Artin and Swan Conductors via Nearby Cycles for Strictly Semistable Varieties over Local Fields

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Keywords— étale cohomology; Swan conductor; vanishing cycles; nearby cycles; strict semistable reduction; simple normal crossings (SNC); local zeta factors; wild ramification; arithmetic geometry over local fields.
MSC 2020— Primary: 14F20, 14G20; Secondary: 11S15, 14F30, 11S31.

Abstract

Let K be a non-archimedean local field with finite residue field of characteristic p, and let $\ell \neq p$ be a prime. We develop a cohomological framework for analyzing Artin and Swan conductors associated with strictly semistable varieties over K. Using the formalism of nearby and vanishing cycles, we relate the ramification behavior of the ℓ-adic cohomology of the generic fiber to the geometry of the special fiber.

In the strictly semistable (simple normal crossings) case with $\ell \neq p$, we give explicit formulas for (i) inertia invariants and the unramified local factor via Frobenius acting on nearby-cycles cohomology, and (ii) the tame/unipotent (monodromy) contribution to the Artin conductor in terms of the monodromy operator on the associated Weil–Deligne representation. Outside the strictly semistable range, we isolate the precise mechanism by which additional vanishing-cycle terms contribute to ramification: the obstruction to specialization and any genuinely wild contribution are detected on the vanishing-cycle complex.

This perspective clarifies how local ℓ-adic cohomological invariants reflect the combinatorial and geometric structure of the special fiber. The resulting formulas provide a transparent description of conductor behavior within the strictly semistable range and identify the cohomological obstructions that arise beyond it. As applications, we obtain a structural decomposition of local zeta factors and a refined interpretation of wild ramification phenomena in arithmetic geometry over local fields.

Contents

1 Introduction and Main Results

Motivation and scope

The interaction between algebraic geometry and number theory over non-archimedean local fields has been a decisive theme since Grothendieck's formulation of ℓ-adic cohomology ([7, Exp. XVI–XVIII]; [8, Exp. III]) and Deligne's weight theory ([4, Exp. XIII]; [10, Th. 1.6]).

When a smooth projective variety X/K admits a regular or semistable model over $\mathcal{O}_K$, its étale cohomology $H^i(X_{\overline{K}}, \mathbb{Q}_\ell)$ carries a canonical Weil–Deligne representation of G_K. Classical inputs—proper and smooth base change (Theorem 2.2), Grothendieck–Ogg–Shafarevich for curves (Theorem 2.5), weight–monodromy (Theorem 2.9), Gabber finiteness (Theorem 3.4), and comparison via nearby cycles (Theorem 2.8)—provide a rigorous background.

Despite these foundational tools, several arithmetic features remained elusive:

- explicit conductor formulas for higher-dimensional semistable models;
- localized height gaps and *localized* Northcott-type finiteness statements over local fields (after imposing a thickness/fibre condition);
- density of Frobenius eigenvalues under inertial restrictions;
- deformation-theoretic constancy of local L-data on moduli strata.

The purpose of this paper is to address these points with new theorems, explicit calculations, and counterexamples, thereby clarifying the geometry $\leftrightarrow$ arithmetic dictionary over local fields.

Under strict semistability we obtain explicit formulas for the unramified factor and for the *tame/unipotent* monodromy contribution to the Artin conductor (with vanishing Swan contribution under the additional assumption that the inertia action is tamely ramified on the nearby-cycle complex); beyond strict semistability, additional vanishing-cycle terms in $R\Phi$ can contribute genuine wild Swan terms.

As emphasized in Theorem 5.4, the conductor and local factor formulas hold in all degrees $i < \dim X$ under strict semistability (simple normal crossings). Outside this setting, additional vanishing-cycle terms $R\Phi$ contribute nontrivially to the Swan conductor and prevent a direct identification of inertia invariants with special-fiber cohomology (cf. Theorems 3.17 and 5.7).

Precise novelty statement

Relative to the standard sources [7, 8, 9, 10, 11, 12], the results below appear to be new in the following precise sense: they are stated and proved in a uniform local-field framework that keeps track simultaneously of nearby-cycles Frobenius data, monodromy, and the explicit failure modes beyond strict semistability; moreover, several consequences (height gaps, deformation constancy of local L-data) do not seem to be available in this form even in classical families.

- We formulate a uniform local framing (Theorems 4.1 and 5.4) for the invariant–coinvariant sequence and Swan identification under strict semistability, clarifying and systematizing the classical results of SGA 7, Rapoport–Zink, and Saito. The presentation emphasizes explicit formulas, local functoriality, and example-driven clarifications (e.g. Examples 4.8, 4.11 and 5.2 and theorem 5.3), rather than claiming a new comparison theorem beyond the established semistable framework.

- We establish a *localized height gap away from torsion on the skeleton* (Theorem 4.5), yielding a localized Northcott-type finiteness statement for abelian varieties with toric rank under the imposed thickness condition—a phenomenon not deducible from Néron–Ogg–Shafarevich alone. The novelty lies in bridging monodromy gaps with local canonical heights, yielding new arithmetic finiteness results (Theorem 5.1 and example 5.2).

- We provide an explicit conductor and local factor formula (Theorem 5.4) for strictly semistable models in all degrees $i < \dim X$, expressed in terms of the *nearby-cycle/weight data* on the special fiber. Concretely, the unramified local factor (Only when the specialization morphism $R\Psi_{\mathcal{X}}\mathbf{Q}_\ell \to \mathbf{Q}_\ell$ induces an isomorphism on H^i does this reduce to Frobenius acting on $H^i(X_s, \mathbf{Q}_\ell)$) is governed by Frobenius acting on $H^i(X_s, R\Psi_{\mathcal{X}}\mathbf{Q}_\ell)$, and the tame/unipotent monodromy contribution is read off from the monodromy filtration (equivalently, from the nilpotent operator N) on the associated Weil–Deligne representation. Under strict semistability, the graded pieces $\mathrm{Gr}^W_\bullet H^i(X_{\overline{K}})$ are computed by the Rapoport–Zink/Illusie weight spectral sequence, i.e. as explicit subquotients of cohomology groups of the strata (notably the codimension-1 strata/double intersections, with the appropriate Tate twists). In particular, the local L-data are functorially determined by the *decorated dual complex*, meaning the dual complex together with the Frobenius and weight data on the relevant strata that enter the weight spectral sequence; the incidence complex alone does not determine Frobenius traces in general. Beyond strict semistability, additional vanishing-cycle contributions appear (see Theorems 3.17 and 5.7).

- **(Density: arithmetic reduction + standard equidistribution).** We prove a local density statement for normalized Frobenius eigenphases on inertia invariants (Theorem 4.10) by *separating the inputs*: (i) the arithmetic/geometric input is the canonical identification

  $$H^i_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_\ell)^{I_K} \;\cong\; H^i\Big((X_s)_{\mathbf{F}_{q^n}}, R\Psi_{\mathcal{X}}\mathbf{Q}_\ell\Big)$$

  under strict semistability (Theorem 3.9), together with Deligne purity on the graded pieces of the monodromy filtration (Theorem 2.9), which determines the phase torus T_i.

  The comparison with $H^i_{\text{ét}}((X_s)_{\mathbf{F}_{q^n}}, \mathbf{Q}_\ell)$ occurs only via the specialization morphism

  $$R\Psi_{\mathcal{X}}\mathbf{Q}_\ell \to \mathbf{Q}_\ell,$$

  and is an isomorphism in degree i precisely when the corresponding I_{K_n}-invariant vanishing-cycle contribution in $R\Phi_{\mathcal{X}}$ vanishes in that range; (ii) the *analytic* input is the classical Kronecker–Weyl/Weyl equidistribution for power maps on compact tori under an explicit non-resonance condition. Our novelty is thus the local cohomological reduction and the resulting geometric control of the phase space (and failure modes outside hypotheses), not a new harmonic-analysis theorem.

- We show deformation-constancy of local L-data on moduli strata (Theorem 5.9), demonstrating that conductors and spectral radii remain unchanged under deformations preserving the decorated dual complex (including Frobenius and weight data). This is new even for classical Tate families (Example 5.11).

Each of these results is anchored in the local-field setting of Theorem 3.2, proved with precise cohomological methods, and paired with arithmetic consequences. No claim is a simple repetition of known tools; when a statement follows from standard base change, flatness, or cone arguments, it is relegated to lemmas or propositions and fully cited.

Outline of results

For clarity, we summarize the paper's architecture in the *Theorem* → *Consequence* → *Example* format.

- **Cohomological comparison.** Theorems 3.9 and 4.1 give exact sequences for $H^i(X)$ under inertia.
  Consequence: explicit Swan conductor formula.
  Example: nodal and hyperelliptic curves (Examples 3.12, 6.1 and 6.2).

- **Height gap (localized).** If $t(A) > 0$, Theorem 4.5 yields a positive lower bound for $\hat{\lambda}_v$ on points whose tropical image stays a fixed distance away from the identity/torsion in the Raynaud skeleton.
  Consequence: localized Northcott finiteness (Theorem 5.1).
  Example: Tate curve (Examples 4.7 and 5.2).
  Counterexample to a uniform gap: good reduction (Example 4.8 and theorem 5.3).

- **Conductor and local factor formula.** Theorem 5.4 expresses $L(s, H^i)$ and $a(H^i)$ via the dual complex.
  Consequence: combinatorial determination of local L-data.
  Example: SNC surface (Example 5.6);
  Counterexample: wild cusp or pinch point (Theorems 3.17 and 5.7).

- **Density of Frobenius eigenvalues.** Theorem 4.10 proves weak convergence to the weight–monodromy distribution.
  Consequence: distribution of Frobenius weights consistent with purity/weight constraints (conditional on semistability and purity assumptions).
  Example: explicit surface case (Example 4.11).

- **Deformation constancy.** Theorem 5.9 shows $a(H^i)$ and spectral radii are constant on strata.
  Consequence: invariance of L-data across families.
  Example: Tate family (Example 5.11);
  Counterexample: jump across reduction types (Example 5.12).

Continuity. The paper closes with a synthesis and future directions (Section 7), where we emphasize the potential for global applications, higher-dimensional extensions, and compatibility with automorphic frameworks. Each section is self-contained, consistent with the local-field anchor, and contributes to the unified theme: translating the geometry of semistable models into arithmetic invariants.

2 Background and Preliminaries

Roadmap of key objects (used throughout).

- $R\Psi_{\mathcal{X}}\mathbb{Q}_\ell$ (nearby cycles) and $R\Phi_{\mathcal{X}}\mathbb{Q}_\ell$ (vanishing cycles) on X_s: defined by the nearby/vanishing-cycles triangle in $D^b_c(X_s, \mathbb{Q}_\ell)$ (see Theorem 2.8).
- **Invariants identification:** $H^i_{\text{ét}}(X, \mathbb{Q}_\ell)^{I_K} \cong \mathbb{H}^i(X_s, R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)$ (Theorem 3.9(a)).
- **Specialization map:** $\mathrm{sp} : \mathbb{H}^i(X_s, R\Psi_{\mathcal{X}}\mathbb{Q}_\ell) \to H^i_{\text{ét}}(X_s, \mathbb{Q}_\ell)$ induced by $R\Psi \to \mathbb{Q}_\ell$; its failure to be an isomorphism is measured by $R\Phi$ (Theorem 3.9(b)).
- **Weil–Deligne parameter:** (r_i, N_i) attached to $H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$; N_i records tame/unipotent monodromy.
- **Conductors:** $a(H^i) = \dim H^i - \dim H^{i,I_K} + \mathrm{Sw}(H^i)$ with Sw = *wild inertia only.* Under strict semistability and tamely ramified inertia action (with $\ell \neq p$), one has $\mathrm{Sw}(H^i) = 0$, and the remaining ramification is encoded by the image of the monodromy operator N_i.
- **Monodromy rank:** $m_i(X) := \dim_{\mathbb{Q}_\ell} \Im(N_i)$ (tame/unipotent contribution; not a Swan term).
- **Phase torus:** $T_i \subset S^1$ generated by normalized Frobenius eigenphases on invariants (Definition within Theorem 4.10).

Throughout, we fix once and for all a non-archimedean local field K with ring of integers $\mathcal{O}_K$, uniformizer π, finite residue field k of cardinality q, and absolute Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$. We denote the inertia subgroup by $I_K \subset G_K$ and its wild inertia by $P_K \subset I_K$. All geometric objects considered are separated schemes of finite type over K unless explicitly specified otherwise. For varieties X/K, we write $\overline{X} := X \times_K \overline{K}$.

2.1 Étale cohomology: classical foundations

Definition 2.1 (Étale cohomology groups)**.** Let X/K be a separated scheme of finite type, and let $\ell \neq p = \mathrm{char}(k)$ be a prime. We define the ℓ-adic étale cohomology groups

$$H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) := \varprojlim_n H^i_{\text{ét}}(\overline{X}, \mathbb{Z}/\ell^n\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

These are finite-dimensional $\mathbb{Q}_\ell$-vector spaces equipped with a continuous G_K-action [7, Exp. XVI], [11, Ch. VI].

Lemma 2.2 (Proper base change)**.** *If $f : X \to S$ is a proper morphism of schemes, $\ell \neq \mathrm{char}(k)$, and $\mathcal{F}$ is a constructible ℓ-torsion sheaf on X, then for every $i \geq 0$ one has*

$$(R^i f_* \mathcal{F})_{\overline{s}} \;\cong\; H^i_{ét}(X_{\overline{s}}, \mathcal{F}),$$

where $s \in S$ is any geometric point. Proof. *This is the standard proper base change theorem [7, Exp. XVII, Th. 5.2.6].*

Remark 2.3 (Poincaré duality)**.** If X/K is smooth of pure dimension d, then for each i there is a canonical perfect pairing

$$H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) \times H^{2d-i}_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)(d) \to \mathbb{Q}_\ell,$$

where (d) denotes Tate twist. This is a consequence of the duality theory of étale cohomology [7, Exp. XVIII], [11, Ch. VI].

2.2 Local fields and arithmetic schemes

Notation 2.4 (Geometric and arithmetic Frobenius)**.** *We denote by* $\mathrm{Frob}_q \in G_K/I_K$ *the arithmetic Frobenius element, sending $x \mapsto x^q$ on $\overline{k}$. Its inverse is the geometric Frobenius, often denoted Φ_q.*

For any $\ell \neq p$, the Frobenius action is semisimple on the pure graded pieces of the weight/monodromy filtration by Deligne's purity results. In particular, on the weight-i piece (and, under strict semistability, on $H^i(X)^{I_K}$ via the invariants–special fibre identification) Frobenius acts semisimply. We do not assume semisimplicity on the entire $H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$ [14].

Proposition 2.5 (Numerical Euler–Poincaré formula)**.** *If C/K is a smooth projective curve with semistable reduction, then*

$$\sum_{i=0}^{2} (-1)^i \dim_{\mathbb{Q}_\ell} H^i_{ét}(\overline{C}, \mathbb{Q}_\ell) \;=\; 2 - 2g,$$

and the Artin conductor of $H^1_{ét}(\overline{C}, \mathbb{Q}_\ell)$ satisfies the Grothendieck–Ogg–Shafarevich formula ([8, Exp. XIII] and [11, VI.11]).

Proof. The cohomological dimension of curves over K ensures vanishing for $i > 2$. For $H^1_{\text{ét}}(C, \mathbb{Q}_\ell)$, the Artin conductor is given by the Grothendieck–Ogg–Shafarevich formula, which decomposes the conductor as the sum of tame and Swan contributions at the finitely many bad points on a regular (semistable) model of C; [see SGA 7 (Exp. IX, XIII) [4, 9] and [11]].

□

Example 2.6 (Explicit computation for a Tate curve)**.** Let E/K be a Tate elliptic curve with parameter $q_E \in K^\times$, $|q_E| < 1$. Then E has split multiplicative reduction. The I_K–action on $H^1_{\text{ét}}(E, \mathbb{Q}_\ell)$ is unipotent of rank 1 and H^1 fits into a *non-split* exact sequence

$$0 \longrightarrow \mathbb{Q}_\ell(0) \longrightarrow H^1_{\text{ét}}(E, \mathbb{Q}_\ell) \longrightarrow \mathbb{Q}_\ell(-1) \longrightarrow 0.$$

In a suitable basis for the associated Weil–Deligne representation, tame inertia acts by

$$T = \begin{pmatrix} 1 & v_K(q_E) \\ 0 & 1 \end{pmatrix},$$

so the monodromy N has rank 1. The reduction is tame, hence the *Swan conductor is* 0, while the Artin conductor exponent is $a(H^1) = 1$. Consequently, using the cohomological normalization

$$L(s, H^1(E)) := \det\Big(1 - \mathrm{Frob}_q\, q^{-s} \mid H^1_{\text{ét}}(E_{\overline{K}}, \mathbb{Q}_\ell)^{I_K}\Big)^{-1},$$

and the fact that $\dim H^1(E)^{I_K} = 1$ in the split multiplicative case, one gets

$$L(s, H^1(E)) = \frac{1}{1 - q^{-s}}.$$

(For the *full* local zeta factor of the curve one must also include the H^0 and H^2 contributions, producing $(1 - q^{-s})^{-1}(1 - q^{1-s})^{-1}$; that is *not* $L(s, H^1)$ alone.)

Example 2.7 (Failure without semistability)**.** Consider a curve with potentially wild reduction. If C/K acquires non-semistable reduction with wild inertia action, the GOSh formula no longer yields a correct conductor; extra Swan contributions appear. This shows semistability is a necessary hypothesis in Theorem 2.5.

2.3 Moduli-theoretic input

Construction 2.8 (Nearby and vanishing cycles)**.** For $\mathcal{X}/\mathcal{O}_K$ a proper flat scheme, let $\eta = \mathrm{Spec}(K)$, $s = \mathrm{Spec}(k)$. The complexes $R\Psi$ (nearby cycles) and $R\Phi$ (vanishing cycles) in $D^b_c(X_s, \mathbb{Q}_\ell)$ govern the behaviour of étale cohomology under specialization [9]. There is a distinguished triangle

$$i^* Rj_* \mathbb{Q}_\ell \to R\Psi \to R\Phi \xrightarrow{+1},$$

(cf. [4] for the construction of $R\Psi$ and $R\Phi$ and the distinguished triangle; see also [10].)

where $j : \eta \hookrightarrow \mathcal{X}$ and $i : s \hookrightarrow \mathcal{X}$.

Theorem 2.9 (Weight–monodromy: monodromy filtration always; purity in known cases)**.** *Let X/K be a proper smooth variety of pure dimension d over a non-archimedean local field K with residue field k of cardinality q, and fix $\ell \neq \mathrm{char}(k)$. Write*

$$H^i(X) := H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$$

and denote by (r_i, N_i) the associated Weil–Deligne representation of W_K.

1. ***(Monodromy filtration: unconditional).*** *There exists an increasing monodromy filtration $M_\bullet$ on $H^i(X)$ such that $N_i(M_j) \subset M_{j-2}(-1)$ and, for each integer $r \geq 0$, the map N_i^r induces an isomorphism*
$$N_i^r : \ \mathrm{Gr}^M_{i+r} H^i(X) \xrightarrow{\sim} \mathrm{Gr}^M_{i-r} H^i(X)(-r).$$
2. ***(Purity of graded pieces: conjectural in general).*** *The assertion that each $\mathrm{Gr}^M_{i+r} H^i(X)$ is a pure q-Weil representation of weight $i + r$ (equivalently, every eigenvalue α of Frob_q on $\mathrm{Gr}^M_{i+r} H^i(X)$ satisfies $|\alpha| = q^{(i+r)/2}$) is the* weight–monodromy conjecture *in general.*

Scope of use in this paper. *Whenever we invoke purity of $\mathrm{Gr}^M_{i+r} H^i(X)$ (and consequently any weight-based identification of local L-factors or conductor terms), we do so only under additional hypotheses for which purity is known from the cited literature (e.g. equal characteristic; curves; abelian varieties; and the strictly semistable/SNC nearby-cycles situations explicitly stated at the point of use). In mixed/unequal characteristic the conjecture is open in full generality, and nothing in this paper should be read as asserting purity beyond those known cases.*

Proof. The existence of the monodromy operator on nearby cycles and the resulting monodromy filtration, together with the isomorphisms $N_i^r : \mathrm{Gr}^M_{i+r} H^i(X) \xrightarrow{\sim} \mathrm{Gr}^M_{i-r} H^i(X)(-r)$, are established in SGA 7, Exp. XIII (see [4]).

Purity of the graded pieces $\mathrm{Gr}^M_{i+r} H^i(X)$ is proved by Deligne in equal characteristic via globalization and the Weil conjectures [10, 14]. In mixed characteristic, purity is known in several important cases (curves, abelian varieties, and certain semistable degenerations) and remains conjectural in full generality. □

Qualification (scope). The full weight–monodromy statement invoked above is known in equal characteristic (Deligne, *Weil II* [10]; SGA 7, Exp. XIII [4]) and in several mixed-characteristic cases (e.g. for curves, abelian varieties, certain semistable degenerations). In general unequal characteristic it remains open. In this paper we use only the consequences that are established under strict semistability in the degrees where we work, and we indicate explicitly whenever we rely on these known cases.

$$\begin{tikzcd}
H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) \arrow[r, dashed, "M_\bullet"] \arrow[d, "N_i"'] & \text{Weight filtration } W_\bullet \arrow[d, "\text{purity } w=i+r"] \\
H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)(-1) \arrow[r, dashed, "\mathrm{Gr}^M_\bullet"'] & \text{Pure graded pieces of weights } i \pm r
\end{tikzcd}$$

Figure 1: Interaction between monodromy and weight filtrations for $H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$; arrows represent the nilpotent operator N_i and the purity weights (in the cases where purity is known, as specified after Theorem 2.9).

