CALCULATION METHODS OF ECONOMIC SYSTEMS

Macro-System Economic Value

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Abstract

This paper develops the Calculation Methods of Economic Systems (CMES) as a rigorous macro-mathematical theory of economic value, grounded in four axiomatic pillars: topology, separability, measurement theory, and network effectiveness, with macro-structure expressed through nested groups of composites. The framework is developed as a purely logical system proceeding from axioms to theorems to corollaries, requiring no empirical calibration. It assumes strictly positive transaction costs, locally finite network topologies, and ratio-scale measurability of transaction value. The central object is the valuation function $\mathrm{Val}(\cdot)$, whose existence is guaranteed by Debreu's (1954) representation theorem and whose additive functional form is derived through topological joint separability (Debreu, 1960). We situate $\mathrm{Val}(\cdot)$ within a self-contained poset and nested-group architecture of economic systems, where every transaction system—from a single firm to an entire global supply chain—is formally represented as a nested transaction set (a poset element) over four canonical role layers: Value-Adding (CE^v), Archived Information and Stored Material for Production (CE^a), Communication (CE^c), and Evaluation (CE^e) elements. All definitions and theorems required for this architecture are fully derived within this manuscript.

The transformation scalar σ, which links any two such systems in the master equation, is shown to equal the total factor productivity ratio under explicit competitive conditions. We introduce and derive the Network Effectiveness metric Ξ_F, which quantifies how efficiently a sub-system integrates with a broader ecosystem, and demonstrate that Ξ_F acts as a structural multiplier on $\mathrm{Val}(\cdot)$. Supply chains are characterized as ordered chains of inclusive nested groups, and market valuations are derived as aggregated $\mathrm{Val}(\cdot)$ integrals over these chains, weighted by Ξ_F.

The theory generates conditional falsifiable predictions at the macroeconomic level. The axiomatic core makes explicit the ratio-scale identification required for multiplying $\mathrm{Val}(CE)$ by transaction counts in the Master Equation (Axiom 8), establishing a commutative monoid structure under concatenation that collapses the admissible transformation group from positive affine to positive similarity mappings. Additive separability is derived through a non-differentiable topological proof route via the Thomsen–Debreu cancellation conditions, with explicit specification of the cancellation order required for the four-dimensional product space. A systematic assessment reveals that the framework's most critical structural tension concerns the gap between interval-scale uniqueness guaranteed by the Debreu–KLST tradition and the ratio-scale identification required by the Master Equation's multiplicative structure—a tension resolved herein through three complementary pathways. The paper further connects the

CMES poset decomposition to Möbius inversion on locally finite posets, axiomatizes $\varXi_F$ via centrality-theoretic frameworks, situates the supply chain decomposition as a CMES-specific analog of Hulten's (1978) theorem, and parameterizes the Archived component's accumulation dynamics to accommodate both Romer (1990) convex compounding and Jones (1995) diminishing returns.

Keywords: transaction value; poset topology; measurement theory; network effectiveness; axiomatic foundations; supply chain valuation; market capitalization; macro-system dynamics; total factor productivity; Möbius inversion; monoid structure; ratio-scale identification; Hulten theorem

Notation and Abbreviations

Symbol	Meaning
G_n	Composite group at hierarchical level n in the economy
N_n^t	Total number of elements in composite group G_n at time t
$N_n^v(t)$	Value-Adding elements in G_n
$N_n^a(t)$	Archived information and stored material elements in G_n
$N_n^c(t)$	Communication elements in G_n
$N_n^e(t)$	Evaluation elements in G_n
CE	Composite element bundle (CE^v, CE^a, CE^c, CE^e)
$\mathrm{Val}(CE)$	Valuation function on composite element bundles
NTr	Number of transactions in a system or period
σ	Transformation scalar linking two transaction systems
$\varDelta$	Value differential $\mathrm{Val}(CE_1) - \mathrm{Val}(CE_2)$
$\varXi_F(G_x \to G_n, t)$	Network Effectiveness of G_x on G_n at time t
$\eta_x^n(t)$	Coupling efficiency between G_x and G_n
α_x^n	Hierarchical adjustment factor
$\mathcal{E}$	Domain of feasible composite element bundles $\subseteq \mathbb{R}_+^4$
$\succcurlyeq$	Complete and transitive valuation ordering on $\mathcal{E}$
w_j	Normalized weight on component $j \in \{v, a, c, e\}$
$s(e,t)$	Multi-role signature of element e at time t
$\rho(e,t)$	Primary role assignment of element e at time t
NE	Exchanged elements (elements crossing the transaction boundary)
S_t	State vector of the CMES system at time t
$\varGamma(S_t)$	Feasibility correspondence mapping states to feasible bundles
$\oplus$	Concatenation operator on composite element bundles
CE^0	Structural zero bundle (empty transaction system)
$\mu(x,y)$	Möbius function on the poset of composite groups

Symbol	Meaning
λ_k	Domar weight / Ξ_F-analog at supply chain stage k

1. Introduction: From Axioms to Mathematical Theorem

1.1 The Central Problem

Economic value, as a mathematical object, must satisfy three requirements to constitute a scientific theory: it must exist as a well-defined function over a specified domain; it must possess a functional form derivable from first principles; and it must generate testable restrictions capable of being falsified. This paper shows, under the stated axioms and scope conditions, that these three requirements are jointly satisfied by the CMES framework when developed from its axiomatic foundations.

The argument proceeds purely deductively from axioms to theorems. We begin with the question: what are the minimal mathematical conditions sufficient to guarantee the existence of a continuous, additively separable valuation function on the space of bilateral exchange? The answer, developed in Sections 2 and 3, is exactly seven axioms—completeness, transitivity, continuity, monotonicity, separability, uncertainty-weighting, and transaction consistency—whose conjunction implies, via Debreu's (1954) representation theorem, that such a function exists and takes the form:

$$\mathrm{Val}(CE) = \sum_{j \in \{v,a,c,e\}} w_j \cdot f_j\left(CE^j\right)$$

where the four component sub-functions f_j correspond to the Value-Adding, Archived Information and Stored Material for Production, Communication, and Evaluation dimensions of any transaction system, and the weights $w_j > 0$ are normalized ($\sum w_j = 1$).

1.2 The Nested-Set Extension

CMES at the bilateral level provides a micro-theory of exchange. The paper's major extension embeds $\mathrm{Val}(\cdot)$ in a poset and nested-group architecture that scales this micro-theory to macro-economic phenomena. The scaling assumes that the valuation ordering $\succcurlyeq$ and the ratio-scale identification of $\mathrm{Val}(\cdot)$ are preserved when composite groups at higher hierarchical levels are formed by set inclusion from lower-level groups. In CMES, the term *composite group* (or *group of composites*) replaces the algebraic notion of "algebraic group": every transaction system at any level of aggregation is represented as a composite group composed of four role-specific subsets. The relationships among composite groups obey precise mathematical constraints (Section 4), and the effectiveness of any sub-system in generating value within a larger system is measured by the Network Effectiveness metric:

$$\Xi_F(G_x \to G_n, t) = \eta_x^n(t) \cdot \left[\mathrm{Ef}_{gx}^{gn}(t) + \mathrm{Ef}_{gx}^{en}(t) + \mathrm{Ef}_{ex}^{gn}(t) + \mathrm{Ef}_{ex}^{en}(t)\right]$$

This metric is a structural multiplier on $\mathrm{Val}(\cdot)$. Two firms operating with identical factor inputs but different Ξ_F values will exhibit divergent market valuations.

1.3 Scope of Theorems

This paper proves the following principal results:

Theorem 2.1 (Existence and Uniqueness of $\mathrm{Val}(\cdot)$): Given Axioms 1–7, a continuous, additively separable valuation function $\mathrm{Val}(\cdot)$ exists and is unique up to a positive affine transformation.

Proposition 2.3b (Ratio-Scale Identification): Under additive conjoint solvability, Archimedean ordering, and a fixed structural zero, the admissible scale transformations collapse to positive similarity mappings, validating $NTr \cdot \mathrm{Val}(CE)$.

Proposition 2.3c (Topological Separability): Additive separability is derived from relational/topological joint separability conditions without invoking differentiability.

Master Equation (Section 3.1): The bilateral transaction identity $NTr_1 \cdot \mathrm{Val}(NE_1) = NTr_2 \cdot \mathrm{Val}(NE_2) \cdot \sigma$.

Theorem 3.1 (σ as TFP ratio): Under the competitive dual-cost regime with explicit normalization.

Theorem 5.1 ($\mathit{\Xi}_F$ as structural multiplier on $\mathrm{Val}(\cdot)$): $V_{\mathrm{eff}}(G_x \to G_n) = \mathit{\Xi}_F(G_x \to G_n) \cdot \mathrm{Val}(CE_x)$.

Theorem 6.1 (Supply Chain Decomposition of $\mathrm{Val}(\cdot)$): For any supply chain $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_N$, the total value generated equals the sum of component-level contributions weighted by the Network Effectiveness of each composite group on its successor.

Theorem 7.1 (Market Capitalization): The market capitalization of a firm within an industry is proportional to $\mathit{\Xi}_F \cdot \mathrm{Val}(CE)$.

Theorem 8.1 (Dynamic Accumulation): The time-evolution of composite element bundles is governed by role-specific accumulation equations derived from the axiomatic structure.

Theorem 4.5 (Möbius Inversion Decomposition): For any locally finite poset of composite groups, the intrinsic value contribution of each group is recovered via Möbius inversion on the CMES incidence algebra.

Theorem 4.6 (Supermodular Comparative Statics): If $\mathrm{Val}(\cdot)$ is supermodular on the nested-group poset, Topkis's theorem delivers monotone comparative statics for all role components.

1.4 Relationship to the "Group Theory and Effectiveness" Foundation

CMES is the mathematically mature, axiomatically complete evolution of the prior "Group Theory and Effectiveness" (GTE) framework. The GTE framework introduced (i) a four-role decomposition of economic elements into value-adding, archived, communication, and evaluation roles; (ii) inclusive and non-inclusive networks of composite groups; (iii) a prototype effectiveness-type metric for interaction across hierarchies; and (iv) conceptual definitions for patterns, forms, and models as levels of structural integration. CMES advances GTE by replacing every conceptual claim with a rigorously proved theorem, grounding each definition in a named mathematical structure from topology, measurement theory, or order theory.

Three specific advances over GTE are central. First, the existence and functional form of Val($\cdot$) is *proved* from axioms, not assumed. Second, the "group" terminology is clarified: CMES operates with *composite groups* as tuple-structured transaction systems ordered by set inclusion—a poset, not an algebraic group—with the algebraic content captured through an explicit commutative monoid of concatenation (Section 2.3e). Third, the effectiveness metric is *axiomatized* via overlap-count primitives with set-theoretically proved boundedness and monotonicity.

The hierarchy of structural integration in GTE—pattern, form, and model—is preserved in CMES through Definitions 4.6–4.8: a pattern is a relatively closed sub-ensemble under an interaction operator; a form is a hierarchical composition of patterns satisfying global invariants; and a model is a time-indexed family of forms together with an evolution operator. The present paper embeds these constructs within the poset and nested-group architecture and proves their algebraic extractability (Theorem 4.4).

1.5 Five Structural Amendments: Overview

A systematic mathematical assessment of the CMES framework reveals that, while its axiomatic core is robust, five structural gaps require explicit resolution. These gaps are identified and addressed in full throughout the paper, with the following overview serving as a roadmap:

Amendment 1 (Ratio-Scale Closure). Debreu's (1960) topological additive representation theorem guarantees interval-scale uniqueness—the valuation function is determined up to a positive affine transformation $V' = \alpha V + \beta$. However, the Master Equation's multiplicative structure requires ratio-scale uniqueness ($V' = \alpha V$, $\alpha > 0$). This gap is closed in Section 2.3b via three complementary pathways: (i) explicit structural zero and concatenation closure (Axiom 8); (ii) identification of the concatenation structure as a commutative monoid satisfying extensive measurement axioms (Helmholtz–Hölder conditions, as formalized in Krantz–Luce–Suppes–Tversky, 1971, Vol. I, Ch. 3); and (iii) Luce and Narens's (1985) classification of 1-point homogeneous, 1-point unique scale types.

Amendment 2 (CES Approximation Order). Additive separability in levels corresponds to the $\rho \to 1$ limit of the CES function—perfect substitutability—not the Cobb-Douglas ($\rho \to 0$) case. Kmenta (1967) shows the Cobb-Douglas form arises as the first-order Taylor expansion of CES in logarithms. Baqaee and Farhi (2019, 2020) demonstrate that under the additive (unit-elasticity) approximation, second-order network amplification effects—which can be 10–100% of first-order predictions—are suppressed. The Uzawa (1962) impossibility theorem further constrains four-input non-nested CES. These issues are addressed in Section 2.3a with explicit domain conditions and nested CES extensions.

Amendment 3 (Ξ_F Axiomatic Grounding). The Network Effectiveness metric, while novel, requires formal axiomatic foundations to be mathematically compelling. Section 5 grounds Ξ_F via Boldi and Vigna's (2014) centrality axioms, connects it to the Leontief inverse multiplier tradition (Acemoglu et al., 2012), and establishes Myerson-value game-theoretic grounding.

Amendment 4 (Möbius Inversion Decomposition). CMES Theorem 6.1 is identified as a CMES-specific analog of Hulten's (1978) theorem. Section 4.5 formalizes the poset decomposition via Rota's (1964) incidence algebra and Möbius inversion, establishing a canonical hierarchical decomposition immune to the SDA non-uniqueness problem (Dietzenbacher and Los, 1998).

Amendment 5 (Archived Dynamics Parameterization). The Romer (1990) linear compounding assumption for informational stocks is refined via Jones (1995) diminishing returns ($\dot{A} = \delta L_A^{\lambda} A^{\phi}$, $\phi < 1$), Crouzet et al. (2022) cross-effects between physical and informational capital, and Stokey–Lucas–Prescott (1989) recursive methods. Section 8 integrates these into the accumulation dynamics.

1.6 What CMES Is and Is Not

Intellectual honesty about the logical architecture of this paper requires clarity on three questions: what is derived from the axioms, what is assumed as a modeling choice, and what is borrowed from other fields. The following inventory is provided so that readers can assess each result on its own terms.

What is derived from the axioms. Theorem 2.1 (additive separability of Val(·)) is a genuine deductive consequence of Axioms 1–5, via the Debreu–Thomsen route. The ratio-scale identification of Section 2.3b is a consequence of a structural zero and an Archimedean concatenation structure; because these are not implied by Axioms 1–5 alone, ratio-scale status is an additional assumption (Axiom 8) rather than a derived result. Proposition 4.1 (conservation of transaction value in inclusive networks) follows by definition of the transformation scalar σ. Theorem 6.1 (supply chain decomposition) is algebra over the set structure. These constitute the deductively secure core of CMES.

What is a modeling choice. The four-component classification (V, A, C, E) is a modeling hypothesis introduced in Section 2.6, not a mathematical consequence of the axioms. The definition of Ξ_F (Section 5) is grounded in an observable set-intersection primitive; the formula is a modeling specification that quantifies integration in a particular way. The constant per-element value assumption that appears in Conditional Result 5.1 is an explicit auxiliary hypothesis.

What is borrowed from other fields. Application 7.1 (market capitalization formula) is the Gordon growth model with Ξ_F as an additional multiplier. The derivation is standard finance applied to the CMES setting; the constant growth rates, constant transaction intensity, and convergence condition $\rho > g$ are auxiliary assumptions that are not derived from the CMES axioms.

What CMES is. CMES is a formal meta-modeling framework: a precisely specified language for describing economic systems, together with structural relationships between its components that hold either by derivation from axioms or by modeling construction. As a framework, it provides: (i) an axiomatic foundation for additive valuation that is derivatively secure; (ii) a structured vocabulary (CE bundles, network effectiveness, poset hierarchy) that enables systematic comparison of economic systems; (iii) a set of structural identities and conditional results that clarify when and how standard results from finance, measurement theory, and topology apply in the CMES setting; and (iv) a set of open problems that mark the frontier of what remains to be shown.

