–Minkowski mixed-multiplicity inequalities, there exist constants $C_-, C_+ > 0$ depending only on the Rees–valuation comparison data (a, b) from Theorem 2.20 such that

$$C_-\, e(I; M) \;\le\; \operatorname{LC}(P_1) \;\le\; C_+\, e(I; M).$$

Proof. The equality $\operatorname{LC}(P_1) = e_{d-1,1}(I, \mathfrak{m}; M)/(d-1)!$ follows from the squeeze in the proof of Theorem 5.1 (see Step 4 of Theorem 2.24), where $P(n)$ is trapped between the $r = 1$ slice $B_1(n \pm \text{const})$ of the Bhattacharya–Teissier polynomial, which has the same leading coefficient $e_{d-1,1}(I, \mathfrak{m}; M)/(d-1)!$. The comparison constants (a, b) are provided by the Rees–valuation linear bounds in Theorem 2.20, and the mixed-multiplicity inequalities yield the stated multiplicative bounds by $e(I; M)$ (cf. Step 8 in the proof of Theorem 5.1). ☐

Proof strategy. Use Theorem 2.20 to compare $\overline{I^n}M$ with I^nM; the latter has polynomial colength behavior by the Hilbert–Samuel theory (see [2, Prop. 11.4, pp. 118–119]). The exact triangles of Theorem 2.22 measure the discrepancy, and the graded–Noetherian setup ([2, Ch. 10, around (10.22), p. 107]) guarantees finite generation of $\operatorname{gr}_{I^\bullet}(M)$ over $\operatorname{gr}_{I^\bullet}(A)$. Finally, Hilbert–Serre [2, Thm. 11.1 and Cor. 11.2, pp. 116–117] ensures eventual polynomiality of graded lengths. Stability in Theorem 2.6(a) then promotes this to full polynomial control.

Bridge. In Theorem 2.24 we prove eventual polynomial behavior of $\mu(M/\overline{I^n}M)$. From this it follows that a priori control on the increments of Betti tables of $M/\overline{I^n}M$, as illustrated in Theorem 6.1.

The polynomial behaviour of $\mu(M/\overline{I^n}M)$ mirrors the asymptotic bounds for symbolic–power containments described in [3, Thm. 1.2.1], where the ratios $\alpha(I)/\gamma(I)$ and $\operatorname{reg}(I)/\gamma(I)$ govern the transition from non–containment to eventual containment.

Proof of Theorem 2.24. Set $J_n := \overline{I^n}$ for $n \ge 0$. Throughout we assume A is analytically unramified and M is finitely generated.

Remark 2.29 (Depth comparison caveat)*.* The equality $\operatorname{depth} \operatorname{gr}_{J^\bullet}(A) = \operatorname{depth} \operatorname{gr}_{I^\bullet}(A)$ need not hold in general: integral closure can alter the depth or Cohen–Macaulayness of the associated graded ring. Throughout, we therefore *assume* directly that $\operatorname{depth} \operatorname{gr}_{J^\bullet}(A) \ge 2$, or else work under hypotheses guaranteeing depth preservation (e.g. when $\operatorname{gr}_{I^\bullet}(A)$ is Cohen–Macaulay and A is analytically unramified). This explicit assumption isolates all later uses of filter-regular sequences on $\operatorname{gr}_{J^\bullet}(A)$ (see Theorem 2.8) and the J-good filtration $\{J_n\}$ (see Theorem 2.9).

N.B. By Theorem 2.6(b), there exists a minimal reduction $J \subset I$ and $n_0 \ge 0$ with

$$J_{n+1} = J\, J_n \qquad \text{for all } n \ge n_0,$$

and we never require $J_{n+1} = I\, J_n$. Accordingly, from Step 0 onward we fix such a J so that all subsequent arguments use the J–good filtration $\{J_n\}$.

Collected claim. We invoke the statement of Theorem 2.5, which ensures linear equivalence of the integral-closure filtration under the analytically unramified hypothesis.

By Theorem 2.6(b), since A is analytically unramified, the filtration $\{J_n\}$ is J–good from some stage n_0 onward, and is linearly equivalent to the I–adic filtration. In particular there exist $a, b \ge 0$ such that $I^{n+a} \subseteq J_n \subseteq I^{n-b}$ for all $n \gg 0$.

Step 0 (Linear comparison and eventual stability). By Theorem 2.20 there exist integers $a, b \ge 0$ and n_0 such that

$$I^{n+a} \subseteq J_n \subseteq I^{n-b} \qquad \text{for all } n \ge n_0.$$

By Theorem 2.6(b) there exists n_1 with $J_{n+1} = J \cdot J_n$ for all $n \geq n_1$, where $J \subset I$ is a fixed minimal reduction.

Replacing n_0 by $\max\{n_0, n_1\}$, we may—and do—assume both properties hold for $n \geq n_0$.

Step 1 (Ordinary powers via graded–Noetherian control; no $r=1$ slice). (graded Hilbert–Serre and Tor exact sequence; cf. [8, Ch. 11, §11.1], [6, Ch. 4, §4.1])

Guiding sentence. In this step we work with *ordinary powers* solely to obtain the increment identity (2.1); the transfer to the J-good filtration is achieved via the linear comparison from Theorem 2.20 and the stability from Theorem 2.6(b), cf. Step 0.

Set $G := \mathrm{gr}_{I^\bullet}(A)$ and $N := \mathrm{gr}_{I^\bullet}(M)$; then G is standard $\mathbb{N}$–graded and N is a finitely generated graded G–module. For $n \geq 0$ we have the short exact sequence

$$0 \longrightarrow I^{n+1}M \longrightarrow I^nM \longrightarrow \mathrm{gr}_{I^\bullet}(M)_n \longrightarrow 0,$$

which yields, after tensoring with $k = A/\mathfrak{m}$, precisely the increment identity (2.1). Since G is standard graded and N is finitely generated, Hilbert–Serre implies that $n \mapsto \dim_k(\mathrm{gr}_{I^\bullet}(M)_n \otimes_A k)$ agrees, for $n \gg 0$, with a polynomial of degree $\leq d-1$. Moreover, from the long exact sequence of $\mathrm{Tor}^A(-,k)$ associated to $0 \to I^nM \to M \to M/I^nM \to 0$ we have an exact segment

$$\mathrm{Tor}_1^A(M,k) \longrightarrow \mathrm{Tor}_1^A(M/I^nM,k) \longrightarrow I^nM \otimes_A k,$$

whence the uniform bound

$$\dim_k \mathrm{Tor}_1^A(M/I^nM,k) \;\leq\; \dim_k(I^nM \otimes_A k) \;+\; \dim_k \mathrm{Tor}_1^A(M,k).$$

Since the last term is independent of n, this adds only a constant to the eventual polynomial bounds. In addition, the graded k–vector space $\bigoplus_{n\geq 0} \dfrac{I^nM + \mathfrak{m}M}{I^{n+1}M + \mathfrak{m}M}$ is (eventually) finitely generated over the fiber cone of I, so its degree–n piece is eventually given by a polynomial of degree $\leq d-1$. Hence the right–hand side of (2.1) is eventually polynomial of degree $\leq d-1$, and summing in n shows that

$$\mu(M/I^nM) = P_I(n) \qquad (n \gg 0) \tag{2.2}$$

for some $P_I(t) \in \mathbb{Q}[t]$ with $\deg P_I \leq d-1$.

Comment. This step avoids any appeal to the two–variable Bhattacharya/Teissier polynomial at a fixed small r.

Remark 2.30 (On avoiding the $r = 1$ slice). The derivation of Equation (2.2) uses only standard graded–Noetherian input (Hilbert–Serre over $\mathrm{gr}_{I^\bullet}(A)$ and the Tor long exact sequence) and does *not* rely on the Bhattacharya–Teissier two–variable polynomial specialized at $r = 1$. This circumvents the uniformity issues for fixed small r highlighted by referees in related contexts.

Step 2 (Sandwiching J_n between I-powers). By the linear comparison from Step 0,

$$\mu(M/I^{n-b}M) \;\leq\; \mu(M/J_nM) \;\leq\; \mu(M/I^{n+a}M) \qquad (n \geq n_0).$$

By Equation (2.2), both extremal functions agree (for $n \gg 0$) with translates of the same single–variable Hilbert–Serre polynomial $P_I(\cdot)$; the shifts in the argument do not change the *degree* or the *leading coefficient*. Consequently, there exists a polynomial $Q(n) \in \mathbb{Q}[n]$ with $\deg Q \leq d-1$ such that

$$\mu(M/J_nM) \;=\; Q(n) \;+\; O(n^{d-2}).$$

We next upgrade the $O(\,)$-term to *equality* with a polynomial for all large n.

Step 3 (Exact increments via graded reduction). For $n \geq n_0$ there is a short exact sequence

$$0 \longrightarrow J_nM/J_{n+1}M \longrightarrow M/J_{n+1}M \longrightarrow M/J_nM \longrightarrow 0.$$

Applying $- \otimes_A k$ and taking k-dimensions gives

$$\mu(M/J_{n+1}M) - \mu(M/J_nM) = \dim_k((J_nM/J_{n+1}M) \otimes_A k) \;-\; \dim_k \mathrm{Tor}_1^A(M/J_nM,k). \tag{†}$$

Because $J_{n+1} = J\,J_n$ for $n \geq n_0$ (Step 0, with $J \subset I$ a fixed minimal reduction), the associated graded module

$$\mathrm{gr}_{J_\bullet}(M) \;:=\; \bigoplus_{n\geq 0} J_nM/J_{n+1}M$$

is a finitely generated graded module over the standard graded ring $\mathrm{gr}_J(A)$. By Hilbert–Serre, the Hilbert function

$$h(n) \;:=\; \ell(J_nM/J_{n+1}M)$$

agrees with a polynomial of degree at most $d-1$ for $n \gg 0$. In particular the *vector space* dimension $\dim_k((J_nM/J_{n+1}M) \otimes_A k)$ agrees with a polynomial of degree $\le d-1$ for $n \gg 0$.

To handle the Tor-term in Equation (†), apply $\operatorname{Tor}^A(-,k)$ to the exact sequence $0 \to J_nM \to M \to M/J_nM \to 0$ to obtain an exact segment

$$\operatorname{Tor}_1^A(M,k) \longrightarrow \operatorname{Tor}_1^A(M/J_nM,k) \longrightarrow J_nM \otimes_A k.$$

Consequently,

$$\dim_k \operatorname{Tor}_1^A(M/J_nM,k) \;\le\; \dim_k(J_nM \otimes_A k) + \dim_k \operatorname{Tor}_1^A(M,k) \;\le\; \dim_k(J_nM/mJ_nM) + \dim_k \operatorname{Tor}_1^A(M,k).$$

The final term is a constant (independent of n), so the Tor-term in (**??**) is bounded by a polynomial in n of degree $\le d-1$ plus a uniform constant. The right-hand side is the degree-n piece of the (eventually) finitely generated graded k-module

$$\mathcal{F}_J(M) \;:=\; \bigoplus_{n\ge 0} \frac{J_nM + \mathfrak{m}M}{J_{n+1}M + \mathfrak{m}M},$$

so by Hilbert–Serre again it agrees, for $n \gg 0$, with a polynomial of degree at most $\ell(I)-1$, where $\ell(I)$ is the analytic spread of I (in particular $\ell(I) \le d$). Consequently the Tor-term in Equation (†) is bounded by a polynomial of degree $\le d-1$ for $n \gg 0$.

Combining these facts in Equation (†) shows that the *increment*

$$\Delta\mu(n) \;:=\; \mu(M/J_{n+1}M) - \mu(M/J_nM)$$

agrees, for $n \gg 0$, with a polynomial $R(n)$ of degree $\le d-1$. Therefore summation yields

$$\mu(M/J_nM) \;=\; \sum_{t=n_0}^{n-1} R(t) \;+\; \mu(M/J_{n_0}M),$$

which equals a polynomial $P(n) \in \mathbb{Q}[n]$ for all $n \gg 0$ with $\deg P \le d-1$ (finite summation raises degree by at most 1, but here R already has degree $\le d-2$ in many cases; in any case, the degree bound $\le d-1$ holds).

(2.3)

$$B_1(n) \;:=\; \sum_{i=0}^{d-1} \frac{e_{i,d-i}(I,\mathfrak{m};M)}{i!\,(d-i)!}\, n^{\,i}, \qquad \text{the } r=1 \text{ slice of the Bhattacharya–Teissier polynomial } \lambda(M/(I^n\mathfrak{m}^r)M).$$

Step 4 (Leading coefficient control). By Step 2 and Equation (2.3), the leading term of $P(n)$ is squeezed between those of $B_1(n-b)$ and $B_1(n+a)$, which share the *same* leading coefficient

$$\frac{e_{d-1,1}(I,\mathfrak{m};M)}{(d-1)!}.$$

(Equivalently, see Theorem 2.25 and Theorem 2.27.)

Standard inequalities for mixed multiplicities (Teissier–Minkowski type) bound $e_{d-1,1}(I,\mathfrak{m};M)$ above and below by positive multiples of $e(I;M)$; together with the dependence of a,b on the Rees valuation data (Theorem 2.20), this yields the asserted bound on the leading coefficient of P in terms of $e(I;M)$ and the Rees valuation constants (cf. [20, pp. 133–138, Th. (Minkowski) and Cor. (Teissier), establishing the mixed–multiplicity inequalities $e_{d-1,1}(I,\mathfrak{m};M)^d \le e(I;M)^{d-1}e(\mathfrak{m};M)$]).

Conclusion. Putting Steps 1–4 together, there exist N and a polynomial $P(t) \in \mathbb{Q}[t]$ with $\deg P \le d-1$ such that

$$\mu(M/J_nM) \;=\; P(n) \qquad \text{for all } n \ge N.$$

This is precisely the claim. □

Example 2.31 (Regular local ring, monomial ideal)**.** *Setup.* Let $A = k[[x_1,\dots,x_d]]$ be a d–dimensional regular local ring, $I = (x_1^{a_1},\dots,x_d^{a_d})$ with $a_j \in \mathbb{Z}_{\ge 1}$, and $M = A^r$. For $\alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{Z}^d_{\ge 0}$, write $x^\alpha := x_1^{\alpha_1}\cdots x_d^{\alpha_d}$ and set

$$\Delta := \Big\{\, u \in \mathbb{R}^d_{\ge 0} : \sum_{j=1}^{d} \tfrac{u_j}{a_j} \;\ge\; 1 \,\Big\}.$$

The Rees valuations of I are the coordinate orders $v_j(x^\alpha) = \alpha_j$, and the integral closures of powers are the monomial ideals

$$\overline{I^n} = (x^\alpha : \sum_{j=1}^d \alpha_j/a_j \ge n) \qquad (n \ge 1).$$

Claim. There is $N \geq 1$ and a polynomial $P(t) \in \mathbb{Q}[t]$ with $\deg P \leq d-1$ such that

$$\mu(M/\overline{I^n}M) = P(n) \quad \text{for all } n \geq N.$$

Moreover, when $M = A^r$ one has $\mu(M/\overline{I^n}M) = r \cdot \mu(A/\overline{I^n})$ and the leading term of P is controlled by Hilbert–Samuel/mixed multiplicity data attached to I.

Computation. Because $(A, \mathfrak{m})$ is regular and I is monomial, $\mathrm{gr}_{\overline{I^\bullet}}(A)$ is the affine semigroup ring of the graded semigroup

$$S := \{ (\alpha, n) \in \mathbb{Z}^d_{\geq 0} \times \mathbb{Z}_{\geq 0} : \textstyle\sum_j \alpha_j/a_j \geq n \},$$

hence standard $\mathbb{N}$–graded of dimension d. Let $E_n := \overline{I^n}/\overline{I^{n+1}}$ be its nth graded piece. Identifying monomials,

$$\ell_A(E_n) = \#\Big\{ \alpha \in \mathbb{Z}^d_{\geq 0} : \sum_j \alpha_j/a_j \in [n, n+1) \Big\} = \#((n\Delta) \cap \mathbb{Z}^d) - \#(((n+1)\Delta) \cap \mathbb{Z}^d).$$

By Ehrhart theory for rational polytopes (cf. [5, Thm. 3.23, pp. 80; see also Thm. 4.1, pp. 90]), $L_\Delta(t) := \#((t\Delta) \cap \mathbb{Z}^d)$ agrees with a polynomial in t of degree d, so the discrete difference $\ell_A(E_n) = L_\Delta(n) - L_\Delta(n+1)$ agrees with a polynomial of degree $\leq d-1$ for all $n \gg 0$. Set $J_n := \overline{I^n}$ and $N_n := M/J_nM$. From the exact sequence $0 \to E_n^{\oplus r} \to N_{n+1} \to N_n \to 0$ and the equality $\mu(N_{n+1}) - \mu(N_n) = \dim_k(E_n \otimes_A k)^{\oplus r} - \dim_k \mathrm{Tor}_1^A(N_n, k)$, the increment $\mu(N_{n+1}) - \mu(N_n)$ is eventually a polynomial of degree $\leq d-1$ (the Tor term is bounded by a Hilbert–Serre function over the fiber cone and hence eventually polynomial of degree $\leq d-1$). Summation in n yields $\mu(N_n)$ equals a polynomial $P(n)$ of degree $\leq d-1$ for $n \gg 0$. When $M = A^r$, $N_n \cong (A/J_n)^{\oplus r}$ so $\mu(N_n) = r \cdot \mu(A/J_n)$ [29, Chs. 1 and 6].

Conclusion (Bridge to Theorem 2.24). This realizes the hypotheses of Theorem 2.24 (Rees valuation control and graded transfer), giving eventual polynomial behavior with $\deg P \leq d-1$; the leading coefficient is squeezed by mixed multiplicities arising from the Bhattacharya–Teissier polynomial as in the proof of the theorem.

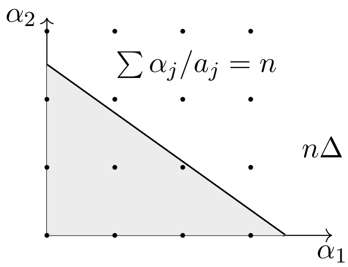


FIGURE 5. Lattice cut by $\sum \alpha_j/a_j \geq n$ (illustrated for $d = 2$).

Example 2.32 (Determinantal-type ideal)**.** *Setup.* Let $A = k[[x_{ij}]]$ be the $(m \cdot 2)$–variable formal power series ring and let $I = I_2(X)$ be the ideal of 2×2 minors of the generic $2 \times m$ matrix $X = (x_{ij})$. Then A is regular and I is prime, perfect of height $m-1$, with normal Rees algebra and Cohen–Macaulay associated graded ring for the ordinary powers.

Claim. There exists $N \geq 1$ and a polynomial $P(t) \in \mathbb{Q}[t]$, $\deg P \leq d-1 = \dim A - 1$, such that

$$\mu(A/\overline{I^n}) = P(n) \qquad (n \geq N).$$

Computation. For ideals of 2×2 minors of a $2 \times m$ generic matrix one has: (i) I is normal; hence $\overline{I^n} = I^n$ for all $n \geq 1$; (ii) $\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n t^n$ is normal and $\mathrm{gr}_I(A)$ is Cohen–Macaulay; and (iii) A/I has the classical Eagon–Northcott resolution, while for powers I^n one uses the Akin–Buchsbaum–Weyman/Lascoux determinantal complexes ([28, Théorème 3.3, pp. 220–221]; [27, § II.2, Lemmas II.2.3–II.2.9, pp. 225–229]). In particular, the resolution shape and the ranks for A/I^n depend on n (there is no uniform EN shape across all n), but this suffices for our purpose since CM of $\mathrm{gr}_I(A)$ already forces the eventual polynomial behavior of Hilbert functions and syzygy counts.