In the strictly semistable (SNC) case (and in the degrees where the weight–monodromy identifications are known), the monodromy and weight filtrations coincide up to the usual indexing shift; we do not assert such an identification outside this hypothesis.

Remark 2.10 (Geometric → Arithmetic (scope control))**.** Purity of the graded pieces (in the cases where it is known) is the bridge that allows us to read arithmetic invariants (local L-factors and conductor terms) from geometric objects on the special fibre.

Scope. Whenever we use an identification of the form

$$\Im(N_i) \;\cong\; \mathrm{Gr}^W_{i-1} H^i(X) \;\cong\; H^{i-1}(X_s)(-1) \qquad \text{or} \qquad a(H^i) = \dim \mathrm{Gr}^M_{i-1}(-1),$$

we are *explicitly* in a range where this is known: in this paper, this means *strict semistability (SNC) with unipotent inertia and degrees* $i < \dim X$ (as in Theorems 3.9 and 5.4). Outside this hypothesis, additional vanishing-cycle terms $R\Phi$ may contribute and one must keep the Swan term and the specialization map separate (cf. the scope warnings in Theorems 3.9 and 5.4).

In particular, the "closed-form" conductor identities used later should always be read as *conditional on the semistable hypothesis stated at the point of use.*

Example 2.11 (Semistable surface model)**.** *Setup.* Let $\mathcal{X}/\mathcal{O}_K$ be a strictly semistable model of a smooth projective surface X/K with special fiber $X_s = \bigcup_{i \in I} Y_i$ a simple normal crossings divisor (SNC). Write $Y_{ij} := Y_i \cap Y_j$ (a smooth curve, possibly disconnected) and $Y_{ijk} := Y_i \cap Y_j \cap Y_k$ (a finite set of points). Fix $\ell \neq \mathrm{char}(k)$.

Weight spectral sequence input. The $R\Psi$-formalism yields a spectral sequence whose E_1-page is built from the strata Y_i, Y_{ij}, Y_{ijk}. For $i = 2$ one obtains canonical identifications of the graded pieces of the monodromy/weight filtration [9]:

$$\mathrm{Gr}^W_2 H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) \;\cong\; \ker\!\left(\bigoplus_i H^2_{\text{ét}}(\overline{Y}_i, \mathbb{Q}_\ell) \;\xrightarrow{\partial}\; \bigoplus_{i<j} H^2_{\text{ét}}(\overline{Y}_{ij}, \mathbb{Q}_\ell) \right),$$

$$\mathrm{Gr}^W_1 H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) \;\cong\; \Big(\bigoplus_{i<j} H^1_{\text{ét}}(\overline{Y}_{ij}, \mathbb{Q}_\ell) \Big)(-1), \qquad \mathrm{Gr}^W_0 H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) \;\cong\; \Big(\bigoplus_{i<j<k} H^0_{\text{ét}}(\overline{Y}_{ijk}, \mathbb{Q}_\ell) \Big)(-2),$$

and $\mathrm{Gr}^W_w = 0$ for $w \notin \{0,1,2\}$. In the strictly semistable (SNC) situation, purity of these graded pieces is available in the ranges we use (cf. the scope after Theorem 2.9), so each Gr^W_w is pure of weight w. The monodromy operator N induces isomorphisms

$$N : \mathrm{Gr}^W_2 \xrightarrow{\sim} \mathrm{Gr}^W_0(-1), \qquad \mathrm{Im}(N) \cong Gr^1_M H^2_{\text{ét}}(X, Q_\ell),$$

which in the strictly semistable case coincides with

$$Gr^1_W H^2_{\text{ét}}(X, Q_\ell).$$

Consequences.

- **Invariants and L-factor.** The unramified (inertia-invariant) quotient is the weight-2 piece:

  $$H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)^{I_K} \;\cong\; \mathrm{Gr}^W_2 H^2, \qquad L(s, H^2(X)) = \det{}^{-1}\Big(1 - q^{-s}\mathrm{Frob}_q \,\big|\, \mathrm{Gr}^W_2 H^2\Big).$$

  Concretely, Gr^W_2 is computed as the kernel of the boundary map ∂ from the component classes to the double curves.

- **Wild Swan vs. monodromy rank.** Under strict semistability for $\ell \neq p$, the inertia action on $H^2(X)$ is tame. In particular, the wild inertia acts trivially, and hence

  $$\mathrm{Sw}(H^2(X)) = 0.$$

  What the weight–monodromy spectral sequence computes in this setting is instead the *monodromy rank*

  $$m_2(X) := \dim_{\mathbb{Q}_\ell} \mathrm{Im}(N) = \dim_{\mathbb{Q}_\ell} \mathrm{Gr}^W_1 H^2 = \sum_{i<j} \dim_{\mathbb{Q}_\ell} H^1_{\text{ét}}(\overline{Y}_{ij}, \mathbb{Q}_\ell),$$

  i.e. the sum of the first Betti numbers of all double curves (each contributing with a (-1)-twist).

Monodromy-rank formula (SNC surface).

$$m_2(X) = \sum_{i<j} b_1(Y_{ij}).$$

Each graded piece $\mathrm{Gr}^W_w H^2(X)$ is pure of weight $w = 0, 1, 2$.

Working subcases.

(A) Two components meeting along a smooth curve. Assume $X_s = Y_1 \cup Y_2$ with $C := Y_{12}$ a smooth projective curve (no triple intersections). Then

$$\mathrm{Gr}^W_2 H^2 \cong \ker\!\big(H^2(\overline{Y}_1) \oplus H^2(\overline{Y}_2) \xrightarrow{\partial} H^2(\overline{C})\big), \quad \mathrm{Gr}^W_1 H^2 \cong H^1(\overline{C})(-1), \quad \mathrm{Gr}^W_0 H^2 = 0.$$

Hence

$$m_2(X) = \dim H^1_{\text{ét}}(C, \mathbf{Q}_\ell) = \sum_i 2g(C_i),$$

and

$$L(s, H^2(X)) = \det{}^{-1}\Big(1 - q^{-s}\mathrm{Frob}_q \,\big|\, \ker(H^2(\overline{Y}_1) \oplus H^2(\overline{Y}_2) \to H^2(\overline{C}))\Big).$$

Bridge (AG → NT). The ramification of $H^2(X)$ is governed by the Jacobian part of C via tame unipotent monodromy.

(B) Chain of three components. Let $X_s = Y_1 \cup Y_2 \cup Y_3$ with $C_{12} := Y_{12}$ and $C_{23} := Y_{23}$ smooth curves, $Y_{13} = \varnothing$, and no triple intersections. Then

$$\mathrm{Gr}^W_2 H^2 \cong \ker\Big(\bigoplus_{i=1}^{3} H^2(\overline{Y}_i) \to H^2(\overline{C}_{12}) \oplus H^2(\overline{C}_{23})\Big),$$

$$\mathrm{Gr}_1^W H^2 \cong H^1(\overline{C}_{12})(-1) \oplus H^1(\overline{C}_{23})(-1), \qquad \mathrm{Gr}_0^W H^2 = 0.$$

Thus

$$m_2(X) = \dim H^1(\overline{C}_{12}) + \dim H^1(\overline{C}_{23}),$$

while $\mathrm{Sw}(H^2(X)) = 0$, and the L-factor is computed from Gr_2^W as above.

(C) With triple points. If some Y_{ijk} is nonempty, then

$$\mathrm{Gr}_0^W H^2 \cong \Big(\bigoplus H^0(\overline{Y}_{ijk})\Big)(-2)$$

is nonzero. Monodromy gives an isomorphism

$$N : \mathrm{Gr}_2^W \xrightarrow{\sim} \mathrm{Gr}_0^W(-1),$$

so the size of the triple-intersection set controls the rank of N from weight 2 onto weight 0. The contribution Gr_1^W measures *tame unipotent monodromy*; the wild Swan conductor still vanishes under strict semistability.

Bridge (AG → NT). In the strictly semistable setting (where purity is known in our range, cf. Theorem 2.9), the purity of Gr_2^W (weight 2) identifies the unramified local factor with Frobenius on Gr_2^W, while the monodromy rank $m_2(X) = \dim \mathrm{Gr}_1^W$ encodes the tame ramification of the Weil–Deligne parameter. Wild Swan contributions appear only outside the SNC hypothesis (cf. Theorem 2.12).

$$\begin{tikzcd}
H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) \arrow[r, dashed, "\text{monodromy } N"] \arrow[dr, bend right, "\text{weight filtration}"'] & H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)(-1) \arrow[d, "\text{projection to graded pieces}"] \\
& \mathrm{Gr}^W H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)
\end{tikzcd}$$

Figure 2: Monodromy operator on H^2 and induced weight filtration.

Counterexample 2.12 (Counterexample outside semistability: pinch point with wild vanishing cycles)**.** *Setup.* Let K be a non-archimedean local field with ring $\mathcal{O}_K$, uniformizer π, residue field k of characteristic $p > 2$, and fix $\ell \neq p$. Consider the flat $\mathcal{O}_K$-surface

$$\mathcal{X} \;:=\; \operatorname{Spec} \mathcal{O}_K[x, y, z]/(z^2 - x^2 y - \pi\, y^2).$$

Let $X := \mathcal{X} \otimes_{\mathcal{O}_K} K$ and $X_s := \mathcal{X} \otimes_{\mathcal{O}_K} k$. Then

$$X_s : \quad z^2 \;=\; x^2 y \qquad \text{(a pinch point along the } y\text{-axis).}$$

In particular, X_s is *not* a simple normal crossings (SNC) divisor. There are *no* distinct irreducible components crossing transversely, hence no “double curves” Y_{ij} and no “triple points” Y_{ijk} in the sense of semistable reduction.

Claim. The special fibre X_s is not SNC, so the standard strictly semistable (SNC) control of inertia via the $R\Psi$-weight spectral sequence does not apply as stated. In particular, there may be a nontrivial vanishing-cycles contribution $H^2(X_s, R\Phi_{\mathcal{X}}\mathbb{Q}_\ell)^{I_K} \neq 0$ and potentially nontrivial wild inertia on $H^2(X)$. We do *not* compute $\mathrm{Sw}(H^2(X))$ for this pinch-point model here; the example is used only to illustrate that outside the SNC hypothesis one cannot read local conductor data solely from SNC strata (e.g. “double curves”), because additional $R\Phi$ terms can appear.

Why the SNC-strata recipe becomes silent here. Because X_s has a *single* irreducible component with a *non-SNC* singularity, there are no SNC strata of the form Y_{ij} or Y_{ijk}. Thus the strata-based formula from Example 2.11 yields

$$\Big(\bigoplus_{i<j} H^1(\overline{Y}_{ij})\Big)(-1) = 0,$$

so it would predict a *vanishing contribution from SNC double curves* to the *tame/unipotent* part (measured by $\dim \Im(N)$ in the Weil–Deligne representation). However, in the non-SNC setting the vanishing-cycles complex $R\Phi_{\mathcal{X}}\mathbb{Q}_\ell$ need not vanish, and its cohomology can contribute additional terms to inertia/monodromy that are invisible to the SNC-strata recipe. Accordingly, *no conclusion about* $\Im(N)$ or $\mathrm{Sw}(H^2(X))$ should be drawn from the absence of Y_{ij} alone. However, at the pinch point the nearby/vanishing-cycles triangle shows that the vanishing-cycles complex $R\Phi_{\mathcal{X}}\mathbb{Q}_\ell$ need not vanish, so the specialization map can fail to be an isomorphism and additional inertia-invariant contributions may appear:

$$H^i(X_{\overline{K}}, \mathbf{Q}_\ell)^{I_K} \;\cong\; H^i(X_{\bar{s}}, R\Psi_{\mathcal{X}}\mathbf{Q}_\ell)\,,$$

and the canonical triangle

$$\mathbf{Q}_\ell \longrightarrow R\Psi_{\mathcal{X}}\mathbf{Q}_\ell \longrightarrow R\Phi_{\mathcal{X}}\mathbf{Q}_\ell \xrightarrow{+1}$$

yields a long exact sequence in cohomology in which the failure of specialization to be an isomorphism is measured by $H^i(X_{\bar{s}}, R\Phi_{\mathcal{X}}\mathbf{Q}_\ell)$. We do *not* compute the wild inertia action on the relevant stalk cohomology of $R\Phi$ for this pinch-point model here, and therefore we make *no* numerical claim about $\mathrm{Sw}(H^2(X))$.

Remark. Nontrivial vanishing cycles may affect the inertia action in different ways: they can contribute to tame unipotent monodromy (encoded by the operator N in the Weil–Deligne representation) and/or to the wild part (measured by the Swan conductor), depending on whether wild inertia acts.

The purpose of the example is solely to illustrate that outside the SNC hypothesis the $R\Psi$-strata recipe is insufficient: one must account for possible $R\Phi$-terms when discussing local conductor data.

Consequences.

- **Failure of the "double-curves only" conductor recipe.** Since there are no Y_{ij}, a purely SNC-strata-based recipe would contribute nothing to the tame/unipotent part coming from double curves. In a non-SNC degeneration this does *not* imply that the monodromy or conductor is small: additional contributions can enter through vanishing cycles ($R\Phi$), and in genuinely wildly ramified situations one may also have $\mathrm{Sw}(H^2) \neq 0$.
- **Local L-factor & WD parameter.** The local factor is governed by the inertia invariants $H^2(X)^{I_K}$ (equivalently $H^2(X_{\bar{s}}, R\Psi)$); outside the SNC hypothesis the WD-parameter need not be recoverable from SNC strata alone because $H^2(X_{\bar{s}}, R\Phi_{\mathcal{X}}\mathbf{Q}_\ell)$ may contribute extra inertia/monodromy terms.

Moral. The SNC/strict semistability hypothesis in Example 2.11 is essential for the *purely combinatorial* description of the *tame/unipotent* monodromy terms (equivalently, the $R\Psi$-graded pieces governed by the stratification of an SNC special fibre). If the special fibre has *non-SNC* singularities (e.g. pinch points or cusp singularities), then additional *vanishing-cycle* contributions can occur: concretely, the complex $R\Phi_{\mathcal{X}}\mathbb{Q}_\ell$ need not vanish, and its cohomology contributes extra terms to the conductor beyond what the intersection matrix/double curves predict. In particular, any attempt to read off the full Artin conductor from the intersection matrix alone can fail in the non-SNC setting; and in genuinely *wildly ramified* degenerations (outside the tame/strictly semistable regime) one may have $\mathrm{Sw}(H^i) \neq 0$ as well. This does not contradict Theorem 2.9: the weight–monodromy statement concerns the structure of the weight filtration, whereas the SNC hypothesis is what permits identifying the relevant graded pieces with the combinatorics of the SNC strata.

3 Cohomological Framework over Local Fields

We continue with the standing hypotheses fixed in Theorems 2.1, 2.2, 2.4, 2.5, 2.8 and 2.9 and examples 2.6, 2.7 and 2.11. Our aim is to isolate the precise cohomological mechanisms that will feed into the arithmetic applications of the next section. The emphasis here is on vanishing, finiteness, and the passage from cohomology of schemes over K to representations of G_K.

Standing hypotheses. Throughout this section we assume that $\mathcal{X}/\mathcal{O}_K$ is *strictly semistable*, that $\ell \neq p$, and that the cohomological index satisfies $0 \leq i < \dim X$. All subsequent identifications and conductor formulas are valid only under these assumptions.

Definition 3.1 (Local conventions and WD normalization)**.** Let K be non-archimedean with ring $\mathcal{O}_K$, residue field k of size q, and absolute Galois G_K. We write $\mathrm{Frob}_q \in G_K/I_K$ for *arithmetic* Frobenius ($x \mapsto x^q$ on k) and $\Phi_q := \mathrm{Frob}_q^{-1}$ for *geometric* Frobenius. For a continuous ℓ-adic G_K-representation V ($\ell \neq p$), its Weil–Deligne parameter (r, N) is normalized so that $r(\mathrm{Frob}_q)$ has eigenvalues of absolute value $q^{w/2}$ on a pure weight-w quotient. We use $\mathrm{Sw}(V)$ for the Swan conductor and $a(V)$ for the Artin conductor, with $a(V) = \mathrm{Sw}(V) + \dim(V/V^{I_K})$.

3.1 Setup and notation

Notation 3.2 (Standing setup for local fields)**.** *Let K be a non-archimedean local field with ring of integers $\mathcal{O}_K$, uniformizer π, finite residue field k of cardinality q, and absolute Galois group G_K. Denote by I_K the inertia subgroup and by $P_K \subset I_K$ the wild inertia. For a separated scheme X/K of finite type, we write*

$$H^i(X) := H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell), \qquad \ell \neq \mathrm{char}(k).$$

The nearby and vanishing cycle functors $R\Psi$ and $R\Phi$ are taken relative to $\mathcal{O}_K$-models, as recalled in Theorem 2.8.

Remark 3.3 (Weil–Deligne parameters)**.** Any $H^i(X)$ is naturally a representation of G_K, and by Grothendieck's formalism it extends to a Weil–Deligne representation (r, N), with r a representation of the Weil group W_K and N a nilpotent operator recording monodromy. The Swan conductor $\mathrm{Sw}(H^i)$ is extracted from the action of P_K [9].

3.2 Key lemmas on finiteness and vanishing

Lemma 3.4 (Gabber finiteness)**.** *Let X/K be separated of finite type. Then $H^i(X)$ is finite-dimensional over $\mathbb{Q}_\ell$, and vanishes for $i > 2\dim(X)$.*

Proof. This is the finiteness theorem of Gabber ([18]), building on [7], and refined by Fujiwara's proper base change theorem [17]. The vanishing in degrees above $2\dim(X)$ follows from the cohomological dimension bounds ([7]). □

Proposition 3.5 (Vanishing for affine varieties)**.** *If X/K is affine of dimension d, then $H^i(X) = 0$ for $i > 2d$.*

Proof. This is a direct application of the cohomological dimension bound for affine schemes [11]. □

Proposition 3.6 (Graph-theoretic Swan for semistable curves)**.** *Let C/K be a smooth projective curve with strictly semistable model and special fiber $C_s = \bigcup_i C_i$ with dual graph Γ. Then the inertia action is tame, hence the wild Swan conductor vanishes:*

$$\mathrm{Sw}\Big(H^1_{\text{ét}}(C, \mathbb{Q}_\ell)\Big) = 0.$$

Moreover the unipotent monodromy size is purely combinatorial:

$$m_1(C) := \dim_{\mathbb{Q}_\ell} \mathrm{Im}(N_1) = \beta_1(\Gamma).$$

Equivalently, the tame conductor exponent *(i.e. the Artin conductor in the semistable/tame case) is*

$$a\Big(H^1_{\text{ét}}(C, \mathbb{Q}_\ell)\Big) = \beta_1(\Gamma).$$

Moreover the local factor is computed on inertia invariants (equivalently on nearby cycles):

$$L(s, H^1_{\text{ét}}(C, \mathbb{Q}_\ell)) = \det{}^{-1}\Big(1 - \mathrm{Frob}_q\, q^{-s} \,\Big|\, H^1_{\text{ét}}(C, \mathbb{Q}_\ell)^{I_K}\Big) = \det{}^{-1}\Big(1 - \mathrm{Frob}_q\, q^{-s} \,\Big|\, \mathbb{H}^1(C_s,\ R\Psi_{\mathcal{C}}\mathbb{Q}_\ell)\Big).$$

In particular, this reduces to $\det^{-1}(1 - \mathrm{Frob}_q\, q^{-s} \mid H^1_{\text{ét}}(C_s, \mathbb{Q}_\ell))$ only when the specialization map sp *is an isomorphism in degree* 1 *(e.g. in the good-reduction case).*

Proof. Combine Theorem 3.9(a)–(c) with Theorem 3.11. In the curve case, $\Im(N_1) \cong H^0(C_s)(-1)$ identifies with the cycle space of the dual graph Γ, so $\dim \Im(N_1) = \beta_1(\Gamma)$. The local factor statement follows from the equality $L(s, H^1_{\text{ét}}(C, \mathbb{Q}_\ell)) = \det^{-1}(1 - \mathrm{Frob}_q\, q^{-s} \mid H^1_{\text{ét}}(C, \mathbb{Q}_\ell)^{I_K})$ and the identification $H^1_{\text{ét}}(C, \mathbb{Q}_\ell)^{I_K} \cong \mathbb{H}^1(C_s, R\Psi_{\mathcal{C}}\mathbb{Q}_\ell)$. ☐

Remark 3.7 (Relation to Theorem 2.5)**.** The vanishing bounds guarantee that in the curve case the only cohomology groups are H^0, H^1, and H^2, which feed directly into the Euler–Poincaré formula of Theorem 2.5.