2. Axiomatic Foundations of the Valuation Function Val$(\cdot)$

2.1 The Mathematical Universe

Definition 2.1 (Composite Element Bundle). Let $\mathcal{E} \subset \mathbb{R}_+^4$ denote the domain of feasible composite element bundles. Each element $CE \in \mathcal{E}$ is a four-vector:

$$CE = (CE^v, CE^a, CE^c, CE^e) \in \mathbb{R}_+^4$$

where $CE^v \geq 0$ measures Value-Adding capacity, $CE^a \geq 0$ measures Archived Information and Stored Material for Production, $CE^c \geq 0$ measures Communication quality, and $CE^e \geq 0$ measures Evaluative governance.

CE^a **— Archived Information and Stored Material for Production.** These are elements that preserve state, history, knowledge, or material stocks for future use. The archived class encompasses two distinct functions:

(i) *Informational archives*: accumulated institutional knowledge, proprietary data, intellectual property, brand equity, certifications, maintenance logs, and traceability documentation.

(ii) *Physical/material archives*: warehouse inventory, stored finished goods reserved for downstream use, spare parts, intermediate goods held for further processing, and any other tangible stocks that remain inside the system boundary until activated in production or exchanged at the transaction boundary.

The term *archived information and stored material* emphasizes both informational persistence and physical productive potential. Physical inventory is classified as CE^a because it represents stored productive potential—held within the system until drawn down for use in production (at which point its value is realized through CE^v) or exchanged as a finished good (at which point it crosses the transaction boundary).

Domain	Value-Adding (v)	Archived (a)	Communication (c)	Evaluation (e)
Computing	CPU/GPU cores	Storage, RAM, logs	Network stack, APIs	OS scheduler, debugger
Manufacturing	Assembly, machining	Inventory, warehouses	Logistics, ERP	Quality control, audits
Trade	Production capacity	Ledgers, reserves	Markets, payment rails	Prices, regulation
Science	Experiments, R&D	Datasets, literature	Journals,	Peer review, ethics

collaboration

Table 2.1: Functional roles appear in every domain.

Definition 2.1a (State Variable and Feasibility Correspondence). The CMES state vector at time t is:

$$EU^{\circ}US_t := \left(CE^a_{\text{phys}}(t),\ CE^a_{\text{info}}(t),\ CE^v(t),\ CE^c(t),\ CE^e(t)\right)$$

EU-US: Established Unification and Unified State Variable in economy synthesized metric used in dynamic models to represent the comprehensive "state" of a complex systems.
Note : For simplification purposes S is used instead of EU-US.

where CE^a_{phys} and CE^a_{info} denote the physical and informational components of the archived stock respectively. The feasibility correspondence $\Gamma: \mathbb{R}^5_+ \rightrightarrows \mathcal{E}$ maps each state to the set of feasible transaction bundles:

$$CE_t \in \Gamma(S_t)$$

We require Γ to be nonempty-valued, compact-valued, and continuous (in the sense of Berge's maximum theorem), which is mathematically necessary for the Bellman contraction mapping of Section 8.

This formalization is the mathematically precise statement of the claim "without archives, each transaction starts anew": if $CE^a_{\text{phys}}(t) = CE^a_{\text{info}}(t) = 0$, then $\Gamma(S_t)$ degenerates to a static, state-independent set—the system is memoryless (Stokey, Lucas, and Prescott, 1989, Ch. 4). Path-dependent accumulation is feasible only when S_t is non-degenerate, confirming that the archived state is a structural necessity, not a modeling convenience.

Remark 2.1a.1 (Alignment with Dynamic Programming). The state-space formulation aligns CMES with the Bellman recursive methodology (Bellman, 1957; Stokey, Lucas, and Prescott, 1989). The contraction mapping theorem guarantees existence and uniqueness of the value function over the product space $\mathbb{R}^2_+ \times \mathbb{R}^3_+$. A key technical challenge is that physical stocks live in a convex cone with linear depreciation dynamics, while knowledge stocks may exhibit non-convex accumulation; verifying Bellman operator closure for this product state space requires the Benveniste–Scheinkman envelope conditions to hold jointly across components.

2.1a Mathematical Necessity of State Persistence

Theorem 2.1a.1 (Necessity of CE^a). *An economic system without a persistent state component CE^a cannot exhibit intertemporal value accumulation.*

Proof. Suppose $CE^a = 0$ identically. Then all state variables are instantaneous: they exist only during the current transaction period and are destroyed upon completion. Consider the cumulative value function:

$$\text{Val}\big(CE(t)\big) = \sum_{j \in \{v,c,e\}} w_j \cdot f_j\left(CE^j(t)\right)$$

Without persistence, $\text{Val}(CE(t+1))$ depends only on exogenous inputs at $t+1$, not on any function of $\text{Val}(CE(t))$. The system is memoryless. But any system capable of compounding returns requires $\text{Val}(CE(t+1)) = g\left(\text{Val}(CE(t)),\text{inputs}(t+1)\right)$ for some non-trivial function g. This requires a state-carrying component—precisely CE^a. More formally, the feasibility correspondence degenerates: $\Gamma(S_t) = \Gamma(0)$ for all t, so the feasible set is time-invariant and cannot encode cumulative capability. ▫

Remark 2.1a.2 (Connection to Inventory Theory). Arrow, Harris, and Marschak (1951) and Scarf's (1960) K-convexity proof of (s, S)-policy optimality establish that inventory persistence is mathematically necessary for intertemporal cost minimization: the optimal policy depends on the current stock level, and the value function inherits convexity from the inventory state. This provides independent confirmation from dynamic programming that CE^a_{phys} must be a genuine state variable.

Definition 2.1b (Dual Dimensions of the Archived State). The archived component decomposes into two sub-stocks with distinct mathematical properties:

Physical Stocks CE^a_{phys}: rivalrous assets subject to linear depreciation and spatial exclusivity. Their accumulation satisfies the law of motion:

$$\dot{CE}^a_{\text{phys}}(t) = x(t) - d(t) - \delta_{\text{phys}} \cdot CE^a_{\text{phys}}(t)$$

where $x(t)$ is gross inflow, $d(t)$ is demand/outflow, and $\delta_{\text{phys}} > 0$ is the physical depreciation rate. This is the CMES formalization of the standard capital accumulation identity in Ramsey–Cass–Koopmans growth theory (Koopmans, 1965).

Informational Stocks CE^a_{info}: non-rivalrous assets exhibiting path-dependent compounding. Following Romer (1990), the benchmark accumulation equation is:

$$\dot{A}_t = \delta H_A A_t$$

where knowledge A_t exhibits increasing returns. In the CMES taxonomy, A_t is the informational component CE^a_{info}. However, Jones (1995) demonstrates that the Romer characterization of strictly convex compounding is too strong: empirical evidence supports a modified law:

$$\dot{A}_t = \delta L_A^\lambda A_t^\phi, \quad \phi < 1$$

where $\phi < 1$ introduces diminishing returns to the existing knowledge stock. CMES adopts the Jones parameterization as the general case, with $\phi = 1$ (Romer compounding) as a special case whose applicability must be justified in specific domains. Weitzman (1998) provides the combinatorial microfoundation: recombinant growth implies that the number of productive idea combinations grows as $C(N, 2) \approx N^2/2$, which bounds the compounding rate and ultimately constrains growth to sub-exponential rates under resource constraints.

Proposition 2.1a.1 (Unified State Evolution). *The CE^a component satisfies a coupled system of differential equations:*

$$\dot{CE}^a(t) = \underbrace{\left[sY(t) - \delta_{\text{phys}} CE^a_{\text{phys}}(t)\right]}_{\text{physical accumulation}} + \underbrace{\left[\delta L_A^{\lambda}\left(CE^a_{\text{info}}(t)\right)^{\phi}\right]}_{\text{informational accumulation}}$$

where $s > 0$ *is the savings rate,* $Y(t)$ *is gross output,* δ_{phys} *is physical depreciation,* $\phi \in (0,1]$ *is the knowledge compounding exponent,* $\lambda \in (0,1]$ *is the research labor elasticity, and* L_A *is labor allocated to knowledge production. The persistent state serves as both the physical raw material for current production and the algorithmic blueprint dictating the efficiency of that production.* ▫

Proposition 2.1a.2 (Physical-Informational Cross-Effects). The archived component's two sub-stocks interact through complementarity. Following Crouzet, Eberly, Eisfeldt, and Papanikolaou (2022), define a non-rivalry parameter $\varrho \in [0,1]$ where $\varrho = 0$ denotes purely rivalrous and $\varrho = 1$ purely non-rivalrous characteristics. The cross-partial derivative:

$$\frac{\partial^2 V_e}{\partial \varrho \, \partial(-\delta)} > 0$$

implies that a reduction in depreciation rate has a larger effect on effective value when the information stock is more non-rivalrous. This complementarity argues for maintaining physical and informational sub-stocks as distinct tracked quantities rather than collapsing them into a single scalar, since their interaction generates value effects beyond what either component produces independently. The CMES formulation in Proposition 2.1a.1 preserves this interaction through the coupled differential system.

Definition 2.2 (Valuation Ordering). Let $\succcurlyeq$ be a binary relation on $\mathcal{E}$, interpreted as a valuation ordering: $CE_1 \succcurlyeq CE_2$ means "CE bundle 1 generates at least as much transaction value as CE bundle 2."

Remark 2.1 (Economic Grounding). The ordering $\succ$ is the aggregate market valuation ordering revealed through market-clearing transaction outcomes, not a psychological preference relation.

The domain $\mathcal{E} = \mathbb{R}^4_+$ is equipped with the standard Euclidean product topology, which is connected, separable, and second-countable.

2.2 The Seven Axioms

Axiom 1 (Completeness). For all $CE, CE' \in \mathcal{E}$, either $CE \succcurlyeq CE'$ or $CE' \succcurlyeq CE$ (or both).

Axiom 2 (Transitivity). For all $CE_1, CE_2, CE_3 \in \mathcal{E}$, if $CE_1 \succcurlyeq CE_2$ and $CE_2 \succcurlyeq CE_3$, then $CE_1 \succcurlyeq CE_3$.

Axiom 3 (Continuity). The preorder $\succcurlyeq$ on $\mathcal{E}$ is continuous in the Euclidean topology: for every bundle CE, the upper and lower contour sets are closed.

Axiom 4 (Component Monotonicity and Essentiality). For each component $j \in \{v, a, c, e\}$ and any fixed values of the remaining three components CE_{-j}, if $x > y$ then $\left(x, CE_{-j}\right) \succ$

(y, CE_{-j}). Each component is *essential*: for each j there exist $x \neq y$ and some fixed CE_{-j} such that $(x, CE_{-j}) \succ (y, CE_{-j})$.

Axiom 5 (Topological Joint Separability / Thomsen Cancellation). For a given local domain of reference bundles $\mathcal{E}_{\text{loc}} \subseteq \mathcal{E}$, any distinct components $i, j \in \{v, a, c, e\}$ and any fixed values of the remaining two components, the induced weak order on $\mathbb{R}_+ \times \mathbb{R}_+$ satisfies Thomsen cancellation: if $(x_1, y_2) \sim_{ij} (x_2, y_1)$ and $(x_2, y_3) \sim_{ij} (x_3, y_2)$, then $(x_1, y_3) \sim_{ij} (x_3, y_1)$.

Remark 2.2.0 (Locality of Axiom 5). The domain restriction $\mathcal{E}_{\text{loc}} \subseteq \mathcal{E}$ permits Axiom 5 to coexist with the nested CES generalizations of Section 2.3a, where variable elasticities of substitution hold globally. Thomsen cancellation delivers additive separability on the local domain; outside $\mathcal{E}_{\text{loc}}$, the CES nesting structure governs the functional form. When $\mathcal{E}_{\text{loc}} = \mathcal{E}$, the global additive form of Theorem 2.1 obtains.

Remark 2.2.1 (Cancellation Order Specification). For an n-dimensional product space, additive representation theorems can require cancellation conditions of different orders depending on n and the regularity conditions imposed. For $n \geq 3$, Debreu's (1960) theorem requires that *every pair* of dimensions (i, j) satisfies Thomsen cancellation—equivalently, pairwise joint separability—with continuity and essentiality furnishing the global extension. Axiom 5 states this requirement explicitly for all $\binom{4}{2} = 6$ pairs of the CMES four-dimensional product space. This is the minimal cancellation condition for global additive representation without differentiability. Higher-order cancellation (as required in finite algebraic conjoint measurement, cf. Fishburn, 1970) is not necessary here because the domain is a connected subset of $\mathbb{R}^4$ with the Euclidean topology, where Thomsen cancellation combined with essentiality and continuity suffices (Debreu, 1960; see also Wakker, 1988). For restricted domains or discrete approximations, higher-order cancellation conditions may be required, a point noted as a scope condition in Section 10.3 (Criterion B).

Axiom 6 (Uncertainty Aversion on Acts; Choquet-Expected-Utility Form). Let $\mathcal{F}$ be the set of acts $f: G \to \mathcal{E}$ on a finite state space G with probability measure P. Let $\succcurlyeq_e$ be a weak order on $\mathcal{F}$ satisfying: (i) comonotonic independence; (ii) monotonicity; (iii) non-degeneracy; and (iv) on constant acts, $\succcurlyeq_e$ coincides with $\succcurlyeq$ on $\mathcal{E}$.

Axiom 7 (Transaction Consistency). Let $\succcurlyeq_T$ be a weak order on transaction systems $(N, CE) \in \mathbb{N} \times \mathcal{E}$. If $CE_1 \sim CE_1'$ and $CE_2 \sim CE_2'$, then $(N_1, CE_1) \succcurlyeq_T (N_2, CE_2)$ iff $(N_1, CE_1') \succcurlyeq_T (N_2, CE_2')$. There exists a positive comparison scalar $\sigma > 0$ consistent with the ordering.

2.2a Representation Discipline and Scale-Type Separation

Definition 2.2a.1 (Order vs. Representation). The valuation ordering $\succcurlyeq$ is a weak order on the bundle domain $\mathcal{E}$. A representation is a map $u: \mathcal{E} \to \mathbb{R}$ such that $CE_1 \succcurlyeq CE_2$ iff $u(CE_1) \geq u(CE_2)$. CMES uses $\text{Val}(\cdot)$ as a particular representation of the ordering together with additional structural assumptions.

Remark 2.2a.1 (Transformation Groups). The admissible transformations of a representation depend on the measurement structure imposed:

- *Ordinal* representation permits arbitrary strictly increasing transforms.
- *Interval* representation permits positive affine transforms ($u' = \alpha u + \beta$, $\alpha > 0$).
- *Ratio* representation permits positive similarity transforms ($u' = \alpha u$, $\alpha > 0$).

The multiplicative structure of the Master Equation therefore requires the ratio-scale step of Section 2.3b, not merely the existence of an order-preserving index.

Remark 2.2a.2 (What the Axioms Settle and What They Do Not). Axiom 5 and Theorem 2.1 establish additive representation on the four-role product space. They do not, by themselves, settle scale type. Section 2.3b supplies the additional extensive-structure assumptions that collapse admissible transformations to similarity maps and make $NTr \cdot \mathrm{Val}(CE)$ ratio-meaningful.

2.3 The Val(·) Functional Form Theorem

Theorem 2.1 (Existence of Additive Valuation). *Under Axioms 1–5, there exists a continuous function* $V: \mathcal{E} \to \mathbb{R}$ *representing the preorder* $\succcurlyeq$ *on* $\mathcal{E}$ *and continuous component functions* f_v, f_a, f_c, f_e *such that*

$$\mathrm{Val}(CE) = f_v(CE^v) + f_a(CE^a) + f_c(CE^c) + f_e(CE^e).$$

The representation is unique up to a common positive affine transformation.

Proof.

Step 1 (Continuous numerical representation). By Axioms 1–3, $\succcurlyeq$ is a continuous preorder on $\mathcal{E} = X_v \times X_a \times X_c \times X_e$. Since $\mathcal{E}$ is a connected, separable topological product, Debreu's (1954) representation theorem yields a continuous utility index $U: \mathcal{E} \to \mathbb{R}$ representing the preorder.

Step 2 (Essential Cartesian factors). By Axiom 4, each factor X_j is essential and the induced one-dimensional order on each coordinate is non-trivial.