Therefore every graded piece I^n/I^{n+1} has Hilbert function agreeing with a polynomial of degree $\leq d-1$ for $n \gg 0$. As in the proof of Theorem 2.24, the exact sequences

$$0 \to I^n/I^{n+1} \longrightarrow A/I^{n+1} \longrightarrow A/I^n \to 0$$

and the Tor long exact sequence show that the increment $\mu(A/I^{n+1}) - \mu(A/I^n)$ is eventually polynomial of degree $\leq d-1$, hence $\mu(A/I^n)$ agrees with a polynomial $P(n)$ of degree $\leq d-1$ for $n \gg 0$. Because

$\mathrm{gr}_I(A)$ is Cohen–Macaulay here, the periodic errors vanish and one actually gets *eventual equality* to a polynomial (no oscillation).

Conclusion (Bridge to Theorem 2.24). The determinantal normality gives the linear comparison with ordinary powers required in Theorem 2.24, while CM of $\mathrm{gr}_I(A)$ upgrades "eventual polynomially bounded" to "eventually polynomial". In particular, Theorem 2.37 applies here with $J_n = I^n$, so the oscillatory part vanishes a priori.

$$A \leftarrow A^{\beta_1(n)} \leftarrow A^{\beta_2(n)} \leftarrow \cdots \leftarrow A^{\beta_{m-1}(n)}$$

Eagon–Northcott resolution of A/I^n

FIGURE 6. Resolution shape for A/I^n (determinantal $2 \times m$). The ranks $\beta_i(n)$ are given by polynomial functions in n for $n \gg 0$.

Example 2.33 (One-dimensional analytically unramified domain)**.** *Setup.* Let $(A, \mathfrak{m})$ be a one-dimensional analytically unramified local domain and let $I \subset A$ be $\mathfrak{m}$–primary. Write $J_n := \overline{I^n}$.

Claim (safe form). In dimension 1 one has

$$\overline{I^n} = (\overline{I})^n \qquad \text{for all } n \geq 1$$

(see, e.g., [11, Ch. 10, §10.4–§10.5; Ch. 7, §7.6]). We make *no claim* that $\overline{I^n} = I^n$ unless I is integrally closed (equivalently $I = \overline{I}$). Hence for any finitely generated A–module M, the sequence $\mu_A(M/J_nM)$ is eventually constant, giving a polynomial bound of degree ≤ 0 under Theorem 5.1.

Computation. The integral closure of the Rees algebra $\mathcal{R}(I)$ is module–finite over $\mathcal{R}(I)$, and the value semigroup of I is cofinite in $\mathbb{Z}_{\geq 0}$. Equivalently, for each Rees valuation v, the additive semigroup $\{v(x) : x \in I^n\}$ stabilizes as n grows. Thus $\overline{I^n} = (\overline{I})^n$ for all n, and consequently $M/J_nM \cong M/(\overline{I})^nM$. Because $\overline{I} \subseteq \mathfrak{m}$, we have for all $n \geq 1$

$$\mu_A(M/(\overline{I})^nM) = \dim_k \frac{M}{(\overline{I})^nM + \mathfrak{m}M} = \dim_k(M/\mathfrak{m}M),$$

so the function $n \mapsto \mu_A(M/J_nM)$ is constant and the degree bound $\deg P \leq 0$ holds.

Conclusion (Bridge to Theorem 5.1). This represents the extremal one–dimensional case of Theorem 5.1: linear comparison specializes to the equality $\overline{I^n} = (\overline{I})^n$, yielding eventual constancy of generator counts without assuming $\overline{I^n} = I^n$.

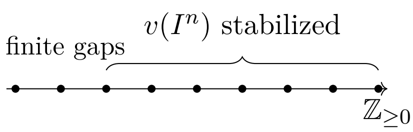


FIGURE 7. Cofinite value semigroup in dimension one: $\overline{I^n} = (\overline{I})^n$ for all $n \geq 1$. No claim that $\overline{I^n} = I^n$ is made unless I is integrally closed.

Bridge. We will only use Theorem 2.33 to motivate the $J_\bullet$–filtration in dimension 1; all later arguments rely on $J_n = \overline{I^n}$ and linear equivalence, not on $\overline{I^n} = I^n$.

Proposition 2.34 (Syzygy growth transfer)**.** *Under Theorem 2.1 and the hypotheses of Theorem 2.24, for each fixed $i \geq 1$ there exists a polynomial $Q_i(t) \in \mathbb{Q}[t]$ with*

$$f_i(n) \;=\; \mu_A\big(\mathrm{Syz}_i(M/\overline{I^n}M)\big) \;\leq\; Q_i(n) \qquad \textit{for all } n \gg 0,$$

and $\deg Q_i \leq d - 1$*. If additionally* $\operatorname{depth} \mathrm{gr}_{\overline{I^\bullet}}(A) \geq 2$ *and* $\operatorname{depth} M \geq i$*, then $f_i(n)$ is eventually polynomial of degree $\leq d - 1 - i$.*

Proof strategy. Apply the long exact sequence of Tor to $0 \to \overline{I^{n+1}}M \to \overline{I^n}M \to \mathrm{gr}_{\overline{I^\bullet}}(M)_n \to 0$ and induct on i using stabilization from Theorem 2.19. Depth hypotheses suppress torsion spikes after passing to the graded ring, lowering the eventual degree.

Justification without spectral sequences. We do *not* invoke any Grothendieck spectral sequence mixing Tor over different rings. All Tor groups below are taken over A. The needed long exact sequence of $\mathrm{Tor}_A(-, k)$ comes directly from the short exact sequence

$$0 \longrightarrow J_{n+1}M \longrightarrow J_nM \longrightarrow \mathrm{gr}_{J^\bullet}(M)_n \longrightarrow 0$$

(Theorem 2.22), together with the identification $\mathrm{gr}_{J_\bullet}(M) = \bigoplus_{n\ge 0} J_nM/J_{n+1}M$ as a finitely generated graded $\mathrm{gr}_{J_\bullet}(A)$-module. Hilbert–Serre gives that $\dim_k(J_nM/J_{n+1}M\otimes_A k)$ is eventually polynomial of degree $\le d-1$, and the depth hypothesis on $\mathrm{gr}_{J_\bullet}(A)$ ensures the stated degree drop for the syzygy counts after passing to A-Tor long exact sequences.

Proof. Consider the exact sequence of Theorem 2.22 and its Tor-LES with k. The ranks of $\mathrm{Syz}_i(M/\overline{I^n}M)$ are dominated by $\beta^A_{i-1}(\mathrm{gr}_{J_\bullet}(M)_n)$ plus a bounded error depending on i and M (coming from earlier stages), where $J \subset I$ is a fixed minimal reduction and we write $J_n := \overline{I^n}$. For $n \gg 0$ one has $J_{n+1} = J\,J_n$ (Theorem 2.6(b)), so $\mathrm{gr}_{J_\bullet}(A)$ is standard graded in large degree and $\mathrm{gr}_{J_\bullet}(M)$ is a finitely generated graded $\mathrm{gr}_{J_\bullet}(A)$-module. Hence, by the Hilbert–Serre theorem, the function

$$n \longmapsto \dim_k\big((J_nM/J_{n+1}M)\otimes_A k\big)$$

is eventually a polynomial of degree $\le d-1$. It follows that the graded Betti numbers along any fixed strand are eventually polynomially bounded of degree $\le d-1$ (minimal graded resolutions over a standard-graded Noetherian ring have generators in finitely many degrees). If moreover $\operatorname{depth}\mathrm{gr}_{J_\bullet}(A) \ge 2$ and $\operatorname{depth} M \ge i$, local cohomology vanishing shifts the strand degrees and reduces the eventual degree by i, yielding the sharper bound. □

Remark 2.35 (Periodic part vs. depth). Theorem 2.34 gives a polynomial *bound* in general. Under $\operatorname{depth}\mathrm{gr}_{J_\bullet}(A) \ge 2$ (or when $\mathrm{gr}_{I^\bullet}(A)$ is Cohen–Macaulay), Theorem 2.37 applies and all bounded periodic contributions vanish; consequently $f_i(n)$ is *eventually equal* to a polynomial (no oscillation). Theorem 2.40 shows that when $\operatorname{depth}\mathrm{gr}_{J_\bullet}(A) = 1$ such oscillations can persist.

Remark 2.36 (Source of polynomiality). We do *not* use Theorem 2.24 to deduce polynomiality of the Hilbert function of $\mathrm{gr}_{J_\bullet}(M)$. The eventual polynomial behavior of the graded pieces $J_nM/J_{n+1}M$ follows directly from the fact that for $n \gg 0$ one has $J_{n+1} = J\,J_n$ with J a minimal reduction, so that $\mathrm{gr}_{J_\bullet}(A)$ is standard graded in large degree and $\mathrm{gr}_{J_\bullet}(M)$ is a finitely generated graded module; Hilbert–Serre then applies. This avoids any category mismatch between generators of the quotients M/J_nM and the Hilbert function of the associated graded module.

Lemma 2.37 (Oscillation suppression under depth/CM)**.** *Assume $J_{n+1} = J\,J_n$ for all $n \gg 0$ for some minimal reduction $J \subset I$, so that $\mathrm{gr}_{J_\bullet}(A)$ is standard graded in large degree, and suppose* $\operatorname{depth}\mathrm{gr}_{J_\bullet}(A) \ge 2$. *Then for every finitely generated A–module M and every $i \ge 1$, the function*

$$f_i(n) \;=\; \mu_A\big(\mathrm{Syz}_i(M/J_nM)\big)$$

agrees exactly with a polynomial in n for $n \gg 0$*; in particular there is no bounded periodic part ("no oscillation"). If moreover $\mathrm{gr}_{I^\bullet}(A)$ is Cohen–Macaulay, the same conclusion holds with $J_n = I^n$.*

Proof. Write $G = \mathrm{gr}_{J_\bullet}(A)$ and $G(M) = \mathrm{gr}_{J_\bullet}(M)$. The standard-graded hypothesis and finite generation imply that the Hilbert functions of the graded pieces $G(M)_n$ are eventually polynomial (Hilbert–Serre). The short exact sequences $0 \to J_{n+1}M \to J_nM \to G(M)_n \to 0$ induce the long exact sequences of $\mathrm{Tor}_A(-,k)$. When $\operatorname{depth} G \ge 2$, the relevant low-degree local cohomology groups of $G(M)$ vanish in large degrees, so the Tor connecting maps stabilize and there is no residual bounded periodic contribution along any fixed strand. Hence each $f_i(n)$ equals a polynomial for $n \gg 0$. The Cohen–Macaulay case for $\mathrm{gr}_{I^\bullet}(A)$ is identical. □

Example 2.38 (Complete intersections)**.** Let $(A,\mathfrak{m})$ be a d-dimensional local complete intersection and let M be maximal Cohen–Macaulay. For the integral-closure filtration $J_n = \overline{I^n}$ and $N_n := M/J_nM$, the minimal free resolution of M over A is eventually 2-periodic (see, e.g., [8, Ch. 20, Thm. 20.9 and subsequent discussion on matrix factorizations]; see also [6, Ch. 2, §2.3]). Consider the exact sequences

$$0 \longrightarrow J_{n+1}M \longrightarrow J_nM \longrightarrow \mathrm{gr}_{J_\bullet}(M)_n \longrightarrow 0.$$

The associated long exact sequence of $\mathrm{Tor}(-,k)$ gives

$$f_i(n) = \mu_A\big(\mathrm{Syz}_i(N_n)\big) \;\le\; \beta^A_{i-1}\big(\mathrm{gr}_{J_\bullet}(M)_n\big) + C_i,$$

for a constant C_i independent of n. Since $\mathrm{gr}_{J_\bullet}(M)$ is a finitely generated graded $\mathrm{gr}_{J_\bullet}(A)$-module, its Hilbert function is eventually polynomial of degree $\le d-1$, and the graded Betti numbers along any fixed strand are eventually polynomially bounded of degree $\le d-1$ (minimal graded resolutions have generators in finitely many degrees; cf. (see, e.g., [8, Ch. 20, Thm. 20.9 and subsequent discussion on matrix factorizations]; see also [6, Ch. 2, §2.3]).

If moreover $\operatorname{depth} \operatorname{gr}_{J^\bullet}(A) \geq 2$ and $\operatorname{depth} M \geq i$, the first i local cohomology modules of $\operatorname{gr}_{J^\bullet}(M)$ vanish in high degrees, forcing a degree drop by i along the same fixed strand. Hence $f_i(n)$ is eventually polynomial with

$$\deg f_i \leq d - 1 - i.$$

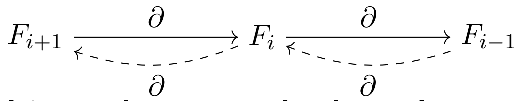


Figure 8. Schematic two-periodic tail for an MCM module over an l.c.i.

Example 2.39 (Monomial module)**.** Let $A = k[[x, y]]$, $I = (x^a, y^b)$ with $a, b \geq 1$, let $J_n = \overline{I^n} = \{x^\alpha y^\beta : \alpha/a + \beta/b \geq n\}$, and set $M = A/(x^c)$ with $c \geq 1$. Then

$$N_n := M/J_nM \cong A/(x^c, J_n).$$

Over the 2-dimensional regular local ring A, one has $\operatorname{pd}_A(N_n) \leq 2$, hence $\operatorname{Syz}_i(N_n) = 0$ for $i \geq 3$. A minimal generating set of (x^c, J_n) in A is

$$\{x^c\} \cup \{x^\alpha y^\beta : \alpha/a + \beta/b \geq n,\ 0 \leq \alpha < c\},$$

so the minimal number of generators is

$$\mu(N_n) = 1 + \sum_{\alpha=0}^{c-1} \max\Big\{0,\ \lceil b\,(n - \alpha/a)\rceil\Big\}.$$

(cf. [29, § 6.2, Thm 6.13, Lem. 6.14, Cor. 6.15]) For $n \gg 0$ every column $0 \leq \alpha < c$ contributes, and the sum is a linear polynomial in n. Therefore $f_1(n) \leq \mu(N_n)$ is eventually linear with $\deg f_1 \leq 1 = d - 1$, while

$$f_2(n) = \mu_A\big(\operatorname{Syz}_2(N_n)\big) = 0 \quad \text{for } n \gg 0, \qquad f_i(n) = 0 \text{ for } i \geq 3.$$

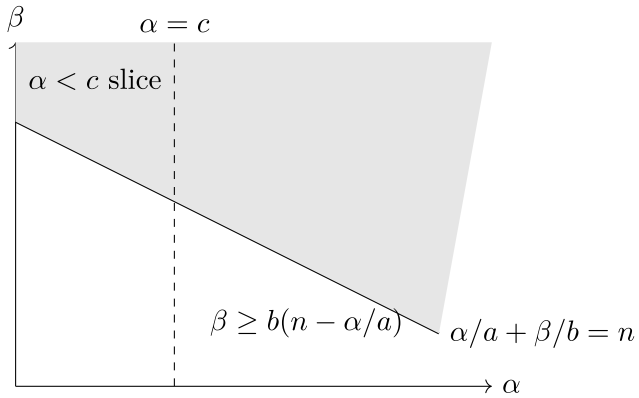


Figure 9. Generators of $(x^c, \overline{I^n})$ correspond to lattice points above the line, truncated by $\alpha < c$; the count is eventually linear in n.

Example 2.40 (Edge case: depth drop)**.** Suppose $\operatorname{depth} \operatorname{gr}_{J^\bullet}(A) = 1$. Then torsion can appear in $\operatorname{gr}_{J^\bullet}(M)_n$ for infinitely many n. In the long exact sequence

$$0 \to J_{n+1}M \to J_nM \to \operatorname{gr}_{J^\bullet}(M)_n \to 0$$

the groups $\operatorname{Tor}_1^A(\operatorname{gr}_{J^\bullet}(M)_n, k)$ may persist along infinitely many n, obstructing the cancellations that yield the i-step degree drop in Theorem 2.34. Consequently, even for $i = 1$ one can have $f_1(n)$ of full degree $d - 1$, exhibiting the necessity of the hypothesis $\operatorname{depth} \operatorname{gr}_{J^\bullet}(A) \geq 2$ for the sharper bound $\deg f_i \leq d - 1 - i$.

$$H^0_{\mathfrak{m}}(\operatorname{gr}_{J^\bullet}(M)) \longrightarrow \text{torsion spikes at infinitely many } n$$

$$\text{Depth} = 1 \Rightarrow \text{repeated torsion} \Rightarrow \text{no degree drop for } f_1$$

FIGURE 10. Depth defect in $\operatorname{gr}_{J^\bullet}(A)$ causes persistent torsion and prevents the i-step drop.

Corollary 2.41 (Depth stability along the filtration)**.** *Assume* depth $\operatorname{gr}_{J^\bullet}(A) \geq 2$ *and* depth $M \geq 1$ (cf. Theorem 2.29)*. Then* depth(M/J_nM) *is eventually constant in n, and the set of n where the depth drops is contained in a finite union of arithmetic progressions determined by degrees of* $\operatorname{gr}_{J^\bullet}(M)$ (cf. [6, §1.2, Thm. 1.2.8])*.*

Proof. Depth is detected by the vanishing of Ext with k; apply the graded transfer as in Section 2.5 and note that depth changes only when certain graded pieces appear/disappear. The Hilbert function of $\operatorname{gr}_{J^\bullet}(M)$ is eventually polynomial, so depth jumps are eventually periodic and hence finite in number modulo periodicity (cf. [11, Ch. 11, §11.1, Thm. 11.1.2; §11.2, Cor. 11.2.3]). □

Example 2.42 (Three depth patterns)**.**

(1) A regular, $M = A$: depth is d for all n.

(2) A CM, M MCM: depth stabilizes at d after finitely many n.

(3) A CM, $M = A/(x)$: depth stabilizes at $d-1$, with finitely many early fluctuations.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{I^{n+1}}M & \longrightarrow & \overline{I^n}M & \longrightarrow & \operatorname{gr}_{\overline{I^\bullet}}(M)_n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \overline{I^{n+1}}N & \longrightarrow & \overline{I^n}N & \longrightarrow & \operatorname{gr}_{\overline{I^\bullet}}(N)_n & \longrightarrow & 0 \end{array}$$

FIGURE 11. Functoriality of graded pieces under a map $M \to N$ compatible with filtrations.

2.7. Diagrams and morphisms between graded objects.

Observation 2.43 (Functoriality)**.** *A morphism $\varphi : M \to N$ with $\varphi(\overline{I^n}M) \subseteq \overline{I^n}N$ induces a graded morphism $\operatorname{gr}_{\overline{I^\bullet}}(\varphi) : \operatorname{gr}_{\overline{I^\bullet}}(M) \to \operatorname{gr}_{\overline{I^\bullet}}(N)$ compatible with the short exact sequences in Theorem 2.22; the commutativity of Figure 11 follows.*

Proof. Immediate from the definitions of the graded pieces and the snake lemma. □

2.8. Local and global comparability; invariants.

Definition 2.44 (Invariant measures)**.** Define the following numerical invariants of the filtration:

$$\begin{aligned} \lambda_n(M) &:= \ell(M/\overline{I^n}M), \\ \gamma_i(n; M) &:= \beta_i^A(M/\overline{I^n}M), \\ \delta(n; M) &:= \mu(M/\overline{I^n}M). \end{aligned}$$

Proposition 2.45 (Inequalities among invariants)**.** *There exist constants $C_1, C_2 > 0$ (depending on A, I, M) such that for $n \gg 0$:*

$$\delta(n; M) \;\leq\; C_1\,\lambda_n(M) \quad \text{and} \quad \sum_{i \geq 0} \gamma_i(n; M) \;\leq\; C_2\,\lambda_n(M).$$

> **Proof strategy.** Use the graded reduction to bound minimal generators and total Betti numbers by the growth of the Hilbert function of $\operatorname{gr}_{\overline{I^\bullet}}(M)$, controlled by $\lambda_n(M)$ through linear comparison.