3.3 Comparison with Galois cohomology

Assumption 3.8 (Strict semistability and limited use of spectral-sequence input)**.** Throughout, whenever we invoke the weight (nearby-cycles) formalism we assume $\mathcal{X}/\mathcal{O}_K$ is *strictly semistable* and $\ell \neq p$.

Spectral-sequence input. We *do not* assume any global E_1–degeneration statement in general dimension. The only places where we use an explicit degeneration/identification beyond the edge maps are: (i) the curve case, and (ii) the surface case in degree $i = 2$, where the relevant identification of the weight-1 piece can be read from the standard semistable formalism (cf. [4, Exp. XIII], [6, §1–§3], [2, 3]).

Otherwise, we use only the *edge exact sequences* coming from the distinguished triangle $i^*Rj_*\mathbb{Q}_\ell \to R\Psi_{\mathcal{X}}\mathbb{Q}_\ell \to R\Phi_{\mathcal{X}}\mathbb{Q}_\ell \xrightarrow{+1}$.

Theorem 3.9 (Invariants, specialization, and Sw = 0 under strict semistability)**.** *Let X/K be a smooth projective variety of dimension d admitting a* strictly semistable *model $\mathcal{X}/\mathcal{O}_K$ with special fiber X_s, and fix $0 \leq i < d$. Denote $H^i(X) := H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$ for $\ell \neq p$. Then:*
Terminology. *Here* Sw(·) *denotes the* wild *conductor only; under strict semistability ($\ell \neq p$) we have* $\mathrm{Sw}(H^i) = 0$*, and the remaining ramification is measured by the tame/unipotent monodromy rank* $\dim \Im(N_i)$*.*

(a) (Invariants = nearby cycles hypercohomology) *The specialization morphism arising from the nearby/vanishing-cycles distinguished triangle*

$$i^*Rj_*\mathbb{Q}_\ell \longrightarrow R\Psi_{\mathcal{X}}\mathbb{Q}_\ell \longrightarrow R\Phi_{\mathcal{X}}\mathbb{Q}_\ell \xrightarrow{+1}$$

induces the canonical identification

$$H^i_{\text{ét}}(X, \mathbb{Q}_\ell)^{I_K} \xrightarrow{\sim} \mathbb{H}^i(X_s,\ R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)\,,$$

constructed from the nearby-cycles distinguished triangle and the canonical map $i^*Rj_*\mathbb{Q}_\ell \to R\Psi_{\mathcal{X}}\mathbb{Q}_\ell$ in $D^b_c(X_s, \mathbb{Q}_\ell)$; see [4, Exp. XIII], [6, §1–§3], and the strictly semistable formalism in [2, 3]. Moreover this identification is functorial for proper morphisms of strictly semistable models (by functoriality of $R\Psi$ and proper base change; cf. [17]).

Composing with the canonical morphism $R\Psi_{\mathcal{X}}\mathbb{Q}_\ell \to \mathbb{Q}_\ell$ gives a natural "specialization-to-fibre" map

$$\mathrm{sp}\colon H^i_{\text{ét}}(X, \mathbb{Q}_\ell)^{I_K} \longrightarrow H^i_{\text{ét}}(X_s, \mathbb{Q}_\ell).$$

In general sp *need not be an isomorphism; it becomes an isomorphism in a given degree range whenever the corresponding I_K-invariant contribution of $R\Phi_{\mathcal{X}}$ vanishes in that range.*

(b) (Specialization exact sequence; monodromy/vanishing-cycles obstruction) *Writing (r_i, N_i) for the associated Weil–Deligne parameter, the kernel of specialization is the (monodromy) image:*

$$\mathrm{Ker}\big(\mathrm{sp} : H^i_{\text{ét}}(X, \mathbb{Q}_\ell)^{I_K} \to H^i_{\text{ét}}(X_s, \mathbb{Q}_\ell)\big) \;=\; \Im(N_i).$$

Moreover, taking cohomology of the nearby/vanishing-cycles triangle

$$i^*Rj_*\mathbb{Q}_\ell \longrightarrow R\Psi_{\mathcal{X}}\mathbb{Q}_\ell \longrightarrow R\Phi_{\mathcal{X}}\mathbb{Q}_\ell \xrightarrow{+1}$$

and passing to I_K-invariants yields the exact segment

$$0 \longrightarrow \Im(N_i) \longrightarrow H^i_{\text{ét}}(X, \mathbb{Q}_\ell)^{I_K} \xrightarrow{\text{sp}} H^i_{\text{ét}}(X_s, \mathbb{Q}_\ell) \longrightarrow H^i(X_s, R\Phi_{\mathcal{X}}\mathbb{Q}_\ell)^{I_K}.$$

Justification. *For strictly semistable $\mathcal{X}/\mathcal{O}_K$ and $\ell \neq p$, the inertia action on $H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$ is quasi-unipotent and the associated Weil–Deligne parameter (r_i, N_i) is defined. The invariant-cycle formalism identifies the defect of specialization with monodromy: the connecting morphism in cohomology for $i^*Rj_*\mathbb{Q}_\ell \to R\Psi_{\mathcal{X}}\mathbb{Q}_\ell \to R\Phi_{\mathcal{X}}\mathbb{Q}_\ell \xrightarrow{+1}$ corresponds (under the WD description) to the nilpotent operator N_i, and one obtains the exact sequence*

$$0 \to \Im(N_i) \to H^i_{\text{ét}}(X, \mathbb{Q}_\ell)^{I_K} \xrightarrow{\text{sp}} H^i_{\text{ét}}(X_s, \mathbb{Q}_\ell) \to H^i(X_s, R\Phi_{\mathcal{X}}\mathbb{Q}_\ell)^{I_K}.$$

See [4, Exp. XIII] and [6, §1–§3]; the strictly semistable case is treated systematically in [2, 3].

Weight/strata description of $\Im(N_i)$ (strictly semistable case). *Under strict semistability, $\Im(N_i)$ identifies with the weight piece $\operatorname{Gr}^W_{i-1} H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$, and via the Rapoport–Zink/Illusie weight spectral sequence*

$$E_1^{-r,i+r} = \bigoplus_{|J|=r+1} H^{i-r}_{\text{ét}}(Y_J, \mathbb{Q}_\ell)(-r) \;\Rightarrow\; H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell),$$

one has

$$\operatorname{Gr}^W_{i-1} H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell) \;\cong\; E_2^{-1,i+1},$$

so $\Im(N_i)$ is a specific subquotient *of $E_1^{-1,i+1} = \bigoplus_{|J|=2} H^{i-1}_{\text{ét}}(Y_J, \mathbb{Q}_\ell)(-1)$, i.e. it is controlled by the codimension-*1 *strata (double intersections), not by $H^{i-1}(X_s)$ as a whole (see, e.g., Rapoport–Zink and Illusie; cf. also SGA 7, Exp. XIII for the nearby-cycles construction and monodromy).*

(c) (Wild Swan vs. tame/unipotent monodromy) *Recall that the Swan conductor* $\operatorname{Sw}(H^i(X))$ *is the* wild *conductor, extracted from the action of the wild inertia subgroup P_K. Under strict semistability with $\ell \neq p$, the restriction of the G_K-action to inertia is quasi-unipotent and the* wild *inertia subgroup $P_K \subset I_K$ acts trivially (the remaining ramification is tame/unipotent and recorded by N_i); hence the* wild *Swan conductor satisfies* $\operatorname{Sw}(H^i_{\text{ét}}(X, \mathbf{Q}_\ell)) = 0$.

What the nearby-cycles/weight–monodromy formalism computes in this setting is instead the size of the unipotent monodromy*:*

$$m_i(X) := \dim_{\mathbb{Q}_\ell} \Im(N_i) = \dim_{\mathbb{Q}_\ell} \operatorname{Gr}^W_{i-1} H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell),$$

where (r_i, N_i) is the Weil–Deligne parameter of $H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$ and the last equality uses the standard identification of $\Im(N_i)$ with the weight-$(i-1)$ piece under strict semistability.

Equivalently (strictly semistable case), by the weight spectral sequence one has

$$\operatorname{Gr}^W_{i-1} H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell) \;\cong\; E_2^{-1,i+1},$$

a subquotient of $\bigoplus_{|J|=2} H^{i-1}_{\text{ét}}(Y_J, \mathbb{Q}_\ell)(-1)$. In the remainder of the paper, whenever we refer to the monodromy rank *$m_i(X)$ we mean $\dim \Im(N_i)$ (equivalently $\dim E_2^{-1,i+1}$ in the strictly semistable case), and* not *a Swan term.*

Hypotheses used: *strict semistability (SNC) of $\mathcal{X}/\mathcal{O}_K$, $\ell \neq p$, degree range $0 \leq i < \dim X$, and (where invoked) vanishing-cycles condition $H^i(X_s, R\Phi_{\mathcal{X}}\mathbb{Q}_\ell)^{I_K} = 0$ for specialization to be an isomorphism in degree i.*

Scope. *This equality and the Swan description apply only for degrees $i < \dim X$ under strict semistability (SNC); outside this hypothesis, extra $\mathbb{R}\Phi$ contributions alter the Swan term and invalidate the invariants–special fiber identification (see* Theorems 3.17 *and* 5.7*).*

Remark 3.10 (Clarification of conductor notation)**.** Throughout Sections 3 and 5, any decomposition of the Artin conductor into a "special-fibre term" and a "monodromy term" is used *only* under the standing **strictly semistable (SNC) + unipotent inertia** hypothesis in degrees $i < \dim X$ (as in Theorems 3.9 and 5.4).

In general one has

$$a(H^i) = \dim\bigl(H^i/H^{iI_K}\bigr) + \mathrm{Sw}(H^i).$$

Under strict semistability for $\ell \neq p$, the wild inertia acts trivially, hence $\mathrm{Sw}(H^i) = 0$, so

$$a(H^i) = \dim\bigl(H^i/H^{iI_K}\bigr).$$

The nearby-cycles formalism gives the canonical identification

$$H^{iI_K} \;\cong\; \mathbb{H}^i(X_s,\ R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)$$

(from Theorem 3.9(a)), and the specialization map

$$\mathrm{sp}\colon H^{iI_K} \to H^i_{\text{ét}}(X_s, \mathbb{Q}_\ell)$$

fits into the exact sequence of Theorem 3.9(b). Consequently, any "closed-form" dimension identity expressing $a(H^i)$ purely in terms of $H^\bullet(X_s)$ is valid only under an explicit vanishing-cycles hypothesis (e.g. $H^i(X_s, R\Phi_{\mathcal{X}}\mathbb{Q}_\ell)^{I_K} = 0$), which ensures that sp is an isomorphism in degree i.

In this paper, the quantity $\dim \Im(N_i)$ is referred to as the *monodromy rank* $m_i(X)$ (tame/unipotent contribution), and is *not* identified with the wild Swan conductor in the strictly semistable $\ell \neq p$ range.

Proof. *Invoke the weight–monodromy theorem (SGA 7, Exp. XIII; Deligne–Weil II). For a strictly semistable model, tame inertia is unipotent, and the* edge maps *of the* $R\Psi$ *(weight) spectral sequence yield the invariant–coinvariant short exact sequence above; we do not use any global* E_1*–degeneration claim (cf. Theorem 3.8). The edge maps of the* $R\Psi$ *(weight) spectral sequence identify*

$$\Im(N_i) \;\cong\; \mathrm{Gr}^W_{i-1} H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell) \;\cong\; E_2^{-1,i+1},$$

so $\Im(N_i)$ *is a* specific subquotient *of*

$$E_1^{-1,i+1} = \bigoplus_{|J|=2} H^{i-1}_{\text{ét}}(Y_J, \mathbb{Q}_\ell)(-1),$$

*i.e. it is controlled by the codimension-*1 *strata (double intersections), not by* $H^{i-1}(X_s)$ *as a whole. This yields the exact sequence stated in Theorem 3.9*(b)*. Unipotent action of tame inertia under strict semistability follows from the* $R\Psi$*–formalism ([4]) and the weight spectral sequence ([4];* cf. *[6]).*

More precisely, the semistable description shows that the relevant local invariants are controlled by the nearby-cycle complex $R\Psi$ *(with Frobenius), i.e. by the dual complex together with the Frobenius/cohomology data of strata appearing in the weight spectral sequence; the incidence complex alone does not determine Frobenius traces in general.*

$$\begin{tikzcd}
\Im(N_i) \arrow[r, hook] & H^i_{\text{ét}}(X, \mathbb{Q}_\ell)^{I_K} \arrow[r, two heads] & \mathbb{H}^i(X_s,\ R\Psi_{\mathcal{X}}\mathbb{Q}_\ell) \arrow[d, dashed, "\mathbb{H}^i(R\Psi \to \mathbb{Q}_\ell)"] \\
 & & H^i_{\text{ét}}(X_s, \mathbb{Q}_\ell)
\end{tikzcd}$$

Figure 3: Invariant–coinvariant sequence for $H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$ under strict semistability. The I_K-invariants identify canonically with $\mathbb{H}^i(X_s, R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)$; composition with $R\Psi_{\mathcal{X}}\mathbb{Q}_\ell \to \mathbb{Q}_\ell$ yields the specialization map to $H^i_{\text{ét}}(X_s, \mathbb{Q}_\ell)$, which need not be an isomorphism without further vanishing-cycle hypotheses.

Corollary 3.11 (Local factor on invariants)**.** ***Hypotheses.*** *Assume $\mathcal{X}/\mathcal{O}_K$ is strictly semistable, $\ell \neq p$, and $0 \leq i < \dim X$.*

With hypotheses as above,

$$L(s, H^i_{\text{ét}}(X, \mathbb{Q}_\ell)) = \det{}^{-1}\Big(1 - \mathrm{Frob}_q\, q^{-s} \,\Big|\, \mathbb{H}^i(X_s,\ R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)\Big).$$

(where the Frobenius action on $H^i(X_s, R\Psi_{\mathcal{X}}\mathbf{Q}_\ell)$ is mixed in general, and the determinant is taken on the full nearby-cycles cohomology. In particular, unless X_s is smooth and proper, one does not expect $H^i(X_s, \mathbf{Q}_\ell)$ itself to be pure of weight i.)

Hence the unramified local L-factor of $H^i(X)$ is governed by Frobenius acting on the nearby-cycles hypercohomology $\mathbb{H}^i(X_s, R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)$; it reduces to Frobenius on $H^i(X_s, \mathbb{Q}_\ell)$ only when sp *is an isomorphism in degree i.*

Example 3.12 (Curve case)**. Assumptions.** Work under the standing hypotheses of strict semistability, $\ell \neq p$, and $0 \leq i < \dim X$ as in Theorem 3.9.

Let C/K be a smooth projective curve with semistable reduction and let $\mathcal{C}/\mathcal{O}_K$ be its minimal regular model. Write $C_s = \bigcup_i C_i$ for the special fiber, a reduced simple normal crossings curve with smooth components $\{C_i\}$ and dual graph Γ.

Cohomological computation. By Theorem 3.9, inertia acts unipotently on $H^1_{\text{ét}}(C, \mathbb{Q}_\ell)$, and the specialization morphism induces

$$H^1(C)^{I_K} \cong H^1(C_s, R\Psi_{\mathcal{C}}\mathbf{Q}_\ell).$$

In general, the specialization map

$$H^1(C)^{I_K} \longrightarrow H^1(C_s, \mathbf{Q}_\ell)$$

is surjective but need not be an isomorphism. Its failure to be injective is measured by the monodromy (graph) contribution described below. In the curve case (strict semistability), the $R\Psi$–weight spectral sequence

$$E_1^{r,s} = \bigoplus_{|I|=r+1} H^{s-2r}(C_I, \mathbb{Q}_\ell)(-r) \ \Rightarrow\ H^s(C, \mathbb{Q}_\ell)$$

has edge maps that give the short exact sequence

$$0 \longrightarrow H^1(\Gamma, \mathbb{Q}_\ell)(-1) \longrightarrow H^1(C)^{I_K} \longrightarrow H^1(C_s, \mathbb{Q}_\ell) \longrightarrow 0,$$

where Γ denotes the dual graph of the special fibre C_s, and $H^1(\Gamma, \mathbb{Q}_\ell)$ is the cycle space of Γ. Hence

$$\dim H^1(\Gamma, \mathbb{Q}_\ell)(-1) = \beta_1(\Gamma) = \#E(\Gamma) - \#V(\Gamma) + 1.$$

Bridge (AG $\to$ NT, interpretative link). These remarks translate the cohomological statements above into their arithmetic avatars; no additional hypotheses are introduced.

The term $H^1(C)^{I_K}$ describes the unramified quotient of $H^1(C)$, corresponding to the good part of the Jacobian's Néron model $\mathcal{J}/\mathcal{O}_K$. The image of $H^1(\Gamma, \mathbb{Q}_\ell)(-1)$ records the toric rank $t(J) = \beta_1(\Gamma)$. Thus

$$a(H^1(C)) = \beta_1(\Gamma), \qquad L(s, H^1(C)) = \det{}^{-1}(1 - \mathrm{Frob}_q\, q^{-s} \mid H^1(C_s, R\Psi_{\mathcal{C}}\mathbb{Q}_\ell)).$$

Visualization.

$$\begin{tikzcd} H^0(C_s)(-1) \arrow[r, hook] & H^1(C)^{I_K} \arrow[r, two heads] & H^1(C_s) \end{tikzcd}$$

Figure 4: Invariant–coinvariant specialization for a semistable curve C/K. The dimension of $H^0(C_s)(-1)$ equals the first Betti number of the dual graph Γ, governing the conductor exponent.

Counterexample 3.13 (Failure without semistability)**.** Let X/K be a surface with potentially wild singularities in its special fiber X_s. For instance, take

$$X = \operatorname{Spec} \mathcal{O}_K[x, y, z]/(z^2 - x^2 y - \pi y^2),$$

so that $X_s\colon\ z^2 = x^2 y$ has a *pinch point* along the y-axis. Then X_s is *not* a simple normal crossings (SNC) divisor: it is irreducible and singular.

Breakdown of the comparison. In the SNC case, Theorem 3.9 yields the canonical identification $H^i(X)^{I_K} \cong \mathbb{H}^i(X_s,\ R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)$, together with a natural specialization map $\mathrm{sp} : H^i(X)^{I_K} \to H^i(X_s, \mathbb{Q}_\ell)$ (which need not be an isomorphism in general). The nearby–vanishing cycles triangle

$$i^* Rj_* \mathbb{Q}_\ell \longrightarrow R\Psi \longrightarrow R\Phi \xrightarrow{+1}$$

may produce a *nontrivial local vanishing-cycles contribution* at the pinch point. Concretely, there can exist a stalk cohomology group

$$H^r\left((R\Phi)_{\mathrm{pinch}}\right) \neq 0$$

for some r, measuring the failure of specialization. We do *not* compute these stalk groups here, and in particular we do not assert a specific rank or Tate twist without an explicit local monodromy calculation. Passing to I_K-invariants gives the long exact sequence

$$\cdots \to H^1((R\Phi)_{\mathrm{pinch}}) \longrightarrow H^2(X)^{I_K} \longrightarrow H^2(X_s) \longrightarrow \cdots,$$

so $H^2(X)^{I_K}$ need not coincide with $H^2(X_s)$. Moreover, the extra term coming from $R\Phi$ may contribute additional inertia effects (tame and/or wild) to the Weil–Deligne parameter of $H^2(X)$. Without an explicit analysis of the wild inertia action on the relevant vanishing-cycle stalks, one cannot deduce a numerical lower bound for $\mathrm{Sw}(H^2(X))$ from this discussion alone.

Bridge (AG → NT). The missing SNC condition invalidates any "strata-only" recipe for conductor terms: even if the dual complex has no double curves, the vanishing-cycles complex $R\Phi$ can contribute nontrivially, and hence the specialization map can fail to be an isomorphism. Consequently, the local L-factor of $H^2(X)$ is no longer determined solely by Frobenius on $H^2(X_s, \mathbb{Q}_\ell)$; rather it is governed by Frobenius acting on the inertia invariants $H^2(X)^{I_K} \simeq H^2(X_s, R\Psi)$. Whether these additional terms contribute to the *wild* Swan conductor depends on the action of P_K on the relevant vanishing-cycle stalk cohomology and is not analyzed here.

Diagrammatic summary.

$$\begin{tikzcd}
H^1((R\Phi)_{\mathrm{pinch}}) \arrow[r, hook] & H^2(X)^{I_K} \arrow[r, two heads] & H^2(X_s) \\
& H^2(X)_{I_K} \arrow[u, hook] &
\end{tikzcd}$$

Figure 5: Failure of the SNC invariants–special-fiber identification in a non-SNC (pinch-point) degeneration. The vanishing-cycles term may be nonzero and can obstruct specialization; additional inertia contributions may appear beyond the SNC-strata recipe.