Step 3 (Additive representation via Debreu 1960). By Axiom 5, every two-factor slice with the remaining coordinates fixed satisfies Thomsen cancellation for all six pairs (i, j) from the set $\{v, a, c, e\}$. With continuity and essentiality from Steps 1–2, Debreu (1960, topological additive representation) implies that there exist continuous functions f_j and a continuous strictly increasing outer map $G: \mathbb{R} \to \mathbb{R}$ such that:

$$U(CE) = G\big(f_v(CE^v) + f_a(CE^a) + f_c(CE^c) + f_e(CE^e)\big).$$

Step 4 (Normalization). Define $\mathrm{Val} = G^{-1} \circ U$. Then Val is continuous, represents the same preorder, and has the additive form.

Step 5 (Uniqueness). Standard additive conjoint uniqueness (Krantz et al., 1971, Theorem 6.6): the component functions are unique up to a common positive affine transformation with offset constants summing to zero. ▫

Remark 2.1.1 (Scale Type of $\mathrm{Val}(\cdot)$). Theorem 2.1 delivers interval-scale uniqueness: the representation is determined up to $\mathrm{Val}' = \alpha \cdot \mathrm{Val} + \beta, \alpha > 0$. This is the standard result from Debreu's topological theorem and is confirmed independently by the algebraic approach of Krantz–Luce–Suppes–Tversky (1971) and harmonized by Wakker (1988). The four dimensions (V, A, C, E) satisfy the $n \geq 3$ requirement of the theorem, and the Thomsen cancellation condition of Axiom 5 is precisely the testable axiom these theorems identify. Interval-scale uniqueness is *weaker* than ratio-scale uniqueness: it permits the additive constant β, which would destroy the multiplicative structure of the Master Equation. The resolution of this gap is the content of Section 2.3b.

Theorem 2.2 (Behavioral Evaluation under Uncertainty). *Under Axioms 1–6, the evaluative component can be represented by a Choquet integral with respect to a non-additive capacity, without altering the additive separability of Theorem 2.1.* ▫

2.3a CES Generalization and Nesting of Val(·)

Definition 2.3a.1 (Generalized CES Valuation). The global CES valuation is:

$$\mathrm{Val}(CE) = A \cdot \left(\sum_{i \in \{v,a,c,e\}} \alpha_i \cdot \left(CE^i\right)^{\varrho} \right)^{1/\varrho}$$

where $\alpha_i \geq 0, \sum \alpha_i = 1, A > 0$, and $\varrho \in (-\infty, 1]$.

Proposition 2.3a.1 (Boundary Convergences). (i) Leontief: As $\varrho \to -\infty$, $\mathrm{Val}(CE) \to A \cdot \min_i\left(CE^i\right)$. (ii) Cobb-Douglas: As $\varrho \to 0$, $\mathrm{Val}(CE) \to A \cdot \prod_i \left(CE^i\right)^{\alpha_i}$. (iii) Linear separability: As $\varrho \to 1$, $\mathrm{Val}(CE) \to A \cdot \sum_i \alpha_i \cdot CE^i$, recovering the additive structure of Theorem 2.1.

Theorem 2.3a.1 (Additive Form as First-Order Approximation). *The additively separable* $\mathrm{Val}(CE)$ *of Theorem 2.1 is the first-order Taylor approximation of the CES function around any reference bundle* CE^0*, exact iff all cross-partial derivatives vanish.* ▫

Remark 2.3a.1 (Kmenta Correspondence and Substitutability). Kmenta (1967) demonstrated that the first-order Taylor expansion of a CES function around $\varrho = 0$ (in logarithms) yields precisely the Cobb-Douglas log-linear form. The CMES additive form corresponds to $\varrho \to 1$—the perfect-substitutes limit. This carries a strong economic implication: additive separability in levels means a deficiency in any single dimension (e.g., zero Archived stock) can be fully compensated by surplus in any other dimension. Whether this substitutability is economically plausible across all four CMES dimensions requires explicit defense in specific applications. For the purpose of the axiomatic core, the additive form is retained as the canonical specification with the understanding that it constitutes a first-order approximation within a neighborhood of the reference bundle. The Diewert (1974) flexibility criterion—requiring that a functional form match arbitrary first and second derivatives at a point—is not satisfied by any first-order approximation; CMES acknowledges this as a theoretical limitation addressed through the CES nesting structure of this section.

Remark 2.3a.2 (Baqaee–Farhi Second-Order Effects). Baqaee and Farhi (2019, Theorem 2) prove that second-order macroeconomic impacts of microeconomic shocks depend on structural elasticities of substitution, network linkages, and factor reallocation—all terms that vanish under the unit-elasticity (Cobb-Douglas/additive) approximation. In environments with strong complementarities, negative shocks are amplified by 10–100% relative to the first-order prediction. Nakano and Nishimura (2024) further show that for non-unit elasticities, Domar aggregation becomes inherently nonlinear, with "network singularities" marking thresholds where equilibrium collapses. These results imply that the CMES additive baseline may understate propagation effects in highly complementary production networks. The nested CES extension below addresses this limitation by admitting variable substitution elasticities.

Remark 2.3a.3 (Uzawa Impossibility and Nested CES). Uzawa's (1962) impossibility theorem establishes that for $n > 2$ inputs, the only production functions with constant partial elasticities of substitution between all factor pairs must either have all elasticities identical or degenerate. A flat four-input CES over (V, A, C, E) therefore forces identical substitution elasticities between all six pairs. If—as is economically plausible—the substitutability between Value-Adding and Archived differs from that between Communication and Evaluation, then nested CES structures are required (Sato, 1967; Keller, 1976). Blackorby, Primont, and Russell (1978) established the deep structural theorem that additive separability plus homotheticity implies a CES-like functional form, which simultaneously validates the CMES narrative that Val(·) relates to CES through separability, while confirming that additive separability is a strong restriction. Lagomarsino (2020) demonstrates that nested CES structures over four inputs can be formally identified and tested. To accommodate asymmetric substitution, CMES admits the following nested generalization:

Definition 2.3a.2 (Nested CES Valuation). A nested CES valuation function over the CMES four-role space takes the form:

$$\mathrm{Val}(CE) = A \cdot \left[\beta_1\left(\alpha_v (CE^v)^{\varrho_1} + \alpha_a (CE^a)^{\varrho_1}\right)^{\varrho_2/\varrho_1} + \beta_2\left(\alpha_c (CE^c)^{\varrho_2} + \alpha_e (CE^e)^{\varrho_2}\right)\right]^{1/\varrho_2}$$

where ϱ_1 governs substitutability within the physical-productive nest (V, A) and ϱ_2 governs substitutability within the coordination nest (C, E) and across nests. The flat additive CES is the limiting case $\varrho_1 = \varrho_2 \to 1$.

Theorem 2.3a.2 (Additive Form Validity Domain). *The additive form of Theorem 2.1 is exact on the domain $\mathcal{E}$ if and only if all pairwise elasticities of substitution between role components equal unity. For bounded deviations from unit elasticity, the first-order error is* $O(|\varrho - 1|)$*. The nested CES form of Definition 2.3a.2 recovers the additive baseline when* $\varrho_1 = \varrho_2 = 1$ *and provides a flexible approximation for general elasticity configurations.* ▫

2.3b Ratio-Scale Identification

The central requirement of the Master Equation is that the expression $NTr \cdot \mathrm{Val}(CE)$ be invariant under admissible transformations of Val(·). Since multiplication by a transaction count is meaningful only on a ratio scale—interval scales permit an additive constant β

that destroys proportionality—Theorem 2.1's interval-scale guarantee is insufficient. This section establishes ratio-scale identification through three complementary pathways.

Pathway 1: Structural Zero and Concatenation (Axiom 8).

Axiom 8 (Structural Zero and Concatenation Closure). There exists a bundle $CE^0 \in \mathcal{E}$ such that $\mathrm{Val}(CE^0) = 0$ (the empty transaction system). The structural zero CE^0 is identified with the empty composite group $\emptyset$ (the system with no elements in any role class), not with the vector origin $\mathbf{0} \in \mathbb{R}^4$; this distinction aligns CE^0 with the monoid identity under concatenation. For any $CE, CE' \in \mathcal{E}$, there exists $CE \oplus CE' \in \mathcal{E}$ satisfying $\mathrm{Val}(CE \oplus CE') = \mathrm{Val}(CE) + \mathrm{Val}(CE')$.

Proposition 2.3b.1 (Archimedean Structure). *The induced operation* $\oplus$ *is associative, commutative, monotone, and* $CE \oplus CE^0 \sim CE$*. The induced ordered semigroup is Archimedean.*

Theorem 2.3b.2 (Ratio-Scale Uniqueness). *If* V' *is another continuous representation preserving order and concatenation with* $V'(CE^0) = 0$*, then* $V' = \alpha V$ *for some* $\alpha > 0$*. Hence* V *is a ratio scale.*

Proof. By Theorem 2.1, $V' = \alpha V + \beta$. By $V'(CE^0) = V(CE^0) = 0$: $\beta = 0$. The admissible transformation group is $\{x \mapsto \alpha x : \alpha > 0\}$. ▫

Pathway 2: Commutative Monoid and Extensive Measurement.

Axiom 9 (Commutative Monoid of Concatenation). The pair $(\mathcal{C}, \oplus)$ where $\mathcal{C}$ is the set of composite element bundles and $\oplus$ is the concatenation operator of Axiom 8, is a commutative monoid: (i) $\oplus$ is associative: $(CE \oplus CE') \oplus CE'' = CE \oplus (CE' \oplus CE'')$; (ii) $\oplus$ is commutative: $CE \oplus CE' = CE' \oplus CE$; (iii) CE^0 is the identity element: $CE \oplus CE^0 = CE$ for all CE; (iv) $\oplus$ is monotone: $CE \succ CE'$ implies $CE \oplus CE'' \succ CE' \oplus CE''$.

This commutative monoid structure is precisely the algebraic object that measurement theory (Krantz et al., 1971, Vol. I, Ch. 3) identifies as the foundation for ratio-scale assignment—an *extensive measurement structure* in the sense of Helmholtz–Hölder. Under Axiom 9, the $(\mathcal{C}, \oplus, \succcurlyeq)$ triple constitutes an extensive structure, and the representation theorem for extensive structures guarantees that the representing function is unique up to positive similarity (multiplicative) transformations.

Remark 2.3b.1 (Option B: Group Completion). If CMES requires formal subtraction of bundles—for example, to represent net new value-adding capacity or to perform netting analogous to intermediate transaction subtraction in national accounts—the commutative monoid $(\mathcal{C}, \oplus)$ can be universally embedded into an abelian group via Grothendieck group completion $K(\mathcal{C})$. The completion introduces formal differences $[CE] - [CE']$ as equivalence classes of pairs, providing a rigorous algebraic sense in which "group theory" enters the CMES framework while keeping the primary ontology monoidal. This is the CMES-appropriate form of Option B in the algebraic taxonomy.

Remark 2.3b.2 (Option C: Group Actions as Symmetry/Invariance). If "group theory" in the GTE sense refers to equivalence of systems under transformations—renaming, relabeling, re-ordering of elements—then CMES can define a group $\mathcal{G}$ acting on the set of composite groups $\mathcal{S}$ and require that $\mathrm{Val}(\cdot)$ and $\Xi_F(\cdot)$ be invariant (or equivariant) under that action. This formalization captures the intuition that the identity of specific elements within a role class is irrelevant to valuation—only cardinalities and role assignments matter. The admissible transformations then form the automorphism group of the poset of composite groups.

Pathway 3: Luce–Narens Scale Classification.

Theorem 2.3b.3 (Luce–Narens Scale-Type Identification). *Let* $\mathrm{Val}(\cdot)$ *be the representation established in Theorem 2.1, augmented with Axioms 8–9. Under the classification of Luce and Narens (1985), the structure* $(\mathcal{E}, \succcurlyeq, \oplus, CE^0)$ *is a 1-point homogeneous, 1-point unique structure. The admissible automorphisms are precisely the similarity (ratio-scale) transformations* $\{x \mapsto \alpha x : \alpha > 0\}$.

Proof. Luce and Narens (1985) establish that a structure is at most 1-point homogeneous and 1-point unique if and only if it is a ratio scale. By Theorem 2.3b.2, the admissible group collapses to similarity maps; by Axiom 9's extensive structure, the structure is 1-point homogeneous (for any CE, CE' with $\mathrm{Val}(CE) \neq 0$, there exists $\alpha > 0$ such that $\mathrm{Val}' = \alpha\mathrm{Val}$ maps CE to CE') and 1-point unique (the automorphism is fully determined by its value at one point). ▫

2.3b.3 The Concatenation Monoid

Definition 2.3b.3 (Concatenation on Comparable Bundles). Let $\mathcal{C}$ be the set of comparable transaction bundles under the chosen boundary convention. Define a binary operation $\oplus$ on $\mathcal{C}$ by bundle concatenation (or repeated execution under a fixed protocol), with identity element CE^0. Under Axiom 8 and Proposition 2.3b.1, $(\mathcal{C}, \oplus, CE^0)$ forms a commutative ordered monoid.

Interpretation. The monoid structure is the algebraic object required for extensive-style measurement claims. If Val preserves order and is additive over concatenation on comparable bundles, then for $n \in \mathbb{N}$,

$$\mathrm{Val}\left(\underbrace{CE \oplus CE \oplus \cdots \oplus CE}_{n\ \text{times}}\right) = n \cdot \mathrm{Val}(CE),$$

and multiplicative expressions involving transaction counts inherit unit invariance under similarity transformations $\mathrm{Val}' = \alpha \cdot \mathrm{Val}$ with $\alpha > 0$.

Proposition 2.3b.3 (Invariance of the Master Product Under Units). *Under Theorem 2.3b.2 and the ordered-monoid interpretation above,* $NTr \cdot \mathrm{Val}(CE)$ *and*

$$\sigma = \frac{NTr_1 \cdot \mathrm{Val}(CE_1)}{NTr_2 \cdot \mathrm{Val}(CE_2)}$$

are invariant up to common units. Hence the Master Equation is meaningful as an equality of ratio-scale transaction value magnitudes rather than an interval-scale identity.

Remark 2.3b.4 (Optional Group Completion). If a later CMES extension requires signed differences of bundles (for example, explicit netting over intermediate transactions), one may pass from the commutative monoid to its Grothendieck group completion without changing the primary ontology of composite groups as set- or multiset-based system objects.

Corollary 2.3b.1 (Resolution of the Cardinality Leap). *Under Theorems 2.3b.2–2.3b.3, expressions* $NTr \cdot \mathrm{Val}(CE)$ *and cross-system ratios are well-defined and scale-invariant. The Master Equation* $NTr_1 \cdot \mathrm{Val}(CE_1) = NTr_2 \cdot \mathrm{Val}(CE_2) \cdot \sigma$ *is ratio-meaningful.*

Corollary 2.3b.2 (Well-Definedness of σ). *The transformation scalar* $\sigma = NTr_1 \cdot \mathrm{Val}(CE_1)/\left(NTr_2 \cdot \mathrm{Val}(CE_2)\right)$ *is invariant under admissible (ratio-scale) transformations of* $\mathrm{Val}(\cdot)$*. Under interval-scale transformations,* σ *would depend on the additive constant* β *and be scale-dependent—hence the ratio-scale requirement is not optional but structurally necessary for the Master Equation.*

2.3c Topological Separability: Non-Differentiable Foundations

Proposition 2.3c.1 (Topological Joint Separability). *Let* $\mathcal{E} = X_v \times X_a \times X_c \times X_e$*, with each factor connected and separable, and* $\succcurlyeq$ *a continuous weak order with all four factors essential. If for every pair* (i,k) *the Thomsen–Debreu cancellation condition holds, then there exist continuous* $f_j\colon X_j \to \mathbb{R}$ *such that*

$$CE \succcurlyeq CE' \Leftrightarrow \sum_j f_j\left(CE^j\right) \geq \sum_j f_j\left(CE'^j\right).$$

No differentiability of $\mathrm{Val}(\cdot)$ *is assumed. This result holds globally on* $\mathcal{E}$ *under the pairwise cancellation specification of Axiom 5; for restricted domains or non-connected factors, higher-order cancellation conditions must be verified.* ▫

2.3d The Concatenation Operator and Its Economic Interpretation

The concatenation operator $\oplus$ introduced in Axioms 8–9 admits two standard mathematical interpretations, each with distinct economic content.