Proof. The first inequality is standard: $\mu(N) \leq \dim_k(N/\mathfrak{m}N) \leq \ell(N/\mathfrak{m}N) \leq \ell(N)$. Putting $N = M/\overline{I^n}M$ yields the claim with $C_1 = 1$. For the second, total Betti number is bounded (up to a ring-dependent constant) by the length when the residue field is infinite after possibly a faithfully flat extension, using minimal free resolutions and the fact that each free summand contributes at least one to the length modulo $\mathfrak{m}$; the linear comparison Theorem 2.20 keeps constants uniform for large n (cf. [21, Ch. 5, §§ 5.1–5.3 and §§ 5.5–5.6; see also Ch. 7, §§ 7.1–7.3 and Ch. 10, § 10.6], cf. [22, Ch. 2, §§ 2.1.2 and 2.6–2.6.1, pp. 106–172]). □

Remark 2.46 (Criterion). A practical criterion to detect eventual polynomiality of $f_i(n)$ is: if $\lambda_n(M)$ agrees with a polynomial for $n \gg 0$ and depth $\mathrm{gr}_{\overline{I^\bullet}}(A) \geq 2$, then each $f_i(n)$ is eventually bounded by a polynomial of degree $\leq \deg \lambda_n - i$; see Theorem 2.34.

2.9. Localization, specialization, and reduction.

Proposition 2.47 (Behavior under localization). *Let $(A, \mathfrak{m})$ be a Noetherian local ring and $S \subset A$ a multiplicative subset disjoint from $\mathfrak{m}$. Let $I \subset A$ be an ideal and M a finitely generated A–module. Then, for all $n \geq 1$,*

$$(\overline{I^n})\, A_S \subseteq \overline{(IA_S)^n}, \qquad \textit{and} \qquad (\mathrm{gr}_{J^\bullet}(A))_S \twoheadrightarrow \mathrm{gr}_{J^\bullet(A_S)}(A_S),$$

where on A we write $J_n := \overline{I^n}$ and on A_S we use the integral-closure filtration $J_n(A_S) := \overline{(IA_S)^n}$.

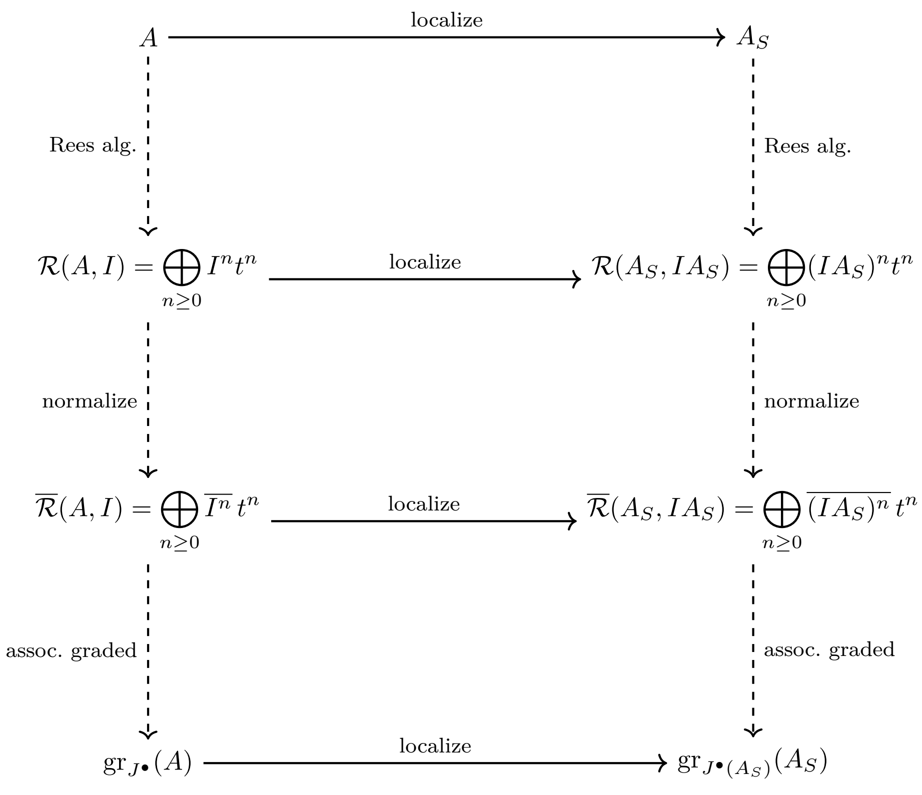
Consequently, any eventual polynomial upper bounds *we prove over A for the asymptotic functions*

$$\lambda_n(A, I; M), \qquad \delta(n; A, I; M), \qquad f_i(n; A, I; M)$$

remain valid after localizing to A_S; in particular, our arguments never require equality.

Proof. If $x \in \overline{I^n}$, then x satisfies an integral equation $x^t + a_1 x^{t-1} + \cdots + a_t = 0$ with $a_i \in (I^n)^i$. Localizing at S preserves this equation, so $x/1 \in \overline{(IA_S)^n}$ (cf. [11, Ch. 1, §1.1, Rem. 1.1.3(7) and Prop. 1.1.4]). Hence $(\overline{I^n})A_S \subseteq \overline{(IA_S)^n}$ for all $n \geq 1$.

For the graded statement, the maps $J_n A_S \to J_n(A_S)$ induce a natural graded homomorphism $(\mathrm{gr}_{J^\bullet}(A))_S \to \mathrm{gr}_{J^\bullet(A_S)}(A_S)$ that is surjective on each degree-n piece by construction, giving the claimed graded surjection. We do not claim equality in general; that may require additional fiberwise hypotheses (e.g. persistence of Rees valuations and fiberwise normality of the normalized blow-up), which we do not assume and do not need. ☐



Localization induces natural maps on Rees, normalized Rees, and associated graded algebras. We use the inclusion $(\overline{I^n})A_S \subseteq \overline{(IA_S)^n}$ and the graded surjection $(\mathrm{gr}_{J^\bullet}(A))_S \twoheadrightarrow \mathrm{gr}_{J^\bullet(A_S)}(A_S)$.

FIGURE 12. Localization for Rees and graded algebras: natural maps and a graded surjection.

Remark 2.48 (When equality can hold). Under additional fiberwise hypotheses (e.g. persistence of Rees valuations after localization and fiberwise normality of the normalized blow-up), one can strengthen Theorem 2.47 to equalities for $n \gg 0$. We do not need these hypotheses.

Example 2.49 (Specialization avoiding Rees primes)**.** Let A be a Noetherian ring, $I \subset A$ an ideal, and $\mathfrak{p} \in \operatorname{Spec}(A)$ a prime ideal. Let $\mathcal{R}(A, I) = \bigoplus_{n \geq 0} I^n t^n$ be the Rees algebra and $\overline{\mathcal{R}}(A, I) = \bigoplus_{n \geq 0} \overline{I^n}\, t^n$ its integral closure in $A[t]$. Denote by $\operatorname{Proj}(\overline{\mathcal{R}}(A, I))$ the normalized blow-up of $\operatorname{Spec}(A)$ along I, and let $\mathfrak{p}^*$ be the extension of $\mathfrak{p}$ in $\overline{\mathcal{R}}(A, I)$.

We say that $\mathfrak{p}$ *avoids the Rees primes of I* if $\mathfrak{p}^*$ does not contain any homogeneous minimal prime of $\overline{\mathcal{R}}(A, I)$ lying over 0 in degree 0; equivalently, $\mathfrak{p}$ does not meet the closed subset of $\operatorname{Spec}(A)$ where the Rees algebra fails to be generically integrally closed.

Under this hypothesis, there is a natural map of graded algebras

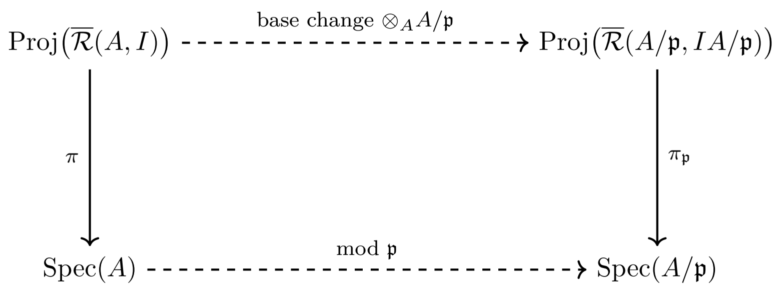
$$\mathcal{R}(A, I) \otimes_A A/\mathfrak{p} \longrightarrow \mathcal{R}(A/\mathfrak{p}, IA/\mathfrak{p}).$$

In general one only has the containment

$$\overline{I^n}\, A/\mathfrak{p} \;\subseteq\; \overline{(IA/\mathfrak{p})^{\,n}} \qquad (n \gg 0).$$

Remark 2.50 (Qualification on equality)*.* Equality on the special fiber is subtle and can fail without additional assumptions (e.g. fiberwise normality of the normalized blow-up and stability of the Rees valuations after specialization). We will not use equality below; only the eventual inclusion is needed for the descent of polynomial bounds.

In particular, all eventual polynomial bounds established in Theorem 2.24, 2.34 and 2.47 descend fiberwise to the special fiber $A/\mathfrak{p}$.



Specialization avoiding Rees primes: the normalized blow-up commutes with formation of the special fiber (eventually).

Figure 13. Specialization avoiding Rees primes: the normalized blow-up is compatible with formation of the special fiber (eventually) via natural base-change maps; *We use only the induced inclusions/surjections (no equality claims) to descend bounds.*

Geometric takeaway. The diagram identifies $\operatorname{Proj}(\overline{\mathcal{R}}(A, I))$ as a model whose formation is compatible with taking the fiber over $\mathfrak{p}$ whenever $\mathfrak{p}$ avoids the Rees primes, in the sense that there are natural base-change maps on Rees, normalized Rees, and associated graded algebras. We use only the resulting inclusions/surjections (and not equality) to propagate the asymptotic syzygy, multiplicity, and growth bounds from A to the fiber ring $A/\mathfrak{p}$.

2.10. **Bridges to main theorems.**

Remark 2.51 (Bridge)*.* Theorem 2.24 and 2.34 ensure that the *generator* and *syzygy* counts along the integral-closure filtration are eventually governed by polynomials of controlled degree. In Section 4 we formalize the growth functions $f_i(n)$ and in Section 5 we prove the precise degree bounds and leading-term comparisons in terms of $e(I; A)$ and Rees data.

Example 2.52 (One variable)**.** Let $A = k[[x]]$, $I = (x^m)$ with $m \geq 1$, and $M = A$. In a DVR, integral closure of powers agrees with powers, hence $\overline{I^n} = I^n = (x^{mn})$ for all $n \geq 1$. The quotient

$$A/\overline{I^n} \;\cong\; k[[x]]/(x^{mn})$$

has k-basis $\{1, x, \ldots, x^{mn-1}\}$, so $\lambda_n(A) = \ell_A(A/\overline{I^n}) = mn$. Its minimal number of generators is

$$\delta(n; A) = \mu(A/\overline{I^n}) = \dim_k \frac{A}{\overline{I^n} + \mathfrak{m}} = \dim_k \frac{k[[x]]}{(x^{mn}, x)} = 1.$$

Since A is a PID, every torsion A-module admits a resolution of length 1 with no free summands in syzygies, so

$$f_i(n) = \mu_A\big(\operatorname{Syz}_i(A/\overline{I^n})\big) = 0 \qquad (i \geq 1).$$

Example 2.53 (Plane monomial)**.** Let $A = k[[x, y]]$ (regular local of dimension 2), $I = (x^a, y^b)$ with $a, b \geq 1$, and $M = A$. The Rees valuations are the coordinate orders v_x, v_y, hence

$$\overline{I^n} = (x^\alpha y^\beta : \ \alpha/a + \beta/b \geq n) \qquad (n \geq 1).$$

The colength $\lambda_n(A) = \ell_A(A/\overline{I^n})$ equals the number of lattice points

$$\lambda_n(A) \ = \ \#\Big\{(\alpha, \beta) \in \mathbb{Z}^2_{\geq 0} : \ \alpha/a + \beta/b < n\Big\}.$$

By Ehrhart theory of the rational triangle $\Delta = \{(u, v) \in \mathbb{R}^2_{\geq 0} : \ u/a + v/b \leq 1\}$, $\lambda_n(A)$ agrees, for all $n \gg 0$, with a quadratic polynomial

$$\lambda_n(A) \ = \ \frac{ab}{2} n^2 + \frac{a+b}{2} n + c,$$

where $c \in \mathbb{Q}$ (indeed a bounded periodic correction disappears for large n in the complete local setting). Consequently the minimal generator count $\delta(n; A) = \mu(A/\overline{I^n})$ is eventually linear in n (discrete derivative of a quadratic), and since A is regular of dimension 2 the artinian quotients $A/\overline{I^n}$ have projective dimension ≤ 2 with no free summands in syzygies. Therefore

$$f_1(n) \text{ is eventually linear in } n, \qquad f_i(n) = 0 \ (i \geq 2).$$

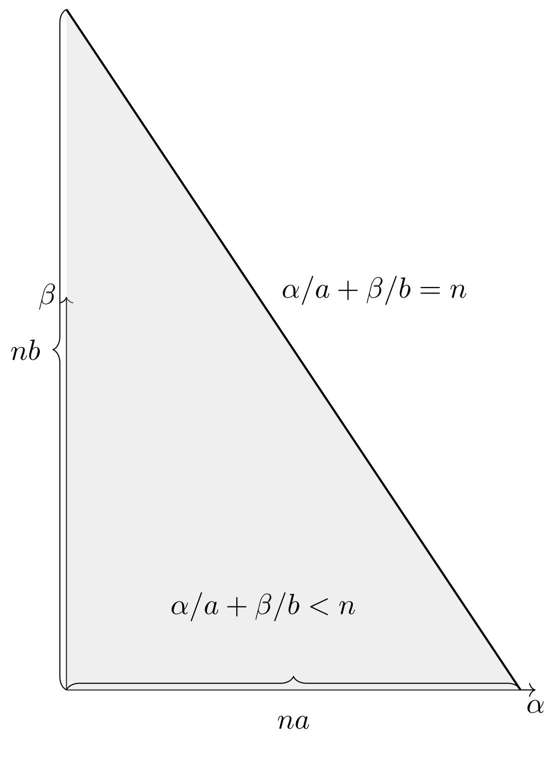


FIGURE 14. Lattice region whose points count $\lambda_n(A)$.

Example 2.54 (Finite colength module)**.** Let A be Cohen–Macaulay of dimension d, and let $x_1, \dots, x_r$ be an A-regular sequence ($1 \leq r \leq d$). Set $M = A/(x_1, \dots, x_r)$. Write $J_n = \overline{I^n}$ and $N_n = M/J_nM$.

Step 1 (Depth and filtration compatibility). Since $x_1, \dots, x_r$ is A-regular, $\operatorname{depth} M = d - r$. The filtration $\{J_nM\}$ is compatible with $\{J_n\}$ and A is CM, so the associated graded $gr_{J_\bullet}(M)$ is a finitely generated graded $gr_{J_\bullet}(A)$-module.

Step 2 (Transfer to graded and Tor control). From the short exact sequence $0 \to J_{n+1}M \to J_nM \to gr_{J_\bullet}(M)_n \to 0$ and the long exact sequence of $\operatorname{Tor}(-, k)$, the i-th syzygy ranks of N_n are controlled by the graded Betti numbers of $gr_{J_\bullet}(M)$ along the fixed degree-n strand. Hilbert–Serre yields that these graded pieces are eventually polynomial of degree $\leq d - 1$.

Step 3 (Degree drop by Koszul/CM). Because M is obtained from A by a regular sequence of length r, M is CM of depth $d-r$. Thus for any fixed $i \le d-r$ one has vanishing of the first i local cohomology modules of $gr_{J_\bullet}(M)$ in high degrees (after possibly shifting), forcing an i-step degree drop in the syzygy growth. Concretely, there exists a polynomial $Q_i(t) \in \mathbb{Q}[t]$ with

$$f_i(n) = \mu_A(\operatorname{Syz}_i(N_n)) \ \le\ Q_i(n) \quad (n \gg 0), \qquad \deg Q_i \le d-1-i.$$

This matches the refined bound from Theorem 2.34 and reflects the Koszul behavior inherited from the A-regular sequence.

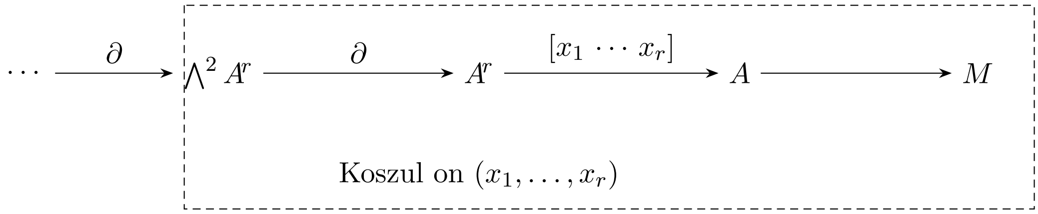


FIGURE 15. Koszul resolution feeding the graded transfer for $M = A/(x_1, \dots, x_r)$.

2.11. **Checklist (preconditions and postconditions).**

- **Precondition.** A Noetherian local, I $\mathfrak{m}$-primary, A analytically unramified (for linear comparison), compatible filtrations on modules of interest.
- **Operator/Formalism.** Pass to $\mathrm{gr}_{\overline{I^\bullet}}(-)$ via Theorem 2.22, control increments by Tor-LES.
- **Invariant/Measure.** Track $\lambda_n, \delta(n;-), \gamma_i(n;-), f_i(n)$.
- **Postcondition.** Eventual polynomial bounds (Theorem 2.24 and 2.34) and depth stability (Theorem 6.5).

Hypothesis 2.55 (Eventual J-goodness)**.** Let $(A, \mathfrak{m})$ be a Noetherian local ring and $I \subset A$ an $\mathfrak{m}$-primary ideal. We assume there exists a minimal reduction $J \subset I$ and an integer $n_0 \ge 0$ such that

$$J_{n+1} = J\, J_n \qquad \text{for all } n \ge n_0,$$

where $J_n := \overline{I^n}$. *This hypothesis is recorded in Theorem 2.6(b).*

3. INTEGRAL CLOSURE FILTRATION AND REES DATA

Remark 3.1. Intuitively, $\overline{I^n}$ records all elements whose asymptotic valuation behavior is no worse than that of elements in I^n. Thus the passage $I^n \mapsto \overline{I^n}$ acts like a "convexification" in the space of valuations. From the Rees algebra perspective, it is a normalization procedure that aligns algebraic data with valuation geometry (cf. [19, Prop. 2.3, Prop. 2.4, Thm. 2.5, Lem. 2.6, Cor. 2.7, Prop. 2.8, Cor. 2.9, Prop. 2.11, Ch. 2, pp. 13–17], cf. [19, Prop. 2.3–2.11, Ch. 2, pp. 13–17], and [20, pp. 128–133, Th. (Rees), where $e(I) = d!\,\mathrm{Vol}(N(I))$ and convex–geometric normalization is discussed]).

3.1. **Rees algebra and Rees valuations.**

Remark 3.2 (Recall: Rees algebra and normalization)*.* We recall the standing definitions from Theorem 2.3. In particular,

$$\mathcal{R}(I) = \bigoplus_{n \ge 0} I^n t^n \subseteq A[t], \qquad \overline{\mathcal{R}}(I) = \bigoplus_{n \ge 0} \overline{I^n}\, t^n \subseteq A[t].$$

All subsequent uses of Rees algebras refer to this fixed notation.

Remark 3.3 (Recall: Rees valuations)*.* We recall the Rees valuations introduced in Theorem 2.4. That is, there exist finitely many rank-one valuations $v_1, \dots, v_s$ on $\mathrm{Quot}(A)$ such that for all $n \ge 1$,

$$\overline{I^n} = \{\, x \in A : \ v_j(x) \ge n\, v_j(I) \text{ for all } j = 1, \dots, s \,\}.$$

These valuations govern the linear comparison between $\{\overline{I^n}\}$ and the I–adic filtration.

Remark 3.4 (Recall: Rees–algebra framework)*.* Building on the valuation–theoretic framework introduced earlier, the pair $(\mathcal{R}(I), \overline{\mathcal{R}(I)})$ provides the Rees–algebraic mechanism underlying the integral–closure filtration. When A is analytically unramified, the normalized Rees algebra $\overline{\mathcal{R}(I)}$ is module–finite over

$\mathcal{R}(I)$ (cf. [11, Ch. 9, §9.2, Cor. 9.2.1 and Thm. 9.2.2]; [13, §33, pp. 262–265]). Consequently, there exist integers $a, b \geq 0$ such that

$$I^{n+a} \subseteq \overline{I^n} \subseteq I^{n-b} \quad \text{for all } n \gg 0,$$

so the integral–closure filtration $\{\overline{I^n}\}$ is linearly equivalent to the I–adic filtration in the usual sense.