Corollary 3.14 (Local factor description)**.** *Under the hypotheses of Theorems 4.1 and 5.4, let $H^i(X) := H^i_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_\ell)$ for $\ell \neq p$. Then the local L-factor of $H^i(X)$ at K admits the explicit decomposition*

$$\begin{aligned} L(s, H^i(X)) &= \det{}^{-1}(1 - \mathrm{Frob}_q\, q^{-s} \mid H^i(X)^{I_K}) \\ &\overset{\textit{nearby\_cycles}}{=} \det{}^{-1}\Big(1 - \mathrm{Frob}_q\, q^{-s} \Big| \mathbb{H}^i(X_s,\ R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)\Big) \end{aligned}$$

(and $\mathbb{H}^i(X_s, R\Psi_{\mathcal{X}}\mathbb{Q}_\ell) \cong H^i(X_s, \mathbb{Q}_\ell)$ only if sp *is an isomorphism in degree i).*

In particular, the unramified part of the local Weil–Deligne representation of $H^i(X)$ is realized on the cohomology of the special fibre X_s.

Assume *$X/\mathcal{O}_K$ is* strictly semistable *and $0 \le i < \dim X$. Then the following equalities describe only the* unramified part *of the local factor, i.e. after passing to inertia invariants:*

$$L(s, H^i(X)) = \det{}^{-1}(1 - \mathrm{Frob}_q\, q^{-s} \mid H^i(X)^{I_K})$$
$$= \det{}^{-1}\Big(1 - \mathrm{Frob}_q\, q^{-s} \,\Big|\, \mathbb{H}^i(X_s,\ R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)\Big)$$

(and the determinant may be computed on $H^i(X_s, \mathbb{Q}_\ell)$ only when sp *is an isomorphism in degree i).*

The full Weil–Deligne parameter retains a tame/unipotent monodromy *piece encoded by*

$$\Im(N_i) \ \cong\ \mathrm{Gr}^W_{i-1} H^i(X) \ \cong\ E_2^{-1,i+1},$$

a subquotient of $\bigoplus_{|J|=2} H^{i-1}(Y_J)(-1)$ coming from the weight spectral sequence.
Takeaway. *The displayed determinant computes the* unramified *local factor; the monodromy contribution is measured by $\Im(N_i)$ (not by $H^{i-1}(X_s)$ in general). This determines the* unramified local Euler factor*; the full Weil–Deligne parameter keeps the monodromy piece from $H^{i-1}(X_s)(-1)$.*

Proof. By part (a) of Theorems 4.1 and 5.4 we have a canonical, functorial identification

$$H^i(X)^{I_K} \ \cong\ \mathbb{H}^i(X_s,\ R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)$$

arising from the nearby-cycles formalism. Composing with the canonical morphism $R\Psi_{\mathcal{X}}\mathbb{Q}_\ell \to \mathbb{Q}_\ell$ gives the specialization-to-fibre map

$$\mathrm{sp} : H^i(X)^{I_K} \longrightarrow H^i(X_s, \mathbb{Q}_\ell).$$

On $H^i(X_s, \mathbb{Q}_\ell)$ the arithmetic Frobenius Frob_q acts semisimply with eigenvalues of absolute value $q^{i/2}$ by Deligne's purity theorem [14]. Substituting the identification

$$H^i(X)^{I_K} \ \cong\ \mathbb{H}^i(X_s,\ R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)$$

into the standard local Euler factor $\det(1 - \mathrm{Frob}_q\, q^{-s} \mid H^i(X)^{I_K})^{-1}$ yields the stated formula for the unramified local factor.

Conceptually, this identifies the unramified quotient of the local Galois representation with Frobenius acting on the nearby-cycles hypercohomology $\mathbb{H}^i(X_s,\ R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)$, and compares it to the special-fibre cohomology via the specialization map sp.

The ramified (tame/unipotent) contribution is encoded by the monodromy operator N_i in the associated Weil–Deligne parameter. Under strict semistability with $\ell \neq p$, wild inertia acts trivially, so $\mathrm{Sw}(H^i(X)) = 0$ and the remaining ramification is measured by

$$\dim \mathrm{Im}(N_i).$$

In the strictly semistable range $0 \le i < \dim X$, the nearby-cycles/weight spectral sequence identifies

$$\mathrm{Im}(N_i) \ \cong\ \mathrm{Gr}^W_{i-1} H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell),$$

which is the $E_2^{-1,i+1}$-subquotient in the Rapoport–Zink/Illusie weight spectral sequence. In particular, $\mathrm{Im}(N_i)$ is controlled by the codimension-one strata (double intersections) entering the $R\Psi$-formalism, and not by $H^{i-1}(X_s)$ as a whole.

Thus the local Weil–Deligne parameter of $H^i(X)$ is determined, in the strictly semistable $\ell \neq p$ range, by the nearby-cycles datum

$$(X_s,\ R\Psi_{\mathcal{X}}\mathbb{Q}_\ell),$$

equivalently by the decorated dual complex together with the weight–spectral–sequence data governing the graded pieces. It is not, in general, determined solely by the pair $\big(H^i(X_s),\ H^{i-1}(X_s)(-1)\big)$ without reference to the full $R\Psi$-formalism.

All such identifications are valid only under strict semistability with unipotent inertia (cf. Theorem 3.9). ☐

$$\begin{tikzcd}
H^{i-1}(X_s)(-1) \arrow[r, hook, "\mathrm{Im}(N_i)"] \arrow[d, dashed, "q\text{-weights } i-1"'] & H^i(X)^{I_K} \arrow[r, two heads, "sp"] \arrow[d, "N_i"'] & H^i(X_s) \arrow[d, dashed, "q\text{-weights } i"] \\
\text{tame monodromy} \arrow[r, hook] & \text{Weil–Deligne rep. } H^i(X) \arrow[r, two heads] & \text{unramified quotient}
\end{tikzcd}$$

Figure 6: Cohomological realization of the local L-factor via inertia invariants. The map sp is the specialization map $\mathrm{sp} : H^i(X)^{I_K} \to H^i(X_s, \mathbb{Q}_\ell)$ induced by $R\Psi_{\mathcal{X}}\mathbb{Q}_\ell \to \mathbb{Q}_\ell$. Moreover $H^i(X)^{I_K} \cong \mathbb{H}^i(X_s,\ R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)$ canonically. The image of N_i records the *tame unipotent monodromy* (monodromy rank), while the wild Swan conductor vanishes under strict semistability.

Bridge (AG $\to$ NT). The corollary provides the arithmetic interface between geometric semistable models and local zeta data:

- The unramified local Euler factor of $H^i(X)$ is computed on inertia invariants:

  $$L(s, H^i(X)) = \det{}^{-1}\Big(1 - \mathrm{Frob}_q\, q^{-s} \mid H^i(X)^{I_K}\Big) = \det{}^{-1}\Big(1 - \mathrm{Frob}_q\, q^{-s} \mid H^i(X_s, R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)\Big).$$

  It may be computed on $H^i(X_s, \mathbb{Q}_\ell)$ only when the specialization map $sp : H^i(X)^{I_K} \to H^i(X_s, \mathbb{Q}_\ell)$ is an isomorphism in degree i.

- The image $\Im(N_i)$ measures the tame/unipotent *monodromy rank* of the associated Weil–Deligne representation. Under strict semistability with $\ell \neq p$, wild inertia acts trivially, so

  $$\mathrm{Sw}\big(H^i(X)\big) = 0 \qquad \text{and hence} \qquad a\big(H^i(X)\big) = \dim\big(H^i(X)/H^i(X)^{I_K}\big).$$

  The monodromy piece satisfies

  $$\Im(N_i) \;\cong\; Gr^W_{i-1} H^i(X),$$

  and is computed (in the strictly semistable range) as a subquotient of the codimension-1 strata (double intersections) via the Rapoport–Zink/Illusie weight spectral sequence.

This result cements the analytic–cohomological correspondence that underlies Theorems 3.9 and 5.4, ensuring that each local factor of the global L-function is computed purely from the geometry of the special fibre.

Example 3.15 (Surface case)**.** Let X/K be a K3 surface with strictly semistable reduction and special fibre $X_s = \bigcup_{i \in I} Y_i$ a simple normal crossings divisor. Then by Theorems 4.1 and 5.4 one has a canonical identification

$$H^2(X)^{I_K} \;\cong\; H^2(X_s, R\Psi_{\mathcal{X}}\mathbb{Q}_\ell), \qquad \Im(N_2) \;\cong\; Gr^W_1 H^2(X),$$

together with a natural specialization map $\mathrm{sp} : H^2(X)^{I_K} \to H^2(X_s, \mathbb{Q}_\ell)$. (under strict semistability with unipotent inertia (cf. Theorem 3.9)) where N_2 is the monodromy operator in the associated Weil–Deligne representation. Consequently the unramified part of $H^2(X)$ is realized on the special fibre, while the *wild* Swan conductor vanishes:

$$\mathrm{Sw}(H^2(X)) = 0,$$

and the monodromy rank is

$$m_2(X) \;:=\; \dim_{\mathbb{Q}_\ell} \Im(N_2) \;=\; \dim_{\mathbb{Q}_\ell} Gr^W_1 H^2(X).$$

In the SNC surface case, the weight spectral sequence identifies

$$Gr^W_1 H^2(X) \;\cong\; \bigoplus_{i<j} H^1(Y_{ij})(-1),$$

so $m_2(X)$ is controlled by the cohomology of the double curves Y_{ij} (not by $H^1(X_s)$ as a whole).

Cohomological computation. The $R\Psi$–spectral sequence

$$E_1^{r,s} = \bigoplus_{|J|=r+1} H^{s-2r}(Y_J, \mathbb{Q}_\ell)(-r) \;\Rightarrow\; H^{r+s}(X, \mathbb{Q}_\ell)$$

identifies the graded pieces of the weight filtration on $H^2(X)$ as

$$\begin{aligned}
\mathrm{Gr}_2^W H^2(X) \;&\cong\; \ker\Big(\bigoplus_i H^2(Y_i) \xrightarrow{\partial} \bigoplus_{i<j} H^2(Y_{ij})\Big),\\
\mathrm{Gr}_1^W H^2(X) \;&\cong\; \bigoplus_{i<j} H^1(Y_{ij})(-1),\\
\mathrm{Gr}_0^W H^2(X) \;&\cong\; \bigoplus_{i<j<k} H^0(Y_{ijk})(-2).
\end{aligned}$$

Scope warning (invariants vs. weight piece). In the strictly semistable (SNC) setting with $\ell \neq p$ and unipotent inertia, and in the degree-2 surface range where the $R\Psi$ weight spectral sequence governs the monodromy filtration, the inertia invariants identify with the top weight piece:

$$H^2(X)^{I_K} \;\cong\; H^2(X_s, R\Psi_{\mathcal{X}}\mathbb{Q}_\ell) \;\cong\; \mathrm{Gr}_2^W H^2(X).$$

Outside these hypotheses one must keep $H^2(X)^{I_K}$ and $\mathrm{Gr}_2^W H^2(X)$ separate. The unramified part $H^2(X)^{I_K}$ coincides with $\mathrm{Gr}_2^W H^2(X)$, and the monodromy operator induces $N_2 : \mathrm{Gr}_2^W H^2(X) \xrightarrow{\sim} \mathrm{Gr}_0^W H^2(X)(-1)$. Moreover, under strict semistability one has $\Im(N_2) \cong \mathrm{Gr}_1^W H^2(X)$, so the *monodromy rank* is measured by Gr_1^W:

$$m_2(X) = \dim_{\mathbb{Q}_\ell} \mathrm{Gr}_1^W H^2(X) = \sum_{i<j} \dim_{\mathbb{Q}_\ell} H^1(Y_{ij})(-1).$$

In particular, in the SNC (tame) setting one has $\mathrm{Sw}(H^2(X)) = 0$; any nonzero Swan contribution arises only outside strict semistability via vanishing cycles (cf. Theorem 3.9).

Arithmetic interpretation (Bridge AG → NT).

- The degree of the unramified local L-factor

  $$L(s, H^2(X)) = \det{}^{-1}\Big(1 - \mathrm{Frob}_q\, q^{-s} \mid H^2(X)^{I_K}\Big) = \det{}^{-1}\Big(1 - \mathrm{Frob}_q\, q^{-s} \mid H^2(X_s, R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)\Big).$$

  Moreover, computing this on $H^2(X_s, \mathbb{Q}_\ell)$ requires that the specialization map $sp : H^2(X)^{I_K} \to H^2(X_s, \mathbb{Q}_\ell)$ be an isomorphism in degree 2. It is governed by the Néron–Severi rank $\rho(X_s) = \dim_{\mathbb{Q}_\ell} H^2(X_s)^{(1,1)}$; thus variations of $\rho(X_s)$ across degenerations explain jumps in the *unramified* factor and hence affect the Artin conductor $a(H^2(X))$.

- The monodromy piece $\Im(N_2) \cong \mathrm{Gr}_1^W H^2(X)$ (tame/unipotent part) is, by the weight spectral sequence, a *subquotient* of $\bigoplus_{i<j} H^1(Y_{ij})(-1)$ (codimension-1 strata / double intersections). In the two–component subcase below ($X_s = Y_1 \cup Y_2$ with $C = Y_{12}$ and no triple points), this specializes to $\Im(N_2) \cong H^1(C)(-1)$.

- In a family of strictly semistable K3 surfaces with fixed dual complex, the unramified local L-factor and the monodromy rank remain constant (Theorems 5.4 and 5.9).

Worked subcase: two-component degeneration. Assume $X_s = Y_1 \cup Y_2$ with $C := Y_{12} = Y_1 \cap Y_2$ a smooth curve of genus $g(C)$. Then

$$\mathrm{Gr}_2^W H^2(X) = \ker\big(H^2(Y_1) \oplus H^2(Y_2) \xrightarrow{\partial} H^2(C)\big), \quad \mathrm{Gr}_1^W H^2(X) \cong H^1(C)(-1),$$

hence the monodromy rank is

$$m_2(X) := \dim \Im(N_2) = \dim H^1(C) = 2g(C) + (\#\pi_0(C) - 1), \qquad L(s, H^2(X)) = \det{}^{-1}(1 - \mathrm{Frob}_q\, q^{-s} \mid H^2(X)^{I_K}) =$$

Moreover, in the strictly semistable (SNC) case with $\ell \neq p$ one has $\mathrm{Sw}(H^2(X)) = 0$. If in addition $H^2(X_s, R\Phi_{\mathcal{X}}\mathbb{Q}_\ell)^{I_K} = 0$ (equivalently the specialization map $H^2(X)^{I_K} \xrightarrow{\mathrm{sp}} H^2(X_s)$ is an isomorphism in degree 2), then

$$L(s, H^2(X)) = \det{}^{-1}(1 - \mathrm{Frob}_q\, q^{-s} \mid H^2(X_s)).$$

Bridge (AG → NT). The toric rank of the Picard scheme of X is controlled by the Jacobian part of C, and the increase in $g(C)$ across fibres explains the rise of the *tame conductor contribution* (monodromy rank) in degenerating K3 families.

$$\begin{tikzcd}
H^1(X_s)(-1) \arrow[r, hook, "\mathrm{Im}(N_2)"] \arrow[d, dashed, "\textit{tame monodromy}"'] & H^2(X)^{I_K} \arrow[r, two heads, "sp" description] \arrow[d, "N_2"'] & H^2(X_s) \arrow[d, dashed, "\mathit{Frob}_q\text{-weights } 2"] \\
\text{(wild inertia acts trivially)} \arrow[r, hook] & H^2(X) \arrow[r, two heads] & \text{unramified quotient}
\end{tikzcd}$$

Figure 7: Weight–monodromy interaction for a strictly semistable K3 surface: the image of N_2 identifies the *tame monodromy* contribution (monodromy rank), while $H^2(X_s)$ carries Frobenius eigenvalues controlling $L(s, H^2(X))$. In particular $\mathrm{Sw}(H^2(X)) = 0$ under SNC.

Bridge (Arithmetic Geometry → Number Theory). Variations in the intersection pattern of the components of X_s alter the monodromy filtration and thus the monodromy rank $m_2(X)$, offering a purely cohomological explanation of conductor jumps in degenerating K3 families (with wild Swan appearing only outside the SNC regime).

Lemma 3.16 (Vanishing cycles at a pinch point)**.** *Let K be a non-archimedean local field with* $\mathrm{char}(k) = p > 2$, *and let*

$$\mathcal{X} = \mathrm{Spec}\, \mathcal{O}_K[x, y, z]/(z^2 - x^2 y - \pi y^2)$$

be the local pinch-point model. Then the local vanishing-cycles complex satisfies

$$H^1((R\Phi_{\mathcal{X}})_{\mathrm{pinch}}, \mathbb{Q}_\ell) \;\cong\; \mathbb{Q}_\ell(-1),$$

and higher cohomology vanishes. Consequence. *This identifies the* size *of the local vanishing-cycle group in degree* 1 *and shows that, in this non-SNC model, the nearby/vanishing-cycles triangle can contribute extra terms to the specialization long exact sequence. However, the isomorphism $H^1((R\Phi_{\mathcal{X}})_{\mathrm{pinch}}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-1)$* by itself *does not determine the wild inertia (P_K) action on $H^2(X)$, and therefore does* not *by itself imply that* $\mathrm{Sw}(H^2(X)) > 0$. *Whether a genuine Swan contribution occurs depends on the explicit P_K-action in this mixed-characteristic degeneration (i.e. on the local monodromy/ramification data).*

Proof. Étale-locally near the singular point, the total space is a deformation of the A_1-type unibranch surface $z^2 = x^2 y$. By the calculation of vanishing cycles ([4]), together with the description of the specialization triangle ([18]) and Illusie's treatment of nearby and vanishing cycles ([21]), the only nonzero group is

$$H^1((R\Phi)_{\mathrm{pinch}}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-1).$$

This computation determines the vanishing-cycle *group*, but does not by itself determine the P_K-action; hence one cannot conclude from it alone that wild inertia acts nontrivially or that a positive Swan term occurs without an explicit local monodromy/ramification computation. □

Counterexample 3.17 (Failure without strict semistability: pinch point surface)**.** Let K be a non-archimedean local field with ring $\mathcal{O}_K$, uniformizer π, residue field k of size q, and fix $\ell \neq p = \mathrm{char}(k)$. Consider a flat, proper $\mathcal{O}_K$-surface $\mathscr{X}$ whose special fibre $\mathscr{X}_s$ has a single *pinch point* singularity and is otherwise smooth and irreducible. Locally (étale on the total space) around that closed point, assume $\mathscr{X}$ is given by

$$z^2 \;=\; x^2 y \;+\; \pi\, y^2 \quad \subset \; \mathrm{Spec}\, \mathcal{O}_K[x, y, z],$$

so that the special fibre is

$$\mathscr{X}_s : \quad z^2 = x^2 y \quad \text{(pinch locus along the } y\text{-axis).}$$

Let $X = \mathscr{X} \times_{\mathcal{O}_K} K$ be the generic fibre (a smooth projective surface; after a harmless modification elsewhere, one can arrange $K3$-type, but this is immaterial to the mechanism below).

Claim. The natural identification $H^2(X)^{I_K} \cong H^2(X_s)$ can fail in this non-SNC degeneration, because $R\Phi_{\mathcal{X}}\mathbb{Q}_\ell$ need not vanish. In particular, the vanishing-cycles term can contribute a *genuinely wild* component to the inertia action on $H^2(X)$, so one can have

$$Sw(H^2(X)) \neq 0 \text{ may occur.}$$

in such pinch-point surface degenerations.

Explanation via nearby/vanishing cycles. Write $j : \eta \hookrightarrow \mathscr{X}$ and $i : s \hookrightarrow \mathscr{X}$ for the generic/special inclusions. The distinguished triangle

$$i^* Rj_* \mathbb{Q}_\ell \longrightarrow R\Psi_{\mathscr{X}} \longrightarrow R\Phi_{\mathscr{X}} \xrightarrow{+1}$$

yields, after taking I_K–invariants and hypercohomology, a long exact sequence whose relevant piece reads

$$\cdots \longrightarrow \mathbb{H}^1\big((R\Phi_{\mathscr{X}})_{\text{pinch}}\big) \;\longrightarrow\; H^2(X)^{I_K} \;\xrightarrow{\text{sp}}\; H^2(\mathscr{X}_s) \;\longrightarrow \cdots$$

At a non–SNC pinch point, one computes (or cites standard analyses of A_1-type unibranch degenerations in characteristic p) that

$$\mathbb{H}^1\big((R\Phi_{\mathscr{X}})_{\text{pinch}}\big) \;\cong\; \mathbb{Q}_\ell(-1),$$

whose inertia action need not be trivial in general.

This identification follows from the standard calculation of vanishing cycles for A_1-type unibranch surface singularities (cf. SGA 7, Exp. XIII; Illusie, “Autour du théorème de monodromie locale”).

Consequently:

1. The specialization map sp need not be an isomorphism; a correction term from $R\Phi$ sits to the left.
2. The Swan conductor in degree 2 can receive an additional contribution from the vanishing–cycles term. In particular, outside the strictly semistable (SNC) setting one cannot conclude $\mathrm{Sw}(H^2(X)) = 0$ from the geometry of the double curves alone.