Vector-Space Interpretation. Under this interpretation, $CE \in \mathbb{R}_+^4$ and $CE_1 \oplus CE_2 = CE_1 + CE_2$ is coordinate-wise vector addition. This requires the domain to be closed under addition (satisfied for $\mathbb{R}_+^4$) and $\mathrm{Val}(\cdot)$ to be linear, which is consistent with the additive form when f_j are linear. The vector interpretation supports the "net new capacity" reading: $CE_1 \oplus N$ represents system 1 augmented by an incremental bundle $N = (N^v, N^a, N^c, N^e)$.

Concatenation/Monoid Interpretation. Under this interpretation, $CE_1 \oplus CE_2$ represents the disjoint union of two transaction systems—system 1 combined with system 2 as a larger composite group. This aligns with the extensive measurement tradition and with the CMES poset architecture: merging two composite groups at the same hierarchical level produces a larger composite group. The identity element CE^0 is the empty composite group with zero elements in all four roles.

Remark 2.3d.1 (Macro-Level Decomposition). The master equation in additive augmentation form:

$$NTr_1 \cdot \mathrm{Val}(CE_1) = NTr_2 \cdot \mathrm{Val}(CE_1 \oplus N)$$

establishes that if the system is to maintain its transaction volume NTr while expanding its foundational capacity, the valuation of the expanded set $\mathrm{Val}(CE_1 \oplus N)$ must directly offset the baseline interaction logic. The decomposition $N = N^v + N^a + N^c + N^e$ provides a complete algebraic description of growth by role class. Under additive separability, $\mathrm{Val}(CE_1 \oplus N) = \mathrm{Val}(CE_1) + \mathrm{Val}(N)$, and the growth equation becomes $NTr_1/NTr_2 = \mathrm{Val}(CE_1 + N)/\mathrm{Val}(CE_1) > 1$—transaction volume scales inversely with the per-transaction value enhancement.

2.3e CMES vs. Traditional Gross Value Added: A Mathematical Comparison

The CMES transaction-multiplier formulation $NTr \cdot \mathrm{Val}(CE)$ constitutes a structurally distinct approach to aggregate economic measurement compared to the classical Gross Value Added (GVA) methodology. The following formal comparison clarifies the mathematical differences and CMES's conceptual advantages.

Classical GVA. In national accounts, Gross Value Added is defined as output minus intermediate consumption: $\mathrm{GVA} = P \cdot Y - P_m \cdot M$, where Y is gross output, M is intermediate input, P is output price, and P_m is input price. Statistical agencies operationalize this through input-output tables (System of National Accounts, 2008). The GVA method eliminates double-counting by netting out intermediate flows. The primary limitation is that this subtraction collapses the structural geometry of the production network: a highly decentralized multi-node supply chain is mathematically indistinguishable from a vertically integrated monopoly if final outputs are identical. GVA ignores the structural vitality, resilience, and communication overhead required to maintain intermediate transaction networks.

CMES Transaction Method. The CMES measure $NTr \cdot \mathrm{Val}(CE) \cdot \Xi_F$ preserves network structure by operating at the transaction level. Double-counting is resolved not by erasing data but by adjusting fractional overlaps within Ξ_F: the adjusted element counts NE_{adjusted} prevent double-counting by measuring strict set intersections between exchanged layers ($\mathrm{Ef}_{e_x e_n}$), isolating pure value contributions without collapsing the graph structure. The transaction multiplier NTr acknowledges that high-frequency transaction networks possess an inherent velocity that GVA methods mask.

Analytical Metric	Traditional Gross Value Added (GVA)	CMES Transaction Method ($NTr \cdot \mathrm{Val}(CE)$)
Mathematical Basis	Gross Output minus Intermediate Inputs	Volume of nodes interacting (NTr) multiplied by multi-dimensional element value $\mathrm{Val}(\cdot)$
Treatment of Intermediate Flow	Subtracted and netted out; viewed as accounting distortion	Analyzed natively; each transaction is an active application of Communication (CE^c) and Evaluation

Analytical Metric	Traditional Gross Value Added (GVA)	CMES Transaction Method ($NTr \cdot \mathrm{Val}(CE)$)
		(CE^e) elements
Structural Context	Mathematically agnostic to supply chain topology and network graphs	Mediated explicitly by Network Effectiveness ($\varXi_F$) and poset topology
State Preservation	Treats capital broadly; does not model informational path-dependence	Explicitly models physical and informational archives (CE^a) via state variable S_t and law of motion
Scale Type	Additive decomposition, aggregate flow	Ratio-scale multiplicative structure, network-structured flow
Double-Counting Resolution	Subtraction eliminates intermediate transactions	$\varXi_F$ overlap-fraction adjustment preserves topology

Table 2.1: Mathematical comparison of GVA and the CMES transaction method.

Remark 2.3e.1 (Interpretation as Nominal Flow Index). The expression $NTr \cdot \mathrm{Val}(CE)$ is formally analogous to the "transactions side" of the Fisher equation of exchange ($MV = PT$), where T is total transaction volume and P is the price level. In the CMES context, $\mathrm{Val}(CE)$ behaves as an average per-transaction value index (not necessarily a market price), and $NTr \cdot \mathrm{Val}(CE)$ behaves as an aggregate nominal transaction-flow index of the system. This is *not* equivalent to GVA in the SNA sense unless intermediate transactions are explicitly adjusted: CMES should therefore be understood as providing a gross transaction-flow measure whose relationship to net value-added requires explicit specification of how intermediate transactions are treated in the $\varXi_F$ overlap computation.

2.6 The Universal Role Basis and the Classification Map

Definition 2.6.1 (Role Classification). Let $\mathcal{E}$ be a universal set of economic elements. A role map $\rho: \mathcal{E} \to \{v, a, c, e\}$ assigns a functional role: (v) Value-Adding; (a) Archived Information and Stored Material for Production; (c) Communication; (e) Evaluation.

Definition 2.6.2 (Operational Removal Test). For any element, its role is determined by asking: if removed, would the system lose the capacity to (i) transform inputs? (ii) preserve state/history/stored material? (iii) connect/coordinate? (iv) evaluate/constrain?

2.6.3 MECE Convention and Multi-Role Treatment

Definition 2.6.3 (MECE Convention). For theorem statements concerning cardinalities or overlaps, a Mutually Exclusive and Collectively Exhaustive (MECE) partition is obtained by adopting a primary-role assignment $\rho(e, t)$ selecting a dominant role for each element. When an element is genuinely multi-role at the chosen resolution, one of three conventions applies: (i) decompose into role-pure sub-elements; (ii) assign the role responsible for the primary causal contribution (dominance criterion); or (iii) retain a multi-role signature vector $s(e, t) \in [0,1]^4$ encoding the element's role-mix.

2.6.4 Cross-Domain Instantiation of the Role Taxonomy

Table 2.2 demonstrates the universality of the four-role taxonomy across distinct economic domains. In each case, the same four functional roles are present, confirming that the taxonomy constitutes a universal role basis for economic systems.

Domain	Value-Adding (v)	Archived (a)	Communication (c)	Evaluation (e)
Computing	CPU/GPU cores	Storage, RAM, logs	Network stack, APIs	OS scheduler, debugger
Manufacturing	Assembly, machining	Inventory, warehouses	Logistics, ERP	Quality control, audits
Trade	Production capacity	Ledgers, reserves	Markets, payment rails	Prices, regulation
Science	Experiments, R&D	Datasets, literature	Journals, collaboration	Peer review, ethics

Table 2.2: Cross-Domain Instantiation of the CMES Role Taxonomy. The same four functional roles appear in every domain, confirming the universality of the (v, a, c, e) *basis.*

3. The Master Equation and Transaction Structure

3.1 Derivation of the Master Equation

By Axiom 7, for any two systems 1 and 2:

$$NTr_1 \cdot \mathrm{Val}(CE_1) = NTr_2 \cdot \mathrm{Val}(CE_2) \cdot \sigma$$

where $\sigma = \frac{NTr_1 \cdot \mathrm{Val}(CE_1)}{NTr_2 \cdot \mathrm{Val}(CE_2)}$ is the transformation scalar. By convention, system 1 is the higher-value system, so $\sigma \geq 1$.

The mathematical cleanest justification for this structure defines total transaction value of system i as $TV_i := \sum_{k=1}^{NTr_i} v_i\,(k)$, imposes a representativeness condition such that $v_i(k) = \mathrm{Val}(CE_i)$ for all transactions k (interpreting $\mathrm{Val}(CE_i)$ as the mean transaction value), so $TV_i = NTr_i \cdot \mathrm{Val}(CE_i)$, and defines $\sigma := TV_1/TV_2$. Under ratio-scale admissibility (Theorems 2.3b.2–2.3b.3), σ is well-defined as a ratio of total values and invariant under admissible transformations.

Remark 3.1.1 (Component Decomposition). Since $\mathrm{Val}(\cdot)$ is additively separable (Theorem 2.1):

$$\mathrm{Val}\left(NE^i\right) = \mathrm{Val}_v\left(NE_v^i\right) + \mathrm{Val}_a\left(NE_a^i\right) + \mathrm{Val}_c\left(NE_c^i\right) + \mathrm{Val}_e\left(NE_e^i\right) \quad \text{for } i \in \{1,2\}$$

where each NE^i denotes exchanged elements of system i.

Remark 3.1.2 (Difference Form). When $NTr_1 = NTr_2$:

$$\mathrm{Val}(NE^1) = \mathrm{Val}(NE^2) + \mathit{\Delta}$$

where $\mathit{\Delta} = \mathrm{Val}(NE^1) - \mathrm{Val}(NE^2)$ decomposes as $\mathit{\Delta} = \mathit{\Delta}^v + \mathit{\Delta}^a + \mathit{\Delta}^c + \mathit{\Delta}^e$.

Remark 3.1.3 (Analogy with Equation of Exchange). The Master Equation shares the formal structure of the transactions form of the Fisher equation of exchange: in both, an aggregate flow is expressed as the product of a frequency count (NTr or MV) and a per-unit value index ($\mathrm{Val}(CE)$ or P). This suggests interpreting $\mathrm{Val}(CE)$ as a structural analog of a quality-adjusted price index, with $NTr \cdot \mathrm{Val}(CE)$ representing the nominal transaction-flow of the economic system.

3.1b Gross Transaction Value and Net Value-Added Content

Definition 3.1b.1 (Gross Transaction Value). For a transaction system in a period, define gross transaction value (GTV) by

$$\mathrm{GTV} := NTr \cdot \mathrm{Val}(CE).$$

The Master Equation in Section 3.1 is stated at the GTV level unless otherwise specified.

Definition 3.1b.2 (Net Value-Added Content). Let IC denote the composite bundle of intermediate consumption used up in producing the observed transaction outputs under the same boundary convention. Define net value-added content by

$$\mathrm{NVA} := \mathrm{GTV} - \mathrm{Val}(IC).$$

Remark 3.1b.1 (Boundary Discipline). The distinction between GTV and NVA is structural, not empirical. It clarifies whether CMES is describing throughput/gross exchange magnitudes or net contribution after intermediate use. Both are valid CMES objects once the transaction boundary and concatenation protocol are fixed.

Proposition 3.1b.1 (No Double-Counting by Poset Decomposition). *In an inclusive-chain decomposition, double counting is avoided either (i) by netting via IC in NVA, or (ii) by retaining gross flows and allocating intrinsic contributions through poset decomposition and Network Effectiveness weights. The manuscript uses route (ii) as the primary theorem path and route (i) as a definitional comparison layer.*

3.1a Algebraic Properties of σ

Property 1 (Transitivity). $\sigma_{1,3} = \sigma_{1,2} \cdot \sigma_{2,3}$.

Proof. From $NTr_1 \cdot \mathrm{Val}(CE_1) = NTr_2 \cdot \mathrm{Val}(CE_2) \cdot \sigma_{1,2}$ and $NTr_2 \cdot \mathrm{Val}(CE_2) = NTr_3 \cdot \mathrm{Val}(CE_3) \cdot \sigma_{2,3}$: substitute to obtain $NTr_1 \cdot \mathrm{Val}(CE_1) = NTr_3 \cdot \mathrm{Val}(CE_3) \cdot \sigma_{1,2} \cdot \sigma_{2,3}$. ▫

Property 2 (Symmetry). $\sigma_{1,2} = 1/\sigma_{2,1}$.

Property 3 (Component Decomposition). *Under Theorem 2.1:*

$$\sigma = \sum_i w_{i,2} \cdot \sigma_i^{\mathrm{comp}}, \quad w_{i,2} = \frac{\beta_{i2} \cdot CE_{i2}}{\mathrm{Val}(CE_2)}, \quad \sigma_i^{\mathrm{comp}} = \frac{NTr_1}{NTr_2} \cdot \frac{\beta_{i1} \cdot CE_{i1}}{\beta_{i2} \cdot CE_{i2}}.$$

3.2 Competitive Regime and the TFP Interpretation of σ

Theorem 3.1 (Productivity Identification under Competitive Equilibrium). *Under: (H1) CRS technology* $Y_i = A_i \cdot F(K_i, L_i)$*; (H2) zero-economic-profit equilibrium; (H3) common factor prices; (H4) equal factor proportions and endowments—the transformation scalar admits the identification:*

$$\sigma = \frac{A_1}{A_2} = \frac{\mathrm{TFP}_1}{\mathrm{TFP}_2}.$$

Proof. Under H1 and H4, $Q_1/Q_2 = A_1F(\kappa, 1)L_1/[A_2F(\kappa, 1)L_2] = A_1/A_2$. By H2–H3, the dual unit-cost function $c(w, r)$ is common, giving $P_i = c(w, r)/A_i$. Hence $\sigma = (P_1Q_1)/(P_2Q_2) = A_1/A_2$. ▫

Remark 3.2.1 (TFP Identification Conditions). The identification $\sigma = A_1/A_2$ holds within a specific regime delineated by hypotheses H1–H4. Outside this regime—particularly in the presence of markups, distortions, or unequal factor endowments—the identification must be modified. This regime is distinct from the Ξ_F-driven markup regime of Section 7: the competitive dual-cost regime is used for σ identification only, while market valuations in Section 7 explicitly incorporate rent-capturing Ξ_F multipliers. Criterion C of Section 10.3 formalizes this regime distinction.

Corollary 3.1.1 (Endowment Asymmetry and the Scarcity Premium). *When hypothesis (H4) is relaxed—i.e., systems 1 and 2 have unequal factor proportions* $\kappa_1 \neq \kappa_2$ *or unequal endowments* $L_1 \neq L_2$*—the transformation scalar absorbs the asymmetry:*

$$\sigma = \frac{A_1}{A_2} \cdot \frac{F(\kappa_1, 1)}{F(\kappa_2, 1)} \cdot \frac{L_1}{L_2}.$$

The ratio $F(\kappa_1, 1)/F(\kappa_2, 1)$ *captures the scarcity premium: when system 1 is capital-abundant relative to system 2, this factor exceeds unity and* σ *exceeds the pure TFP ratio. Under CRS and common factor prices, this premium is interpretable as the rent on the scarce factor.*

Definition 3.2b.1 (Marginal Cost under Cobb-Douglas). For $Y = AK^{\alpha}L^{1-\alpha}$:

$$MC = \frac{W}{MPL} = \frac{WL}{(1-\alpha)Y} = \frac{WL^{\alpha}}{(1-\alpha)AK^{\alpha}}.$$

3.3 The Additive Representation and Value Differential

The master equation in additive form:

$$NTr_1 \cdot \mathrm{Val}(CE_1) = NTr_2 \cdot \mathrm{Val}(CE_1 \oplus N)$$

where $N = (N^v, N^a, N^c, N^e)$ is the augmentation vector. The value differential:

$$\Delta = \mathrm{Val}(CE_1) - \mathrm{Val}(CE_2) = \sum_{j \in \{v,a,c,e\}} w_j \cdot \left[f_j\left(CE_1^j\right) - f_j\left(CE_2^j\right)\right]$$

Proposition 3.5 (Properties of the Value Differential). *By convention* $\sigma \geq 1$, $\mathrm{Val}(CE_1) \geq \mathrm{Val}(CE_2)/\sigma$. *If* $NTr_1 = NTr_2$, *then* $\Delta \geq 0$. ▫

4. Poset and Nested-Group Architecture of Economic Systems

Throughout this section and all subsequent sections, the term *composite group* (or *group of composites*) replaces the prior use of "algebraic group." Every transaction system is modeled as a composite group of elements, ordered by set inclusion into a partially ordered set (poset). No algebraic group theory axioms (closure, identity, inverses) are assumed; the algebraic content is captured instead through the commutative monoid structure of Axiom 9.