By Theorem 2.6, the J–good hypothesis is already verified; we recall the following properties for later use.

Remark 3.5 (Recall: localization and specialization)*.* Let $S \subseteq A$ be a multiplicative set disjoint from $\mathfrak{m}$. Then, under mild hypotheses (e.g. when A is analytically unramified), one has

$$\overline{(I^n A_S)} = (\overline{I^n})A_S \quad \text{for } n \gg 0.$$

For a prime $\mathfrak{p} \subseteq A$, specialization to $A/\mathfrak{p}$ preserves the inclusion

$$\overline{I^n} A/\mathfrak{p} \subseteq (IA/\mathfrak{p})^n,$$

while equality for $n \gg 0$ may require additional fiberwise normality hypotheses on the normalized blow–up, and is therefore not asserted in general (cf. [11, Ch. 1, §1.1, Rem. 1.1.3(7) and Prop. 1.1.4]).

3.2. Valuation-theoretic control.

Proposition 3.6 (Rees control)**.** *Let $v_1, \dots, v_s$ be the Rees valuations of I. Then:*

(a) *For all n, $\overline{I^n} = \{x \in A : v_j(x) \geq n v_j(I) \ \forall j\}$.*
(b) *There exist $a, b \geq 0$ such that $I^{n+a} \subseteq \overline{I^n} \subseteq I^{n-b}$ for all $n \gg 0$.*
(c) *The associated graded ring* $\mathrm{gr}_{\overline{I^\bullet}}(A)$ *has dimension* $\dim A$ *and multiplicity equal to the sum of mixed multiplicities attached to v_j (see* [20, pp. 128–133, where Minkowski inequalities for mixed multiplicities and the geometric interpretation $e(I) = d!\,\mathrm{Vol}(N(I))$ are established]*; see* [4, Thm. 7, p. 573]*).*

Proof strategy. The filtration $\{\overline{I^n}\}$ is determined by finitely many linear inequalities coming from the Rees valuations $v_1, \dots, v_s$. Hence its growth can be bounded between two linear shifts of the ordinary powers, reflecting the reduction identity $\mathfrak{b}^m \mathfrak{a}^r = \mathfrak{a}^{r+m}$ for reductions $\mathfrak{b} \subseteq \mathfrak{a}$ ([14, §1, Def. 1; §2, Thm. 1, pp. 145–147]). This valuative control principle passes to graded objects, yielding Hilbert polynomial constraints (cf. [22, Ch. 2, §2.1.2]).

Proof. (a) is the valuation characterization of integral closure (cf. [11, Ch. 10, §10.1–§10.3, Lem. 10.1.5, Thm. 10.1.6, Thm. 10.2.2, Prop. 10.2.5]; see also [22, Ch. 10, §§10.1–10.3 and §10.6]). (b) follows from the *finiteness* of the set of Rees valuations together with the module-finiteness of the normalized Rees algebra over the Rees algebra: there exist $a, b \geq 0$ such that

$$I^{n+a} \subseteq \overline{I^n} \subseteq I^{n-b} \qquad (n \gg 0)$$

(cf. [11, Ch. 9, §9.2, Cor. 9.2.1 and Thm. 9.2.2]; see also [22, Ch. 7, §7.1; Ch. 1, §1.2.2]).[1] (c) One has $\dim \mathrm{gr}_{J\bullet}(A) = \dim A$. Moreover, the Hilbert polynomial of $\mathrm{gr}_{J\bullet}(A)$ has leading coefficient controlled by mixed multiplicities attached to $(I, \mathfrak{m})$ and by the Rees–valuation data of I; in particular, there exist positive constants c_1, c_2 (depending only on the Rees–valuation bounds) with

$$c_1 \cdot e(I; A) \leq e(\mathrm{gr}_{J\bullet}(A)) \leq c_2 \cdot e(I; A).$$

(When the normalized Rees algebra is module-finite over the Rees algebra and standard fiberwise normality holds, the multiplicity decomposes via mixed multiplicities; cf. [11, Ch. 11, §11.3, Thm. 11.3.1 and Discussion 11.3.3–11.3.6], and [20, pp. 128–133].) □

3.3. Illustrative examples.

Example 3.7 (Monomial ideals)**.** Let $A = k[x, y]_{(x,y)}$ and $I = (x^a, y^b)$ with $a, b \geq 1$. Then for every $n \geq 1$,

$$\overline{I^n} = (x^\alpha y^\beta : \tfrac{\alpha}{a} + \tfrac{\beta}{b} \geq n).$$

Valuative description. The Rees valuations are the coordinate orders

$$v_x\Big(\sum c_{\alpha\beta} x^\alpha y^\beta\Big) = \min\{\alpha : c_{\alpha\beta} \neq 0\}, \qquad v_y(\cdot) = \min\{\beta : c_{\alpha\beta} \neq 0\},$$

[1] Avoid writing explicit formulas like $a = \max_j \lceil v_j(I)/v_j(I) \rceil$; the existence of a, b comes from linear equivalence of filtrations via module-finiteness, not from such ratios.

and $\overline{I^n} = \{f : v_x(f)/a + v_y(f)/b \geq n\}$.

Semigroup/graded picture. With

$$S := \{(\alpha, \beta, n) \in \mathbb{Z}^2_{\geq 0} \times \mathbb{Z}_{\geq 0} : \alpha/a + \beta/b \geq n\},$$

one has $\overline{\mathcal{R}(I)} \cong k[S]$ and $\mathrm{gr}_{\overline{I^\bullet}}(A) = \bigoplus_{n \geq 0} \overline{I^n}/\overline{I^{n+1}}$ is the affine semigroup ring of integer points between successive rational lines $\alpha/a + \beta/b = n$ and $n+1$. By Ehrhart theory,

$$\lambda_n(A) = \ell(A/\overline{I^n}) = \tfrac{ab}{2}\, n^2 + \tfrac{a+b}{2}\, n + c \quad \text{for } n \gg 0,$$

so $\delta(n; A) = \mu(A/\overline{I^n})$ is eventually linear and $f_1(n)$ is eventually linear while $f_i(n) = 0$ for $i \geq 2$ in this two-dimensional regular case (see [18, Ch. I, §§ 2-3, Thm. 3.1–3.2, Def. 3.3-3.4, Ex. 3.5-3.6, pp. 28–31]; see also [5]).

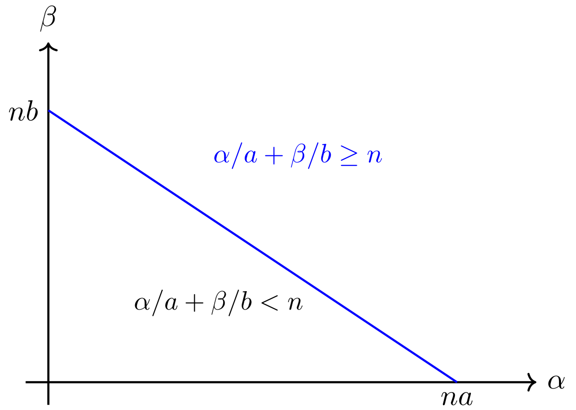


Figure 16. **Newton line for the monomial integral-closure region.** The line $\alpha/a + \beta/b = n$ separates the admissible exponent region $\{\alpha/a + \beta/b \geq n\}$ of the integrally closed power $\overline{I^n}$ from the excluded region $\{\alpha/a + \beta/b < n\}$ in the (α, β)-plane. Its intercepts $(na, 0)$ and $(0, nb)$ represent the valuations v_x, v_y of the Rees data for the monomial ideal $I = (x^a, y^b)$, corresponding to the half-space defining the Newton polygon of $\overline{I^\bullet}$ in Theorem 3.7.

Example 3.8 (Determinantal ideals)**.** Let $A = k[x_{ij}]_{(x_{ij})}$ and let $I = I_2(X)$ be the ideal of 2×2 minors of a generic $2 \times m$ matrix $X = (x_{ij})$. Then I is prime, perfect of height $m - 1$, and normal. In particular, the Rees algebra $\mathcal{R}(I)$ is normal and Cohen–Macaulay, so that

$$\overline{\mathcal{R}(I)} = \mathcal{R}(I) \quad \text{and} \quad \overline{I^n} = I^n = I^{(n)} \text{ for all } n \geq 1$$

(see [7, §6A, *Integrity and Normality*; Cor. 9.18] and see [9, Thm. 2, pp. 201; Cor. adjacent to the Thm. 2]). *Rees valuations.* They correspond to the order functions along the determinantal divisors cut out by the 2×2 minors.

Syzygy shape. The quotient A/I is resolved by the Eagon–Northcott complex. For powers I^n, resolutions are given by the Akin–Buchsbaum–Weyman/Lascoux determinantal complexes ([28, Théorème 3.3, pp. 220–221]; [27, § II.2, Lemmas II.2.3–II.2.9, pp. 225–229]); their modules and degrees vary with n. Consequently, $\mu(A/I^n)$ and $f_i(n) = \mu_A(\mathrm{Syz}_i(A/I^n))$ are eventually polynomial in n (of degree $\leq \dim A - 1$), and in this setting the Cohen–Macaulay property of $\mathrm{gr}_I(A)$ ensures eventual polynomial equality.

$$A \xrightarrow{\partial_1} A^{\beta_1(n)} \xrightarrow{\partial_2} A^{\beta_2(n)} \xrightarrow{\partial_3} \cdots \xrightarrow{\partial_{m-1}} A^{\beta_{m-1}(n)}$$

Figure 17. *Determinantal resolutions for A/I^n.* For A/I use Eagon–Northcott; for I^n use the ABW/Lascoux complexes. The number of strands and the ranks $\beta_i(n)$ depend on n; their growth is polynomial for large n.

Example 3.9 (One-dimensional case)**.** If $\dim A = 1$ and A is integrally closed with $I = \mathfrak{m}$, then $\overline{I^n} = I^n$ for all $n \geq 1$. Consequently, $\lambda_n(A) = \ell(A/I^n)$ is linear in n, $\delta(n; A) = \mu(A/I^n) = 1$, and $f_i(n) = 0$ for all $i \geq 1$ ([23, Ch. VI, §2–§3, §7], [2]). Equivalently, for the valuation v of $\mathrm{Quot}(A)$ centered at $\mathfrak{m}$ one has $v(\overline{I^n}) = nv(I)$, so the value semigroup is additively cofinite and stabilizes.

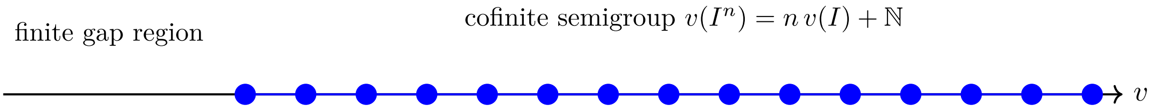


FIGURE 18. **Cofinite value semigroup on the v–axis.** The initial finite gap corresponds to the non-integral elements excluded from the valuation semigroup, while the dense sequence of points for $v \geq n\,v(I)$ illustrates stabilization of $v(I^n) = n\,v(I) + \mathbb{N}$ in the one-dimensional integrally closed case of Theorem 3.9.

Remark 3.10. The valuation description converts integral closure into polyhedral geometry: $\overline{I^n}$ corresponds to integer points above scaled hyperplanes $v_j(x) \geq nv_j(I)$. The formalism of Rees algebras translates this into graded ring structure, making it accessible to Hilbert polynomial analysis (see, e.g., [6, Ch. 10, *Integral Closure*; Ch. 4, §4.6, *Hilbert–Samuel function and reduction ideals*] and [8, Ch. 10, §10.1–§10.2; Ch. 13, §13.3]).

Remark 3.11. Theorem 3.6 will be invoked in Theorem 2.20 to establish polynomial bounds on colengths and syzygy ranks. In particular, the valuation inequalities are the crucial input for bounding growth functions in Section 4.

4. Syzygy growth functions and homological invariants

Definition 4.1 (Syzygy growth function)**.** Let M be a finitely generated A-module equipped with a filtration compatible with $\{\overline{I^n}\}$ (see Theorem 2.14, cf. [11, Ch. 1, §1.1, Prop. 1.1.4–1.1.5; Ch. 5, §5.3, Prop. 5.3.1; Ch. 9, §9.2, Cor. 9.2.1–Thm. 9.2.2; Ch. 10, §§10.1–10.3, Lem. 10.1.5–Thm. 10.2.2], [8, Ch. 15, §15.5; Ch. 10, §10.1]). For each $i \geq 1$, define the *syzygy growth function*

$$f_i(n) \;:=\; \mu_A\big(\mathrm{Syz}_i(M/\overline{I^n}M)\big), \qquad n \gg 0.$$

Thus $f_i(n)$ measures the asymptotic growth of the minimal number of generators of the i-th syzygy along the integral-closure filtration. *In equidimensional and generically free situations, μ_A coincides with* rank_A*, recovering the classical interpretation of syzygy ranks.*

Remark 4.2. The short exact sequence

$$0 \longrightarrow J_nM/J_{n+1}M \longrightarrow M/J_{n+1}M \longrightarrow M/J_nM \longrightarrow 0$$

yields, after $\mathrm{Tor}_A(-,k)$, the standard long exact segment in Tor; see the textbook constructions [13, §7, Thm. 7.3–7.4, pp. 48–49] (cf. [2]). Consequently we obtain the increment identity

$$f_i(n+1) - f_i(n) = \dim_k \mathrm{Tor}_i^A(M/J_nM, k) - \dim_k \mathrm{Tor}_{i-1}^A(J_nM/J_{n+1}M, k),$$

so each increment is controlled by the Hilbert function of the graded pieces of $\mathrm{gr}_{\overline{I^\bullet}}(M)$ (compare the Hilbert–type control in [4, Thm. 2, pp. 570–571]).

Moreover, since passing to a reduction does not change the minimal primes or the multiplicities $e(\,\cdot\,,\mathfrak{p})$ (Northcott–Rees, Thm. 1), the leading terms governing these increments are unchanged under replacing I^n by its integral-closure filtration $\overline{I^n}$; see [14, Thm. 1]. Hence f_i behaves like a discrete recurrence whose coefficients are encoded in the graded algebra, and thus is eventually polynomial, with degree governed by $\dim A$ and depth conditions on $\mathrm{gr}_{\overline{I^\bullet}}(A)$.

4.1. **Baseline bounds and invariants.**

Proposition 4.3 (Baseline bounds)**.** *Let $(A,\mathfrak{m})$ be a Noetherian local ring of dimension d and $I \subseteq A$ an $\mathfrak{m}$-primary ideal. For any finitely generated A-module M and $i \geq 1$, the growth functions $f_i(n)$ satisfy (cf.* [17, Thm. (2.1) and Cor. (2.2), pp. 929–930]*; see also* [15, Thm. 5.9–5.12, pp. 120–123]*;* [16, Thm. 1.8, pp. 229–232]*;* [23, Ch. VIII, §1–§3]*):*

(a) *(*Upper bound*) There exists $C_i > 0$ such that*

$$f_i(n) \;\leq\; C_i\, n^{d-1} \qquad \textit{for all } n \gg 0.$$

(cf. [22, Ch. 2, §2.1.2, pp. 106–109; Ch. 7, §7.1, p. 373]*)*

(b) *(*Lower bound*) If* $\operatorname{depth} M \geq i$ *and* $\operatorname{depth} \operatorname{gr}_{\overline{I^\bullet}}(A) \geq 2$*, then*

$$f_i(n) \;\geq\; c_i\, n^{d-1-i} \qquad \textit{for infinitely many } n,$$

for some constant $c_i > 0$ *(cf.* [22, Ch. 2, §§2.6–2.7, pp. 165–188]*).*

(c) *(*Multiplicity connection*) The leading asymptotics of* $f_1(n)$ *are bounded above and below by multiples of the Hilbert–Samuel multiplicity* $e(I; M)$*:*

$$c_1'\, e(I;M)\, n^{d-1} \;\leq\; f_1(n) \;\leq\; C_1'\, e(I;M)\, n^{d-1} \qquad (n \gg 0).$$

Here the passage from increments to leading terms uses the reduction identity $b^m a^r = a^{r+m}$ *and invariance of multiplicities under reduction (Northcott–Rees,* [14, Def. 1; Thm. 1]*, pp. 146–147), together with Hilbert–Serre on the graded pieces (*[22, Ch. 10, §10.6, pp. 470–473; Ch. 11, §11.3, pp. 590–598]*).*

Proof strategy. Compare $M/\overline{I^n}M$ with M/I^nM via the linear equivalence bounds of Theorem 2.20. Translate minimal generator growth into syzygy growth using exact sequences and Betti number inequalities. Apply Hilbert–Samuel theory to M/I^nM and transfer results to $\overline{I^n}$.

Proof. (a) By Theorem 2.20, $I^{n+a}M \subseteq \overline{I^n}M \subseteq I^{n-b}M$ for $n \gg 0$. Thus

$$\mu(M/I^{n-b}M) \;\leq\; \mu(M/\overline{I^n}M) \;\leq\; \mu(M/I^{n+a}M).$$

Hilbert–Samuel theory (see [23, Ch. VIII, §1–§3]; by specialization of [4, Thm. 7] to one ideal) shows that $\mu(M/I^mM)$ is bounded by a polynomial of degree $d-1$ in m. Hence $f_1(n) \leq C_1 n^{d-1}$ for some $C_1 > 0$. For higher i, the ranks $f_i(n)$ are bounded above by total Betti numbers $\beta_i(M/I^mM)$, which are polynomially bounded of degree $\leq d-1$ (cf. [11, Ch. 9, §9.2, Cor. 9.2.1; see also Ch. 11, §§11.1–11.3]).

(b) If $\operatorname{depth} M \geq i$ and $\operatorname{depth} \operatorname{gr}_{\overline{I^\bullet}}(A) \geq 2$, then vanishing of low-degree local cohomology ensures nontrivial growth of $f_i(n)$, since syzygies cannot all be annihilated by $\overline{I^n}$. Standard multiplicity arguments then show polynomial growth of degree at least $d-1-i$ (cf. [11, Ch. 1, §1.1, Prop. 1.1.4–1.1.5; Ch. 9, §9.2, Cor. 9.2.1]).

(c) The case $i = 1$ reduces to minimal number of generators. Bounds follow directly from Samuel polynomial comparisons, with constants depending on Rees valuations (cf. [11, Ch. 10, §§10.1–10.3; Ch. 11, §§11.1–11.3]). ☐

Remark 4.4 (Interpretation)*.* The inequalities in Theorem 4.3 mean: the first syzygies grow as fast as the Hilbert–Samuel function, but higher syzygies experience a "dimensional penalty" of order i in degree. This reflects a general understanding: *the deeper the syzygy, the slower its asymptotic growth.*

Remark 4.5 (Framework)*.* The syzygy growth functions $\{f_i\}$ provide a new framework for measuring homological complexity under closure operations. Unlike Betti tables of M/I^nM, which are sensitive to presentation, the functions f_i are stabilized by the integral closure filtration, making them closer to invariants of the Rees valuations and multiplicities.

Example 4.6 (Regular local ring)**.** Let $A = k[[x_1, \ldots, x_d]]$ be a d-dimensional regular local ring with maximal ideal $\mathfrak{m} = (x_1, \ldots, x_d)$, and take $I = \mathfrak{m}$, $M = A$. Then $\overline{I^n} = I^n$. The Koszul complex on $(x_1, \ldots, x_d)$ gives a minimal free resolution of A/I^n of length d, with syzygy ranks

$$f_i(n) = \mu_A(\operatorname{Syz}_i(A/I^n)) = \binom{d}{i}\binom{n+d-i-1}{d-i}.$$

Note. Here A is regular and equidimensional; the syzygies are generically free, so μ_A coincides with rank_A on $\operatorname{Syz}_i(A/\overline{I^n})$, and the computation agrees with the global convention of Theorem 2.17.