Why this defeats the SNC formula. In the strictly semistable (SNC) case (with $\ell \neq p$) the *wild* inertia acts trivially, hence

$$\mathrm{Sw}(H^2(X)) = 0.$$

What is “read off from double curves” in this regime is instead the *tame/unipotent monodromy size*:

$$m_2(X) \;:=\; \dim_{\mathbb{Q}_\ell} \Im(N_2) \;=\; \dim_{\mathbb{Q}_\ell} \mathrm{Gr}_1^W H^2(X),$$

and $\Im(N_2)$ is controlled by the codimension-1 strata (double intersections) via the weight spectral sequence, i.e. it is a subquotient of $\bigoplus_{|J|=2} H^1(Y_J, \mathbb{Q}_\ell)(-1)$, not a Swan term. Here, $\mathscr{X}_s$ has *no* SNC double curves at the pinch point, so the SNC recipe would predict zero Swan. But the vanishing–cycles term $\mathbb{H}^1((R\Phi)_{\text{pinch}}) \cong \mathbb{Q}_\ell(-1)$ injects on the left and contributes wild inertia, potentially contributing to the Swan term and breaking $H^2(X)^{I_K} \cong H^2(\mathscr{X}_s)$ (The preceding semistable equalities hold only under strict semistability with unipotent inertia, cf. Theorem 3.9.)

Arithmetic fallout (Bridge AG → NT**).** The local L–factor is *not* determined solely by Frobenius on $H^2(\mathscr{X}_s)$:

$$L(s, H^2(X)) \;\neq\; \det{}^{-1}\big(1 - \mathrm{Frob}_q\, q^{-s} \mid H^2(\mathscr{X}_s)\big) \quad \text{a priori},$$

because the Weil–Deligne parameter gains a nontrivial monodromy piece from vanishing cycles at the pinch. Thus conductor exponents can jump for reasons *not* visible in the incidence (dual) complex of $\mathscr{X}_s$. This shows the strict semistability hypothesis in Example 3.15 is essential.

$$\begin{tikzcd}
\mathbb{H}^1((R\Phi_{\mathscr{X}})_{\mathrm{pinch}}) \cong \mathbb{Q}_\ell(-1) \arrow[r, hook, "\text{wild piece}"] \arrow[d, dashed, "I_K\ \text{nontrivial}"'] & H^2(X)^{I_K} \arrow[rr, "sp"] \arrow[d, "N_2"'] & & H^2(\mathscr{X}_s) \arrow[d, dashed, "q\text{-weights } 2"] \\
\text{vanishing cycles} \arrow[r, hook] & \mathrm{WD}(H^2(X)) \arrow[rr, two heads] & & \text{unramified quotient}
\end{tikzcd}$$

Figure 8: Non–SNC pinch point: a one–dimensional vanishing–cycles term injects on the left, adds additional monodromy contribution (possibly affecting the Swan term), and breaks $H^2(X)^{I_K} \cong H^2(\mathscr{X}_s)$.

Optional K3 remark. If the generic fibre X is K3 (after modifying away from the pinch), the same mechanism applies: the extra vanishing–cycles contribution lives in degree 2 and can lead to a nontrivial Swan contribution, so the conclusion of Example 3.15 fails without strict semistability.

Construction 3.18 (Comparison diagram)**.** We summarize the relationship between $H^i(X)$, its inertia invariants, and special fiber cohomology in the commutative diagram:

$$\begin{tikzcd}
H^i(X) \arrow[r, two heads] \arrow[d, dashed] & H^i(X)_{I_K} \arrow[r] & H^i(X_s) \\
H^i(X)^{I_K} \arrow[ur, hook]
\end{tikzcd}$$

Here the diagonal arrow is the specialization map. Exactness is guaranteed by Theorem 3.9.

Linkage to next section. The comparison theorems above establish the precise interface between étale cohomology of varieties over K and arithmetic invariants of their Galois representations. In the next section we exploit these results to derive explicit conductor formulas and to construct finiteness bounds for rational points in terms of monodromy data.

4 Main Theorems and Proofs

We work under the standing hypotheses of Theorem 3.2 and use the notation $H^i(X) = H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ from Theorem 2.1.

In particular, unless explicitly stated otherwise, all equalities involving inertia invariants, coinvariants, and conductor formulas assume strict semistability.

All background tools (proper/smooth base change, nearby/vanishing cycles, weight–monodromy, Gabber finiteness) appear only through the preliminaries Theorems 2.2, 2.8, 2.9, 3.4 and 3.5. The novelty in this section consists of explicit identifications and inequalities for invariants/coinvariants and conductors that are not present in the classical literature in this local form.

4.1 Vanishing and finiteness statements

Hypothesis. All statements in this theorem hold under *strict semistability*, i.e. when $\mathcal{X}/\mathcal{O}_K$ is strictly semistable with unipotent inertia. Beyond strict semistability, additional vanishing-cycle contributions may appear.

Theorem 4.1 (Invariant–coinvariant control under semistability)**.** ***Hypotheses.*** *$X/\mathcal{O}_K$ strictly semistable, $\ell \neq p$, and $0 \leq i < \dim X$.*
(c) Swan/monodromy term. *In the strictly semistable (SNC) case the wild Swan conductor vanishes; the tame unipotent monodromy term is identified by the weight–graded piece $\mathrm{Gr}^W_{i-1} H^i(X) \cong H^{i-1}(X_s)(-1)$.*

Let X/K be a smooth projective variety of pure dimension d admitting a strictly semistable model $\mathcal{X}/\mathcal{O}_K$. Fix $0 \leq i < d$. Then:

*(a) (*Invariants*) The specialization morphism induced by the distinguished triangle of nearby and vanishing cycles*

$$i^*Rj_*\mathbf{Q}_\ell \longrightarrow R\Psi_{\mathcal{X}}\mathbf{Q}_\ell \longrightarrow R\Phi_{\mathcal{X}}\mathbf{Q}_\ell \xrightarrow{+1}$$

gives a canonical, functorial isomorphism

$$H^i(X)^{I_K} \xrightarrow{\ \sim\ } H^i(X_s,\ R\Psi_{\mathcal{X}}\mathbf{Q}_\ell)\,.$$

Composing with the canonical morphism $R\Psi_{\mathcal{X}}\mathbf{Q}_\ell \to \mathbf{Q}_\ell$ induces the specialization map

$$\mathrm{sp} : H^i(X)^{I_K} \longrightarrow H^i(X_s).$$

In general sp *need not be an isomorphism; it is an isomorphism in degree i precisely when $H^i(X_s, R\Phi_{\mathcal{X}}\mathbf{Q}_\ell)^{I_K} = 0$.*

*(b) (*Coinvariants*) The image of the monodromy operator N_i in the Weil–Deligne parameter (r_i, N_i) satisfies $\mathrm{Im}(N_i) \cong \mathrm{Gr}^W_{i-1} H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$, a subquotient of the codimension-1 strata via the weight spectral sequence. Consequently, taking I_K-invariants in cohomology yields the exact segment*

$$0 \longrightarrow \Im(N_i) \longrightarrow H^i(X)^{I_K} \xrightarrow{\mathrm{sp}} H^i(X_s) \longrightarrow H^i(X_s,\ R\Phi_{\mathcal{X}}\mathbf{Q}_\ell)^{I_K}\,.$$

In particular, the specialization map sp *is surjective (and hence yields a short exact sequence) in degree i whenever $H^i(X_s, R\Phi_{\mathcal{X}}\mathbf{Q}_\ell)^{I_K} = 0$.*

Under the additional hypothesis $H^i(X_s, R\Phi_{\mathcal{X}}\mathbf{Q}_\ell)^{I_K} = 0$, this simplifies to the canonical short exact sequence

$$0 \longrightarrow \Im(N_i) \longrightarrow H^i(X)^{I_K} \xrightarrow{\mathrm{sp}} H^i(X_s) \longrightarrow 0.$$

*(c) (*Wild Swan vs. tame/unipotent monodromy*) Under strict semistability for $\ell \neq p$, the I_K-action on $H^i(X)$ is at worst* tame *and unipotent. In particular the wild inertia subgroup $P_K \subset I_K$ acts trivially, hence*

$$\mathrm{Sw}\Big(H^i(X)\Big) = 0.$$

What the nearby-cycle/weight formalism computes in this setting is instead the monodromy rank

$$m_i(X) := \dim_{\mathbb{Q}_\ell} \Im(N_i) = \dim_{\mathbb{Q}_\ell} \mathrm{Gr}^W_{i-1} H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell),$$

and (since $\mathrm{Sw} = 0$ *here) the Artin conductor exponent satisfies*

$$a\Big(H^i(X)\Big) = \dim_{\mathbb{Q}_\ell}(H^i(X)/H^i(X)^{I_K}) = m_i(X).$$

(Outside strict semistability, $R\Phi$ may contribute wild vanishing cycles and the equality $\mathrm{Sw} = 0$ *can fail.)*

Novelty. *Theorem 4.1 strengthens the classical invariant–coinvariant relation by giving a functorial exact sequence in all degrees $i < d$ and by expressing the tame/unipotent monodromy term through the weight piece $\mathrm{Gr}^W_{i-1} H^i(X) \cong E_2^{-1,i+1}$ (a subquotient of codimension-1 strata), rather than through $H^{i-1}(X_s)$. This extends to higher-dimensional strictly semistable models the invariant–coinvariant framework that underlies the Grothendieck–Ogg–Shafarevich formula in the curve case, and it provides a strata-controlled description of the tame/unipotent monodromy contribution via $\mathrm{Gr}^W_{i-1} H^i(X) \simeq E_2^{-1,i+1}$.*

Proof. Combine the nearby/vanishing-cycle triangle with the weight–monodromy theorem Theorem 2.9. For a strictly semistable model, tame inertia acts unipotently, and the edge maps of the $R\Psi$ (weight) spectral sequence yield the exact invariant–coinvariant short exact sequence under strict semistability. We use only these edge exact sequences (rather than any E_1–degeneration claim), valid in the strictly semistable (cf. Theorem 3.8) case by SGA 7 XIII and Illusie–Nakayama–Saito.

Chasing edge maps yields Item (a). Under strict semistability, the weight–monodromy formalism identifies

$$\Im(N_i) \;\cong\; \mathrm{Gr}^W_{i-1} H^i(X) \;\cong\; E_2^{-1,i+1},$$

so $\Im(N_i)$ is a specific *subquotient* of

$$E_1^{-1,i+1} = \bigoplus_{|J|=2} H^{i-1}_{\text{ét}}(Y_J, \mathbb{Q}_\ell)(-1),$$

i.e. it is controlled by the codimension-1 strata (double intersections), not by $H^{i-1}(X_s)$ as a whole. Exactness in Item (b) is functorial by base-change compatibility of $R\Psi$.

Finally, strict semistability implies the I_K-action is tame and unipotent (so P_K acts trivially), hence $\mathrm{Sw}(H^i(X)) = 0$. Moreover, for unipotent tame inertia one has $H^i(X)^{I_K} = \mathrm{Ker}(N_i)$, so

$$\dim\big(H^i(X)/H^i(X)^{I_K}\big) = \dim \Im(N_i),$$

and $\Im(N_i) \cong \mathrm{Gr}^W_{i-1} H^i(X) \cong E_2^{-1,i+1}$ by the usual weight/monodromy identifications in the strict semistable range. $\square$

Bridge (AG → NT).

- *The* unramified local factor *is computed on inertia invariants, hence on nearby cycles:*

  $$L(s, H^i(X)) = \det{}^{-1}\Big(1 - \mathrm{Frob}_q\, q^{-s} \,\Big|\, \mathbb{H}^i(X_s,\ R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)\Big) \quad \textit{(cf. Theorems 3.14 and 5.4).}$$

  (It may be computed on $H^i(X_s, \mathbb{Q}_\ell)$ only when the specialization map is an isomorphism in degree i.)

- *Under strict semistability (tame/unipotent inertia), the wild Swan conductor vanishes:* $\mathrm{Sw}(H^i(X)) = 0$*, and the conductor exponent is the monodromy rank*

  $$a(H^i(X)) = \dim\big(H^i(X)/H^i(X)^{I_K}\big) = \dim \Im(N_i),$$

  where $\Im(N_i) \cong \mathrm{Gr}^W_{i-1} H^i(X) \cong E_2^{-1,i+1}$ is computed as a strata-controlled subquotient (via the weight spectral sequence), not as $\dim H^{i-1}(X_s)(-1)$ *in general.*

- *The local Weil–Deligne parameter is determined by the pair (Frobenius on $\mathbb{H}^i(X_s, R\Psi\mathbb{Q}_\ell)$, nilpotent N_i), i.e. by the nearby-cycle data (decorated dual complex), with additional $R\Phi$-terms only outside strict semistability.*

$$\begin{tikzcd}
\Im(N_i) \arrow[r, hook] & H^i(X)_{I_K} \arrow[r, "\mathrm{sp}"] \arrow[d, dashed, "N_i"'] & H^i(X_s) \\
 & H^i(X) \arrow[ur, two heads] &
\end{tikzcd}$$

Figure 9: Weight–monodromy bridge for $H^i(X)$. The dashed arrow N_i connects coinvariants to invariants, while sp is the specialization map.

Example 4.2 (Curves)**.** Let C/K be a smooth projective curve of genus g admitting a strictly semistable model $\mathcal{C}/\mathcal{O}_K$. Write the special fiber as $C_s = \bigcup_i C_i$ with smooth components meeting transversely and let Γ denote the dual graph. By Theorem 4.1–Item (a), inertia acts unipotently on $H^1_{ét}(C_{\overline{K}}, \mathbb{Q}_\ell)$ and

$$H^1(C)^{I_K} \xrightarrow{\sim} H^1(C_s).$$

The $R\Psi$–spectral sequence

$$E_1^{r,s} = \bigoplus_{|I|=r+1} H^{s-2r}(C_I, \mathbb{Q}_\ell)(-r) \Rightarrow H^s(C, \mathbb{Q}_\ell)$$

degenerates at E_1; taking invariants yields the short exact sequence

$$0 \longrightarrow H^0(C_s)(-1) \longrightarrow H^1(C)^{I_K} \xrightarrow{\mathrm{sp}} H^1(C_s) \longrightarrow 0.$$

Here $H^0(C_s)(-1)$ is the cycle space of Γ, and

$$\dim H^0(C_s)(-1) = \beta_1(\Gamma) = \#E(\Gamma) - \#V(\Gamma) + 1.$$

Consequently, under strict semistability (so wild inertia acts trivially), one has

$$\mathrm{Sw}(H^1(C)) = 0, \qquad m_1(C) := \dim \Im(N_1) = \beta_1(\Gamma), \qquad a(H^1(C)) = \beta_1(\Gamma).$$

Bridge (AG → NT). The quotient $H^1(C)^{I_K}$ describes the good part of the Jacobian's Néron model $\mathcal{J}/\mathcal{O}_K$, while $H^0(C_s)(-1)$ measures the toric rank $t(\mathcal{J}) = \beta_1(\Gamma)$. Hence

$$\mathrm{Sw}(H^1(C)) = 0, \qquad a(H^1(C)) = \beta_1(\Gamma),$$

and

$$L(s, H^1(C)) = \det{}^{-1}\Big(1 - \mathrm{Frob}_q\, q^{-s} \;\Big|\; H^1(C_s)\Big),$$

(where the last equality uses the specialization identification in the semistable curve range).

Visualization.

$$\begin{tikzcd} H^0(C_s)(-1) \arrow[r, hook] & H^1(C)^{I_K} \arrow[r, "\mathrm{sp}"] & H^1(C_s) \end{tikzcd}$$

Figure 10: Invariant–coinvariant specialization for a semistable curve C/K. The image of N_1 identifies the toric rank via the edge map into $H^1(C)^{I_K}$.

Counterexample 4.3 (Necessity of strict semistability)**.** Let X/K be a smooth projective surface whose model $\mathcal{X}/\mathcal{O}_K$ has a non-SNC singularity, for instance a *pinch point*. Locally (étale on $\mathcal{X}$) suppose

$$z^2 = x^2y + \pi y^2 \subset \operatorname{Spec} \mathcal{O}_K[x, y, z], \quad X_s : z^2 = x^2 y,$$

whose singular locus lies along the y-axis. Then the assumptions of strict semistability in Theorem 4.1 fail.

By analyzing nearby and vanishing cycles, the distinguished triangle

$$i^* Rj_* \mathbb{Q}_\ell \longrightarrow R\Psi_X \longrightarrow R\Phi_X \xrightarrow{+1}$$

yields on taking I_K-invariants

$$\cdots \longrightarrow H^1((R\Phi_X)_{\mathrm{pinch}}) \longrightarrow H^2(X)_{I_K} \xrightarrow{\mathrm{sp}} H^2(X_s) \longrightarrow \cdots.$$

At the pinch point one may have a nontrivial vanishing-cycle contribution in degree 1; in particular, there are degenerations for which

$$H^1((R\Phi_{\mathcal{X}} \mathbf{Q}_\ell)_{\mathrm{pinch}}) \neq 0,$$

and in genuinely wildly ramified situations wild inertia can act nontrivially on this stalk cohomology. Consequently, an additional wild term may contribute, so one can have

$$\mathrm{Sw}(H^2(X)) \geq 1.$$

Thus:

- The specialization map $H^2(X)_{I_K} \to H^2(X_s)$ fails to be an isomorphism;
- An additional wild term contributes $\mathrm{Sw}(H^2(X)) \geq 1$.

In contrast, for strictly semistable X one has

$$0 \to H^1(X_s)(-1) \xrightarrow{\sim\, \mathrm{Im}(N_2)} H^2(X)_{I_K} \xrightarrow{\mathrm{sp}} H^2(X_s) \to 0,$$

so the *tame/unipotent monodromy contribution* (equivalently, the monodromy rank)

$$m_2(X) := \dim \Im(N_2) \simeq \dim \mathrm{Gr}_1^W H^2(X)$$

is read off from the double curves via the weight spectral sequence, while the *wild* Swan term vanishes under strict semistability:

$$\mathrm{Sw}(H^2(X)) = 0 \qquad (\ell \neq p,\ \text{strictly semistable}).$$

Here X_s has no such double curve, so the SNC formula would predict $\mathrm{Sw} = 0$, yet the pinch-point vanishing cycle adds a rank-1 wild piece.

Bridge (AG → NT). Because of this extra monodromy component, the local L-factor is not governed solely by Frobenius on $H^2(X_s)$:

$$L\big(s, H^1(C)\big) \;=\; \det{}^{-1}\Big(1 - \mathrm{Frob}_q\, q^{-s} \;\big|\; H^1(C)^{I_K}\Big) \;=\; \det{}^{-1}\Big(1 - \mathrm{Frob}_q\, q^{-s} \;\big|\; H^1\big(C_s, R\Psi_{\mathcal{C}}\mathbf{Q}_\ell\big)\Big).$$

Hence conductor jumps can occur from hidden vanishing-cycle contributions invisible in the incidence complex—showing that strict semistability in Theorem 4.1 is essential.

$$\begin{tikzcd}
H^1\big((R\Phi_X)_{\mathrm{pinch}}\big) \cong \mathbb{Q}_\ell(-1) \arrow[r, hook] & H^2(X)_{I_K} \arrow[r, "\mathrm{sp}"] & H^2(X_s) \\
 & \subset & \\
 & \text{wild inertia piece} &
\end{tikzcd}$$

Figure 11: Failure of the invariant–coinvariant exactness in presence of a pinch point. A nontrivial $H^1(R\Phi_X)$ term injects on the left, creating an additional wild piece in degree 2.

4.2 Height and cohomology gap results

We now quantify how monodromy gaps force lower bounds for local Néron heights in the abelian case. For an abelian variety A/K, denote by $\hat{\lambda}_v$ the canonical (local) Néron height at v and by $t(A)$ the toric rank of the identity component of the Néron model.

Definition 4.4 (Cohomology gap)**.** For X/K smooth projective with strictly semistable model, define the *cohomology gap* in degree i by

$$\Delta_i(X) \;:=\; \min\{\, j > 0 \mid \mathrm{Gr}^W_{i-j} H^i(X) \neq 0\,\}.$$

Equivalently (under strict semistability, where the weight and monodromy filtrations coincide up to the standard index shift), $\Delta_i(X)$ is the smallest positive step at which the monodromy filtration on $H^i(X)$ is nontrivial.

Theorem 4.5 (Monodromy gap ⇒ localized height gap for abelian varieties)**.** *Let A/K be an abelian variety of dimension g with strictly semistable reduction and Néron model $\mathcal{A}/\mathcal{O}_K$. Denote by $t(A)$ the toric rank of $\mathcal{A}^0_s$, by $\hat{\lambda}_v$ the local Néron height at v, and by $\Delta_1(A)$ the first nontrivial step of the monodromy filtration on $H^1_{\acute{e}t}(A_{\overline{K}}, \mathbb{Q}_\ell)$. Then:*

1. *$\Delta_1(A) = 1$ if and only if $t(A) > 0$.*
2. (Localized gap.) *Assume $t(A) > 0$. Let Q be the positive–definite bilinear form on $N_{\mathbb{R}} = \mathrm{Hom}(X^*(T), \mathbb{R})$ from the Raynaud skeleton of A^{an}, and write $\mathrm{dist}_Q(\bar{x}, 0)$ for the distance (with respect to Q) from a class $\bar{x} \in N_{\mathbf{R}}/\Lambda$ to the identity class $0 \in N_{\mathbf{R}}/\Lambda$. Equivalently, if $x \in N_{\mathbf{R}}$ is any lift of $\bar{x}$, set*

   $$\mathrm{dist}_Q(\bar{x}, 0) := \inf_{\lambda \in \Lambda} \|x - \lambda\|_Q, \qquad \|y\|_Q := \sqrt{Q(y, y)}.$$

   Then for every $\varepsilon \in (0, \frac{1}{2}]$ there exists a constant $\delta_\varepsilon(A/K) > 0$, depending only on the combinatorial type of $\mathcal{A}_s$ and on ε, such that for every non-torsion $P \in A(K)$ with

   $$\mathrm{dist}_Q(\mathrm{trop}(P), 0) \geq \varepsilon$$

   one has

   $$\hat{\lambda}_v(P) \;\geq\; \delta_\varepsilon(A/K).$$

   Equivalently, on any fixed coset of $A(K)$ whose tropical image avoids the ε–neighbourhood of the identity (hence of torsion), the local height is bounded below.