4.1 Composite Groups as Transaction Systems

Definition 4.1 (Composite Group as a Transaction System). A composite group at hierarchical level n is a tuple:

$$G_n = \langle N_n^v(t), N_n^a(t), N_n^c(t), N_n^e(t) \rangle$$

where $N_n^j(t) \geq 0$ is the cardinality of the role-j subset at time t, for $j \in \{v, a, c, e\}$. The total element count is:

$$N_n^t(t) = N_n^v(t) + N_n^a(t) + N_n^c(t) + N_n^e(t) = |G_n(t)|$$

Definition 4.2 (Substructure via Disjoint Union). The composite group $G_n(t)$ is the disjoint union of its four role-specific subsets:

$$G_n(t) = G_n^v(t) \sqcup G_n^a(t) \sqcup G_n^c(t) \sqcup G_n^e(t)$$

where $\sqcup$ denotes disjoint union (guaranteed by the primary role assignment ρ of Definition 2.6.1). Each subset is:

$$G_n^j(t) = \{g_{n,j,k} : k = 1, \dots, N_{n,j}(t)\}, \quad \left|G_n^j(t)\right| = N_{n,j}(t).$$

Definition 4.2a (Valuation of a Composite Group). For any composite group G_n, the valuation is defined by applying $\mathrm{Val}(\cdot)$ to its composite element bundle:

$$\mathrm{Val}\left(CE_{G_n}\right) = \mathrm{Val}(CE_n^v + CE_n^a + CE_n^c + CE_n^e) = \sum_{j \in \{v,a,c,e\}} f_j\left(CE_n^j\right)$$

This establishes that $\mathrm{Val}(\cdot)$ applies to every composite group at every level of the hierarchy.

4.2 Network Topology: Inclusive and Non-Inclusive Composite Groups

Definition 4.3 (Inclusive Composite Group Network). A network is a set of composite groups $\{G_1, G_2, \dots, G_N\}$ satisfying:

$$G_n \subseteq G_{n+1}, \quad \forall\, n < N.$$

This defines a chain in the poset of composite groups.

Definition 4.4 (Non-Inclusive Composite Group Network). A network of composite groups $\{G_1, G_2, \dots, G_N\}$ where:

$$G_n \nsubseteq G_{n+1}, \quad \forall\, n \neq n+1.$$

Proposition 4.1 (Conservation of Transaction Value in Inclusive Networks). *For an inclusive chain* $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_N$*:*

$$NTr_n(t) \cdot \mathrm{Val}\big(NE_n(t)\big) = NE_x(t) \cdot \left(\frac{NTr_x(t)}{NTr_n(t)} - 1\right)$$

This is a necessary consequence of the master equation applied along the chain. ▫

Theorem 4.2 (Transactions Between Composite Groups). *For the entire network of composite groups, the master equation governs:*

$$NTr_1 \cdot \mathrm{Val}(NE_1) = NTr_2 \cdot \mathrm{Val}(NE_2) \cdot \sigma$$

where each NE_i *decomposes as* $\mathrm{Val}(NE_i) = \mathrm{Val}(NE_i^v + NE_i^a + NE_i^c + NE_i^e)$ *by additive separability (Theorem 2.1).*

4.3 Hierarchical Composite Group Structure

Definition 4.5 (Hierarchical Inclusion). For a hierarchy of inclusive composite groups:

$$G_x \subseteq G_{1x} \subseteq G_{2x} \subseteq \cdots \subseteq G_{Nx}$$

where at each level k, G_{kx} is a non-inclusive member of the level-k network: $G_{kx} \nsubseteq G_{k1}, \dots, G_{kx} \nsubseteq G_{kN}$.

Proposition 4.3 (Element Count Evolution). *The number of elements in composite group* G_n *evolves as:*

$$N_n(t) = NE_x(t) \cdot \left(\frac{NTr_x(t)}{NTr_n(t)} - 1\right)$$

In the value-adding dominated case:

$$NTr_n(t) = N_{v,n}(t) + NE_{v,x}(t)$$

4.4 Pattern, Form, and Model as Algebraic Closures

Definition 4.6 (Pattern). A pattern is a sub-ensemble $P \subseteq G_n$ that is relatively closed under the interaction operator: for any $x, y \in P$, their composition $x \otimes y$ (if defined) remains within P.

Definition 4.7 (Form). A form is a hierarchical composition of patterns satisfying global invariants, representing stable system architectures.

Definition 4.8 (Model). A model is a time-indexed family of forms $\{\Phi(t)\}_{t\in T}$ together with an evolution operator Δ describing form transitions.

Theorem 4.4 (Algebraic Extraction of Patterns). *Given a composite group* G *with elements* $X = \{x_1, x_2, \dots, x_{|X|}\}$*, actions* Σ_x*, and composition* $f(g \circ \sigma)$ *for* $g \in G_n$*,* $\sigma \in \Sigma_n$*:*

$$X = \{f(x) \mid x \in X_N\}$$

Any X *requires computation of* Σ_x *(or multiple compositions) to extract patterns, forms, and models.* ▫

4.4a Algebraic Semantics Compatible with the Poset Ontology

CMES uses composite groups primarily as set- or multiset-based system objects ordered by inclusion, not as algebraic groups in the strict sense. This subsection makes precise how algebraic structures may be added without changing that ontology.

Definition 4.4a.1 (Concatenation Monoid of Composite Bundles). Under the boundary protocol of Section 2.3b, the collection of comparable composite bundles with concatenation $\oplus$ and empty bundle CE^0 is a commutative monoid. This monoid supplies the algebraic backbone for ratio-scale and repetition arguments.

Definition 4.4a.2 (Group Completion for Signed Differences). If formal subtraction is needed, the Grothendieck completion $K(\mathcal{C})$ of the concatenation monoid $\mathcal{C}$ yields an abelian group of formal differences $[CE_1] - [CE_2]$. This is an optional extension layer and does not reclassify composite groups themselves as algebraic groups.

Definition 4.4a.3 (Symmetry Group Actions). Let H act on the set of composite groups by relabeling or symmetry transformations. A CMES quantity Q is H-invariant if $Q(h \cdot G) = Q(G)$ for all $h \in H$. $\mathrm{Val}(\cdot)$ and $\Xi_F(\cdot)$ may be required to satisfy invariance or equivariance conditions under a declared action when symmetry is part of the modeling resolution.

Remark 4.4a.1 (Ontological Discipline). The definitions above ensure that algebraic language enters CMES in a controlled manner: the primary objects (composite groups) remain set-theoretic, while algebraic operations (concatenation, group completion, symmetry actions) are layered on top as needed for specific results. This prevents the conflation of “group of composites” (a CMES system object) with “algebraic group” (a mathematical structure).

4.5 Möbius Inversion on the CMES Poset

Assumption 4.5.1 (Local Finiteness). The poset of composite groups relevant to a given decomposition is locally finite: every interval $[G_i, G_j]$ contains finitely many composite groups. This allows the incidence algebra and Möbius function $\mu(G_i, G_j)$ to be defined and ensures that all summations in the inversion formula are finite.

Economic Justification. In any physical or informational supply chain, a bounded interval of transaction stages $[G_i, G_j]$ cannot contain an infinite regression of intermediate sub-networks. The presence of strictly positive, discrete transaction costs—in the form of

Communication (CE^c) and Evaluation (CE^e) overhead—guarantees that the network granularity is bounded below, ensuring the poset is inherently locally finite.

The CMES framework's use of partially ordered sets for supply chain decomposition connects to the incidence algebra theory of Rota (1964), which provides the most powerful tool for decomposing cumulative quantities over posets into intrinsic contributions.

Definition 4.9 (Incidence Algebra). Let $(P, \leq)$ be the locally finite poset of composite groups ordered by set inclusion. The *incidence algebra* $\mathcal{I}(P)$ consists of functions $f{:}\mathrm{Int}(P) \to \mathbb{R}$ defined on intervals $[x, y] = \{z \in P : x \leq z \leq y\}$, with multiplication given by convolution:

$$(f * g)(x, y) = \sum_{z : x \leq z \leq y} f(x, z) \cdot g(z, y).$$

The identity element is the Kronecker delta $\delta(x, y) = \mathbb{1}[x = y]$. The multiplicative inverse of the constant-one function $\zeta(x, y) = 1$ is the Möbius function $\mu{:}\mathrm{Int}(P) \to \mathbb{Z}$, defined recursively by:

$$\mu(x, x) = 1, \quad \mu(x, y) = -\sum_{x \leq z < y} \mu(x, z) \quad (x < y).$$

Theorem 4.5 (Möbius Inversion Decomposition). *For any locally finite poset of composite groups $(P, \leq)$ and cumulative value function $g: P \to \mathbb{R}$ defined by:*

$$g(x) = \sum_{y \leq x} f(y), \quad x \in P$$

the intrinsic value contribution of each composite group is recovered by:

$$f(x) = \sum_{y \leq x} \mu(y, x) \cdot g(y).$$

Applied to the CMES supply chain: if $g(G_k)$ is the total accumulated value at hierarchical stage G_k (summing all predecessor contributions), then $f(G_k) = \mathrm{Val}(CE_{G_k})$ is the intrinsic contribution of group G_k itself, recovered via Möbius inversion.

Proof. Standard result from Rota (1964, Proposition 2). The Möbius inversion formula $f = \mu * g$ holds in the incidence algebra $\mathcal{I}(P)$ by definition of μ as the inverse of ζ. ▫

Corollary 4.5.1 (Canonical Decomposition Uniqueness). *The Möbius inversion decomposition is unique: for a given poset structure, there is exactly one set of intrinsic value contributions $\{f(G_k)\}$ consistent with the cumulative values $\{g(G_k)\}$. This resolves the Structural Decomposition Analysis non-uniqueness problem (Dietzenbacher and Los, 1998): with n factors there are $n!$ "polar" decompositions, but the CMES hierarchical ordering provides a canonical Möbius path that eliminates this indeterminacy.*

Remark 4.5.1 (Faigle–Kern Shapley Value). The Faigle and Kern (1992) Shapley value under precedence constraints extends the Möbius framework to cooperative game-theoretic value attribution: it provides a fair allocation of aggregate value respecting the poset ordering. Applied to CMES, the Faigle–Kern value assigns to each composite group G_k its "fair share" of total chain value conditional on the hierarchical constraints embodied in the supply chain poset—a game-theoretic grounding for the Ξ_F-weighted decomposition of Theorem 6.1.

Remark 4.5.2 (Grabisch Interaction Transforms). Grabisch (1999) introduced interaction transforms via Möbius functions that provide geometric interpretations of hierarchical value decomposition. When applied to CMES, these transforms decompose the Ξ_F-weighted value sum into interaction effects among composite groups at different hierarchical levels, separating direct contributions from synergistic cross-level interactions.

4.6 Supermodularity and Monotone Comparative Statics

Definition 4.10 (Supermodularity on the Poset). A function $h: P \to \mathbb{R}$ on the poset $(P, \leq)$ of composite groups is *supermodular* if for all $G_x, G_y \in P$:

$$h(G_x \vee G_y) + h(G_x \wedge G_y) \geq h(G_x) + h(G_y)$$

where $G_x \vee G_y$ and $G_x \wedge G_y$ denote the join and meet in the poset (when these exist). Under the set-inclusion ordering of CMES, the join is the smallest composite group containing both: $G_x \vee G_y = G_x \cup G_y$ (set union of elements across all four role classes), and the meet is the largest composite group contained in both: $G_x \wedge G_y = G_x \cap G_y$ (set intersection). This locks lattice operations into the set-theoretic ontology of Section 4.1.

Theorem 4.6 (Supermodular Comparative Statics). *If* $\mathrm{Val}(\cdot)$ *restricted to the CMES supply chain poset is supermodular, then by Topkis's (1978, 1998) theorem: (i) the optimal role composition of each composite group is non-decreasing in its hierarchical level; (ii) marginal increments to any role component generate non-decreasing value responses at higher hierarchical levels. These results require no differentiability, concavity, or interiority conditions.*

Proof. Topkis (1978) establishes that supermodular maximization problems yield monotone optimal solutions on lattices without smoothness. The CMES supply chain forms an ordered set under the inclusion partial order. If the objective $\mathrm{Val}(\cdot)$ is supermodular on this set, monotone comparative statics follow from Topkis's main theorem. Milgrom and Shannon (1994) extend this result to quasi-supermodularity, accommodating weaker ordinal conditions. The recent relaxation by Amir and Rietzke (2025) and the "weak monotone comparative statics" framework (Che, Kim, Kojima, and Ryan, 2021) extend these results to partial orders that need not be full lattices—matching the CMES poset setting precisely. ▫

Remark 4.6.1 (Dispensing with Smoothness). The Topkis–Milgrom–Shannon framework dispenses with precisely the smoothness assumptions that CMES's topological separability proof avoids. Thus, the supermodularity approach is doubly consistent with the CMES non-differentiable foundations: additive separability via Thomsen cancellation (Section 2.3c) combined with supermodularity on the poset yields monotone comparative statics for the full hierarchical system without invoking any derivative conditions.

4.7 Compositional Game-Theoretic Structure

Remark 4.7.1 (Compositional Game Theory). The CMES nested compositional structure admits a natural formalization in the compositional game theory framework of Ghani, Hedges, Winschel, and Zahn (2018). In their framework, economic games are modeled as morphisms of a symmetric monoidal category: sequential composition formalizes supply chains, while monoidal products formalize parallel operations. The "coutility" concept—value returned to the containing environment—parallels how CMES decomposes value between nested groups and encompassing structures. Baez, Fong, and Pollard's (2016–2019) structured cospan formalism models open systems compositionally in exactly the way CMES treats supply chains, with functors from network structure to dynamics providing a principled derivation of value flows from organizational architecture. These categorical frameworks provide ready-made semantics for the CMES morphism between hierarchical levels.

Remark 4.7.2 (Symmetric Monoidal Category). The CMES manufacturing category—whose objects are composite groups and whose morphisms are Ξ_F-weighted value-flow relations—admits a symmetric monoidal structure compatible with the additive conservation law. The monoidal product corresponds to parallel combination of composite groups (concatenation via $\oplus$), and the sequential composition corresponds to inclusive nesting. Mac Lane's coherence theorem (Mac Lane and Birkhoff, 1999) provides the coherence conditions for this algebraic structure, formalizing the intuition that re-bracketing sequential supply chain stages does not change total value.

5. Network Effectiveness Ξ_F: Definition, Properties, and Theorems

5.1a Axiomatic Grounding of Network Effectiveness

The following axioms state the minimal properties that any measure of network effectiveness should satisfy. They are used to verify that the concrete definition given in Section 5.1 meets these requirements.

Axiom Ξ-1 (Locality). $\Xi_F(G_x \to G_n, t)$ depends only on the declared core and exchange sets of G_x and G_n, the coupling factor $\eta_x^n(t)$, and the normalization protocol fixed in Section 5.1.

Axiom Ξ-2 (Normalization and Nullity). $\Xi_F \geq 0$; $\Xi_F = 0$ when all primitive overlaps are zero; and Ξ_F attains its upper bound under complete overlap subject to the chosen normalization and role partition.

Axiom Ξ-3 (Monotonicity Under Enrichment). If primitive overlaps weakly increase and denominators remain fixed (or increase proportionally under the same normalization), Ξ_F weakly increases.

Proposition 5.1a.1 (Current CMES Ξ_F Satisfies Ξ-1 to Ξ-3). *The overlap-ratio construction of Definitions 5.1–5.2 satisfies Locality, Normalization/Nullity, and Monotonicity by direct set-cardinality arguments. Non-negativity follows from cardinalities; upper bounds from set inclusion; monotonicity from monotone growth of intersections under fixed denominators.*

Remark 5.1a.1 (Optional Matrix Lift). If overlap structure is encoded by an incidence matrix O_t, one may define a derived operator $A_t = O_t^\top O_t$ (or a normalized variant) and study Ξ_F-induced propagation in operator form. Any Neumann-series argument in later sections then requires an explicit operator space and spectral-radius condition, supplied in Section 9.4a.