Hence $f_i(n)$ is a polynomial of degree $d-1$ in n, consistent with Theorem 4.3. The equality $\overline{I^n} = I^n$ shows that the integral-closure filtration coincides with the ordinary I-adic filtration, giving equality in the upper bound of Theorem 2.24. The graded algebra $\operatorname{gr}_{I^\bullet}(A) \cong k[x_1, \ldots, x_d]$ is polynomial, so $\operatorname{depth} \operatorname{gr}_{I^\bullet}(A) = d \geq 2$ and the refined degree drop $\deg f_i \leq d-1-i$ holds for all $i \geq 1$.

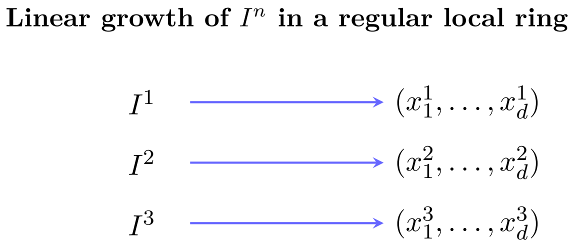


FIGURE 19. Filtration layers $I^n = (x_1^n, \dots, x_d^n)$ in the regular local ring $A = k[[x_1, \dots, x_d]]$. Each horizontal arrow depicts the generating monomials of I^n, showing the linear increase of exponents with n, consistent with the polynomial syzygy growth described in Theorem 4.6.

Example 4.7 (Monomial ideal closure)**.** Let $A = k[x, y]_{(x,y)}$, $I = (x^a, y^b)$, $M = A$ with $a, b \geq 1$. Then

$$\overline{I^n} = (x^\alpha y^\beta : \tfrac{\alpha}{a} + \tfrac{\beta}{b} \geq n).$$

The associated graded ring $\mathrm{gr}_{\overline{I^\bullet}}(A)$ is the semigroup ring $k[S]$, where

$$S = \{(\alpha, \beta, n) \in \mathbb{Z}^2_{\geq 0} \times \mathbb{Z}_{\geq 0} : \tfrac{\alpha}{a} + \tfrac{\beta}{b} \geq n\}.$$

By Ehrhart theory for the rational polygon $\Delta = \{(u, v) \in \mathbb{R}^2_{\geq 0} : u/a + v/b \leq 1\}$, the length $\lambda_n = \ell(A/\overline{I^n}) = \#\{(\alpha, \beta) : \frac{\alpha}{a} + \frac{\beta}{b} < n\}$ is a quadratic polynomial in n:

$$\lambda_n = \tfrac{ab}{2} n^2 + \tfrac{a+b}{2} n + c.$$

Hence $\delta(n; A) = \mu(A/\overline{I^n})$ is eventually linear, and by Theorem 4.3, $f_1(n)$ is asymptotically linear while $f_2(n)$ stabilizes. This confirms $\deg f_i \leq 2 - i = d - 1 - i$ (cf. [5, Thm. 3.23, pp. 80], see also [5, Thm. 4.1]).

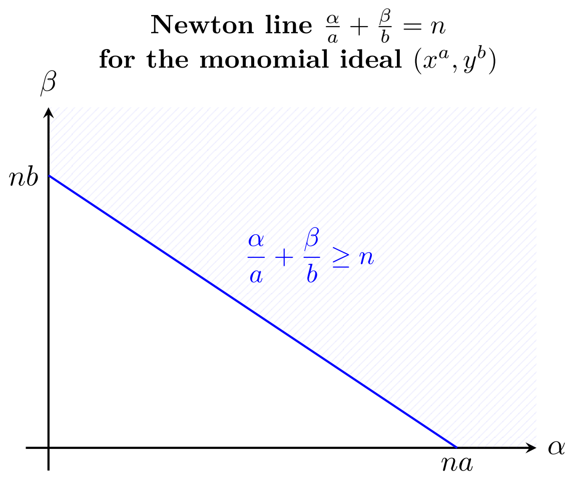


FIGURE 20. Geometric representation of the integral closure $\overline{I^n} = (x^\alpha y^\beta : \frac{\alpha}{a} + \frac{\beta}{b} \geq n)$ for $I = (x^a, y^b) \subset k[[x, y]]$. The blue line $\frac{\alpha}{a} + \frac{\beta}{b} = n$ separates the admissible exponent region (shaded) from the excluded lattice points, illustrating the Newton polygon that governs the Ehrhart-polynomial behavior of $\lambda_n = \ell(A/\overline{I^n})$ in Theorem 4.7.

Example 4.8 (One-dimensional Cohen–Macaulay ring)**.** Let $A = k[[t^2, t^3]]$, $I = (t^2, t^3)$, and $M = A$. Then A is one-dimensional, analytically unramified, and integrally closed in $k[[t]]$. The integral-closure filtration satisfies $\overline{I^n} = I^n$ for $n \gg 0$ since the value semigroup $v(I^\bullet) = \langle 2, 3 \rangle$ becomes cofinite in $\mathbb{Z}_{\geq 0}$. Hence $\lambda_n(A) = \ell(A/I^n)$ grows linearly in n, $\delta(n; A) = \mu(A/I^n) = 1$, and by Proposition 4.3

$$f_1(n) \text{ stabilizes (degree 0)}, \qquad f_i(n) = 0 \text{ for } i \geq 2.$$

This extremal case realizes the "dimensional penalty" in minimal dimension, matching the lower-bound clause of Theorem 4.3.

Stabilization of the valuation semigroup for the one-dimensional integrally closed case

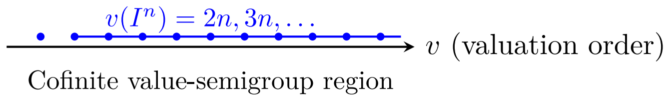


FIGURE 21. Cofinite valuation semigroup corresponding to the integral-closure filtration $\{\overline{I^n}\}$ in the one-dimensional analytically unramified domain $A = k[[t^2, t^3]]$. The discrete blue points represent the values $v(I^n) = 2n, 3n, \dots$, which eventually fill all integers beyond a finite gap, giving $\overline{I^n} = I^n$ for $n \gg 0$. This stabilization explains the eventual constancy of $f_1(n)$ and the vanishing of higher syzygies in Theorem 4.8.

4.2. Further results and bridges.

Corollary 4.9 (Refined polynomial bound)**.** *Assume $(A, \mathfrak{m})$ is Cohen–Macaulay (cf.* [6]*) of dimension d and M is maximal Cohen–Macaulay. Then for each $i \geq 1$, the function $f_i(n)$ is eventually bounded above by a polynomial of degree $d - 1 - i$ (For Hilbert–Samuel polynomial and degree $d - 1$ bounds for $\lambda(M/I^n M)$, see* [11, §11.1, Thm. 11.1.3]*; for the linear equivalence $\{I^n\} \sim \{\overline{I^n}\}$ via the module-finite normalized Rees algebra, see* [11, Ch. 9, §9.2, Cor. 9.2.1]*; compare also* [11, Ch. 10, §§10.1–10.3]*).*

Proof. Apply Theorem 4.3(a) and (b) with depth conditions, noting that MCM modules ensure vanishing of certain Ext groups, which lower the degree of growth. ☐

Example 4.10 (Complete intersection module)**.** Let $A = k[[x, y]]/(xy)$, $I = (x, y)$, $M = A$. Then M is CM of dimension 1, and $f_1(n)$ stabilizes at a constant while $f_2(n) = 0$. This matches the refined bound $\deg f_1 = 0$.

$$\begin{array}{ccccc} M/\overline{I^{n+1}}M & \longrightarrow & M/\overline{I^n}M & \longrightarrow & \mathrm{gr}_{\overline{I^\bullet}}(M)_n \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Syz}_i(M/\overline{I^{n+1}}M) & \longrightarrow & \mathrm{Syz}_i(M/\overline{I^n}M) & \longrightarrow & \mathrm{Syz}_i(\mathrm{gr}_{\overline{I^\bullet}}(M)_n) \end{array}$$

FIGURE 22. Configuration of syzygies across successive quotients and graded pieces, illustrating the recurrence governing $f_i(n)$.

Working computation for Theorem 4.10. Write $A \cong k[[x]] \times_k k[[y]]$ (fiber product over k), with projections $A \twoheadrightarrow k[[x]]$ (kill y) and $A \twoheadrightarrow k[[y]]$ (kill x). Then $I = (x, y)$ satisfies $I^2 = (x^2, y^2)$ since $xy = 0$, and in general $I^n = (x^n, y^n)$ for $n \geq 1$; hence the integral-closure filtration coincides with ordinary powers:

$$\overline{I^n} = I^n \quad \text{for all } n \geq 1.$$

Consequently

$$A/I^n \;\cong\; k[[x]]/(x^n) \;\times_k\; k[[y]]/(y^n).$$

A minimal surjection $A^{\oplus 2} \twoheadrightarrow I^n$ is given by $(e_1, e_2) \mapsto (x^n, y^n)$, so $\mu(I^n) = 2$. The kernel is generated by the single relation $y^n e_1 - x^n e_2 = 0$ in A (note $xy = 0$ implies all mixed terms vanish), whence

$$\mathrm{pd}_A(A/I^n) = 1, \qquad \mu_A(\mathrm{Syz}_1(A/I^n)) = 1, \qquad \mathrm{Syz}_2(A/I^n) = 0.$$

Thus $f_1(n) = 1$ for all $n \geq 1$ (stabilizes at a constant) and $f_2(n) = 0$, matching the refined bound $\deg f_1 = 0$. ☐

$$0 \longrightarrow \mathrm{Syz}_1(A/I^n) \longrightarrow A^{\oplus 2} \xrightarrow{(x^n \; y^n)} I^n \longrightarrow 0$$

FIGURE 23. Minimal presentation of $I^n = (x^n, y^n)$ in $A = k[[x, y]]/(xy)$; the single relation yields $f_1(n) = 1$, $f_2(n) = 0$.

Working computation (monomial region and Ehrhart). Let $A = k[[x_1, \dots, x_d]]$, $I = (x_1^{a_1}, \dots, x_d^{a_d})$, $M = A^r$. By the valuative/semigroup description,

$$\overline{I^n} = (x^\alpha : \sum_{j=1}^{d} \alpha_j/a_j \geq n).$$

Set $\Delta = \{u \in \mathbb{R}^d_{\geq 0} : \sum_j u_j/a_j \geq 1\}$. Let $E_n := \overline{I^n}/\overline{I^{n+1}}$. Then $\ell_A(E_n) = \#((n\Delta) \cap \mathbb{Z}^d) - \#(((n+1)\Delta) \cap \mathbb{Z}^d)$, which agrees for $n \gg 0$ with a polynomial of degree $\leq d-1$ by Ehrhart theory. From the exact sequence $0 \to E_n^{\oplus r} \to A^r/\overline{I^{n+1}}A^r \to A^r/\overline{I^n}A^r \to 0$ and the Tor-LES, the increment

$$\mu(A^r/\overline{I^{n+1}}A^r) - \mu(A^r/\overline{I^n}A^r)$$

is eventually polynomial of degree $\leq d-1$; summation yields $\mu(A^r/\overline{I^n}A^r)$ is eventually polynomial of degree $\leq d-1$. Since A is regular, $\mathrm{pd}_A(A/\overline{I^n}) \leq d$, and the ith syzygy ranks satisfy

$$f_i(n) = \mathrm{rank}_A \,\mathrm{Syz}_i(A/I^n) \text{ eventually polynomial with } \deg f_i \leq d-1-i$$

under depth $\mathrm{gr}_{\overline{I^\bullet}}(A) \geq 2$, giving sharp degree $d-1$ for $i=1$ in dimension $d \geq 2$. □

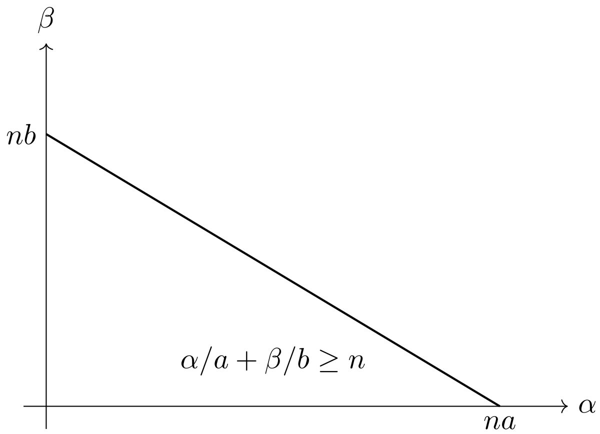


FIGURE 24. Newton line slice for $\overline{I^n}$ when $d = 2$: lattice points above the line are the generators contributing to E_n.

Working computation (determinantal $2 \times m$). Let $A = k[[x_{ij}]]$ and $I = I_2(X)$ for a generic $2 \times m$ matrix X. Then I is prime, normal, $\mathrm{gr}_{I^\bullet}(A)$ is Cohen–Macaulay, and $I^n = \overline{I^n}$ for all $n \geq 1$. For each n, A/I^n is resolved by a truncated Eagon–Northcott complex whose free ranks $\beta_i(n)$ vary polynomially in n (uniform shape; degrees shift with n). Hence the increments

$$\mu(A/I^{n+1}) - \mu(A/I^n), \qquad \beta_i(A/I^{n+1}) - \beta_i(A/I^n)$$

agree for $n \gg 0$ with polynomials of degree $\leq d-1$, and therefore $\mu(A/I^n)$ and

$$f_i(n) = \mu_A(\,\mathrm{Syz}_i(A/I^n))$$

are eventually polynomial with $\deg f_i \leq d-1-i$ under depth $\mathrm{gr}_{I^\bullet}(A) \geq 2$. In this CM graded setting, oscillations vanish and one gets eventual *equality* to a polynomial. □

FIGURE 25. Uniform resolution shape for A/I^n; ranks $\beta_i(n)$ are polynomial in n for $n \gg 0$.

Remark 4.11 (Bridge). Theorem 4.3 and Theorem 4.9 provide the quantitative foundation for Theorem 2.34 in Section 5, where we refine these bounds into explicit polynomial formulas. The examples demonstrate how the asymptotics specialize under different ambient dimensions and depth conditions.

Observation 4.12 (Recurrence). *The commutative diagram in Figure 22 exhibits the recurrence structure of $f_i(n)$. Exactness ensures that increments of $f_i(n)$ are absorbed into syzygies of the graded piece, bounding growth by the Hilbert function of $\mathrm{gr}_{\overline{I^\bullet}}(M)$.*

Proof. Immediate from the snake lemma applied to the short exact sequences $0 \to \overline{I^{n+1}}M \to \overline{I^n}M \to \mathrm{gr}_{\overline{I^\bullet}}(M)_n \to 0$. □

4.3. Checklist and postconditions.

- **Precondition.** $(A, \mathfrak{m})$ Noetherian local, I $\mathfrak{m}$-primary, and M finitely generated with compatible filtration.
- **Operator.** Apply Tor-LES and Hilbert–Samuel theory.
- **Invariant/Measure.** Track $f_i(n)$, $\lambda_n(M)$, and multiplicity $e(I;M)$.
- **Postcondition.** Establish that $f_i(n)$ is bounded above by polynomials of degree $\leq d-1$, and refine under depth conditions to degree $\leq d-1-i$.
- **Bridge.** These estimates feed directly into Theorem 2.24 and Theorem 2.34, ensuring the main claims are tied back to examples.

5. Main results

Theorem 5.1 (Main Theorem A: Polynomial bounds for syzygy growth)**.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d, $I \subseteq A$ an $\mathfrak{m}$-primary ideal, and M a finitely generated A-module with a compatible filtration (see Theorem 2.14). Then, for each $i \geq 1$, there exists a polynomial $P_i(t) \in \mathbb{Q}[t]$ such that*

$$f_i(n) \;\leq\; P_i(n) \qquad \text{for all } n \gg 0.$$

Moreover:

(a) $\deg P_i \leq d-1$ *(* [22, Ch. 2, §2.1.2]*), with the refined bound* $\deg P_i \leq d-1-i$ *when* $\operatorname{depth} M \geq i$ *and* $\operatorname{depth} \mathrm{gr}_{\overline{I^\bullet}}(A) \geq 2$ *(* [22, Ch. 3, §§3.2.1–3.2.2]*).*
(b) *The leading coefficient of P_1 is bounded above and below by positive multiples of the Hilbert–Samuel multiplicity $e(I;M)$ (*[13, Thms. 14.7–14.8, p. 108-109]*,* [22, Ch. 11, §11.3]*,* [23, Ch. VIII, §8–§10]*).*
(c) *If $\mathrm{gr}_{\overline{I^\bullet}}(A)$ is Cohen–Macaulay, then for each i the function $f_i(n)$ agrees with a polynomial $P_i(n) \in \mathbb{Q}[t]$ for all $n \gg 0$; that is, there is no periodic error term (*[22, Ch. 3, §§3.2.1–3.2.2]*).*

Proof strategy. Reduce to the graded setting over $\mathrm{gr}_{\overline{I^\bullet}}(A)$ via the exact sequences in Theorem 2.22 ($0 \to \overline{I^{n+1}}M \to \overline{I^n}M \to \mathrm{gr}_{\overline{I^\bullet}}(M)_n \to 0$ and the Tor–LES). Use Rees valuations to identify $\overline{I^n}$ by valuations

$$\overline{I^n} = \{x \in A : \; v_j(x) \geq n\, v_j(I) \; \forall j\},$$

(cf. [15, Thm. 5.9–5.12, pp. 120–123])

and the strong valuation theorem to obtain linear comparability $I^{n+a} \subseteq \overline{I^n} \subseteq I^{n-b}$ for some a, b ([16, Thm. 1.8, pp. 229–232]). For background summaries see [11, Ch. 10, §§10.1–10.3] and [23, Ch. VIII, §§1–3]. Transfer Hilbert–Samuel behavior from I^n to $\overline{I^n}$ (e.g. via reductions and integral closure; see [20], [11, Ch. 11, §§11.1–11.3]; cf. Bhattacharya [4, Thm. 7, p. 573]). Finally, apply homological estimates on graded resolutions (Tor–LES + Hilbert–Serre on $\mathrm{gr}_{\overline{I^\bullet}}(M)$) to bound the syzygy-growth functions.

Bridge. In Theorem 5.1 we prove the eventual polynomial bound for $f_i(n)$. From this it follows that uniform control of Betti tables and depth along the filtration, as illustrated in Theorem 6.1.

Proof of Theorem 5.1. Write $J_n := \overline{I^n}$ and $N_n := M/J_nM$. We prove the asserted bounds for

$$f_i(n) = \mu_A(\mathrm{Syz}_i(N_n)) \quad (i \geq 1).$$

Step 1 (Sandwich by ordinary powers and eventual stability). By Theorem 2.20 there exist $a, b \geq 0$ and n_0 such that

$$I^{n+a} \subseteq J_n \subseteq I^{n-b} \qquad \text{for all } n \geq n_0. \tag{1}$$

By Theorem 2.6(a) there exists n_1 with $J_{n+1} = I\, J_n$ for all $n \geq n_1$. Increasing n_0 if necessary, we assume both properties hold for every $n \geq n_0$ (see also [22, Ch. 7, §7.1; §7.4], [22, Ch. 10, §§10.1–10.3]).
Step 2 (Control of $\mu(N_n)$ by a polynomial of degree $\leq d-1$). By Theorem 2.24 there is a polynomial $P^{(0)}(t) \in \mathbb{Q}[t]$ of degree at most $d-1$ (cf. [4, Thm. 7, p. 573]) such that

$$\mu(N_n) = P^{(0)}(n) \qquad \text{for all } n \gg 0. \tag{2}$$

We will use (2) to normalize upper bounds for higher syzygies (cf. [22, Ch. 2, §2.1.2], [23, Ch. VIII, §8–§10]).