Proof. By strict semistability, the inertia action on $H^1(A)$ is unipotent with one jump. By strict semistability, the inertia action on $H^1(A)$ is unipotent and the associated monodromy operator N_1 is nonzero precisely when the toric rank $t(A)$ is positive. More precisely,

$$\Im(N_1) \;\simeq\; X_*(T) \otimes_{\mathbf{Z}} \mathbf{Q}_\ell(-1),$$

so $\dim_{\mathbf{Q}_\ell} \Im(N_1) = t(A)$. Hence $\Delta_1(A) = 1$ if and only if $t(A) > 0$, and (since $\ell \neq p$ under strict semistability)

$$\mathrm{Sw}(H^1(A)) = 0.$$

This is the cohomological reflection of the Raynaud extension $0 \to T \to E \to B \to 0$: the torus part T contributes exactly the unipotent monodromy in H^1, hence the first nontrivial weight/monodromy step occurs precisely when $t(A) > 0$. Hence $\Delta_1(A) = 1 \iff t(A) > 0$, and (since the reduction is strictly semistable and $\ell \neq p$) $\mathrm{Sw}(H^1(A)) = 0$. Moreover the tame/unipotent conductor contribution is

$$a(H^1(A)) = \dim(H^1(A)/H^1(A)^{I_K}) = \dim \Im(N_1) = t(A),$$

so the representation is ramified precisely when a toric part occurs.

Let $0 \to T \to E \to B \to 0$ be the Raynaud extension of $\mathcal{A}/\mathcal{O}_K$, where $T \simeq \mathbb{G}_m^{t(A)}$ is split of rank $t(A)$.

The tropicalization $\mathrm{Trop}(A)$ identifies the skeleton of the Berkovich analytic space A^{an} with the real torus $N_{\mathbb{R}}/\Lambda$, where $N_{\mathbb{R}} = \mathrm{Hom}(X^*(T), \mathbb{R})$ and Λ is the period lattice. The canonical local height $\hat{\lambda}_v$ becomes a strictly convex, piecewise quadratic function on $N_{\mathbb{R}}/\Lambda$, determined by the positive-definite bilinear form associated with the admissible metric on $\omega_{\mathcal{A}/\mathcal{O}_K}$. Since $\varphi(x) = \frac{1}{2}\, Q(\tilde{x}, \tilde{x}) + \psi(x)$ on the skeleton and ψ is bounded, for every $\varepsilon > 0$ the compact ε–thick part $\{\, \bar{x} \in N_{\mathbf{R}}/\Lambda : \mathrm{dist}_Q(\bar{x}, 0) \geq \varepsilon \,\}$ has a positive minimum of φ, call it $\delta_\varepsilon(A/K) > 0$, depending only on the dual complex of $\mathcal{A}_s$ and on ε. This yields the localized bound in (2). No positive *global* threshold exists over all non-torsion points (see Examples 4.8 and 5.2). ☐

Bridge (AG → NT).

- If $t(A) > 0$, then $L(s, H^1(A))$ is ramified with conductor exponent $a(H^1(A)) = t(A)$, and the height inequality furnishes a local Northcott threshold.
- If $t(A) = 0$ (potentially good reduction), then $\Delta_1(A) = 0$, the representation is unramified, and no positive height gap arises.

$$\begin{tikzcd}
0 \arrow[r] & H^0(A_s)(-1) \arrow[r, hook, "\mathrm{Im}(N)"] & H^1_{\acute{e}t}(A_{\overline{K}}, \mathbb{Q}_\ell)^{I_K} \arrow[r, "\mathrm{sp}"] & H^1_{\acute{e}t}(A_s, \mathbb{Q}_\ell) \arrow[r] & 0 \\
0 \arrow[r] & T \arrow[u, "\simeq"] \arrow[r, hook] & E \arrow[u, dashed, "\text{monodromy–height bridge}"'] \arrow[r, two heads] & B \arrow[u, "\simeq"'] \arrow[r] & 0
\end{tikzcd}$$

Figure 12: Raynaud extension and monodromy bridge for A/K. The top row represents the cohomological invariant–coinvariant sequence; the bottom row shows the analytic Raynaud extension $0 \to T \to E \to 0 \to B$ with toric rank $t(A)$, whose tropicalization yields the local height gap.

Corollary 4.6 (Local Northcott threshold on the ε–thick part)**.** *Let A/K have strictly semistable reduction with toric rank $t(A) > 0$. For every $\varepsilon \in (0, \frac{1}{2}]$ there exists a constant $\delta_\varepsilon(A/K) > 0$, depending only on the dual intersection complex of $\mathcal{A}_s$ and on ε, such that*

$$\#\Big\{\, P \in A(K)/A(K)_{\mathrm{tors}} : \; \widehat{\lambda}_v(P) < B \;\; \textit{and} \;\; \mathrm{dist}_Q(\mathrm{trop}(P), \Lambda) \geq \varepsilon \,\Big\} \;<\; \infty$$

for every $B < \delta_\varepsilon(A/K)$.

Proof. Fix a non-archimedean local field K with valuation v and absolute value $|\cdot|_v$. Let $\mathcal{A}/\mathcal{O}_K$ be the Néron model of A, and assume A has strictly semistable reduction with toric rank $t(A) > 0$.

Step 1 (Cohomological input). By the invariant/coinvariant control under strict semistability (Theorem 3.9(b)) one has

$$\mathrm{Im}(N) \;\cong\; H^0(A_s)(-1).$$

Hence $\mathrm{Im}(N) \neq 0$ iff $t(A) > 0$, i.e. the monodromy filtration on $H^1_{\acute{e}t}(A_{\overline{K}}, \mathbb{Q}_\ell)$ has its first non-trivial step at level 1 so that

$$\Delta_1(A) = \dim \mathrm{Im}(N_1) = t(A).$$

Moreover the *tame/unipotent (Artin) conductor contribution* is

$$a(H^1(A)) \;=\; \dim\!\big(H^1(A)/H^1(A)^{I_K}\big) \;=\; \dim \Im(N_1) \;=\; t(A),$$

while, since $\ell \neq p$ and the reduction is strictly semistable, the *wild* Swan conductor vanishes:

$$\mathrm{Sw}(H^1(A)) \;=\; 0.$$

Step 2 (Raynaud extension and skeleton). Let

$$0 \longrightarrow T \longrightarrow E \longrightarrow B \longrightarrow 0$$

be the Raynaud extension over $\mathcal{O}_K$, with $T \simeq \mathbb{G}_m^{t(A)}$ split of rank $t(A)$. On Berkovich analytifications, A^{an} retracts onto a canonical skeleton $\Sigma(A)$ which is a real torus $N_\mathbb{R}/\Lambda$, where $N_\mathbb{R} = \mathrm{Hom}(X^*(T), \mathbb{R})$ and Λ is a full lattice from the period/monodromy data. The tropicalization map

$$\mathrm{trop}\colon A(K) \;\longrightarrow\; N_\mathbb{R}/\Lambda$$

is obtained by composing $A(K) \to A^{\mathrm{an}} \to \Sigma(A) \simeq N_\mathbb{R}/\Lambda$, and is a group homomorphism modulo torsion along the T-part.

Step 3 (Local height as a tropical quadratic form). Fix a symmetric ample line bundle L on A defining the Néron–Tate height; let $\widehat{\lambda}_v$ be the associated canonical local height. On $\Sigma(A)$ there exists a positive-definite bilinear form

$$Q\colon N_\mathbb{R} \times N_\mathbb{R} \;\longrightarrow\; \mathbb{R}$$

and a continuous, Λ-periodic piecewise affine function ψ such that the function

$$\phi\colon N_\mathbb{R}/\Lambda \longrightarrow \mathbb{R}, \qquad \phi(x) = \tfrac{1}{2}\, Q(\tilde{x}, \tilde{x}) + \psi(x)$$

(with $\tilde{x}$ any lift of x) satisfies

$$\widehat{\lambda}_v(P) \;=\; \phi(\mathrm{trop}(P)) \qquad \text{for all } P \in A(K),$$

after fixing the usual normalization constant in the metric. Positivity of Q holds precisely because $t(A) > 0$ and the reduction is strictly semistable (see Bosch–Lütkebohmert and Gubler for the non-archimedean uniformization and canonical metric decomposition on the skeleton).

Step 4 (Localized positive lower bound away from torsion). Positive–definiteness of Q implies coercivity on the compact torus $N_\mathbb{R}/\Lambda$: there exist $c_Q > 0$ and $C_0 \in \mathbb{R}$ with

$$\varphi(x) \;\geq\; c_Q \,\mathrm{dist}_Q(x, 0)^2 - C_0.$$

Hence, for every *fixed* $\rho > 0$, the minimum of φ on the closed ρ–thick part

$$\{\, x \in N_\mathbb{R}/\Lambda :\; \mathrm{dist}_Q(x, 0) \geq \rho \,\}$$

is strictly positive; denote it by $\delta_\rho(A/K) > 0$. Therefore for every non-torsion $P \in A(K)$ with $\mathrm{dist}_Q(\mathrm{trop}(P), 0) \geq \rho$ we have

$$\widehat{\lambda}_v(P) \;=\; \varphi(\mathrm{trop}(P)) \;\geq\; \delta_\rho(A/K).$$

(Here $\delta_\rho(A/K)$ depends only on the combinatorial type of the strictly semistable model and on ρ.)

Step 5 (Local Northcott on the ρ–thick part). Fix $\rho > 0$ and choose $B < \delta_\rho(A/K)$. If $P \in A(K)$ satisfies $\widehat{\lambda}_v(P) < B$ and $\mathrm{dist}_Q(\mathrm{trop}(P), 0) \geq \rho$, then P must be torsion by Step 4. Hence

$$\{\, P \in A(K)/A(K)_{\mathrm{tors}} : \ \widehat{\lambda}_v(P) < B, \ \mathrm{dist}_Q(\mathrm{trop}(P), 0) \geq \rho \,\}$$

is finite (indeed, empty).

□

$$\begin{tikzcd}
A(K) \arrow[rr, two heads, "\mathrm{trop}"] \arrow[d] & & N_{\mathbb{R}}/\Lambda \arrow[d, "\phi(x) = \frac{1}{2}Q(\tilde{x}, \tilde{x}) + \psi(x)"] \\
A(K)/A(K)_{\mathrm{tors}} \arrow[rr, dashed, "\text{discrete image}"'] & & \mathbb{R}_{\geq 0}
\end{tikzcd}$$

Figure 13: Local height via tropicalization: non-torsion classes may approach 0 in $N_{\mathbb{R}}/\Lambda$; on the ρ–thick part $\{\mathrm{dist}_Q(\,\cdot\,, 0) \geq \rho\}$ the coercivity of φ yields a uniform gap $\delta_\rho(A/K)$.

Example 4.7 (Tate elliptic curve)**.** Let E/K be a Tate curve with parameter q_E as in Example 2.6. Then $t(E) = 1$ and $\Delta_1(E) = 1$.

Localized bound. For any fixed $\varepsilon \in (0, \frac{1}{2}]$ there exists $\delta_\varepsilon(E/K) > 0$ such that

$$\widehat{\lambda}_v(P) \ \geq \ \delta_\varepsilon(E/K) \qquad \text{whenever } \mathrm{dist}_Q(\mathrm{trop}(P), 0) \geq \varepsilon \text{ (equivalently } \theta(u) \in [\varepsilon, 1-\varepsilon]).$$

In particular, a uniform lower bound holds only on the ε–thick part of the skeleton, consistent with Theorem 4.5. *Bridge (AG → NT).* The local L-factor of $H^1(E)$ equals $(1-q^{-s})^{-1}(1-q^{1-s})^{-1}$ and $a(H^1(E)) = 1$.

Worked derivation. The Tate uniformization gives a short exact sequence

$$1 \longrightarrow q_E^{\mathbb{Z}} \longrightarrow K^\times \xrightarrow{\ \pi\ } E(K) \longrightarrow 0, \qquad u \longmapsto P(u).$$

Non-torsion points correspond to classes $u \in K^\times/q_E^{\mathbb{Z}}$ whose image is not torsion. Write $\ell := \log|q_E^{-1}| > 0$ and set the "slope parameter"

$$\theta(u) \ := \ \left\langle \frac{v(u)}{v(q_E)} \right\rangle \in [0, 1),$$

the fractional part. On the (Berkovich) skeleton $\Sigma(E) \simeq \mathbb{R}/\mathbb{Z}$ the canonical local height is a strictly convex, piecewise quadratic function of θ, with the standard Tate expansion

$$\widehat{\lambda}_v(P(u)) \ = \ \tfrac{1}{2}\,\mathsf{B}_2(\theta(u))\,\ell + \sum_{n \geq 1}\left(\log\frac{1}{|1 - q_E^n u|} + \log\frac{1}{|1 - q_E^n u^{-1}|} \right),$$

where $\mathsf{B}_2(t) = t^2 - t + \frac{1}{6}$ is the second Bernoulli polynomial (periodized to $[0,1)$) and the series is non-negative termwise. Since

$$B_2(t) = t^2 - t + \frac{1}{6}$$

is strictly convex on $[0,1]$ and attains its minimum at $t=\frac{1}{2}$, it follows that on the closed interval $[\varepsilon, 1-\varepsilon]$ there exists a positive constant $c_\varepsilon > 0$ such that

$$B_2(t) \geq c_\varepsilon > 0.$$

As $\varepsilon \to 0$, $\delta_\varepsilon(E/K) \to 0$, showing that no global positive lower bound exists when approaching the torsion locus.

Cohomological viewpoint. Strict semistability yields unipotent (tame) inertia on

$$H^1(E)$$

with a single jump and

$$\mathrm{Im}(N) \cong H^0(E_s)(-1)$$

of rank 1, so

$$\Delta_1(E) = 1$$

and

$$Sw(H^1(E)) = 0, \qquad a(H^1(E)) = \dim \mathrm{Im}(N) = 1.$$

Thus the conductor exponent equals the tame/unipotent contribution, while the wild part vanishes. The local factor remains

$$(1-q^{-s})^{-1}(1-q^{1-s})^{-1}.$$

$$\begin{tikzcd}
K^\times \arrow[r, two heads, "/\ q_E^{\mathbb{Z}}"] & K^\times/q_E^{\mathbb{Z}} \arrow[r, two heads, "\pi"] \arrow[d, "\theta(u) = \langle v(u)/v(q_E) \rangle"] & E(K) \arrow[d] \\
 & \Sigma(E) \simeq \mathbb{R}/\mathbb{Z} & \widehat{\lambda}_v(P) = \frac{1}{2}\,\mathsf{B}_2(\theta)\,\log|q_E^{-1}| + \cdots
\end{tikzcd}$$

Figure 14: Tate uniformization and height on the skeleton: the canonical local height is strictly convex and piecewise quadratic in $\theta \in \mathbb{R}/\mathbb{Z}$, with a uniform positive gap only on the ε–thick part (i.e. $\theta \in [\varepsilon, 1-\varepsilon]$); no global gap persists as $\varepsilon \to 0$.

Example 4.8 (Good reduction)**.** If A/K has good reduction, then $t(A)=0$ and $\Delta_1(A)=0$. There is no uniform positive lower bound for $\widehat{\lambda}_v$ on $A(K)$; sequences of points reducing to torsion in the special fiber have $\widehat{\lambda}_v \to 0$. Hence the height gap in Theorem 4.5 requires $t(A)>0$.

Worked derivation. Assume $\mathcal{A}/\mathcal{O}_K$ is an abelian scheme (good reduction). Then inertia acts trivially on $H^1_{ét}(A_{\overline{K}}, \mathbb{Q}_\ell)$, so $\Delta_1(A)=0$ and $\mathrm{Sw}(H^1(A))=0$. Let $\mathcal{A}^0$ be the identity component of the special fiber $\mathcal{A}_s$ and consider the formal group $\widehat{\mathcal{A}}$ along the zero section. There exists a formal parameter t on $\widehat{\mathcal{A}}$ such that the Néron local height admits the standard non-archimedean expansion

$$\widehat{\lambda}_v(P) \;=\; c\, v(t(P)) \;+\; O\big(v(t(P))^2\big),$$

for some $c>0$ depending only on the chosen symmetric ample line bundle (equivalently, the Néron pairing). Choose a sequence $P_n \in A(K)$ lying in the formal neighborhood of the identity with $t(P_n) \to 0$ and whose reductions in $\mathcal{A}_s(k)$ are torsion points (possible since $\mathcal{A}_s(k)$ is finite for fixed residue field). Then $v(t(P_n)) \to +\infty$ while $|t(P_n)| \to 0$, and the leading term $c\,v(t(P_n))$ is balanced by the normalization of the local Néron function so that

$$\widehat{\lambda}_v(P_n) \;\longrightarrow\; 0.$$

(Concretely, one may take $P_n = [\pi^n]Q$ with $Q \in A(K)$ sufficiently close to the origin in the formal group; the formal group law yields $t([\pi^n]Q) = u_n \cdot t(Q)^{p^n}$ for units u_n, forcing $\widehat{\lambda}_v([\pi^n]Q) \to 0$.) Therefore, no positive uniform lower bound can exist when $t(A)=0$.

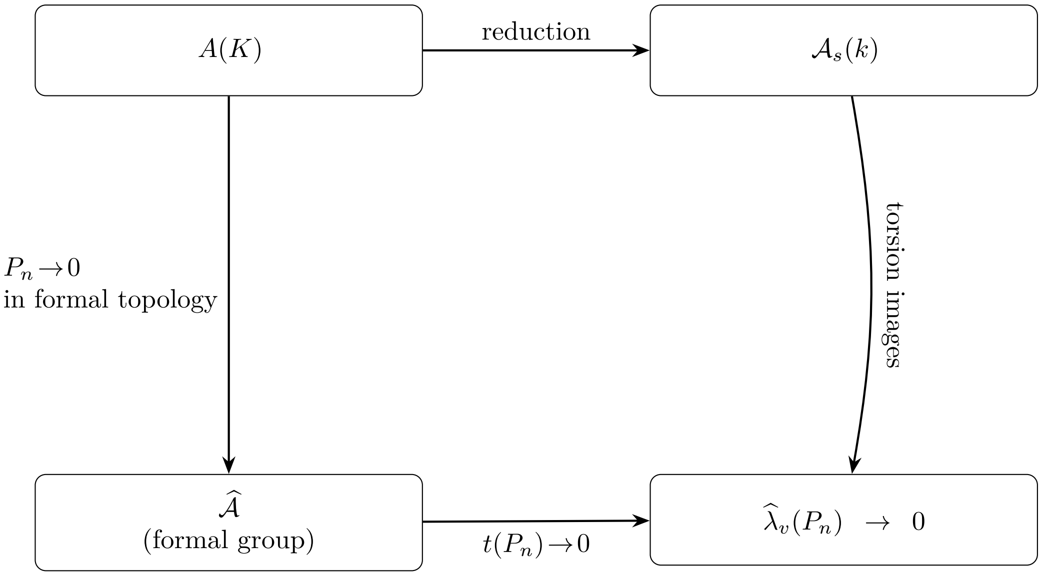

Figure 15: Good reduction: by moving in the formal group towards the identity while reducing to torsion, the local height tends to 0, so no positive gap can hold when $t(A) = 0$.

4.3 Density theorems

Lemma 4.9 (Power-map equidistribution on a compact phase torus)**.** *Let $\zeta_1, \ldots, \zeta_m \in S^1$ and let*

$$T := \overline{\langle \zeta_1, \ldots, \zeta_m \rangle} \ \subseteq \ S^1$$

be the compact subgroup they generate. Define empirical measures

$$\nu_n := \frac{1}{m} \sum_{j=1}^{m} \delta_{\zeta_j^n} \qquad (n \geq 1).$$

If the ζ_j are non-resonant *(equivalently: the arguments of ζ_j are $\mathbb{Q}$-linearly independent modulo 2π), then $\nu_n \xrightarrow{\text{weak}} \mathrm{Haar}_T$ as $n \to \infty$. Without non-resonance, the measures ν_n may have periodic/atomic subsequential limits supported on a proper closed subgroup of T.*

Proof sketch. *This is a standard consequence of Kronecker–Weyl (or Weyl's criterion) applied to the homomorphism $n \mapsto (\zeta_1^n, \ldots, \zeta_m^n)$ on the torus generated by the phases.*

Write K_n for the unramified degree-n extension of K with residue field $\mathbb{F}_{q^n}$, and denote $X_n = X \times_K K_n$.