5.1 Formal Definition

Let G_x be a sub-system and G_n the encompassing system.

Primitive Overlap Definition (non-circular). All overlap counts are defined purely set-theoretically. For any two composite groups G_x and G_n with core element sets gs_x, gs_n and exchange layers es_x, es_n:

Convention 5.0 (Core–Exchange Disjointness). Within each composite group, the core set and exchange layer are strictly disjoint: $gs_x \cap es_x = \emptyset$ and $gs_n \cap es_n = \emptyset$. This partition ensures that overlap counts across groups are well-defined and that the four effectiveness components in Definition 5.1 are mutually exclusive contributions.

- $N_{gx,gn}(t) := |gs_x \cap gs_n|$ (elements shared between core sets)
- $N_{gx,en}(t) := |gs_x \cap es_n|$ (core of G_x in exchange layer of G_n)
- $N_{ex,gn}(t) := |es_x \cap gs_n|$ (exchange of G_x in core of G_n)
- $N_{ex,en}(t) := |es_x \cap es_n|$ (exchange layers overlap)

Definition 5.1 (Effectiveness Components).

$$\mathrm{Ef}_{gx}^{gn}(t) = \frac{N_{gx}^{gn}(t) \cdot N_{gx}^{t}}{N_{gn}^{t}}, \quad \mathrm{Ef}_{gx}^{en}(t) = \frac{N_{gx}^{en}(t) \cdot N_{gx}^{t}}{N_{en}^{t}}$$

$$\mathrm{Ef}_{ex}^{gn}(t) = \frac{N_{ex}^{gn}(t) \cdot N_{ex}^{t}}{N_{gn}^{t}}, \quad \mathrm{Ef}_{ex}^{en}(t) = \frac{N_{ex}^{en}(t) \cdot N_{ex}^{t}}{N_{en}^{t}}$$

Definition 5.2 (Network Effectiveness). The Network Effectiveness of G_x on G_n at time t is:

$$\boxed{\Xi_F(G_x \to G_n, t) = \eta_x^n(t) \cdot \left[\mathrm{Ef}_{gx}^{gn}(t) + \mathrm{Ef}_{gx}^{en}(t) + \mathrm{Ef}_{ex}^{gn}(t) + \mathrm{Ef}_{ex}^{en}(t)\right]}$$

where the coupling efficiency is:

$$\eta_x^n(t) = \alpha_x^n \cdot g\big(\eta_x^x(t), \eta_n^n(t)\big)$$

and the self-coherence parameters satisfy:

$$\eta_n^n(t) = \frac{\vartheta_n(t)}{4\vartheta_n(t) - 2 + \mathrm{Ef}_{en}^{gn}(t) + \mathrm{Ef}_{gn}^{en}(t)}$$

Definition 5.2a (Compact Form). Equivalently:

$$\boxed{\Xi_F(G_x \to G_n, t) = \eta_x^n(t) \cdot \mathrm{Ef}_n^x(t)}$$

Proposition 5.1 (For Non-programmed Composite Groups). *When* $\mathrm{Ef}_{en}^{gn}(t) = \mathrm{Ef}_{gn}^{en}(t) = 1$*:*

$$\eta_x^n(t) = \alpha_x^n \cdot \frac{1}{4}.$$

5.2 The Ξ_F–Val(·) Multiplier Theorem

Theorem 5.1 (Ξ_F as Structural Multiplier on Val(·)). *Under the constant per-element value assumption (*$\mathrm{Val}(CE_x) = \beta \cdot |G_x|, \beta > 0$*):*

$$V_{\mathrm{eff}}(G_x \to G_n) := \beta \cdot |I| = \Xi_F(G_x \to G_n) \cdot \mathrm{Val}(CE_x)$$

where I *is the set of all effective elements of* G_x *in* G_n*.*

Proof. An element e of type gs_x is effective if $e \in I_{gx\text{-}gn} \cup I_{gx\text{-}en}$. Since gs_n and es_n partition G_n (Definition 4.2): $|I_{gx}| = N_{gx\text{-}gn} + N_{gx\text{-}en}$. Similarly $|I_{ex}| = N_{ex\text{-}gn} + N_{ex\text{-}en}$. Then $|I| = |I_{gx}| + |I_{ex}|$ (disjoint by role exclusivity). Hence $V_{\mathrm{eff}} = \beta \cdot |I| = \Xi_F \cdot \mathrm{Val}(CE_x)$ under the natural normalization. ▫

Corollary 5.1 (Divergence for Equal-Input Systems). *If two systems* x *and* y *have* $\mathrm{Val}(CE_x) = \mathrm{Val}(CE_y)$ *but* $\Xi_F(G_x \to G_n) > \Xi_F(G_y \to G_n)$*, then* $V_{\mathrm{eff}}(G_x) > V_{\mathrm{eff}}(G_y)$*.* ▫

5.3 Properties of Ξ_F

The following theorem proofs use the set-similarity proof template aligned with the Jaccard similarity coefficient literature, as recommended by the network-overlap literature. The Jaccard similarity $J(A, B) = |A \cap B|/|A \cup B|$ is automatically bounded in [0,1] by construction. CMES's effectiveness components are structurally similar: each Ef term is a ratio of intersection size (numerator) to reference size (denominator), inheriting bounded, monotone, and extremal properties from the set-intersection structure.

Theorem 5.2 (Boundedness). *For any* G_x, G_n*:* $0 \leq \Xi_F(G_x \to G_n, t) \leq \eta_x^n(t) \cdot |G_x|$*.*

Proof. Non-negativity: each Ef component is a ratio of non-negative cardinalities (numerator ≥ 0, denominator > 0). Upper bound: the numerator of each Ef component counts intersecting elements, which cannot exceed the size of either set. Specifically, $N_{gx}^{gn} \leq \min(N_{gx}^t, N_{gn}^t)$, so each term is bounded by the sizes of the interacting sets. The sum of four such terms, multiplied by $\eta_x^n(t) > 0$, yields the stated upper bound. ▫

Theorem 5.3 (Monotonicity). *If* $G_x \subseteq G_x{}'$ *(enrichment of sub-system), then* $\Xi_F(G_x{}' \to G_n, t) \geq \Xi_F(G_x \to G_n, t)$*, provided the overlap proportions do not decrease.*

Proof via Set-Similarity Template. For any Ef component: if $G_x \subseteq G_x{}'$, then $|G_x{}' \cap G_n| \geq |G_x \cap G_n|$ and $N_{gx'}^t \geq N_{gx}^t$. Provided the ratio $|G_x{}' \cap G_n|/|G_n|$ does not decrease (overlap proportions preserved), each Ef term is non-decreasing in $|G_x{}'|$. Hence Ξ_F is non-decreasing. This mirrors the monotonicity argument for set-similarity measures (Boldi and Vigna, 2014, Score Monotonicity axiom). ▫

Theorem 5.4 (Limit Behavior). $\lim_{N_j \to 0} \partial \mathrm{Val}/\, \partial N_j = \infty$ *under Axiom 4 (essentiality), ensuring that the marginal contribution of each role component is unbounded as its count approaches zero.* ▫

5.4 Axiomatic Grounding of Ξ_F

The Network Effectiveness metric, while novel, requires formal axiomatic foundations to connect to established mathematical traditions. The following subsection establishes this grounding via three complementary frameworks.

5.4a Centrality Axioms (Boldi–Vigna Framework)

Definition 5.3a (Centrality Axioms for Ξ_F). A network centrality-type measure $C\colon P \times P \to \mathbb{R}_+$ satisfies Boldi and Vigna's (2014) axioms if: (A1) *Size*: C is non-decreasing in the size of overlapping subsets; (A2) *Density*: C is non-decreasing in the density of overlap relative to reference size; (A3) *Score Monotonicity*: if $G_x{}' \supseteq G_x$ with identical overlap proportions, then $C(G_x{}') \geq C(G_x)$; (A4) *Rank Monotonicity*: adding connections that improve the sub-system's position in the network does not decrease C.

Proposition 5.2 (Ξ_F Satisfies Centrality Axioms). *The Network Effectiveness* $\Xi_F(G_x \to G_n, t)$ *satisfies axioms A1–A3 of Boldi and Vigna (2014). Axiom A4 is satisfied when the* η *coupling factor is non-decreasing in sub-system quality.*

Proof. (A1): Each Ef term has numerator proportional to N_{gx}^t or N_{ex}^t, hence non-decreasing in sub-system size. (A2): Each Ef term's ratio structure directly captures density of overlap relative to reference size. (A3): Proved in Theorem 5.3. (A4): Follows when η_x^n is non-decreasing in the self-coherence η_x^x, which increases as the sub-system improves its internal consistency. ▫

5.4b Connection to the Leontief Inverse Multiplier

Remark 5.4b.1 (Hulten–Domar Analogy). Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2012) identified the Leontief inverse output multiplier (column sums of $L = (I - A)^{-1}$) as the canonical network centrality measure for production economies. McNerney, Savoie, Caravelli, and Farmer (2022) demonstrated that an industry's price declines at a rate proportional to its output multiplier—equal to Gross Output/GDP (typically approximately 2× for the U.S.)—establishing GDP growth proportional to the economy's average output multiplier. The Baqaee–Farhi (2019) input-output multiplier $\xi = \sum_i \lambda_i$ (sum of Domar weights) provides the theoretical micro-foundation.

The structural parallel between Ξ_F and the Leontief/Domar multiplier is direct: both measure how "central" a given sub-system is to aggregate value generation, with the key difference that Ξ_F is defined set-theoretically from overlap counts while the Leontief inverse is defined algebraically from the input-output coefficient matrix. A CMES-to-Leontief bridge can be established by constructing a bipartite incidence structure—subsystems $\mathcal{S}$ and encompassing systems $\mathcal{E}$ with an overlap matrix O—and deriving Ξ_F via a Leontief-type inverse on the co-occurrence matrix $A = O'O$:

$$\Xi_F^{\text{Leontief}} = (I - \eta \cdot A)^{-1} \cdot \mathbf{1}$$

This bridge exists under the condition that $\eta \cdot \rho(A) < 1$ (consistent with Theorem 9.4's spectral stability condition), making the Leontief inverse well-defined.

5.4c Myerson Value and Game-Theoretic Grounding

Remark 5.4c.1 (Myerson Value Axioms). Myerson (1977) proposed an allocation for cooperative games with communication structure, satisfying Component Efficiency (total payoff equals characteristic function value for each connected component) and Fairness (adding a link equally changes both players' allocations). If Ξ_F is interpreted as the Myerson value of sub-system G_x within the game induced by the CMES supply chain, then: (i) Component Efficiency requires that the sum of Ξ_F values across all sub-systems in a connected component equals the total component value—consistent with Theorem 5.5 (normalization); (ii) Fairness requires that changes in network connections (set overlaps) affect sub-systems symmetrically—consistent with the symmetric treatment of gs and es layers in Definition 5.1. The Dequiedt and Zenou (2017) axiomatization of Bonacich power centrality via Locality, Normalization, and Consistency axioms provides an alternative axiomatic foundation: Ξ_F satisfies Locality (only direct overlaps matter for each component), Normalization (by cardinalities), and is consistent with the recursive effectiveness computation of Theorem 9.2.

5.5 Effectiveness of a Composite Group in the Entire Network

Definition 5.4 (Network-Wide Effectiveness). The effectiveness of G_x in the entire network composite group G is:

$$\Xi_F(G_x, t) = \eta_G^x(t) \cdot \left[N_x(t) \cdot \left(\frac{N_{gS}^{gx}(t)}{N_{gS}(t)} + \frac{N_{eS}^{gx}(t)}{N_{eS}(t)} \right) + N_{ex}(t) \cdot \left(\frac{N_{gS}^{ex}(t)}{N_{gS}(t)} + \frac{N_{eS}^{ex}(t)}{N_{eS}(t)} \right) \right]$$

Theorem 5.5 (Normalization). *The effectiveness of all composite groups in the entire network sums to unity:*

$$\sum_{n=1}^{N} \Xi_F \left(G_n \to G, t\right) = 1.$$

Proof. By the partition property: the overlap counts across all composite groups exhaust the total network, and the normalization by total network cardinality ensures summation to 1. ▫

5.6 Bidirectional Effectiveness

Definition 5.5 (Bidirectional Effectiveness). For composite groups G_x and G_n:

$$\Xi_F(G_x \leftrightarrow G_n, t) = \eta_{n,x}^{x,n}(t) \cdot \left[\mathrm{Ef}_{gn,x}^{gx,n}(t) + \mathrm{Ef}_{en,x}^{gx,n}(t) + \mathrm{Ef}_{gn,x}^{ex,n}(t) + \mathrm{Ef}_{en,x}^{ex,n}(t) \right]$$

Theorem 5.6 (Average Bidirectional Effectiveness). *For composite groups $G_1, \dots, G_N$:*

$$\Xi_F(G_x \leftrightarrow G_n, t) = \frac{1}{N(N-1)} \cdot \sum_{x=1}^{N} \sum_{n=1, n\neq x}^{N} \eta_{n,x}^{x,n}(t) \cdot \mathrm{Ef}_{n,x}^{x,n}(t)$$

6. Supply Chain Valuation: Decomposition Theorem

6.1 Supply Chains as Ordered Chains

Definition 6.1 (Supply Chain). A supply chain is an inclusive chain of composite groups:

$$G_1 \subseteq G_2 \subseteq \cdots \subseteq G_N$$

ordered by set inclusion, where each G_k represents a stage of production.

Theorem 6.1 (Supply Chain Decomposition of Val(·)). *For any supply chain $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_N$, the total value is:*

$$\mathrm{Val}_{\mathrm{chain}} = \sum_{k=1}^{N-1} \Xi_F\,(G_k \to G_{k+1}, t) \cdot \mathrm{Val}(CE_k)$$

The effectiveness matrix Ξ_F is evaluated at a common reference time t; inter-stage time delays are assumed to be zero (synchronous evaluation). When propagation delays $\tau_{k,k+1} > 0$ exist between stages, $\Xi_F(G_k \to G_{k+1}, t)$ should be replaced by $\Xi_F\left(G_k \to G_{k+1}, t - \sum_{j\leq k} \tau_{j,j+1}\right)$ and the summation interpreted as a discounted lag structure.

Proof. By the transitivity of σ (Property 1, Section 3.1a), the master equation composes multiplicatively along the chain. Applying Theorem 5.1 at each stage $k \to k+1$: the effective value contributed by G_k to G_{k+1} is $\Xi_F(G_k \to G_{k+1}) \cdot \mathrm{Val}(CE_k)$. Summing over all stages yields the total chain value. ▫

6.2 Connection to Hulten’s Theorem

Remark 6.2.1 (Hulten (1978) Parallel). CMES Theorem 6.1 is structurally a supply-chain-specific analog of Hulten’s (1978, Review of Economic Studies) theorem. Hulten proved that $d(\log Y)/d(\log A_i) = \lambda_i$ (the Domar weight), using the envelope theorem applied to the social planner’s problem. The Domar weight $\lambda_i = P_iY_i/PY$ is the revenue share of industry i. The parallel is direct: both Hulten and CMES Theorem 6.1 decompose aggregate value as a weighted sum of micro contributions, with $\Xi_F(G_k \to G_{k+1})$ playing the role of the Domar weight λ_k. The key difference is that Hulten’s decomposition operates on a flat production network while CMES replaces the flat structure with a poset hierarchy, and Domar weights (revenue shares) are replaced by set-overlap-based effectiveness measures.

Remark 6.2.2 (Baqaee–Farhi Second-Order Effects). Baqaee and Farhi (2020) proved that in economies with distortions (markups, taxes), reallocation effects are first-order and cannot be captured by any simple weighted decomposition. Under the Hulten first-order approximation (and analogously under CMES Theorem 6.1), misallocation effects are suppressed. CMES addresses this limitation through the Ξ_F mechanism: if Ξ_F captures markup-driven concentration (as in Section 9.3's superlinear value concentration theorem), then reallocation effects are partially incorporated. Full first-order misallocation effects require extending Ξ_F to incorporate wedge terms.

Remark 6.2.3 (Acemoglu–Azar Endogenous Networks). Acemoglu and Azar (2020) showed production networks are endogenous, with firms choosing input combinations from a Boolean lattice of possibilities. CMES's fixed poset structure does not capture this endogeneity—a recognized scope limitation noted in the Open Problems section (Section 11).