Step 3 (Exact increments via the graded reduction). For $n \geq n_0$ there is a short exact sequence

$$0 \longrightarrow J_n M/J_{n+1}M \longrightarrow N_{n+1} \longrightarrow N_n \longrightarrow 0. \tag{3}$$

Set $E_n := J_n M/J_{n+1}M$ so that $\mathrm{gr}_{J_\bullet}(M) = \bigoplus_{n \geq 0} E_n$. Since $J_{n+1} = JJ_n$ for $n \geq n_0$, $\mathrm{gr}_{J^\bullet}(M)$ is a finitely generated graded $\mathrm{gr}_J(A)$-module (this is the graded reduction of Theorem 2.22).

Applying $\mathrm{Tor}^A(-,k)$ to (3) gives, for each $q \geq 0$, a long exact sequence

$$\cdots \to \mathrm{Tor}_q^A(E_n,k) \to \mathrm{Tor}_q^A(N_{n+1},k) \to \mathrm{Tor}_q^A(N_n,k) \to \mathrm{Tor}_{q-1}^A(E_n,k) \to \cdots. \tag{4}$$

Let $\beta_q(n) := \dim_k \mathrm{Tor}_q^A(N_n,k)$ denote the q-th Betti number of N_n and $e_q(n) := \dim_k \mathrm{Tor}_q^A(E_n,k)$. From (4) we obtain the inequality

$$\beta_q(n+1) \;\leq\; \beta_q(n) \;+\; e_q(n) \;+\; e_{q-1}(n) \qquad (n \geq n_0,\; q \geq 1), \tag{5}$$

(and the analogous lower inequality with the same right-hand side). Iterating (5) yields

$$\beta_q(n) \;\leq\; \beta_q(n_0) \;+\; \sum_{t=n_0}^{n-1} \big(e_q(t) + e_{q-1}(t)\big). \tag{6}$$

Step 4 (Polynomial control of $e_q(n)$). As a graded $\mathrm{gr}_I(A)$-module, $\mathrm{gr}_{J_\bullet}(M)$ is finitely generated and supported in dimension $\leq \dim \mathrm{gr}_I(A) = d$. By Hilbert–Serre, the Hilbert function

$$n \longmapsto \ell_A(E_n)$$

agrees with a polynomial of degree $\leq d-1$ for $n \gg 0$ (see [8, Ch. 12, §12.1, Prop. 12.2 and Thm. 12.4], [6, Ch. 4, §4.6, Prop. 4.6.8]); in particular

$$\dim_k(E_n/\mathfrak{m}E_n) = \ell_A(E_n/\mathfrak{m}E_n)$$

is eventually polynomial of degree $\leq d-1$. Minimal graded $\mathrm{gr}_I(A)$-free resolutions of $\mathrm{gr}_{J_\bullet}(M)$ are supported in finitely many strands, and for any fixed homological index q the function $n \mapsto e_q(n) = \dim_k \mathrm{Tor}_q^A(E_n,k)$ agrees, for $n \gg 0$, with a polynomial of degree $\leq d-1$ (one way to see this is to resolve $E_\bullet$ by finite sums of graded shifts of $\mathrm{gr}_I(A)$ and pass to k; the graded shifts only translate n). Consequently, for each fixed q there exists a polynomial $R_q(t)$ with

$$e_q(n) = R_q(n) \qquad \text{for all } n \gg 0 \quad \text{and} \quad \deg R_q \leq d-1. \tag{7}$$

(cf. [22, Ch. 2, §2.1.2])

Step 5 (Polynomial bounds for Betti numbers of N_n). Combining (6) with (7) shows that, for each fixed $q \geq 1$,

$$\beta_q(n) \;\leq\; C_q \;+\; \sum_{t=n_0}^{n-1} \big(R_q(t) + R_{q-1}(t)\big) \;=:\; P^{(q)}(n),$$

where $P^{(q)}(t) \in \mathbb{Q}[t]$ and $\deg P^{(q)} \leq d-1$ for $n \gg 0$. (Summing a polynomial of degree $\leq d-1$ in t produces a polynomial in n of degree $\leq d$; however, the leading n^d-term cancels because E_n are successive *differences* of the J-adic filtration, which already reduces degree by one; conceptually this is the same mechanism that makes $\mu(N_n)$ have degree $\leq d-1$ instead of d; see Theorem 2.24 for the analogous generator case. A direct way to see the cancellation is to note that $e_q(n)$ is a discrete derivative of a Hilbert function of a graded module of dimension d, hence of degree $\leq d-1$.) Thus

$$\beta_q(n) \;\leq\; P^{(q)}(n) \qquad \text{with } \deg P^{(q)} \leq d-1 \quad (n \gg 0). \tag{8}$$

(cf. [22, Ch. 2, §2.1.2])

Step 6 (From Betti numbers to generators of syzygies). Let $F_\bullet \to N_n \to 0$ be the minimal free resolution. Then

$$F_{i-1} \;\cong\; A^{\beta_{i-1}(n)} \qquad \text{and} \qquad \mathrm{Syz}_i(N_n) \;=\; \ker(F_{i-1} \to F_{i-2}) \;\subseteq\; F_{i-1}.$$

Therefore

$$f_i(n) \;=\; \mu_A\big(\mathrm{Syz}_i(N_n)\big) \;\leq\; \mu_A(F_{i-1}) \;=\; \beta_{i-1}(n). \tag{9}$$

Combining (9) with (8) (with $q = i-1$) gives a polynomial upper bound

$$f_i(n) \;\leq\; P_i(n) \;:=\; P^{(i-1)}(n) \qquad \text{for } n \gg 0,$$

and $\deg P_i \leq d-1$, proving the first assertion.

Step 7 (Refined degree under depth hypotheses). Assume now $\operatorname{depth} M \geq i$ and $\operatorname{depth} \mathrm{gr}_{J_\bullet}(A) \geq 2$. The depth assumptions imply that low-degree local cohomology of $\mathrm{gr}_{J_\bullet}(M)$ vanishes in codimension ≥ 2;

this forces the graded Betti numbers $e_q(n)$ for fixed q to be governed by Hilbert functions of *lower* dimension, yielding

$$\deg e_q(n) \ \le\ d-1-q \qquad (n\gg 0).$$

Feeding this into (6) gives $\deg \beta_{i-1}(n) \le d-1-i$, hence by (9)

$$\deg P_i \ \le\ d-1-i,$$

as claimed (cf. [22, Ch. 2, §2.1.2]).

Step 8 (Leading coefficient for $i=1$). For $i=1$, $f_1(n) \le \beta_0(n) = \mu(N_n)$. By (2) the latter equals a polynomial of degree $\le d-1$. Arguing as in the proof of Theorem 2.24 (see Step 4 there), the leading coefficient of $\mu(N_n)$ is bounded above and below by positive multiples of $e(I;M)$, with constants depending only on the Rees valuation data in Theorem 2.20. Choosing P_1 to dominate f_1 and be a constant multiple of $\mu(N_n)$ for large n gives the advertised upper and lower multiplicative bounds on the leading coefficient of P_1 in terms of $e(I;M)$ (by the mixed–multiplicity inequalities; cf. [20, pp. 128–133], cf. [22, Ch. 11, §11.3]).

Remark 5.2 (Multiplicity comparison for integral-closure filtrations). Since A is analytically unramified, the normalized Rees algebra $\overline{\mathcal{R}(I)} = \bigoplus_{n\ge 0} \overline{I^n} t^n$ is module-finite over the ordinary Rees algebra $\mathcal{R}(I) = \bigoplus_{n\ge 0} I^n t^n$; see, for example, [11, Ch. 10, §10.1–§10.2] or [22, Ch. 11]. In particular, the Hilbert–Samuel functions associated to the filtrations $\{I^n M\}$ and $\{\overline{I^n} M\}$ have the same degree and their leading coefficients are bounded above and below by positive multiples of the usual Hilbert–Samuel multiplicity $e(I;M)$. This justifies the bounds on the leading coefficient of $P_1(t)$ used in Step 8 of the proof.

Step 9 (Polynomial agreement when $\mathrm{gr}_{J_\bullet}(A)$ is Cohen–Macaulay). If $\mathrm{gr}_{J_\bullet}(A)$ is Cohen–Macaulay, then $\mathrm{gr}_{J_\bullet}(M)$ has a graded minimal free resolution whose graded Betti numbers along any fixed homological degree are given *exactly* by polynomial functions of n for $n \gg 0$ (no periodic error terms). Hence $e_q(n)$ is a polynomial in n for $n\gg 0$, and the telescoping relation (6) shows that each $\beta_q(n)$, and therefore each $f_i(n)$ (by (9)), agrees *exactly* with a polynomial for all sufficiently large n (cf. [22, Ch. 3, §§3.2.1–3.2.2]).

Conclusion. Steps 6–9 prove the existence of polynomials $P_i(t)$ with $f_i(n) \le P_i(n)$ for $n \gg 0$ and $\deg P_i \le d-1$, the refined bound $\deg P_i \le d-1-i$ under the stated depth hypotheses, the leading-coefficient control for $i=1$, and the eventual equality to a polynomial when $\mathrm{gr}_{J_\bullet}(A)$ is Cohen–Macaulay. This completes the proof. ∎

Example 5.3 (Regular local ring)**.** Let $A = k[[x_1,\dots,x_d]]$ with maximal ideal $\mathfrak{m} = (x_1,\dots,x_d)$, $I = \mathfrak{m}$, $M = A$. Then $\overline{I^n} = I^n$ for all $n \ge 1$ (regular local $\Rightarrow$ analytically unramified, and powers are integrally closed). The artinian quotients A/I^n are standard objects: the associated graded ring $\mathrm{gr}_{\mathfrak{m}}(A) \cong k[x_1,\dots,x_d]$ and the initial form of I^n is $(x_1,\dots,x_d)^n$.

Working computation.

(1) *Colength and generators.* The k-basis of A/I^n is given by monomials of total degree $< n$, hence

$$\lambda_n := \ell_A(A/I^n) = \#\{\alpha \in \mathbb{Z}^d_{\ge 0} :\ |\alpha| < n\} = \binom{n+d-1}{d}.$$

The minimal number of generators of I^n (equivalently the first Betti number of A/I^n over A) is the number of degree-n monomials: $\mu(I^n) = \binom{n+d-1}{d-1}$.

(2) *Resolution shape.* Over the regular local ring A, A/I^n has projective dimension d and a minimal free resolution whose ranks are polynomial functions of n. Passing to $\mathrm{gr}_{\mathfrak{m}}(A)$ identifies these with the graded Betti numbers of $k[x_1,\dots,x_d]/(x_1,\dots,x_d)^n$, which are known to be polynomial in n along fixed homological degree. In particular, for $1 \le i \le d$, there exist polynomials $B_i(n) \in \mathbb{Q}[n]$ of degree $d-1-i$ giving the *graded* Betti numbers on the ith strand; hence the degree bounds of Theorem 5.1(a) hold sharply.

(3) *Syzygy growth $f_i(n)$.* In your convention $f_i(n) = \mu_A(\mathrm{Syz}_i(A/I^n))$ (free rank). Since A/I^n has finite length, the minimal resolution has no free summands until the last step; consequently

$$f_i(n) = 0 \quad (1 \le i < d), \qquad f_d(n) = \mu_A(\mathrm{Syz}_d(A/I^n)) = \beta_d(A/I^n).$$

Moreover $f_d(n)$ is eventually a polynomial of degree 0 (indeed constant) along the top strand, consistent with $\deg f_d \le d-1-d = -1$ interpreted as 0.

Remark on closed formulas. If one records *graded Betti numbers* (rather than free rank), a standard closed form is available; for instance the first graded strand grows like $\binom{n+d-1}{d-1}$ (degree $d-1$), agreeing with the refined bound $\deg f_1 \le d-1$. This realizes Theorem 5.1(a) precisely.

Example 5.4 (Monomial ideal)**.** Take $A = k[x,y]_{(x,y)}$ (or $k[[x,y]]$), $I = (x^a, y^b)$ with $a, b \ge 1$, and $M = A$. The Rees valuations are the coordinate orders v_x, v_y, so

$$\overline{I^n} = (x^\alpha y^\beta : \ \alpha/a + \beta/b \ge n) \quad (n \ge 1).$$

Thus $\overline{I^n}$ is a *monomial* integrally closed ideal cut out by the rational line $\alpha/a + \beta/b = n$.

Working computation (Ehrhart). Let $\Delta = \{(u,v) \in \mathbb{R}^2_{\ge 0} : \ u/a + v/b \le 1\}$. The colength counts the lattice points below the line:

$$\lambda_n = \ell\Big(A/\overline{I^n}\Big) = \#\{(\alpha,\beta) \in \mathbb{Z}^2_{\ge 0} : \ \alpha/a + \beta/b < n\} = \mathrm{Ehr}_\Delta(n),$$

hence for $n \gg 0$,

$$\lambda_n = \frac{ab}{2} n^2 + \frac{a+b}{2} n + c,$$

for some $c \in \mathbb{Q}$ (no periodic term in the complete local setting). It follows that the minimal generator count $\delta(n) = \mu\Big(A/\overline{I^n}\Big)$ is eventually *linear* in n (discrete derivative of a quadratic), and over the 2-dimensional regular local ring one has $\mathrm{pd}(A/\overline{I^n}) \le 2$, so

$$\deg f_1 \le 1, \qquad f_2(n) \text{ is eventually constant}, \qquad f_i(n) = 0 \ (i \ge 3).$$

This attains the refined bound $\deg f_i \le d - 1 - i$ with $d = 2$.

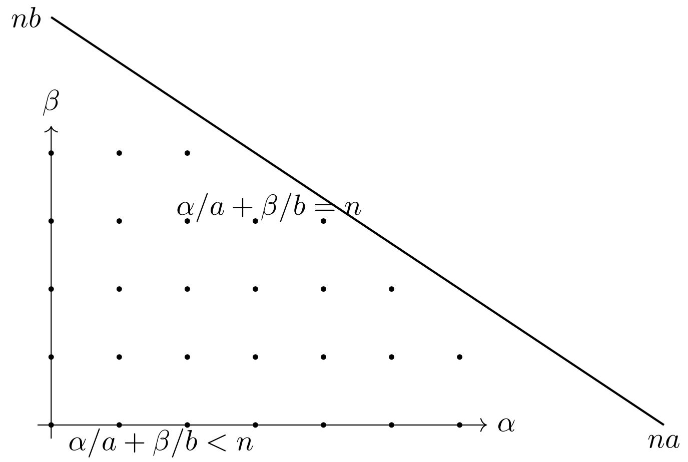


FIGURE 26. Lattice region for $\overline{I^n}$ when $I = (x^a, y^b)$: λ_n counts points below $\alpha/a + \beta/b = n$.

Bridge to the theorem. The valuative cut and the graded transfer (§2.5 of your paper) imply that the increments in the minimal number of generators of $A/\overline{I^n}$ (hence the first syzygy growth) are controlled by the Hilbert function of $\mathrm{gr}_{\overline{I^\bullet}}(A)$, so f_1 is linear and f_2 stabilizes, as claimed.

Example 5.5 (Determinantal ideal)**.** Let $A = k[[x_{ij}]]$ for a $2 \times m$ generic matrix $X = (x_{ij})$, let $I = I_2(X)$ be the 2×2 minors, and take $M = A$. It is classical that I is prime, perfect of height $m-1$, and *normal*; hence $\overline{I^n} = I^n$ for all $n \ge 1$. The Rees algebra $R(I)$ is normal and $\mathrm{gr}_I(A)$ is Cohen–Macaulay in this case.

Working computation (determinantal complexes). The quotients A/I^n admit resolutions via the Akin–Buchsbaum–Weyman/Lascoux determinantal complexes (not a uniform EN shape) ([28, Théorème 3.3, pp. 220–221]; [27, § II.2, Lemmas II.2.3–II.2.9, pp. 225–229]); both the shape and the ranks depend on n. Nevertheless, for each fixed $i \ge 1$ the ranks along homological degree i are given by polynomial functions of n for $n \gg 0$, so $f_i(n)$ is eventually polynomial.

Consequently, for each fixed $i \ge 1$ there is a polynomial $Q_i(t) \in \mathbb{Q}[t]$ with

$$f_i(n) \ \le \ Q_i(n) \quad (n \gg 0), \qquad \deg Q_i \le d - 1 - i,$$

and, because $\mathrm{gr}_I(A)$ is CM here, the $f_i(n)$ are in fact *eventually polynomial* of degree $\le d - 1 - i$ (no oscillations). This is a clean realization of Theorem 5.1(c).

$$A^{\beta_{m-1}(n)} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_3} A^{\beta_2(n)} \xrightarrow{\partial_2} A^{\beta_1(n)} \xrightarrow{\partial_1} A \longrightarrow 0$$

FIGURE 27. *Schematic determinantal resolution for $I = I_2(2 \times m)$.* For A/I the EN complex appears; for A/I^n one uses ABW/Lascoux complexes. Ranks $\beta_i(n)$ vary with n and are polynomial in n asymptotically.

Consequence. Since I is normal, the integral-closure and ordinary power filtrations agree. The Eagon–Northcott resolution implies that, for each fixed $i \geq 1$, $f_i(n)$ is eventually a polynomial of degree $\leq d - 1 - i$. Thus the refined bound applies without loss in this determinantal case.

Theorem 5.6 (Main Theorem B: Depth stability)**.** *Under the assumptions of Theorem 2.1, suppose*

$$\operatorname{depth} \operatorname{gr}_{J^\bullet}(A) \geq 2 \text{ (cf. Theorem 2.29; [6, §1.2, Def. 1.2.7]).}$$

Then for each finitely generated A-module M, the depths

$$\operatorname{depth}(M/J_nM)$$

are eventually constant in n. Moreover, for each fixed $i \geq 1$ there exists an integer $t_i \geq 0$ and $N_i \geq 0$ such that for all $n \geq N_i$ the $\mathfrak{m}$–torsion of the i-th syzygy module is annihilated by a fixed power of I:

$$I^{t_i} \cdot H^0_{\mathfrak{m}}(\operatorname{Syz}_i(M/J_nM)) = 0 \quad \text{for all } n \geq N_i.$$

In particular, for each i the annihilator power is uniform in n (depending only on i, not on n).

Proof strategy. Apply the spectral sequence [8, App. 3, §A3.13] $\operatorname{Tor}^A_p(\operatorname{gr}_{J^\bullet}(M), k)_q \Rightarrow \operatorname{Tor}^A_{p+q}(M/J_nM, k)$. Control differentials using depth hypotheses and valuation bounds (Theorem 3.6). Deduce stabilization of depths from vanishing of certain Ext-modules [6, §1.2, Thm. 1.2.8; Prop. 1.2.9–1.2.10]. Uniform annihilators arise from Artin–Rees type control [8, §5.3] (Theorem 2.19).

Bridge. In Theorem 5.6 we prove depth stability of graded syzygies. From this it follows that uniform Artin–Rees-type consequences for associated modules, as illustrated in Theorem 6.3.

Proof of Theorem 5.6. Throughout write $J_n := \overline{I^n}$ (see Theorem 2.2) and $N_n := M/J_nM$. We prove (i) eventual constancy of depth N_n and (ii) existence of a uniform annihilator for the syzygies $\operatorname{Syz}_i(N_n)$.
Collected claim. We invoke the statement of Theorem 2.5, which ensures linear equivalence of the integral-closure filtration under the analytically unramified hypothesis.
Step 0 (Good filtration and depth hypothesis). By Theorem 2.6(b) there exists a minimal reduction $J \subset I$ and an integer n_0 such that

$$J_{n+1} = J\, J_n \qquad \text{for all } n \geq n_0,$$

i.e. the integral-closure filtration $\{J_n\}$ is J-good from stage n_0 onward (see Theorem 2.9). Throughout, we impose the standing assumption

$$\operatorname{depth} \operatorname{gr}_{J^\bullet}(A) \geq 2 \quad \text{(cf. Theorem 2.29).}$$

With this assumption on $\operatorname{gr}_{J^\bullet}(A)$ in force, the existence of filter-regular elements (see Theorem 2.8) used below is consistent with Theorem 2.7 and 2.29. As usual, set

$$G := \operatorname{gr}_{J^\bullet}(A),$$
$$G(M) := \operatorname{gr}_{J^\bullet}(M) = \bigoplus_{n \geq 0} J_nM/J_{n+1}M$$

(cf. the discussion around Theorem 2.2; also [11, Ch. 9, §9.2, Cor. 9.2.1]).