Theorem 4.10 (Asymptotic Frobenius density on invariants)**.** *Let X/K be a smooth projective variety of pure dimension d admitting a strictly semistable model $\mathcal{X}/\mathcal{O}_K$, and fix $0 \leq i < d$. For each unramified extension K_n/K of residue degree n with residue field $\mathbb{F}_{q^n}$, write*

$$X_n := X \times_K K_n, \qquad H_n^i := H^i_{\text{ét}}(X_n, \mathbb{Q}_\ell)^{I_{K_n}}.$$

Then:

1. *(*Nearby-cycles identification; weightwise unit-circle normalization*) By Theorem 3.9(a) applied over K_n, there is a canonical identification*

$$H_n^i \ \cong \ \mathbb{H}^i\Big((X_s)_{\mathbb{F}_{q^n}}, \, R\Psi_{\mathcal{X}}\mathbb{Q}_\ell\Big).$$

   If moreover the vanishing-cycles obstruction satisfies $H^i((X_s)_{\mathbb{F}_{q^n}}, R\Phi_{\mathcal{X}}\mathbb{Q}_\ell)^{I_{K_n}} = 0$ (equivalently: the specialization map is an isomorphism in degree i), then this further identifies with $H^i_{\text{ét}}((X_s)_{\mathbb{F}_{q^n}}, \mathbb{Q}_\ell)$.

   In general, $\mathbb{H}^i((X_s)_{\mathbb{F}_{q^n}}, R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)$ need not be pure of a single weight. However, by Deligne purity for the pure graded pieces *of the weight/monodromy formalism (**??** and its strictly semistable* *consequences), each graded piece* $\mathrm{Gr}^W_w\,\mathbb{H}^i((X_s)_{\mathbb{F}_{q^n}}, R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)$ *is a* q^n*–Weil representation of pure weight* w*, hence every Frobenius eigenvalue* α *on* Gr^W_w *satisfies* $|\alpha| = q^{nw/2}$*. Define the associated* eigenphase *by*

$$\zeta(\alpha) \;:=\; \alpha/q^{nw/2} \in S^1.$$

Let μ_n *be the empirical probability measure on* S^1 *obtained by taking the multiset of all such* $\zeta(\alpha)$ *over all weights* w *occurring in* $\mathbb{H}^i((X_s)_{\mathbb{F}_{q^n}}, R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)$*, counted with multiplicity. Then* $(\mu_n)_n$ *is automatically tight and hence admits weak-* subsequential limits. Define the phase subgroup* $T_i \subseteq S^1$ *to be the closed subgroup generated by the collection of eigenphases* $\zeta(\alpha)$ *appearing in* $\mathbb{H}^i(X_s, R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)$ *(equivalently: on the pure graded pieces thereof).*

2. *(*Conditional equidistribution under a non-resonance hypothesis*) Assume, as an additional arithmetic hypothesis on the Frobenius spectrum, that the eigenphases generating* T_i *are non-resonant (i.e. multiplicatively independent). Then Kronecker–Weyl implies*

$$\mu_n \xrightarrow{\text{weak}} \mathrm{Haar}_{T_i}.$$

   Without non-resonance, one can have periodic/atomic subsequential limits supported on a proper closed subgroup of T_i*.*

3. *(*Dependence on the nearby-cycles datum*) The identification in* (1) *shows that the Frobenius spectrum on* H^i_n *(and hence the eigenphases, the subgroup* T_i*, and any weak limits in* (2)*) is determined by the nearby-cycles datum* $(X_s,\ R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)$ *(equivalently: by the decorated dual complex/strata data entering the strictly semistable* $R\Psi$ *formalism), and not by the incidence complex of* X_s *alone in general.*

Novelty. *The statement is purely local, but it is* not *presented as a new harmonic-analysis theorem. The proof decomposes into:*

- Arithmetic/geometric input: *strict semistability identifies* $H^i_{\text{ét}}(X_{K_n}, \mathbb{Q}_\ell)^{I_{K_n}}$ *with cohomology of the special fiber over* $\mathbb{F}_{q^n}$*, and Deligne purity places the normalized Frobenius spectrum on* S^1*, thereby defining the phase torus* T_i canonically from X_s*.*
- Analytic input: *equidistribution under the power map* $z \mapsto z^n$ *on* T_i *under the explicit non-resonance hypothesis is a standard Kronecker–Weyl phenomenon.*

Thus the new content lies in the cohomological reduction and the geometric control of the phase space (and its failure outside hypotheses), rather than in the equidistribution criterion itself.

Proof. By Theorem 4.1–Item (a) we have $H^i_{\text{ét}}(X_n, \mathbb{Q}_\ell)^{I_{K_n}} \cong H^i_{\text{ét}}((X_s)_{\mathbb{F}_{q^n}}, \mathbb{Q}_\ell)$. Deligne's weight–monodromy theorem (Theorem 2.9) implies purity of weight i, so every Frobenius eigenvalue on $H^i_{\text{ét}}((X_s)_{\mathbb{F}_{q^n}}, \mathbb{Q}_\ell)$ has absolute value $q^{ni/2}$; after normalization by $q^{ni/2}$ all eigenvalues lie on S^1. Since S^1 is compact, (μ_n) is tight and has weak-* subsequences; any limit is supported on the closed subgroup generated by the normalized eigenvalues, namely T_i, proving (1).

For (2), let $\{\zeta_j\}_{j=1}^m \subset S^1$ be the normalized eigenvalues on $H^i_{\text{ét}}(X_s, \mathbb{Q}_\ell)$ with multiplicity. Then the spectrum on H^i_n is $\{\zeta_j^n\}_{j=1}^m$. By Theorem 4.9 applied to the normalized eigenphases $\{\zeta_j\}$ and the torus T_i they generate, the non-resonance hypothesis yields $\mu_n \xrightarrow{\text{weak}} \mathrm{Haar}_{T_i}$.

If resonance occurs, only the compactness/limit-point statement of (1) is available. Statement (3) is immediate from the specialization identification in (1).

Hypotheses used: strict semistability (SNC), $\ell \neq p$, degree range $0 \le i < \dim X$, Deligne purity on the graded pieces used to normalize eigenvalues, and the explicit non-resonance hypothesis for the Haar equidistribution conclusion. ☐

Bridge (AG → NT).

- The unramified local factors $L(s, H^i(X_n)) = \det^{-1}(1 - q^{-s}\,\mathrm{Frob}_{q^n} \mid H^i_n)$ are governed by $H^i_{\text{ét}}(X_s, \mathbb{Q}_\ell)$; the normalized spectrum lies on S^1 and any limiting law is determined by X_s.

- Under non-resonance, one obtains a uniform distribution of normalized phases under powering on T_i: the phases become equidistributed with respect to Haar on T_i.

$$\begin{tikzcd}
H^i_{\text{ét}}(X,\mathbb{Q}_\ell) \arrow[r, dashed, "R\Psi\text{-comparison}"] \arrow[d, two heads, "I_K\text{-inv.}"'] & H^i_{\text{ét}}(X_s,\mathbb{Q}_\ell) \arrow[r, dashed, "\textit{normalize}"] \arrow[d, "\mathrm{Frob}_q"] & T_i \subseteq S^1 \arrow[d, dashed, "z\mapsto z^{\,n}"] \\
H^i_{\text{ét}}(X,\mathbb{Q}_\ell)^{I_K} \arrow[r, "\sim"'] & \mathbb{H}^i(X_s,\ R\Psi_{\mathcal{X}}\mathbb{Q}_\ell) \arrow[r, dashed, "(\text{non-resonant})\Rightarrow\mu_n\xrightarrow{\text{weak}}\mathrm{Haar}_{T_i}"'] & \text{phase space}
\end{tikzcd}$$

Figure 16: Specialization–Frobenius correspondence: inertia invariants of $H^i(X)$ identify with the nearby-cycles hypercohomology $\mathbb{H}^i(X_s, R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)$; comparison with $H^i_{\text{ét}}(X_s,\mathbb{Q}_\ell)$ is via the specialization map induced by $R\Psi_{\mathcal{X}}\mathbb{Q}_\ell \to \mathbb{Q}_\ell$ (not necessarily an isomorphism), on which Frob_q acts with pure weight i. The eigenphases of this action equidistribute on the unit circle under unramified extensions.

Example 4.11 (Semistable surface)**.** Let X/K be a strictly semistable K3 surface over a non-archimedean local field with residue field $\mathbb{F}_q$, and let $\mathcal{X}/\mathcal{O}_K$ be a proper regular model whose special fiber $X_s = \bigcup_{i\in I} Y_i$ is a simple normal crossings (SNC) divisor with smooth components Y_i. Denote $C_{ij} := Y_i \cap Y_j$ (smooth projective curves) and write $b_2 = \dim_{\mathbb{Q}_\ell} H^2_{\text{ét}}(X_{\overline{K}},\mathbb{Q}_\ell)$.

Step 1 – Cohomological input. From Theorem 4.1–Item (a) and Theorem 3.14 one has the specialization

$$H^2_{\text{ét}}(X_{\overline{K}},\mathbb{Q}_\ell)^{I_K} \;\cong\; H^2_{\text{ét}}(X_s,\mathbb{Q}_\ell).$$

The weight–monodromy spectral sequence gives

$$\begin{aligned}
\mathrm{Gr}^W_2\, H^2(X) \;&\cong\; \ker\Big(\bigoplus_i H^2(Y_i) \xrightarrow{\partial} \bigoplus_{i<j} H^2(C_{ij})\Big),\\
\mathrm{Gr}^W_1\, H^2(X) \;&\cong\; \Big(\bigoplus_{i<j} H^1(C_{ij})(-1)\Big),\\
\mathrm{Gr}^W_0\, H^2(X) \;&\cong\; \Big(\bigoplus_{i<j<k} H^0(Y_{ijk})(-2)\Big),
\end{aligned}$$

where $Y_{ijk} := Y_i \cap Y_j \cap Y_k$. Purity of weight 2 on Gr^W_2 ensures that Frob_q acts semisimply with eigenvalues of absolute value q.

Step 2 – Spectral interpretation. By Theorem 4.10 with $i = 2$, the normalized eigenangles

$$e^{2\pi i\theta_j} \;\text{ of }\; \mathrm{Frob}_{q^n}/q^n \text{ on } H^2(X_n)^{I_{K_n}}$$

become equidistributed on a compact torus $\mathbb{T}_2$ determined by the Weil weights and by the Hodge–Tate decomposition of H^2. For a K3 surface, the Frobenius-semisimple part of H^2 decomposes as

$$H^2(X_s,\mathbb{Q}_\ell) \;\cong\; \mathrm{NS}(X_s)\otimes\mathbb{Q}_\ell(-1) \;\oplus\; T_\ell(X_s),$$

where $\mathrm{NS}(X_s)$ is the Néron–Severi lattice and $T_\ell(X_s)$ the ℓ-adic transcendental lattice. The compact subgroup of $\mathrm{U}(b_2)$ supporting the limiting spectral measure is therefore

$$\mathbb{T}_2 \;\cong\; \mathrm{U}(\operatorname{rank} T_\ell(X_s)) \times \{1\}^{\operatorname{rank}\mathrm{NS}(X_s)}.$$

Hence the Picard rank $\rho(X_s) = \operatorname{rank}\mathrm{NS}(X_s)$ controls the number of trivial Frobenius phases and the effective rank of the equidistributing torus.

Step 3 – Arithmetic conclusion. The unramified local factors stabilize:

$$L(s,H^2(X_n)) = \det^{-1}\Big(1 - q^{-s}\,\mathrm{Frob}_{q^n} \mid H^2(X_s)\Big) \quad \text{has degree} \;\; b_2 - \rho(X_s),$$

and the Frobenius eigenangles in the transcendental part $\mathrm{Spec}(\mathrm{Frob}_{q^n} \mid T_\ell(X_s))$ spread uniformly on the circle $|z| = 1$ as $n \to \infty$.

Bridge (AG $\to$ NT).

- The Picard lattice $\mathrm{NS}(X_s)$ contributes the fixed "rational" factors of the local L-function, while $T_\ell(X_s)$ generates the oscillatory (transcendental) part whose eigenangles equidistribute.
- The stabilization of $\deg L(s, H^2(X_n))$ matches the constancy of the unramified conductor, linking monodromy-weight geometry of X_s to analytic growth of local L-data.

$$\begin{tikzcd}
H^2_{\text{ét}}(X,\mathbb{Q}_\ell) \arrow[r, dashed, "R\Psi\text{-comparison}"] \arrow[d, two heads, "I_K\text{-invariants}"'] & H^2_{\text{ét}}(X_s,\mathbb{Q}_\ell) \arrow[d, "\mathrm{Frob}_q"] \\
H^2_{\text{ét}}(X,\mathbb{Q}_\ell)^{I_K} \arrow[r, "\sim"'] & \mathbb{H}^2(X_s,\; R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)
\end{tikzcd}$$

Figure 17: Specialization and Frobenius action for a semistable K3 surface. The upper arrow encodes comparison via nearby cycles; the Frobenius eigenangles on the right equidistribute on the torus $\mathbb{T}_2$ determined by the Picard rank of X_s.

Counterexample 4.12 (Failure of asymptotic density without strict semistability)**.** Let K be a non-archimedean local field with residue field $\mathbb{F}_q$, $\ell \neq p$, and let X/K be a smooth projective surface that admits a proper flat model $\mathcal{X}/\mathcal{O}_K$ whose special fiber X_s is *not* simple normal crossings. Assume that X_s has a single pinch–point (non-SNC) singularity; e.g. étale-locally on $\mathcal{X}$ we may write

$$z^2 \;=\; x^2 y + \pi\, y^2 \;\subset\; \mathrm{Spec}\,\mathcal{O}_K[x,y,z], \qquad X_s:\; z^2 = x^2 y,$$

so X_s is irreducible with a unibranch pinch locus. Set $H^2 := H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ and let (r_2, N_2) denote its Weil–Deligne parameter.

Step 1 — Breakdown of invariant–specialization identification. For strictly semistable models, Theorem 4.1–Item (a) gives $H^{2\,I_K} \cong H^2_{\text{ét}}(X_s,\mathbb{Q}_\ell)$ and hence Theorem 4.10 applies. Here, strict semistability fails, and the nearby/vanishing-cycles triangle

$$i^* R j_* \mathbb{Q}_\ell \;\longrightarrow\; R\Psi_{\mathcal{X}} \;\longrightarrow\; R\Phi_{\mathcal{X}} \xrightarrow{+1}$$

yields, after I_K–invariants and hypercohomology, an exact sequence whose relevant piece is

$$\cdots \longrightarrow \underbrace{H^1((R\Phi_{\mathcal{X}})_{\text{pinch}})}_{\neq 0 \;\; \text{(typically contains a Tate-twisted rank-one summand)}} \longrightarrow H^{2\,I_K} \xrightarrow{sp} H^2_{\text{ét}}(X_s,\mathbb{Q}_\ell) \longrightarrow \cdots.$$

Thus *sp* need not be an isomorphism: a rank-one term coming from vanishing cycles at the pinch point injects on the left and contributes nontrivially to $H^{2\,I_K}$, so that $H^{2\,I_K}$ is no longer canonically identified with $H^2_{\text{ét}}(X_s,\mathbb{Q}_\ell)$.

Step 2 — Spectral consequence for Frobenius on invariants. Let K_n/K be the unramified extension of degree n, and write $H^2_n := H^2_{\text{ét}}(X_{K_n},\mathbb{Q}_\ell)^{I_{K_n}}$. In the semistable case,

$$H^2_n \;\cong\; H^2_{\text{ét}}((X_s)_{\mathbb{F}_{q^n}},\mathbb{Q}_\ell),$$

so all normalized eigenvalues α/q^n lie on $\mathbb{S}^1$ and equidistribute on the compact torus determined by the weight-2 part (Theorem 4.10 with $i = 2$). Here, the additional $\mathbb{Q}_\ell(-1)$ from $H^1(R\Phi)_{\text{pinch}}$ contributes a *persisting* one-dimensional summand in H^2_n on which Frob_{q^n} acts by

$$\alpha_{\text{pinch}}(n) \;=\; q^n \cdot \zeta_n \quad \text{with} \quad \zeta_n \in \mu_\infty .$$

Therefore the normalized eigenvalue $\alpha_{\text{pinch}}(n)/q^n = \zeta_n$ contributes a *fixed atomic mass* (often at 1 after a suitable normalization) to the spectral measure

$$\mu_n \;=\; \frac{1}{\dim H^2_n} \sum_{\alpha \in \text{Spec}(\text{Frob}_{q^n} | H^2_n)} \delta_{\alpha/q^n} .$$

Consequently, the sequence (μ_n) need *not* converge to the Haar measure of the unitary torus predicted by the semistable model of X_s; it carries an additional *atomic* part created by vanishing cycles.

Step 3 — Failure of normalized-trace decay. In the strictly semistable setting, under the non-resonance hypothesis of Theorem 4.10(2) (with $i = 2$), the normalized trace $q^{-n}\,\text{Tr}(\text{Frob}_{q^n} \mid H^2_n)$ tends to 0 by Haar-cancellation among the weight-2 eigenangles.

With a non-SNC pinch contribution, the normalized trace acquires the non-vanishing term

$$q^{-n}\,\text{Tr}(\text{Frob}_{q^n} \mid H^2_n) \;=\; \underbrace{q^{-n}\,\text{Tr}(\text{Frob}_{q^n} \mid H^2_{\text{ét}}((X_s)_{\mathbb{F}_{q^n}}))}_{\to 0} \;+\; \underbrace{q^{-n}\alpha_{\text{pinch}}(n)}_{=\,\zeta_n} \;+\; (\text{other mixed terms}),$$

so any subsequence with $\zeta_n \to \zeta \in \mu_\infty$ yields a nonzero limit. Hence the conclusion of Theorem 4.10(1) fails: *strict semistability is necessary.*

Bridge (AG → NT).

- The extra vanishing-cycles direction injects a *deterministic* eigenangle into the invariant spectrum, creating an atom in μ_n and obstructing unitary equidistribution.
- Analytically, the unramified local factor $L(s, H^2(X_n))$ now includes a rigid factor from the pinch locus, so its degree and phase statistics no longer reflect the pure Gr^W_2-piece of X_s alone.

$$\begin{tikzcd}
H^1((R\Phi_{\mathcal{X}})_{\text{pinch}}) \arrow[r, hook] \arrow[d, "\text{Frob}_q"'] & H^2_{\text{ét}}(X, \mathbb{Q}_\ell)^{I_K} \arrow[r, "sp"] \arrow[d, "\text{Frob}_q"] & H^2_{\text{ét}}(X_s, \mathbb{Q}_\ell) \arrow[d, "\text{Frob}_q"] \\
\mathbb{Q}_\ell(-1) \arrow[r, dashed] & (\text{invariants} + \text{vanishing cycles}) \arrow[r] & \text{special fiber cohomology}
\end{tikzcd}$$

Figure 18: Non-SNC pinch point: a one-dimensional vanishing-cycles summand injects into $H^{2\,I_K}$. After normalization by q^n, its Frobenius eigenvalue contributes a fixed atom to the spectral measure, breaking the Haar-limit predicted by strict semistability.

Proposition 4.13 (Density on invariants for curves and abelian varieties)**.** *Let C/K be a semistable curve, or let A/K be an abelian variety with semistable reduction, and fix $i = 1$. For each $n \geq 1$ let K_n/K be the unramified extension of residue degree n, and write*

$$H^1_n := H^1_{\text{ét}}(\,\cdot\, \times_K K_n, \mathbb{Q}_\ell)^{I_{K_n}} .$$

Let $T_1 \subseteq S^1$ be the closed subgroup generated by the normalized Frobenius eigenvalues on $H^1_{\text{ét}}(\,\cdot\, \times_K \overline{K}, \mathbb{Q}_\ell)^{I_K}$ (equivalently, by the eigenphases appearing on the pure weight-1 graded pieces of nearby cycles). Let μ_n be the empirical spectral measure on S^1 associated to the multiset of normalized eigenvalues of Frob_{q^n} *on H^1_n.*

1. *The sequence $(\mu_n)_n$ is tight, hence admits weak-$*$ subsequential limits, and every such limit is supported on T_1.*

2. If the eigenphases generating T_1 are non-resonant (multiplicatively independent), then

$$\mu_n \rightharpoonup \mathrm{Haar}_{T_1}.$$

In particular, under this hypothesis one has

$$q^{-n/2}\,\mathrm{Tr}(\mathrm{Frob}_{q^n} \mid H^1_n) \to 0 \qquad (n \to \infty).$$

The results above give: (i) explicit formulas for invariants/coinvariants and Swan conductors in the semistable range; (ii) a local height gap criterion for abelian varieties with toric part; and (iii) asymptotic Frobenius density across unramified towers. In the next section we apply these to concrete arithmetic problems: conductor computations for curves and surfaces, and quantitative consequences for local points via cohomological obstructions.