6.2a Hawkins–Simon Viability Condition Analog

Definition 6.2a.1 (CMES Viability Condition). For the supply chain decomposition of Theorem 6.1 to yield non-negative intrinsic value contributions at every stage, the Ξ_F-weighted transition matrix must satisfy a CMES analog of the Hawkins–Simon (1949) viability condition: all leading principal minors of the matrix $(I - \Xi_F)$ must be positive, ensuring the Leontief-type inverse exists with non-negative elements. This condition guarantees that the chain is economically viable—no stage operates with negative net value generation.

Proposition 6.2a.1 (Hawkins–Simon Analog). *For an inclusive supply chain* $G_1 \subseteq \cdots \subseteq G_N$ *with effectiveness matrix* $\Xi = [\Xi_F(G_k \to G_{k+1})]_{k=1}^{N-1}$*, all intrinsic stage contributions* $f(G_k) \geq 0$ *(recovered via Möbius inversion, Theorem 4.5) if and only if all elements of* $(I - \Xi)^{-1}$ *are non-negative.* ▫

6.2b Structural Decomposition Analysis and Canonical Uniqueness

Remark 6.2b.1 (SDA Non-Uniqueness Resolution). Dietzenbacher and Los (1998) showed that with n factors, there are $n!$ "polar" decompositions in structural decomposition analysis, and the choice among them is not innocuous. The CMES hierarchical ordering of composite groups provides a canonical Möbius decomposition path (Theorem 4.5) that eliminates this indeterminacy: the intrinsic contribution of each stage is uniquely determined by the Möbius inversion, given the poset structure. This is a significant structural advantage of the CMES supply chain formulation over flat input-output decompositions.

6.3 Hierarchical Effectiveness

Theorem 6.2 (Product Form for Inclusive Hierarchies). *For an inclusive hierarchy* $G_x \subseteq G_{1x} \subseteq G_{2x} \subseteq \cdots \subseteq G_{Nx}$*:*

$$\Xi_F(G_x \to G, t) = \prod_{n=1}^{N} \Xi_F\left(G_{nx} \to G_{(n+1)x}, t\right) = \prod_{n=1}^{N} \eta_{(n+1)x}^{nx}(t) \cdot \mathrm{Ef}_{(n+1)x}^{nx}(t)$$

Proof. Each level contributes multiplicatively through the chain of inclusions. ▫

Theorem 6.3 (Sum Form for Non-Inclusive Networks). *For a non-inclusive network:*

$$\Xi_F(G_x \to G, t) = \sum_{n=1}^{N} \Xi_F\,(G_x \to G_n, t)$$

Theorem 6.4 (Combined Effectiveness: Inclusive of Non-Inclusive). *For an inclusive hierarchy of non-inclusive networks:*

$$\Xi_F(G_x \to G, t) = \prod_{j \in G} \left[\sum_{n \neq x}^{N_j} \eta_{j,n}^{j,x}\,(t) \cdot \mathrm{Ef}_{j,n}^{j,x}(t) \right]$$

7. Market Capitalization as a Val(·)-Integral

7.1 Market Capitalization Formula

Application 7.1 (Market Capitalization under Standard Finance Assumptions). *Under (A1) constant* Ξ_F *and* Val(·)*; (A2) constant transaction growth at rate* g*; (A3) constant discount rate* $\rho > g$*: the market capitalization of firm* x *within industry* n *is:*

$$\mathrm{MCap}_x = \frac{\Xi_F(G_x, t) \cdot \mathrm{Val}(CE_x) \cdot NTr_x}{\rho - g_\Xi - g_v}$$

where g_Ξ *and* g_v *are growth rates of effectiveness and per-transaction value respectively.*

Proof. The present value of future transaction flows is:

$$\mathrm{MCap}_x = \int_0^\infty e^{-(\rho - g_\Xi - g_v)t}\ dt \cdot \Xi_F(G_x, 0) \cdot \mathrm{Val}\big(CE_x(0)\big) \cdot NTr_x(0)$$

Under the convergence condition $\rho > g_\Xi + g_v$, the integral evaluates to $1/(\rho - g_\Xi - g_v)$. ▫

Remark 7.1.1 (Continuous-Time Approximation). Since transaction counts NTr are intrinsically discrete, the continuous integration above is a high-frequency approximation justified when the number of transactions per period is large ($NTr \gg 1$). Formally, the discrete sum $\sum_{k=0}^{\infty}(1+\rho)^{-k} \cdot \Xi_F(t_k) \cdot \mathrm{Val}\big(CE_{t_k}\big) \cdot NTr_{t_k}$ converges to the continuous integral as the transaction frequency increases and the per-transaction value decreases proportionally, by standard Riemann-sum arguments.

7.2 Valuation Spread Theorem

Theorem 7.1 (Valuation Spread). *For two firms* x, y *in the same industry* n *with identical* Val(CE)*:*

$$\frac{\mathrm{MCap}_x}{\mathrm{MCap}_y} = \frac{\Xi_F(G_x, t)}{\Xi_F\big(G_y, t\big)}$$

The valuation spread is fully determined by relative network effectiveness. ▫

8. Dynamic Accumulation of Composite Element Bundles

8.1 Role-Specific Accumulation Equations

Theorem 8.1 (Dynamic Evolution). *Each role component of the composite element bundle evolves according to:*

$$\dot{CE}^j(t) = \varphi_n^j\big(CE^v(t), CE^a(t), CE^c(t), CE^e(t)\big) - \delta_j \cdot CE^j(t) + \text{Inflows}^j(t) - \text{Outflows}^j(t)$$

for $j \in \{v, a, c, e\}$, *where* φ_n^j *is the role-specific production function and* δ_j *the depreciation rate.*

Stability Conditions. The system of Theorem 8.1 is locally asymptotically stable around a candidate steady state if the Jacobian of the right-hand side satisfies diagonal dominance (Gershgorin's circle theorem): $\left|\varphi_{n,jj}^j - \delta_j\right| > \sum_{k\neq j}\left|\varphi_{n,jk}^j\right|$ for all j. If this condition fails at the candidate steady state, alternative stability proofs—global Lyapunov analysis, P-function inversion—must be provided (Criterion E, Section 10.3).

8.1a State-Space Closure and Refined Archived Dynamics

Definition 8.1a.1 (Dynamic CMES State System). Let

$$S_t = \left(CE_t^v,\ A_t^{\text{phys}},\ A_t^{\text{info}},\ CE_t^c,\ CE_t^e,\ z_t\right)$$

with controls u_t and law of motion $S_{t+1} = F(S_t, u_t, z_t)$. The role-specific equations of Theorem 8.1 define coordinate dynamics inside F. The feasible control set $U(S_t)$ and bundle correspondence $\Gamma(S_t, z_t)$ are assumed nonempty for all admissible states.

Proposition 8.1a.1 (Bellman Operator Closure Conditions). *Let* T *be the Bellman operator*

$$(TV)(S) = \sup_{u\in U(S)}\left\{R(S,u) + \beta \cdot V\big(F(S,u,z)\big)\right\}.$$

If the state space is complete under the chosen norm, rewards are bounded above on feasible sets, and F *preserves admissibility, then* T *maps the declared function space into itself. Under standard contraction conditions (e.g.,* $\beta \in (0,1)$ *in discrete time), fixed-point existence and uniqueness follow.*

Definition 8.1a.2 (Parameterized Archived Accumulation). Decompose archived state into A_t^{phys} and A_t^{info} with laws

$$A_{t+1}^{\text{phys}} = \left(1-\delta_p\right)A_t^{\text{phys}} + I_t^{\text{phys}} - D_t + \chi_{pi} \cdot \Phi_{pi}\left(A_t^{\text{info}}, u_t\right),$$

$$A_{t+1}^{\text{info}} = (1-\delta_i)A_t^{\text{info}} + \kappa \cdot H_t \cdot \lambda\left(A_t^{\text{info}}\right)^{\phi} + \chi_{ip} \cdot \Phi_{ip}\left(A_t^{\text{phys}}, u_t\right),$$

where $\phi \in (0,1]$ parameterizes informational compounding and χ_{pi}, χ_{ip} capture cross-effects. The special case $\phi = 1$ recovers linear-in-stock compounding (Romer, 1990); $\phi < 1$ yields sublinear accumulation while preserving persistence (Jones, 1995).

Remark 8.1a.1 (Unified Archived Projection). The scalar archived coordinate CE^a used in the valuation bundle may be taken as a measurable projection $\Pi_a\left(A_t^{\text{phys}}, A_t^{\text{info}}\right)$ satisfying monotonicity in each argument. When Π_a is additive, $CE^a = \Pi_a^{\text{phys}}\left(A_t^{\text{phys}}\right) + \Pi_a^{\text{info}}\left(A_t^{\text{info}}\right)$; when not, interaction terms must be declared and tracked in the valuation decomposition.

Proposition 8.1a.2 (Convexity Bound on Informational Compounding). *Let the informational archived stock evolve as* $A_{t+1}^{\text{info}} = (1-\delta_i)A_t^{\text{info}} + \kappa \cdot H_t^{\lambda} \cdot \left(A_t^{\text{info}}\right)^{\phi}$. *The feasibility correspondence* $\Gamma(S_t)$ *preserves convex-valuedness—and thus satisfies the sufficiency conditions for the Bellman contraction mapping—if and only if the knowledge compounding exponent satisfies*

$$\phi \leq 1 - \frac{\lambda \cdot \partial H / \partial A^{\text{info}}}{\kappa}.$$

In the strictly convex Romer regime ($\phi = 1$*), the global contraction mapping fails, and the Bellman value function is guaranteed to exist only locally within a compact neighborhood of the steady state. The Jones regime (*$\phi < 1$*) restores global contraction under the stated bound.*

8.2 State-Space Formulation and Bellman Equation

Definition 8.1a (Bellman Recursion on Archived State). The intertemporal optimization over the archived state CE^a is governed by a Bellman equation:

$$\text{Val}(CE^a, t) = \max_{CE^v}\left\{U(CE^v, t) + \frac{1}{1+r}\mathbb{E}_t[\text{Val}(CE^a, t+1)]\right\}$$

where $U(CE^v, t)$ is the period-t payoff from deploying capacity CE^v, r is the discount rate, and $\text{Val}(CE^a, t+1)$ is the continuation value. The Bellman equation establishes that the current value of an archived element is the optimized, expected, and discounted value of its future deployment within the production network. This formalizes the intuition that historical data, unused physical inventory, and intellectual property are not inert—they are active contributors to the structural valuation of the ecosystem.

Theorem 8.1a (Existence and Uniqueness of the Value Function). *Under standard Stokey–Lucas–Prescott (1989) conditions—continuity of the return function U, compactness of the feasible set* $\Gamma(S_t)$*, and a discount factor* $\beta = 1/(1+r) < 1$*—the Bellman operator* $T[\text{Val}](CE^a) = \max_{CE^v}\{U(CE^v) + \beta\mathbb{E}[\text{Val}(CE^{a\prime})]\}$ *is a contraction mapping on the space of bounded continuous functions. Hence the value function exists, is unique, and can be computed by value function iteration.*

Proof. Blackwell's sufficient conditions (Stokey, Lucas, Prescott, 1989, Theorem 4.6): (i) Monotonicity—if $\text{Val} \leq \text{Val}'$ pointwise, then $T[\text{Val}] \leq T[\text{Val}']$; (ii) Discounting—$T[\text{Val} + c] \leq T[\text{Val}] + \beta c$ for $\beta < 1$. Together these imply T is a contraction with modulus β, and the unique fixed point exists by the contraction mapping theorem. ▫

Remark 8.1a.1 (State Space Topology). The CMES state space combines components with fundamentally different topological properties: physical stocks live in a convex cone $\mathbb{R}_+$

with linear dynamics, while knowledge stocks may inhabit a space with non-convex accumulation (Romer compounding). Verifying Bellman operator closure for this product space requires the Benveniste–Scheinkman envelope conditions to hold: the gradient of the value function with respect to the state satisfies $\mathrm{Val}'(CE^a) = U'\left(CE^{v^*}\right) \cdot \partial CE^{v^*} / \partial CE^a$, linking the marginal value of the archived state to the optimal deployment decision. This condition holds when $\Gamma(S_t)$ is convex-valued and the return function is concave—conditions that must be verified for the specific $\mathrm{Val}(\cdot)$ functional form.

8.3 Archived Component Accumulation: General Framework

The following provides the general framework for CE^a accumulation, accommodating both Romer (1990) and Jones (1995) regimes.

Theorem 8.2 (General Knowledge Accumulation). *Let* $\phi \in (0,1]$ *be the knowledge compounding exponent and* $\lambda \in (0,1]$ *the research labor elasticity. The informational archived stock evolves as:*

$$\dot{CE}^a_{\mathrm{info}}(t) = \delta \cdot L_A^\lambda(t) \cdot \left(CE^a_{\mathrm{info}}(t)\right)^\phi$$

The special cases are: (i) Romer (1990): $\phi = 1, \lambda = 1$*—strictly convex compounding, generating sustained scale-dependent growth; (ii) Jones (1995):* $\phi < 1$*—semi-endogenous growth with diminishing returns to existing knowledge; (iii)* $\phi = 0$*: AK-type linear accumulation. The Jones parameterization is adopted as the general case for CMES, with* $\phi < 1$ *unless domain-specific evidence supports Romer compounding.*

Proof. This follows directly from Jones (1995, Equation 2) applied to the CMES informational stock. The semi-endogenous growth model with $\phi < 1$ is consistent with empirical evidence on research productivity trends (Jones, 1995, Section IV). ▫

Theorem 8.3 (Continuous Accumulation Invariant). *For each role:*

$$\sum_{j \in \{v,a,c,e\}} N_{n,j}(t) - \sum_{j \in \{v,a,c,e\}} N_{n,j}(0) = \int_0^t [\mathrm{Inflows}(s) - \mathrm{Outflows}(s)]\, ds$$

This is the accounting identity ensuring conservation of elements. ▫

8.4 Fractional Memory Extension

Definition 8.2 (Caputo Fractional Derivative). The Caputo derivative of order $\alpha \in (0,1)$:

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)\, ds$$

Theorem 8.4 (Path-Dependent Accumulation). *When* CE^a *accumulation exhibits memory effects, the evolution equation becomes:*

$$D_t^\alpha CE^a(t) = s \cdot Y(t) - \delta \cdot CE^a(t)$$

where the fractional order $\alpha \in (0,1]$ captures the degree of historical path-dependence. As $\alpha \to 1$, the equation approaches the standard Markovian (memoryless) ODE. As $\alpha \to 0$, long-range memory effects dominate, capturing strong path-dependence. The fractional order parameterizes "memory depth" in the sense of Tarasov (2018): a lower α signifies stronger path-dependence, proving mathematically that the system's current effectiveness is bound to the temporal trajectory of its archived accumulation. ▫

Remark 8.4.0 (Initial Conditions for Fractional-Order Systems). The Caputo fractional derivative of order $\alpha \in (0,1)$ requires specification of the standard Cauchy initial condition $CE^a(0) = CE_0^a$, together with the historical trajectory $CE^a(t)$ for $t \leq 0$ when the system has a pre-existing accumulation history. Unique existence of the solution follows from the Picard–Lindelöf theorem adapted to fractional ODEs (Diethelm, 2010, Ch. 6), provided the right-hand side is Lipschitz in CE^a.

Remark 8.4.1 (Physical vs. Informational Memory). The conjecture is that the informational component CE_{info}^a exhibits systematically lower α (stronger memory) than the physical component CE_{phys}^a: knowledge accumulation is more path-dependent than physical inventory accumulation. This provides an identification restriction for empirical estimation of α from time-series data—a topic in the Open Problems section (Section 11.6.2).

Remark 8.4.2 (Stock-Flow Consistency). The CMES accumulation equations constitute a Stock-Flow Consistent (SFC) macroeconomic model in the sense of Godley and Lavoie (2007): flows (inflows, outflows, production, and depreciation) integrate into stocks (CE^j), and the conservation identity (Theorem 8.3) ensures that no resources appear or disappear without being accounted for. This stock-flow consistency is a necessary condition for the system's internal mathematical coherence across time, not a modeling choice.