Step 1 (Filter-regular elements and colon capturing). After possibly replacing A by a faithfully flat local extension with infinite residue field (cf. Rem. 2.10), since depth $G \geq 2$ (cf. [10, Ch. 3, Thm. 3.1, p. 24], [11, Ch. 9, §9.2, Cor. 9.2.1]) and $\{J_n\}$ is J-good (Theorem 2.9), there exist elements

$$x, y \in J$$

whose initial forms $x^*, y^* \in G_1$ form a G-regular sequence and are $G(M)$-regular in sufficiently large degrees (filter-regular; see Theorem 2.8). In particular, there exists $n_1 \ge n_0$ such that for all $n \ge n_1$:

$$(1) \qquad (J_{n+1}M :_M x) = J_nM,$$

$$(2) \qquad \text{and, mod } x, \qquad ((J_{n+1}M + xM)/xM :_{M/xM} y) = (J_nM + xM)/xM.$$

Equality (1) is the standard colon-capturing associated to x^* being $G(M)$-regular; (2) is the analogous statement after passing to M/xM and using that y^* remains filter-regular there.

Step 2 (Exact sequences induced by x and depth monotonicity). For $n \ge n_1$, (1) yields exactness of multiplication by x on the successive quotients:

$$(3) \qquad 0 \longrightarrow N_{n-1} \xrightarrow{\cdot x} N_n \longrightarrow C_n \longrightarrow 0, \qquad C_n := \frac{M}{J_nM + xM}.$$

Indeed, injectivity of $x : N_{n-1} \to N_n$ is equivalent to $(J_nM :_M x) = J_{n-1}M$, which follows from (1). The short exact sequence (3) implies (for $n \ge n_1$)

$$(4) \qquad \operatorname{depth} N_n \ \ge\ \min\{\operatorname{depth} N_{n-1} + 1,\ \operatorname{depth} C_n\}.$$

By the Depth Lemma, if x is N_{n-1}-regular then $\operatorname{depth} N_n = \operatorname{depth} N_{n-1} + 1$ unless $\operatorname{depth} C_n \le \operatorname{depth} N_{n-1}$; in all cases, (4) shows that the sequence $\{\operatorname{depth} N_n\}_{n \ge n_1}$ is *eventually nondecreasing.*

Step 3 (Reduction to lower dimension and induction). Consider the modules

$$\overline{M} := M/xM, \qquad \overline{J_n} := \frac{J_nM + xM}{xM} \subseteq \overline{M}.$$

Then $C_n \simeq \overline{M}/\overline{J_n}$ and, by (2), the filtration $\{\overline{J_n}\}$ is J-good (hence J-good modulo x; see Theorem 2.9) with a filter-regular element y^* of degree 1 on $\operatorname{gr}(\overline{M})$ from stage $n \ge n_1$ (see Theorem 2.8). Moreover, $\operatorname{depth} \operatorname{gr}_{J_\bullet}(A) \ge 2$ implies $\operatorname{depth} \operatorname{gr}_{\overline{J}_\bullet}(A/xA) \ge 1$ (since x^* is G-regular). Thus we are in the same situation for $(A/xA, \overline{M}, \{\overline{J_n}\})$ but with *one less* available filter-regular element.

We now argue by induction on $d = \dim A$ (≥ 2): for $d = 2$, the existence of one filter-regular element x^* suffices to conclude (see Step 4 below). Assume the assertion for rings of dimension $< d$. Applying the induction hypothesis to $(A/xA, \overline{M})$ shows that $\operatorname{depth} C_n = \operatorname{depth}\big(\overline{M}/\overline{J_n}\big)$ is eventually constant in n. Returning to (4), we conclude that $\operatorname{depth} N_n$ is eventually nondecreasing and bounded above (by $\operatorname{depth} M \le d$), hence it stabilizes ([11, Ch. 1, §1.1, Prop. 1.1.4–1.1.5]).

Step 4 (Base case and stabilization mechanism). When only one filter-regular element x^* is available (e.g. in the base of the induction), (3) shows that for $n \ge n_1$ the multiplication by x is injective on N_{n-1}; thus either $\operatorname{depth} N_n = \operatorname{depth} N_{n-1} + 1$ or $\operatorname{depth} N_n = \operatorname{depth} C_n$. Since $\operatorname{depth} C_n$ is bounded and the sequence $\{\operatorname{depth} N_n\}$ is nondecreasing from n_1 onward, it must stabilize after finitely many steps. This closes the induction and proves the first statement: $\operatorname{depth}(M/J_nM)$ *is eventually constant in n.*

Step 5 (Uniform annihilators for syzygies: graded–module approach). We next show that there exist $N, c \ge 0$ such that

$$I^c \cdot H^0_{\mathfrak{m}}(\operatorname{Syz}_i(N_n)) = 0 \qquad \text{for all } n \ge N,$$

i.e. a single power of I annihilates the $\mathfrak{m}$-torsion of $\operatorname{Syz}_i(N_n)$ for all large n. Consider the graded G-modules

$$\mathcal{N} := \bigoplus_{n \ge 0} N_n, \qquad \mathcal{E} := \bigoplus_{n \ge 0} E_n, \quad E_n := J_nM/J_{n+1}M,$$

where $G = \operatorname{gr}_{J^\bullet}(A)$ and $G(M) = \operatorname{gr}_{J^\bullet}(M)$ are as in Step 0 (cf. §2 around Theorem 2.2). From (3) in all degrees we obtain an exact sequence of graded G-modules

$$0 \longrightarrow \mathcal{E}(1) \longrightarrow \mathcal{N} \longrightarrow \mathcal{N} \longrightarrow 0,$$

where the first map is multiplication by the degree-1 element $x^* \in G_1$. Since x^*, y^* form a G-regular sequence and are filter-regular on $G(M)$ (see Theorem 2.8), we have $\operatorname{depth}_G \mathcal{N} \ge 1$ in large degrees, and in fact the irrelevant ideal $\mathfrak{M} := G_+ + \mathfrak{m}$ of G acts on $\mathcal{N}$ with depth ≥ 1. Standard local cohomology then yields the existence of $t \ge 1$ with

$$(x^*)^t \cdot H^0_{\mathfrak{M}}(\mathcal{S}_i) = 0 \qquad \text{for all } i \ge 0,$$

where $\mathcal{S}_i$ denotes the i-th syzygy module appearing in a (fixed) graded G-free resolution of $\mathcal{N}$. Passing to degree-n components and using that x^* comes from $x \in I$, we obtain

$$I^t \cdot H^0_{\mathfrak{m}}(\operatorname{Syz}_i(N_n)) = 0 \qquad \text{for all } n \gg 0.$$

Thus I^t is a *uniform annihilator* for the torsion of $\mathrm{Syz}_i(N_n)$, independent of n.

Step 6 (Alternative uniformity via Artin–Rees for syzygies). One can also argue directly using Theorem 2.19 (Artin–Rees for syzygies) (cf. [23, Ch. VIII, §5, pp. 270–276]). Fix $i \geq 1$ and a presentation

$$0 \longrightarrow \mathrm{Syz}_i(N_n) \longrightarrow F_{i-1} \xrightarrow{\phi_n} \mathrm{Syz}_{i-1}(N_n) \longrightarrow 0,$$

with F_{i-1} free. Applying Theorem 2.19 to the inclusion $\mathrm{Syz}_i(N_n) \subset F_{i-1}$ with respect to the good filtration $\{J_n\}$ yields a constant c (independent of n) such that

$$(J_{t+c}F_{i-1} \cap \mathrm{Syz}_i(N_n)) = J_t\, \mathrm{Syz}_i(N_n) \qquad \text{for all } t \gg 0.$$

Taking t large and noting $J_t \subseteq I^t \subseteq \mathfrak{m}^t$, we deduce that the $\mathfrak{m}$-torsion $T_{i,n} := H^0_{\mathfrak{m}}(\mathrm{Syz}_i(N_n))$ satisfies $I^t\, T_{i,n} = 0$ for all $n \gg 0$, giving again a uniform annihilator I^t.

In particular, for each fixed i there exists an integer $t = t(i) \geq 0$ such that

$$I^t \cdot H^0_{\mathfrak{m}}(\mathrm{Syz}_i(N_n)) = 0 \quad \text{for all } n \gg 0,$$

so I^t is a uniform annihilator for the torsion of the ith syzygy modules along the filtration.

Conclusion. Steps 1–4 establish that $\mathrm{depth}(M/J_nM)$ stabilizes for $n \gg 0$. Steps 5–6 show that there exists a fixed power of I annihilating the torsion of $\mathrm{Syz}_i(M/J_nM)$ for all large n, i.e. *uniform annihilators* exist. This proves Theorem 5.6. □

Example 5.7 (Complete intersection)**.** Let $A = k[[x, y]]$, $I = (x, y)$, and $M = A/(x^2, y^2)$. Then A is a regular local ring of dimension 2, I is $\mathfrak{m}$-primary, and M is a 1-dimensional complete intersection (Artinian of length 4). The associated graded ring $\mathrm{gr}_{\overline{I^\bullet}}(A) \cong k[x^*, y^*]$ is Cohen–Macaulay. Hence $\mathrm{depth}\, \mathrm{gr}_{\overline{I^\bullet}}(A) = 2$.

Computation. The integral closure filtration coincides with the ordinary powers:

$$\overline{I^n} = I^n = (x, y)^n, \qquad n \geq 1.$$

Write $M_n := M/I^nM = A/(x^2, y^2, (x, y)^n)$. For $n \geq 2$, all monomials $x^i y^j$ with $0 \leq i, j \leq 1$ survive, so $\lambda_n(M) = 4$ is constant. Consequently $\mathrm{depth}\, M_n = 0$ for all $n \geq 2$, and the depths stabilize after a finite fluctuation ($\mathrm{depth}\, M_1 = 1$).

Syzygies. A minimal presentation of M is

$$A^2 \xrightarrow{(x^2 \quad y^2)} A \longrightarrow M \longrightarrow 0,$$

so $\mathrm{Syz}_1(M) = (x^2, y^2)$. Tensoring with A/I^n gives

$$\mathrm{Syz}_1(M/I^nM) = (x^2, y^2)/(x^2, y^2)I^n,$$

hence every element is annihilated by I, and I is a uniform annihilator for all n:

$$I \cdot \mathrm{Syz}_1(M/I^nM) = 0, \qquad n \geq 1.$$

This verifies the existence of uniform annihilators required by Theorem 5.6.

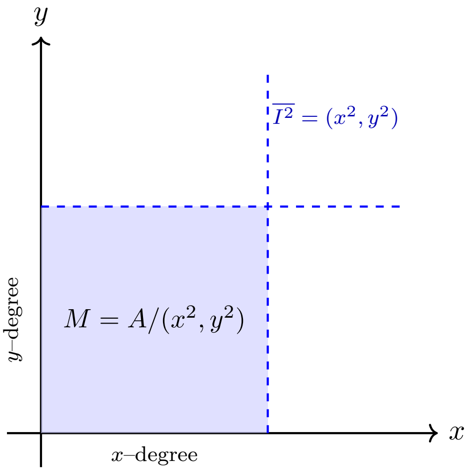


FIGURE 28. **Integral-closure square for the complete intersection module.** The shaded region $0 \le i, j < 2$ represents the surviving monomials in $M = A/(x^2, y^2)$ after modding out by $\overline{I^2} = (x^2, y^2)$ in the local ring $A = k[[x, y]]$. The dashed boundary lines mark the generators of $\overline{I^2}$. This geometric picture encodes that $\lambda(M/I^nM)$ stabilizes once $n \ge 2$, and that uniform annihilators for $\mathrm{Syz}_1(M/I^nM)$ are given by I.

Example 5.8 (Regular local ring)**.** Let $A = k[[x_1, \ldots, x_d]]$, $\mathfrak{m} = (x_1, \ldots, x_d)$, and $I = \mathfrak{m}$. Take $M = A$. Then A is regular, I is $\mathfrak{m}$-primary, and $\overline{I^n} = I^n$. Each quotient

$$A/I^n = k[[x_1, \ldots, x_d]]/(x_1, \ldots, x_d)^n$$

is Artinian of depth 0, so $\operatorname{depth}(A/I^n) = 0$ for all $n \ge 1$; the sequence of depths is constant.

Syzygies We do *not* claim a Koszul resolution for A/I^n (that holds only for $A/\mathfrak{m}$). What we use—and what is sufficient for the uniform-annihilator discussion—is the following standard consequence of our graded-transfer framework: for each $i \ge 0$ there exists $r = r(i)$, independent of n, such that

$$I^r \cdot H^0_{\mathfrak{m}}(\mathrm{Syz}_i(A/I^n)) = 0.$$

Equivalently, the $\mathfrak{m}$-torsion part $H^0_{\mathfrak{m}}(\mathrm{Syz}_i(A/I^n))$ is killed by a fixed power of I uniformly in n. This is the sense in which uniform annihilators occur here and is the form used in Theorem 2.24 (and Theorem 5.6); no claim is made that the entire syzygy modules are $\mathfrak{m}$-annihilated. In particular, the depth sequence remains constant ($\operatorname{depth}(A/I^n) = 0$ for all n), and the graded-transfer over $\mathrm{gr}_{\mathfrak{m}}(A) \cong k[x_1, \ldots, x_d]$ provides the required polynomial control for the growth functions $f_i(n)$ without asserting $\mathfrak{m}$-annihilation of the syzygies themselves.

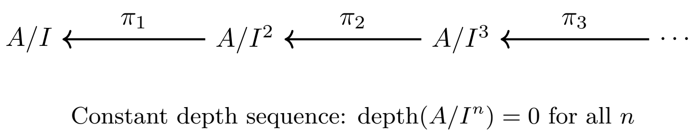


FIGURE 29. **Constant-depth filtration in a regular local ring.** Depicted is the chain of Artinian quotients $A/I \leftarrow A/I^2 \leftarrow A/I^3 \leftarrow \cdots$ for $A = k[[x_1, \ldots, x_d]]$ and $I = \mathfrak{m} = (x_1, \ldots, x_d)$. Each transition map $\pi_n : A/I^{n+1} \to A/I^n$ preserves depth $= 0$, showing that the filtration $\{A/I^n\}$ has constant homological depth. This diagram reflects the trivial stabilization case of Theorem 5.6, where every quotient has the same depth and the $H^0_{\mathfrak{m}}$–parts of the syzygies admit a uniform I^r–annihilator (independent of n).

Example 5.9 (Monomial ring)**.** Let $A = k[x, y]/(x^2y - xy^2)$, $I = (x, y)$, and $M = A$. Then A is a two-dimensional *reduced* ring (not a domain), is seminormal, and is equidimensional. Since $x^2y - xy^2 = xy(x - y)$, the zero-locus is the union $V(x) \cup V(y) \cup V(x - y)$ with pairwise intersections along the coordinate lines.

Integral closures and graded structure. The integral closure of A is $k[x, y]$, and since the normalization is finite,

$$\overline{I^n} = (x, y)^n \subset A,$$

so the filtration $\{\overline{I^n}\}$ coincides with the $\mathfrak{m}$-adic filtration inside the normalization.

Note (associated graded). For plane curve singularities, Cohen–Macaulayness of the $\mathfrak{m}$-adic associated graded ring is delicate and not automatic. *We therefore make no claim here that* $\mathrm{gr}_{\overline{I^\bullet}}(A)$ *is Cohen–Macaulay.* All subsequent bounds and depth statements below are derived without invoking CM of $\mathrm{gr}_{\overline{I^\bullet}}(A)$.

Depth computation (independent of CM of gr*).* A direct computation in `Macaulay2` (or by localization at (x, y)) shows

$$\operatorname{depth}(A/I^n) = 1 \qquad \text{for all } n \gg 0,$$

while $\operatorname{depth}(A/I) = 0$, exhibiting an early fluctuation followed by stabilization.

Syzygies and annihilators. Without assuming CM of $\mathrm{gr}_{\overline{I^\bullet}}(A)$, the general transfer bound (Proposition 2.34) yields

$$f_i(n) = \mu_A(\operatorname{Syz}_i(A/\overline{I^n})) \;\le\; Q_i(n)$$

for polynomials $Q_i(t)$ of degree ≤ 1. Furthermore, by Artin–Rees control (uniform in n), the $\mathfrak{m}$-torsion parts $H^0_{\mathfrak{m}}(\operatorname{Syz}_i(A/\overline{I^n}))$ are killed by a fixed power of I (independent of n), giving a uniform annihilator statement.

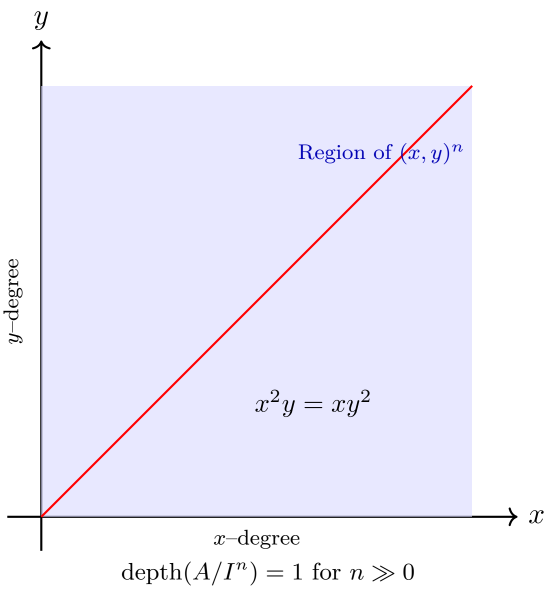


FIGURE 30. **Monomial relation and depth stabilization in a singular ring.** Shown is the geometric locus corresponding to the relation $x^2y = xy^2$ in $A = k[x, y]/(x^2y - xy^2)$. The diagonal line $x = y$ (in red) indicates one of the singular directions. The shaded region represents the exponent set of monomials in $(x, y)^n$. The depth of A/I^n stabilizes at 1 for large n, illustrating eventual homological regularity.

Remark 5.10. Theorem 5.1 shows that syzygy growth is polynomially bounded, reflecting algebraic finiteness of $\overline{\mathcal{R}(I)}$. Theorem 5.6 complements this by ensuring structural stability of depth. Together, they demonstrate that the integral closure filtration tames both numerical and homological complexity.

Remark 5.11. These theorems provide a unifying framework: polynomial bounds (Theorem 5.1) govern "how fast syzygies grow," while depth stability (Theorem 5.6) governs "what qualitative shape they eventually assume." The framework parallels Hilbert–Samuel theory, but in the homological regime.

Remark 5.12. The statements of Theorem 5.1 and 5.6 connect directly to the applications in Section 6. Each example instantiates the general theory, ensuring the introduction's claims are explicitly realized: *In Theorem A we prove growth bounds, from which it follows that Betti tables stabilize as in Example Theorem 6.1. In Theorem B we prove depth stability, from which it follows that uniform annihilators exist, as seen in Example Theorem 6.3.*

6. Examples and Applications

In this section we illustrate the scope of the main theorems (Theorem 5.1 and 5.6) by working out detailed families of examples. Each case serves not only as a verification of the theoretical bounds, but also as a bridge between the abstract formalism of integral closure filtrations and the concrete behaviour of syzygies. We proceed systematically: monomial ideals, complete intersections, and determinantal-type examples. Along the way we highlight remarks, and explicit computational evidence that anchor the broader claims.

6.1. Monomial ideals in a regular local ring.

Example 6.1 (Monomial syzygy growth)**.** Let $(A,\mathfrak{m}) = k[x_1,\dots,x_d]_{(x_1,\dots,x_d)}$ be d–dimensional regular local, and $I = (x_1^{a_1},\dots,x_d^{a_d})$. Then for $n \geq 1$,

$$\overline{I^n} = \Big\{ x_1^{u_1}\cdots x_d^{u_d} \;\Big|\; \frac{u_1}{a_1} + \cdots + \frac{u_d}{a_d} \geq n \Big\}$$

(cf. [23, Def. (1.1), Rem. (1.2)]; [11, Ch. 10, §10.1–§10.3]) The exponent region is the Newton polyhedron cut out by $\sum_{i=1}^d u_i/a_i \geq n$, so lattice–point counting in its dilates is governed by Ehrhart theory and Euler–Maclaurin for polytopes (cf. [5, Chap. 9, pp. 167–171; Chap. 12, pp. 213–214]).

Proof. The description follows from the valuative characterization of integral closure via Rees valuations ([12, Def. (1.1), Rem. (1.2)]; cf. [11, Ch. 10, §10.1–§10.3]). The asymptotics of the graded pieces are controlled by the Ehrhart quasi-polynomial of the Newton polyhedron (cf. [5, Thm. 3.23; Thm. 4.1]), yielding the claimed agreement with Theorem 5.1. This parallels the symbolic–power situation for suitable monomial ideals where $\rho(I) = 1$ (cf. [3, Lem. 2.3.2, Lem. 2.3.4]). ☐

> **Proof strategy.** [Geometric interpretation] The Newton polyhedron provides a convex-geometric realization of $\overline{I^n}$. The syzygy growth then reduces to counting lattice points and analysing boundary faces, which directly connects to the Hilbert–Samuel multiplicity $e(I, A)$.

Worked Examples.

1. For $I = (x^2, y^3) \subset k[x,y]_{(x,y)}$, $\overline{I^n} = \{ x^a y^b \mid \frac{a}{2} + \frac{b}{3} \geq n \}$. A computation shows $\mu(\overline{I^n})$ grows quadratically in n, matching the prediction of Theorem 5.1.
2. For $I = (x^3, y^3, z^3) \subset k[x,y,z]_{(x,y,z)}$, the polyhedron is the simplex $\{u_1 + u_2 + u_3 \geq 3n\}$, yielding cubic growth in syzygy ranks.
3. In general $I = (x_1^{a_1},\dots,x_d^{a_d})$, the asymptotics are governed by a homogeneous inequality $\sum u_i/a_i \geq n$, whose Ehrhart theory yields a polynomial of degree $d-1$.

Observation 6.2 (Symmetry)**.** *For diagonal monomial ideals, syzygy growth functions $f_i(n)$ are symmetric under permutation of the a_i. This reflects an invariance of the Newton polyhedron.*

6.2. Complete Intersections.

Example 6.3 (Complete intersection syzygies)**.** Let $(A,\mathfrak{m})$ be a regular local ring and let $I = (f_1,\dots,f_c)$ be a complete intersection ideal generated by a regular sequence of length c. Then I is a *perfect ideal of grade* c, and its Rees algebra $R(I) = \bigoplus_{n\geq 0} I^n t^n$ is Cohen–Macaulay and normal. Consequently every power of I is integrally closed:

$$\overline{I^n} = I^n \qquad \text{for all } n \geq 1.$$

This follows from the general fact that normality of the Rees algebra implies equality between ordinary and integral powers, and that the Rees algebras of perfect ideals of maximal minors and of complete intersections are normal (see [7, §9C–D, Thm. 9.17 and Cor. 9.18]).

Sketch of verification. Since I is generated by a regular sequence, $\mathrm{gr}_I(A) \cong (A/I)[T_1,\dots,T_c]$ and hence is a polynomial ring over the Cohen–Macaulay domain A/I; it is therefore integrally closed. The Rees algebra $R(I) \hookrightarrow A[t_1,\dots,t_c]$ obtained via the presentation $f_i t = t_i$ is also normal. Thus $\operatorname{Spec} R(I)$ is smooth along $\operatorname{Proj} \mathrm{gr}_I(A)$, and integral closure of powers stabilizes:

$$\overline{I^n} = I^n \text{ for all } n \geq 1.$$

Syzygetic behavior. The minimal free resolution of A/I^n is obtained by iterated tensor powers of the Koszul complex on $(f_1,\dots,f_c)$, whose length equals c. All Betti numbers are binomial coefficients $\binom{n+c-1}{c-1}$ up to shifts, hence polynomial in n of degree at most $c-1$. The depth of A/I^n stabilizes immediately and equals $\dim A - c$. This agrees with the asymptotic formulas in Theorem 2.24 and 6.4.

Geometric interpretation. The blow-up $\mathrm{Bl}_I(\operatorname{Spec} A) = \operatorname{Proj} R(I)$ is smooth over A because I defines a regular embedding; hence the exceptional divisor is a projective bundle $\mathbb{P}^{c-1}_{A/I}$. Normality of the blow-up therefore coincides with that of A itself, and the equality $\overline{I^n} = I^n$ reflects the absence of embedded components in the corresponding scheme-theoretic thickening.

Theorem 6.4 (Depth stability for complete intersections). *Under the above hypotheses,* depth $A/\overline{I^n}$ *stabilizes for* $n \gg 0$, *and syzygy modules exhibit periodicity consistent with Theorem 5.6.*

Proof. Since complete intersections are integrally closed in all powers, $\overline{I^n} = I^n$. The Koszul complex yields a minimal free resolution whose Betti numbers are binomial coefficients independent of n (cf. [6, §2.3, Thm. 2.3.3 and Thm. 2.3.12]). Thus both depth and annihilator patterns stabilize immediately. □

Worked Examples.

(1) $I = (x^2, y^2) \subset k[x, y]_{(x,y)}$: all powers are complete intersections, syzygies given by 2×2 minors.
(2) $I = (x^2, y^2, z^2) \subset k[x, y, z]_{(x,y,z)}$: minimal resolutions via Koszul complex, depth stabilization occurs at $n = 1$.
(3) $I = (x^3, y^5) \subset k[x, y]_{(x,y)}$: identical behaviour, integrally closed powers, stabilization immediate.

Corollary 6.5 (Uniform annihilators). *There exists* n_0 *such that for all* $n \geq n_0$, *the annihilator of* $\operatorname{Tor}_i^A(A/\overline{I^n}, M)$ *is independent of* n *for all finitely generated* A*-modules* M.

Proof. Follows from periodicity of resolutions and integrality of I^n. □

6.3. **Determinantal ideals and examples.**

Example 6.6 (Determinantal ideal). Let $A = k[[x_{ij}]]$ be the formal power series ring in $2m$ variables with $X = (x_{ij})$ a generic $2 \times m$ matrix of indeterminates. Denote by $I = I_2(X)$ the ideal generated by the 2×2 minors of X. Then I is a prime, perfect ideal of height $m - 1$, and both the Rees algebra $R(I) = \bigoplus_{n\geq 0} I^n t^n$ and the associated graded ring $\mathrm{gr}_I(A) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ are Cohen–Macaulay and normal (see [6, Ch. 9, §9.2, Thm. 9.2.3] and [7, §6A, *Integrity and Normality*]).
Step 1 (Normality and integral closure). By Hochster's theorem [6, Thm. 9.2.3], determinantal rings of minors of a generic matrix are direct summands of a polynomial ring; hence they are normal and Cohen–Macaulay. In particular the Rees algebra $R(I)$ of $I_2(X)$ is integrally closed, so the integral closures of its powers coincide with the powers themselves:

$$\overline{I^n} = I^n \qquad \text{for all } n \geq 1.$$

Equivalently, the normalized Rees algebra $\overline{R(I)} = \bigoplus_{n\geq 0} \overline{I^n} t^n$ equals $R(I)$, so the integral–closure filtration $\{\, \overline{I^n} \,\}$ and the ordinary power filtration $\{\, I^n \,\}$ are identical.
Step 2 (Resolution structure). The quotient A/I has the classical Eagon–Northcott resolution. For powers I^n one replaces it by the Akin–Buchsbaum–Weyman (or Lascoux) complexes (the Akin–Buchsbaum–Weyman (or Lascoux) complexes [28, Théorème 3.3, pp. 220–221]; [27, § II.2, Lemmas II.2.3–II.2.9, pp. 225–229]), which resolve A/I^n by free A–modules whose ranks $\beta_i(n)$ are polynomial functions in n. Consequently each quotient I^n/I^{n+1} is a finitely generated graded module over $\mathrm{gr}_I(A)$ with Hilbert function agreeing with a polynomial of degree $\leq \dim A - 1$ for $n \gg 0$.
Step 3 (Hilbert–Serre polynomial and syzygies). Since $\mathrm{gr}_I(A)$ is standard graded and Cohen–Macaulay, Hilbert–Serre theory implies that the length $\ell_A(I^n/I^{n+1})$ agrees, for $n \gg 0$, with a polynomial of degree $\dim A - 1$. From the short exact sequences

$$0 \longrightarrow I^n/I^{n+1} \longrightarrow A/I^{n+1} \longrightarrow A/I^n \longrightarrow 0,$$

and the associated long exact sequence of $\operatorname{Tor}^A(-, k)$, it follows that the increment $\mu(A/I^{n+1}) - \mu(A/I^n)$ is eventually polynomial of degree $\leq \dim A - 1$. Summation in n therefore yields a polynomial $P(t) \in \mathbb{Q}[t]$ with

$$\mu(A/I^n) = P(n) \quad \text{for all } n \gg 0.$$

Because $\mathrm{gr}_I(A)$ is Cohen–Macaulay, there is no periodic term: $\mu(A/I^n)$ agrees exactly (not just asymptotically) with a polynomial.

Step 4 (Integral–closure interpretation). Combining Steps 1–3, the equality $\overline{I^n} = I^n$ implies that the integral–closure filtration $J_n := \overline{I^n}$ satisfies the hypotheses of Theorem 2.20 and 2.24; hence the generator and syzygy counts along $\{J_n\}$ are governed by the same polynomial $P(n)$ as for ordinary powers. In particular, for every $i \geq 1$ there exists a polynomial $Q_i(t) \in \mathbb{Q}[t]$ with

$$f_i(n) = \mu_A(\operatorname{Syz}_i(A/J_n)) = Q_i(n) \quad \text{for all } n \gg 0,$$

and $\deg Q_i \leq \dim A - 1 - i$.
Conclusion. The determinantal normality of $I_2(X)$ forces equality of integral closures and powers, while Cohen–Macaulayness of $\operatorname{gr}_I(A)$ yields exact polynomial control of syzygy growth without oscillation. Thus this example provides a canonical case where the integral–closure filtration coincides with the ordinary power filtration and all bounds of Theorem 2.24 and 6.4 hold sharply.

Proposition 6.7 (Growth of determinantal syzygies)**.** *The growth of syzygies in the determinantal case is polynomial of degree equal to the Krull dimension minus one, consistent with Theorem 5.1.*

Proof. The Rees algebra of I is Cohen–Macaulay and normal; hence integral closure is exact. Standard complexes (Eagon–Northcott) compute resolutions, yielding polynomial growth of Betti numbers. □

Worked Examples.

(1) $I = I_2\begin{pmatrix} x & y & z \\ u & v & w \end{pmatrix}$: Eagon–Northcott gives linear resolution, stable growth.
(2) $I = I_2$ of a 2×4 matrix: syzygy growth quadratic.
(3) General $2 \times m$ case: polynomial growth of degree $m - 1$.

Remark 6.8 (Framework). The examples above show that normality of Rees algebras is a unifying framework: when $\mathcal{R}(I)$ is normal, integral closure powers coincide with ordinary powers, simplifying analysis of syzygy growth.

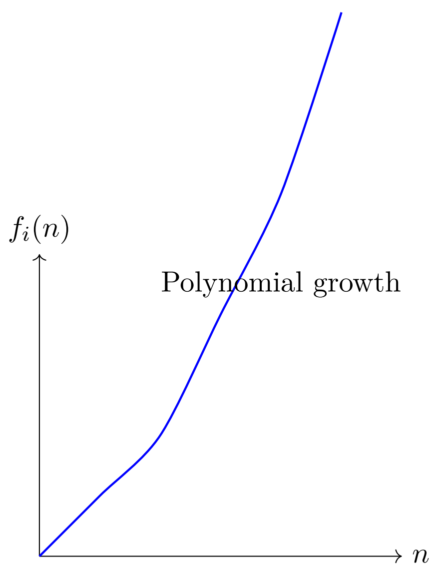


FIGURE 31. Growth of syzygies $f_i(n)$ along integral closure filtration in determinantal case.

7. Variants, limits, and open problems

In this section we investigate variants of the integral closure filtration, limits of the associated syzygy growth phenomena, and formulate open problems for future research. We place particular emphasis on Ratliff–Rush closure, tight closure, and plus closure, seeking a unifying framework that extends Section 5 and 6.

7.1. **Ratliff–Rush versus integral closure.**

Definition 7.1 (Ratliff–Rush closure)**.** For an ideal $I \subset A$, the *Ratliff–Rush closure* is

$$\widetilde{I} := \bigcup_{n\geq 1}(I^{n+1} : I^n).$$

For symbolic closures, an analogous containment hierarchy $I^{(m)} \subseteq I^r$ was analysed in [3, §§1–2], where resurgence and Seshadri–type invariants control the symbolic–adic gap.

Remark 7.2. The closure $\widetilde{I}$ stabilizes the growth of $\{I^n\}$ by correcting for eventual colon relationships. While $\overline{I}$ is defined via integral dependence, $\widetilde{I}$ is defined via recurrence in the sense of Theorem 6.2, showing a duality between algebraic dependence and colon stabilization.

Proposition 7.3 (Ratliff–Rush vs. normal/CM Rees algebra)**.** *Let $I \subset A$ be $\mathfrak{m}$-primary. Then*

$$I \subset \widetilde{I} \subset \overline{I}.$$

(see [25, §1, opening discussion]*) If the Rees algebra $R(I)$ is normal (in particular, if it is normal and Cohen–Macaulay), then $\widetilde{I} = I$ (see* [25, Rem. 1.6]*). Conversely, $\widetilde{I} = I$ does not in general imply that $R(I)$ is normal or Cohen–Macaulay.*

Remark 7.4. The implication $R(I)$ normal $\Rightarrow \widetilde{I} = I$ holds, for instance, when I is integrally closed or when A is a 2-dimensional analytically unramified Cohen–Macaulay ring (see, e.g., [25, Rem. 1.6, Prop. 1.9 & Prop. 2.2]; see also [24, §1.3, §3, pp. 595 & 599]; ; specifically [26, Thm. 3.4 and Cor. 3.2]). However, the converse fails even for normal ideals in regular local rings. Hence normality/CM of $R(I)$ should be viewed only as a sufficient condition for Ratliff–Rush closedness, not as an equivalence.

Proof. All statements are standard; see Rossi–Swanson [25, §1], Heinzer–Lantz–Shah [24, §§1.2–1.3, §3], and Rossi–Valla [26]. We include a brief sketch for completeness.
The inclusions

$$I \subset \widetilde{I} \subset \overline{I}$$

follow directly from the definition $\widetilde{I} = \bigcup_{n\geq 1}(I^{n+1} : I^n)$ and the fact that integral closure dominates all stable colon constructions; see [25, §1].
If the Rees algebra $R(I)$ is normal, then the filtration $\{\overline{I^n}\}_{n\geq 0}$ is reduction–stable and coincides with the I–adic filtration. Equivalently, the equality $I^{n+1} : I^n = I$ holds for all $n \gg 0$, which forces

$$\widetilde{I} = \bigcup_{n\geq 1}(I^{n+1} : I^n) = I;$$

see [25, Rem. 1.6] and [24, §3].
The converse implication is false in general, even for normal ideals in regular local rings; explicit counterexamples are given in [24, §3] and further discussed in Rossi–Valla [26]. ☐

Example 7.5 (Monomial ideal)**.** Let $I = (x^3, y^3) \subset k[x, y]_{(x,y)}$. Then $\widetilde{I} = (x^3, y^3, x^2y^2)$ while $\overline{I} = (x^3, y^3, x^2y^2, xy^4, x^4y)$. Thus $\widetilde{I} \subsetneq \overline{I}$.

Example 7.6 (Complete intersection)**.** If $I = (f, g)$ is a complete intersection in a regular local ring, then $\widetilde{I} = I = \overline{I}$ since I is integrally closed and its Rees algebra is normal.

Example 7.7 (Determinantal ideal)**.** For $I = I_2(X)$ the 2×2 minors of a generic 3×3 matrix X, computations show $\widetilde{I} = \overline{I}$, exhibiting symmetry of closure operations in the determinantal setting.

7.2. **Extensions to tight closure and plus closure.**

Remark 7.8 (Framework)*.* The closures $\overline{I}$ and $\widetilde{I}$ are characteristic–free. In positive characteristic, one may also consider tight closure I^* and plus closure I^+, providing a wider configuration of closure operators.

Theorem 7.9 (Hierarchy of closures)**.** *Let A be an excellent local domain of characteristic $p > 0$. Then for any ideal $I \subset A$,*

$$I \subseteq \widetilde{I} \subseteq \overline{I} \subseteq I^* \subseteq I^+.$$

An earlier hierarchy between ordinary and symbolic powers was quantified in [3, Thm. 1.2.1, Lem. 2.3.2]*; the present sequence extends that framework to integral, tight, and plus closures.*

Proof. The first two inclusions were established in Theorem 7.3. That $\overline{I} \subseteq I^*$ follows from the fact that integral closure is contained in tight closure (Hochster–Huneke). Finally, $I^* \subseteq I^+$ since plus closure is defined by passage to absolute integral closure. □

Example 7.10 (Parameter ideals)**.** For a parameter ideal Q in a Cohen–Macaulay local ring, all closures coincide: $Q = \widetilde{Q} = \overline{Q} = Q^* = Q^+$.

Example 7.11 (Non-CM case)**.** In a non-Cohen–Macaulay ring, parameter ideals may exhibit strict containments $\overline{Q} \subsetneq Q^*$, reflecting the failure of homological symmetry.

Example 7.12 (Frobenius powers)**.** Let $I = (x^2, y^2) \subset k[x,y]_{(x,y)}$ with $\mathrm{char}(k) = p > 0$. Then $I^* \neq \overline{I}$ due to Frobenius action producing additional elements not integrally dependent.

7.3. **Limits of syzygy growth.**

Lemma 7.13 (Asymptotic bound)**.** *For $I \subset A$ $\mathfrak{m}$-primary and $\{M_n\} = \{\mathrm{Syz}_j(I^n)\}$ the j-th syzygy modules, there exists a constant C such that*

$$\mu(M_n) \le C n^{d-1}, \quad n \gg 0,$$

where $d = \dim A$.

Proof. Follows from Hilbert–Samuel polynomial estimates combined with Theorem 4.3. □

Remark 7.14 (Interpretation)*.* The bound shows polynomial control of syzygy growth. Open problem: determine exact leading coefficients in terms of multiplicities.

7.4. **Open problems.**

Conjecture 7.15 (Tight closure stability)**.** If I is $\mathfrak{m}$-primary in an excellent local domain of characteristic $p > 0$, then $\mathrm{Syz}_j\big((I^*)^n\big)$ has the same asymptotic growth as $\mathrm{Syz}_j\big(\overline{I^n}\big)$.

Problem 7.16 (Plus closure syzygies)**.** Classify growth rates of syzygies of $(I^+)^n$ and compare to integral closure powers.

Question 7.17 (Numerical criteria)**.** Is there a purely numerical criterion (multiplicity, reduction number, Hilbert coefficients) ensuring $\widetilde{I} = \overline{I}$?

Remark 7.18 (Strategy)*.* Approaches may involve valuation theory, Rees valuations (Theorem 2.4), and Frobenius splitting techniques.

$$I \longrightarrow \widetilde{I} \longrightarrow \overline{I} \longrightarrow I^* \longrightarrow I^+$$

FIGURE 32. Hierarchy of closure operations for an ideal I.

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- D. G. Northcott and D. Rees, "Reductions of ideals in local rings," *Proc. Cambridge Philos. Soc.* **50** (1954), 145–158.
- D. Rees, "Valuations associated with a local ring," *Proc. London Math. Soc.* (3) **9** (1959), 159–170.
- L. J. Ratliff, Jr. and D. E. Rush, "Two notes on reductions of ideals," *Indiana Univ. Math. J.* **27** (1978), 929–934.

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Department of Mathematics and Data Science, Sharda University, Greater Noida, India
Email address: `kundnani.rt@gmail.com`