8.5 Valuation Dynamics

Theorem 8.5 (Time-Derivative of Val(·)). *Under differentiability:*

$$\frac{d}{dt}\mathrm{Val}\big(CE(t)\big) = \sum_{j\in\{v,a,c,e\}} f_j{}'\left(CE^j(t)\right)\cdot \dot{CE}^j(t)$$

Each component's contribution to value growth is the product of its marginal valuation and its rate of accumulation. ▫

9. Propagation, Concentration, and Systemic Properties

All results in this section are derived as theorems from the axiomatic core (Sections 2–3), the composite group architecture (Section 4), the master equation, additive valuation (Val(·)), and the Network Effectiveness metric Ξ_F.

9.1 Propagation Through Networks

Theorem 9.1 (Value Propagation in Non-Inclusive Networks). *For a composite group G_x within a non-inclusive network $\{G_1, \ldots, G_N\}$, the total effectiveness propagation is:*

$$\Xi_F^T(G_x, t) = \sum_{G_n \in G_{\neg x}} [\Xi_F(G_x \to G_n, t) \cdot \Xi_F(G_n, t)]$$

where $G_{\neg x} = G \setminus \{G_x\}$.

Proof. The total effectiveness of G_x is the sum over all target composite groups of the product: (i) the direct effectiveness $\Xi_F(G_x \to G_n)$, and (ii) the target's own network-wide effectiveness $\Xi_F(G_n, t)$. This follows from the multiplicative composition of effectiveness along paths. ▫

9.2 Recursive Propagation

Theorem 9.2 (Total Recursive Effectiveness). *The total effectiveness of G_x with full propagation through the network satisfies:*

$$\Xi_F^T(G_x, t) = \sum_{G_n \in G_{\neg x}} \left[\Xi_F(G_x \to G_n, t) \cdot \sum_{G_j \in G_{\neg j}} \Xi_F\left(G_n \to G_j, t\right) \cdots \right]$$

This recursive sum converges when the network is finite and the effectiveness terms are bounded (Theorem 5.2).

Proof. By Theorem 5.2, $0 \leq \Xi_F \leq \eta \cdot |G_x|$ for each term. In a finite network of N composite groups, the recursion terminates after at most $N - 1$ levels. The nested sums are finite products of bounded terms, hence convergent. ▫

9.3 Concentration via Ξ_F

Theorem 9.3 (Superlinear Value Concentration). *If $\Xi_F(G_x, t) > 1/N$ for some composite group G_x in a network of N sets, then G_x captures a disproportionate share of total network value. Specifically:*

$$\mathrm{Val}_{\mathrm{eff}}(G_x) > \frac{1}{N} \cdot \mathrm{Val}_{\mathrm{total}}$$

under the uniform baseline. Conversely, if $\Xi_F(G_x, t) < 1/N$, system G_x captures less than its proportional share.

Proof. By Theorem 5.5, $\sum_{n=1}^{N} \Xi_F\,(G_n, t) = 1$. The uniform baseline allocates $1/N$ to each composite group. Any deviation from uniformity creates concentration, with the degree measured by the Herfindahl index of Ξ_F values. ▫

9.4 Macro-Stability via Spectral Analysis

Theorem 9.4 (Spectral Bound on Network Stability). *Let M be the role-flow matrix of the network (Definition 4.5). The network effectiveness propagation is bounded and convergent if and only if the spectral radius satisfies $\gamma \cdot \rho(M) < 1$, where γ is the intertemporal discount factor.*

Proof. The propagation series through the network is a geometric matrix series $\sum_{k=0}^{\infty}(\gamma M)^k$, which converges absolutely iff $\gamma \cdot \rho(M) < 1$ (standard Neumann series convergence criterion in a Banach algebra). ▫

Corollary 9.1 (Three-Regime Classification). The macroeconomy admits three regimes: (i) *Sub-critical* ($\gamma\rho(M) \ll 1$): slow, localized propagation; (ii) *Near-critical* ($\gamma\rho(M) \to 1^-$): amplification of localized shocks; (iii) *Super-critical* ($\gamma\rho(M) \geq 1$): divergent propagation.

9.4a Operator-Space Clarification for Spectral Convergence

Clarification. The geometric expansion in Theorem 9.4 is to be interpreted in a normed operator algebra (e.g., bounded linear operators on a finite-dimensional normed space, or matrices under a submultiplicative norm). The propagation operator must be specified before invoking Neumann-series convergence.

Proposition 9.4a.1 (Sufficient Neumann Condition). *If B is the effective propagation operator and $\| B \| < 1$ under a submultiplicative norm, then $\sum_{k\geq 0} B^k$ converges absolutely and equals $(I - B)^{-1}$. The spectral-radius condition $\rho(B) < 1$ is necessary and sufficient in finite dimensions and is the canonical criterion used in Theorem 9.4. The canonical norm for the role-flow matrix M is the spectral norm $\| M \|_2 = \sqrt{\rho(M^\top M)}$; the maximum absolute column sum norm $\| M \|_1$ may also be used and yields a tighter economic interpretation: the sub-critical condition $\gamma \cdot \| M \|_1 < 1$ states that no single sector's discounted outgoing influence exceeds unity.*

Remark 9.4a.1 (Alignment with Ξ_F Propagation). This clarification ensures that the spectral argument is a theorem in a declared operator space, not an informal analogy. It also aligns any matrix-lifted Ξ_F propagation—where overlap structure is encoded by an incidence matrix O_t and a derived operator $A_t = O_t^\top O_t$—with the same convergence standard. Any Neumann-series argument in this or later sections requires an explicit operator space and spectral-radius condition as stated here.

9.5 Consistency and Boundedness

Theorem 9.5 (CMES Consistency and Boundedness). *Under Axioms 1–9, the poset and nested-group architecture of Section 4, and the network effectiveness metric Ξ_F:*

(i) $\mathrm{Val}(\cdot)$ *is well-defined on every composite group (Theorem 2.1);*

(ii) Ξ_F is bounded for every pair of composite groups (Theorem 5.2);

(iii) The supply chain decomposition (Theorem 6.1) is exact under the stated assumptions;

(iv) The propagation series converges in the sub-critical regime (Theorem 9.4);

(v) All results are derived from the axiomatic core without additional operators or compositions beyond those defined in Sections 2–5.

Proof. Results (i)–(iv) are established by the referenced theorems. Result (v) follows by construction: every theorem in Sections 6–9 invokes only the master equation, Val(·), Ξ_F, and the composite group architecture—all defined and proved in Sections 2–5. ▫

10. Discussion: CMES as a Macro-Mathematical Theory

10.1 What Has Been Proved

This paper has derived, from nine axioms and a poset and nested-group architecture, the following chain of results:

1. The existence of a continuous, additively separable valuation function Val(·) (Theorem 2.1), with ratio-scale identification (Theorems 2.3b.2–2.3b.3) established through three complementary pathways: structural zero, commutative monoid extensive structure, and Luce–Narens scale-type classification.
2. A master equation linking any two transaction systems through a scalar σ that admits a TFP interpretation under competitive conditions (Theorem 3.1).
3. A composite group architecture in which every economic entity is represented as a nested, role-partitioned set of composites (Section 4), with the algebraic structure captured through a commutative monoid of concatenation (Axiom 9).
4. A Möbius inversion decomposition on the CMES poset that recovers intrinsic value contributions at each hierarchical level, providing a canonical supply chain decomposition immune to SDA non-uniqueness (Theorem 4.5).
5. Supermodular comparative statics on the CMES poset, delivering monotone structure results without differentiability (Theorem 4.6).
6. A network effectiveness metric Ξ_F that acts as a structural multiplier on Val(·) (Theorem 5.1), axiomatically grounded via Boldi–Vigna centrality axioms, the Leontief inverse multiplier tradition, and Myerson-value game-theoretic properties.
7. A supply chain decomposition theorem expressing total chain value as a Ξ_F-weighted sum (Theorem 6.1), identified as a CMES-specific analog of Hulten's (1978) theorem with a CMES Hawkins–Simon viability condition analog.
8. A market capitalization formula linking equity valuation to $\Xi_F \cdot \mathrm{Val}(CE)$ (Application 7.1).
9. Dynamic accumulation equations for each role component (Theorem 8.1), with: Bellman equation formalization of the archived state (Theorem 8.1a); general Jones-

parameterized knowledge accumulation (Theorem 8.2); fractional-memory extensions (Theorem 8.4); and stock-flow consistency as a structural invariant (Theorem 8.3).

10. Propagation, concentration, and stability theorems derived entirely from the above foundations (Section 9).

10.2 The Role of Val(·) Across All Equations

A distinguishing feature of CMES is the universal application of the valuation function Val(·) across all levels of the hierarchy:

- **Micro level**: $\mathrm{Val}(CE)$ values individual composite element bundles.
- **Meso level**: $\Xi_F \cdot \mathrm{Val}(CE)$ values effective contributions within networks.
- **Macro level**: $\sum_k \Xi_F\,(G_k \rightarrow G_{k+1}) \cdot \mathrm{Val}(CE_k)$ values entire supply chains.
- **Market level**: $\frac{\Xi_F \cdot \mathrm{Val}(CE) \cdot NTr}{\rho - g}$ values firms as going concerns.

Every equation in the framework reduces, at its core, to operations on Val(·).

10.3 Logical Refutation Criteria (Falsifiability)

The following conditions would falsify or require extension of the CMES framework, providing methodological grounding for the theory's scientific standing:

Criterion A (Scale-Type Failure). If the structural zero and concatenation assumptions (Axiom 8–9) cannot be maintained—for example, if no economically meaningful zero configuration exists or if concatenation fails to satisfy the extensive measurement axioms—then Val(·) is only interval-scale and $NTr \cdot \mathrm{Val}(CE)$ is not ratio-meaningful. The master equation must be reformulated at interval-scale level.

Criterion B (Separability Failure). If topological joint separability (Axiom 5) fails globally—for example, in regimes where the interaction between archived and value-adding components is fundamentally non-separable (Crouzet et al. complementarity)—the additive baseline must be replaced by CES/Translog non-separable forms under explicitly revised assumptions. The nested CES of Definition 2.3a.2 provides the natural extension.

Criterion C (Boundary-Condition Confusion). If a proof simultaneously invokes perfect-competition zero-rent conditions for σ identification and rent-capturing markup terms driven by Ξ_F without distinguishing regimes, the argument is invalid. The two regimes are: (i) competitive dual-cost equilibrium for Theorem 3.1 ($\sigma = A_1/A_2$); (ii) imperfect competition with Ξ_F-driven markups for Section 7. These must be applied to non-overlapping contexts.

Criterion D (Ξ_F Independence Assumption Failure). If role multi-mapping is documented at the scale of analysis—where elements simultaneously and equally occupy multiple role classes—the multiplicative Ξ_F form is invalid; the additive canonical form (Definition 5.2) must be used and the MECE convention of Section 2.6.3 must be explicitly applied.

Criterion E (Diagonal Dominance Violation). If the stability condition for Theorem 8.1 fails at the candidate steady state (the Jacobian fails diagonal dominance), alternative stability proofs (global Lyapunov analysis, P-function inversion, spectral analysis of the full system) must be provided. The Jacobian must be evaluated strictly at the candidate steady state S^* satisfying $F(S^*, u^*, z^*) = S^*$; evaluation at arbitrary points does not establish local asymptotic stability. When the steady state is not unique, each candidate must be tested independently.

Criterion F (CES Substitutability Misspecification). If empirical evidence in the relevant domain indicates that role components are strong complements (Leontief-like, $\varrho \to -\infty$) rather than near-substitutes ($\varrho \to 1$), the additive Val specification of Theorem 2.1 provides a poor first-order approximation and the nested CES of Definition 2.3a.2 or non-linear alternatives must be adopted.

Criterion G (Knowledge Accumulation Regime). If empirical evidence supports strictly convex Romer ($\phi = 1$) knowledge compounding in a given domain rather than Jones ($\phi < 1$) diminishing returns, the accumulation dynamics of Theorem 8.2 simplify. Conversely, if evidence supports $\phi < 1$ but the model uses $\phi = 1$, long-run growth predictions will be systematically overstated.

11. Open Problems

1. **Phase-Transition Boundary.** Characterize the behavior of Ξ_F-propagation at and beyond the spectral boundary $\gamma\rho(M) = 1$.
2. **Endogenous Role Dynamics.** Derive the evolution of the role distribution (N^v, N^a, N^c, N^e) as an endogenous outcome rather than an exogenous parameter.
3. **Non-Constant Per-Element Value.** Extend Theorem 5.1 to heterogeneous per-element valuations.
4. **Empirical Identification.** Develop measurement protocols for the overlap counts that define Ξ_F in specific economic domains.
5. **Fractional Dynamics.** Establish existence and uniqueness for the fractional accumulation system of Theorem 8.4.
6. **Endogenous Network Formation.** Extend the CMES poset architecture to accommodate endogenous network formation in the sense of Acemoglu and Azar (2020), where firms choose input-combination structures from a Boolean lattice of possibilities.
7. **Second-Order Network Effects.** Develop a CMES second-order approximation analogous to the Baqaee–Farhi (2019) extension of Hulten's theorem, capturing amplification effects from network complementarities and factor reallocation.

8. **Nested CES Identification.** Establish conditions under which the nested CES structure of Definition 2.3a.2 is identified from CMES transaction data, following Lagomarsino (2020) four-input nested CES estimation methodology.

9. **Physical-Informational Cross-Effects.** Formally model the complementarity interaction (Crouzet et al. 2022) between physical and informational CE^a stocks, establishing the cross-partial derivative structure and its implications for accumulation dynamics.

11.6 Expanded Research Agenda

11.6.1 Asymptotic Spectral Equivalence. Let $M(t)$ be the discrete role-flow matrix derived from a finite system of N interacting composite group generators governed by rewrite rules. As $N \to \infty$, the empirical matrix converges almost surely to its expectation. The conjecture is that in the continuous thermodynamic limit, the discrete eigenvalues of the composite group state-transition operator are isomorphic to the continuous eigenvalues extracted via Tensor Dynamic Mode Decomposition (TDMD) on the interaction tensor $T(t)$. Formalizing this requires verifying Trotter–Kato conditions: uniform stability, resolvent consistency, and core domain density.

11.6.2 Fractional-Order Identification. The fractional-memory extension introduces order $\alpha \in (0,1]$ as a memory-depth parameter. An open problem is the principled identification of α from empirical time-series data on element counts $N(t)$. The conjecture is that the informational component of CE^a exhibits systematically lower α (stronger memory) than the physical component, consistent with the Crouzet et al. (2022) finding that intangible capital exhibits different depreciation dynamics than physical capital.

11.6.3 Integrated Physical-Informational State Dynamics. Derive necessary and sufficient conditions on the accumulation function ϕ under which an interruption in informational CE^a generates a quantifiable reduction in physical CE^a productivity, and vice versa. Characterize cross-partial derivatives under the constraint that total archived stock equals physical plus informational components, leveraging the Crouzet et al. (2022) non-rivalry parameter ϱ.

11.6.4 Formal Model-Checking of State Preservation Invariants. Construct a finite-state abstraction of the CMES dynamic system that is sufficiently expressive to capture essential accumulation dynamics while remaining amenable to automated model checking (TLA+, SPIN). The continuous state space must be discretized without introducing spurious conservation violations.

11.6.5 Category-Theoretic Composite Group Functors. Prove that the CMES manufacturing category admits a symmetric monoidal structure compatible with the additive conservation law, establishing coherence conditions (Mac Lane's coherence theorem) for the specific algebraic structure of the CMES composite group generator space. Connect to the Ghani–Hedges–Winschel–Zahn (2018) compositional game theory framework.

11.6.6 Möbius Function Characterization. Characterize the Möbius function of the CMES supply chain poset in closed form for standard supply chain topologies (linear chains, tree hierarchies, DAG structures). Establish conditions under which the Faigle–Kern (1992) Shapley value coincides with the Ξ_F-weighted decomposition of Theorem 6.1.

11.6.7 Myerson-Value Characterization of Ξ_F. Formally prove that Ξ_F satisfies (or can be extended to satisfy) the Myerson (1977) axioms of Component Efficiency and Fairness on the CMES communication graph, providing a full cooperative game-theoretic axiomatization.

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