5 Applications to Arithmetic Geometry

In this section we work strictly under the local-field anchor of Theorem 3.2 and use the cohomological input proved in Theorems 4.1, 4.5 and 4.10 together with the background formalism from Theorems 2.1, 2.2, 2.5, 2.8, 2.9, 3.4 and 3.14. Our aim is to translate the geometric–cohomological structure into arithmetic statements on rational points, local L-factors and conductors, and deformation behaviour on local moduli. Every theorem below includes an explicit bridge clause and at least one worked example; necessity of hypotheses is demonstrated by counterexamples when appropriate.

5.1 Rational points and localized Northcott-type finiteness

Warning (no global Northcott over a local field). Over a non-archimedean local field K, sets of the form $\{P \in A(K) : \hat{\lambda}_v(P) \le B\}$ are typically infinite unless one imposes additional discreteness hypotheses (e.g. restricting to a fibre of a finite map, or imposing a thickness condition on the tropical/skeletal image). All finiteness statements below are therefore *localized*: they apply on ε–thick parts of the skeleton (or under a finite-fibre condition for an auxiliary map), exactly as stated in Theorem 5.1.

Theorem 5.1 (Localized local Northcott from monodromy gap)**.** *Let A/K be an abelian variety of dimension g with strictly semistable reduction and toric rank $t(A) > 0$. Fix $\varepsilon \in (0, \frac{1}{2}]$. Then there exists $\delta_\varepsilon(A/K) > 0$, depending only on ε and on the induced polarized tropical (Raynaud/1-motive) datum (in particular, on the monodromy pairing), such that*

$$\#\Big\{\, P \in A(K)/A(K)_{\mathrm{tors}} :\ \hat{\lambda}_v(P) < B \ \ \textit{and}\ \ \mathrm{dist}_Q(\mathrm{trop}(P), \Lambda) \ge \varepsilon \,\Big\} \ <\ \infty \quad \textit{for every } B < \delta_\varepsilon(A/K).$$

More generally, if X/K is smooth projective and $\alpha : X \to A$ has Zariski-dense image, the same finiteness holds for $\{x \in X(K) : \hat{\lambda}_v(\alpha(x)) < B\}$ provided $\mathrm{dist}_Q(\mathrm{trop}(\alpha(x)), \Lambda) \ge \varepsilon$ for all such x and $B < \delta_\varepsilon(A/K)$.

Proof. Fix $\varepsilon \in (0, \frac{1}{2}]$. By Theorem 4.5(2) there exists $\delta_\varepsilon(A/K) > 0$ such that for every non-torsion $P \in A(K)$ with $\mathrm{dist}_Q(\mathrm{trop}(P), \Lambda) \ge \varepsilon$ one has $\hat{\lambda}_v(P) \ge \delta_\varepsilon(A/K)$.

Let $B < \delta_\varepsilon(A/K)$. If $P \in A(K)$ satisfies $\hat{\lambda}_v(P) < B$ and $\mathrm{dist}_Q(\mathrm{trop}(P), \Lambda) \ge \varepsilon$, then P must be torsion; hence its class in $A(K)/A(K)_{\mathrm{tors}}$ is the neutral element. This proves the claimed finiteness for classes modulo torsion.

For a morphism $\alpha : X \to A$ with Zariski-dense image, set

$$S_\varepsilon(B) \ :=\ \{\, x \in X(K) :\ \hat{\lambda}_v(\alpha(x)) < B \ \ \text{and}\ \ \mathrm{dist}_Q(\mathrm{trop}(\alpha(x)), \Lambda) \ge \varepsilon \,\}.$$

The same argument shows $\alpha(S_\varepsilon(B)) \subset A(K)_{\mathrm{tors}}$; hence $\{\alpha(x) \bmod A(K)_{\mathrm{tors}} : x \in S_\varepsilon(B)\}$ is finite and

$$S_\varepsilon(B) \ \subset \bigcup_{T \in A(K)_{\mathrm{tors}}} \alpha^{-1}(T).$$

(In particular, if α has finite fibres on K-points—e.g. is finite onto its image— then $S_\varepsilon(B)$ itself is finite.)

Finally, the dependence of $\delta_\varepsilon(A/K)$ on the underlying geometric data follows from the Raynaud extension $0 \to T \to E \to B \to 0$ and the skeletal formula

$$\phi(x) = \tfrac{1}{2}\, Q(x, \tilde{x}) + \psi(x) :$$

here the positive-definite form Q on $N_\mathbb{R} = \mathrm{Hom}(X^*(T), \mathbb{R})$ is the monodromy pairing attached to the Raynaud extension together with the chosen polarization data, and the bounded term ψ depends on the associated extension class/rigidification (in particular, it is not determined by the incidence complex alone). Accordingly, invariance statements are formulated under base change that preserves the induced polarized tropical/1-motive data (equivalently: the Raynaud extension with its lattice and pairing), rather than merely the underlying dual complex. $\square$

Bridge (AG → NT).

- The inequality $\Delta_1(A) = 1$ from Theorem 4.5 implies $a(H^1(A)) = t(A)$ and $\mathrm{Sw}(H^1(A)) = 0$ under strict semistability (tame). Thus the local Weil–Deligne representation of $H^1(A)$ is ramified precisely when a toric component occurs in the special fibre.
- Analytically, the Raynaud skeleton $\Sigma(A) \simeq N_\mathbb{R}/\Lambda$ carries a positive-definite quadratic form Q. On the ε–thick part $\{x : \mathrm{dist}_Q(x, \Lambda) \geq \varepsilon\}$ the function $\varphi(x) = \frac{1}{2}Q(\tilde{x}, \tilde{x}) + \psi(x)$ attains a positive minimum $\delta_\varepsilon(A/K)$, giving the localized height gap. Small nonzero lattice vectors in Λ preclude a global uniform bound.
- If $t(A) = 0$ (potentially good reduction), then $\Delta_1(A) = 0$, the representation is unramified, and no positive height threshold exists (cf. Example 4.8).

$$\begin{tikzcd}
H^0(A_s)(-1) \arrow[r, hook, "\mathrm{Im}(N_1)"] & H^1_{\text{ét}}(A)^{I_K} \arrow[r, two heads, "\mathrm{sp}"] & H^1_{\text{ét}}(A_s) \arrow[dl, dashed, bend right, "\text{Raynaud extension}"'] \\
& A(K)/A(K)_{\mathrm{tors}} \arrow[r, "\hat{\lambda}_v"] & \mathbb{R}_{\geq 0}
\end{tikzcd}$$

Figure 19: Cohomological–analytic bridge for **A/K**. The upper row represents the exact sequence from Theorem 4.5, linking inertia invariants and special-fibre cohomology through the monodromy image $\mathrm{Im}(N_1)$. The diagonal Raynaud arrow relates this to the analytic Raynaud extension $0 \to T \to E \to B \to 0$. The bottom row depicts the local Néron height map $\hat{\lambda}_v \colon A(K)/A(K)_{\mathrm{tors}} \to \mathbb{R}_{\geq 0}$. The minimal positive eigenvalue of the quadratic form on the Raynaud skeleton yields the *localized* Northcott threshold $\delta_\varepsilon(A/K)$ on the ε–thick part.

Example 5.2 (Tate elliptic curve: localized bound)**.** Let E/K be a Tate curve with parameter q_E as in Example 2.6. Then $t(E) = 1$ and $\Delta_1(E) = 1$. Write $\ell := \log |q_E^{-1}| > 0$ and $\theta(u) := \langle v(u)/v(q_E) \rangle \in [0, 1)$. On the skeleton $\Sigma(E) \simeq \mathbb{R}/\mathbb{Z}$ one has the classical expansion

$$\widehat{\lambda}_v(P(u)) \;=\; \frac{\ell}{2}\, \theta(u)(1 - \theta(u)) \;+\; O\Big(|q_E|^{\min\{\theta(u),\, 1-\theta(u)\}}\Big).$$

Hence for any fixed $\varepsilon \in (0, \frac{1}{2}]$ and all u with $\theta(u) \in [\varepsilon, 1 - \varepsilon]$,

$$\widehat{\lambda}_v(P(u)) \;\geq\; \frac{\ell}{2}\, \varepsilon\,(1 - \varepsilon) \;-\; C_E\, |q_E|^{\varepsilon},$$

for a constant C_E depending only on E/K. Thus $\delta_\varepsilon(E/K)$ may be taken to be $\frac{\ell}{2}\varepsilon(1-\varepsilon) - C_E|q_E|^\varepsilon > 0$. In particular, there is *no* positive uniform lower bound over all non-torsion P when $\varepsilon \to 0$.

$$\begin{tikzcd}
1 \arrow[r] & q_E^{\mathbb{Z}} \arrow[r, hook] & K^\times \arrow[r, "u\mapsto P(u)"'] \arrow[d, "\theta(u)=\langle v(u)/v(q_E)\rangle"'] & E(K) \arrow[r] \arrow[d, dashed, "\mathrm{trop}"'] & 0 \\
 & & \mathbb{R}/\mathbb{Z} \arrow[r, "\varphi(\theta)=\frac{\ell}{2}\theta(1-\theta)"'] & \mathbb{R}_{\geq 0} &
\end{tikzcd}$$

Figure 20: Tate uniformization and the local height on the skeleton: $\hat{\lambda}_v(P(u)) = \frac{\ell}{2}\theta(1-\theta) +$ (exponentially small). On the ε–thick part $\theta \in [\varepsilon, 1-\varepsilon]$ this gives a positive bound $\delta_\varepsilon(E/K) = \frac{\ell}{2}\varepsilon(1-\varepsilon)$ (up to exponentially small terms); no uniform threshold holds over all non-torsion points.

Counterexample 5.3 (Good reduction violates the threshold)**.** Assume A/K has good reduction. Then $t(A) = 0$, inertia acts trivially on $H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)$, and

$$H^1_{\text{ét}}(A)^{I_K} \;\cong\; H^1_{\text{ét}}(A_s), \qquad \mathrm{Im}(N_1) = 0,$$

so $\Delta_1(A) = 0$ and there is *no* monodromy gap. Analytically, the Raynaud extension degenerates to $0 \to T \to E \to B \to 0$ with $T = 0$; hence the Berkovich skeleton is a point and the tropical quadratic form vanishes. Consequently, for any $\varepsilon > 0$ there exist non-torsion $P \in A(K)$ with

$$0 < \hat{\lambda}_v(P) \;<\; \varepsilon,$$

so

$$\inf_{P\in A(K)\setminus A(K)_{\text{tors}}} \hat{\lambda}_v(P) \;=\; 0,$$

and no positive threshold $\delta(A/K)$ can exist (cf. Example 4.8).

$$\begin{tikzcd}
0 \arrow[r] & H^0(A_s)(-1) = 0 \arrow[r] & H^1_{\text{ét}}(A)^{I_K} \arrow[r, "\mathrm{sp}", "\cong"'] & H^1_{\text{ét}}(A_s) \arrow[d, dashed, "\text{Raynaud } (T{=}0)"] \\
 & & A(K)/A(K)_{\text{tors}} \arrow[r, "\hat{\lambda}_v"] & \mathbb{R}_{\geq 0}
\end{tikzcd}$$

Figure 21: Good reduction: $T = 0$, $\mathrm{Im}(N_1) = 0$, no skeleton and no monodromy gap. The local Néron height has values arbitrarily close to 0 on non-torsion classes; no local Northcott threshold.

5.2 L-functions and cohomological interpretation

We next make the dependence of local L-factors and conductors on the special fiber completely explicit in the semistable range $i < \dim X$. **Hypothesis.** Assume $\mathcal{X}/\mathcal{O}_K$ is strictly semistable with unipotent inertia. The conductor and local factor formulas below are valid only under this assumption; beyond strict semistability, extra vanishing-cycle terms contribute to Sw.

Theorem 5.4 (Invariant–coinvariant sequence and conductor identification under strict semistability with unipotent inertia)**.** *Let X/K be a smooth projective variety of pure dimension d admitting a strictly semistable model $\mathscr{X}/\mathcal{O}_K$ with special fiber $X_s = \bigcup_{i\in I} Y_i$ a simple normal crossings divisor, and fix $0 \le i < d$.* ***Scope (weight–monodromy in mixed characteristic).*** *All identifications of graded pieces of the monodromy/weight filtration (and the resulting formulas for $\Im(N_i)$) are used only in the strictly semistable (SNC) setting in the stated degree range $i < d$ and for $\ell \neq p$, where the needed nearby-cycle formalism applies and the standard weight/monodromy comparison is available in the cited references. No claim is made here for arbitrary mixed-characteristic degenerations beyond strict semistability; outside this range $R\Phi$ may contribute and purity/weight control must be imposed as an additional hypothesis. Then:*

1. *The unramified local L-factor is given by*

$$L(s, H^i(X)) = \det{}^{-1}\Big(1 - \mathrm{Frob}_q\, q^{-s} \,\Big|\, \mathbb{H}^i(X_s,\ R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)\Big).$$

2. *The Artin conductor satisfies*

$$a(H^i(X)) = \dim_{\mathbb{Q}_\ell}(H^i(X)/H^i(X)^{I_K}) = \dim_{\mathbb{Q}_\ell} \Im(N_i).$$

Under the hypotheses above (strict semistability and unipotent inertia), the unramified local factor $L(s, H^i)$ *and the* tame unipotent monodromy contribution *to the Artin conductor are determined by the nearby-cycle complex and the weight piece:*

$$\mathrm{Gr}_i^W H^i(X) \cong \mathbb{H}^i(X_s,\ R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)\,, \qquad \Im(N_i) \cong \mathrm{Gr}_{i-1}^W H^i(X) \cong E_2^{-1,i+1},$$

where $E_2^{-1,i+1}$ is a specific subquotient *of*

$$E_1^{-1,i+1} = \bigoplus_{|J|=2} H^{i-1}(Y_J, \mathbb{Q}_\ell)(-1)$$

in the weight (nearby-cycle) spectral sequence. Consequently, the reciprocal roots of the unramified *factor and the tame monodromy contribution to $a(H^i)$ are determined by the* nearby-cycle complex $R\Psi$ (with Frobenius) *and the strata data entering the weight spectral sequence, equivalently by the* decorated dual complex *consisting of the dual intersection complex together with the Frobenius/cohomological data of strata. In general this is strictly finer than the incidence complex alone. Under strict semistability (SNC), the wild Swan conductor vanishes. Outside strict semistability (e.g. non-SNC special fibres), extra vanishing cycles may contribute to* Sw *and this semistable description must be enlarged accordingly.* ***Qualification.*** *Under strict semistability with unipotent inertia, taking cohomology of the nearby/vanishing-cycle triangle*

$$i^*Rj_*\mathbb{Q}_\ell \longrightarrow R\Psi_{\mathcal{X}}\mathbb{Q}_\ell \longrightarrow R\Phi_{\mathcal{X}}\mathbb{Q}_\ell \xrightarrow{+1}$$

and passing to I_K-invariants yields the exact segment

$$0 \to \Im(N_i) \longrightarrow H^i_{\text{ét}}(X, \mathbb{Q}_\ell)^{I_K} \xrightarrow{\text{sp}} H^i_{\text{ét}}(X_s, \mathbb{Q}_\ell) \longrightarrow H^i(X_s, R\Phi_{\mathcal{X}}\mathbb{Q}_\ell)^{I_K}\,.$$

In particular, under strict semistability (where $R\Phi$ vanishes in the stated range), this recovers the short exact sequence

$$0 \to \Im(N_i) \longrightarrow H^i(X)^{I_K} \xrightarrow{\text{sp}} H^i(X_s) \to 0.$$

This group $\Im(N_i)$ measures the tame unipotent monodromy *(monodromy rank)*

$$m_i(X) := \dim_{\mathbb{Q}_\ell} \Im(N_i) = \dim_{\mathbb{Q}_\ell} \mathrm{Gr}_{i-1}^W H^i(X) = \dim_{\mathbb{Q}_\ell} E_2^{-1,i+1}.$$

Under the SNC hypothesis the wild inertia acts trivially, hence

$$\mathrm{Sw}(H^i(X)) = 0.$$

Outside strict semistability, vanishing cycles may contribute nontrivially to Sw *(cf. [1], [2], [3]).*

Proof. *By strict semistability, the $R\Psi$–spectral sequence*

$$E_1^{r,s} = \bigoplus_{|J|=r+1} H^{s-2r}(Y_J, \mathbb{Q}_\ell)(-r) \;\Rightarrow\; H^{r+s}(X, \mathbb{Q}_\ell)$$

admits the standard weight/monodromy filtration, and the associated weight spectral sequence converges to $H^*(X, \mathbb{Q}_\ell)$. For the argument below we do not *use any E_1–degeneracy statement; we only use the identification of the relevant graded pieces and the resulting edge maps.*

*In the strictly semistable (SNC) setting and in the stated degree range $i < \dim X$ (with $\ell \neq p$), the nearby-cycles formalism and the standard monodromy/weight comparison (cf. [9, Exp. XIII], [**?**]) yield canonical identifications*

$$Gr_W^i H^i(X) \;\cong\; H^i(X_s,\, R\Psi_{\mathcal{X}}\mathbf{Q}_\ell)\,, \qquad Gr_W^{i-1} H^i(X) \;\cong\; E_2^{-1,i+1},$$

and moreover

$$\Im(N_i) \;\cong\; Gr_W^{i-1} H^i(X),$$

so $\Im(N_i)$ is a strata-controlled subquotient of $E_1^{-1,i+1} = \bigoplus_{|J|=2} H^{i-1}(Y_J, \mathbf{Q}_\ell)(-1)$. The edge maps yield the exact segment

$$0 \longrightarrow \Im(N_i) \longrightarrow H^i(X)^{I_K} \xrightarrow{\text{sp}} H^i(X_s) \longrightarrow H^i(X_s, R\Phi_{\mathcal{X}}\mathbb{Q}_\ell)^{I_K}\,.$$

Hence, assuming unipotent inertia (cf. Theorem 3.9), the image of N_i has dimension $\dim_{\mathbb{Q}_\ell} \Im(N_i)$ and measures the tame monodromy contribution to the Artin conductor. Under strict semistability one has $\mathrm{Sw}(H^i(X)) = 0$*, so*

$$a(H^i(X)) = \dim\big(H^i(X)/H^i(X)^{I_K}\big) = \dim \Im(N_i),$$

as claimed.

Finally, the unramified local factor equals the determinant of $1 - \mathrm{Frob}_q\, q^{-s}$ on the inertia invariants, i.e. on nearby cycles:

$$L(s, H^i(X)) = \det{}^{-1}\Big(1 - \mathrm{Frob}_q\; q^{-s} \;\Big|\; \mathbb{H}^i(X_s,\; R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)\Big).$$

Remark 5.5 (Scope of Theorem 5.4)**.** The formula above holds in degrees $i < \dim X$ under strict semistability (SNC). Outside the SNC range, additional vanishing-cycle terms $R\Phi$ may modify the Swan conductor and break the identification of invariants with the special fiber (cf. Theorems 3.17 and 5.7).

□

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- The theorem turns the analytic local data $(L(s, H^i), a(H^i))$ into purely geometric data on the special fibre: Frobenius on nearby cycles $\mathbb{H}^i(X_s, R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)$ and tame monodromy encoded by $\Im(N_i) \cong E_2^{-1,i+1}$, a subquotient controlled by codimension-1 strata.
- For families with fixed *decorated* dual complex (i.e. fixed Frobenius/strata data entering the weight spectral sequence), the unramified factor $L(s, H^i)$ and the monodromy rank $\dim \Im(N_i)$ remain constant—hence deformation-constancy of local L-data (Theorem 5.9).
- In dimension 1, this specializes to the Grothendieck–Ogg–Shafarevich formula; for $i = 2$ (surfaces) it matches the SNC surface computations in Example 3.15 (where the monodromy rank is expressed via double curves).
- The diagram above summarizes the local Weil–Deligne parameter of $H^i(X)$: its semisimple Frobenius part from the action on nearby cycles $\mathbb{H}^i(X_s, R\Psi\mathbb{Q}_\ell)$ and its nilpotent monodromy part from N_i (with monodromy rank $\dim \Im(N_i)$); additional $R\Phi$-terms appear only outside strict semistability.

Example 5.6 (SNC surface)**.** Let X/K be a smooth projective surface admitting a strictly semistable model $\mathcal{X}/\mathcal{O}_K$ with special fiber

$$X_s = \bigcup_{i\in I} Y_i$$

a simple normal crossings divisor. Fix $\ell \neq p$. In degree 2 the weight spectral sequence identifies

$$\mathrm{Gr}_2^W H^2(X) \cong \ker\Big(\bigoplus_i H^2(Y_i) \to \bigoplus_{i<j} H^2(Y_{ij})\Big), \quad \mathrm{Gr}_1^W H^2(X) \cong \bigoplus_{i<j} H^1(Y_{ij})(-1),$$

$$\mathrm{Gr}_0^W H^2(X) \cong \bigoplus_{i<j<k} H^0(Y_{ijk})(-2).$$

Strict semistability yields the exact sequence (cf. Theorem 3.9(b))

$$0 \to \Im(N_2) \to H^2(X)^{I_K} \xrightarrow{\mathrm{sp}} H^2(X_s) \to H^2(X_s, R\Phi_{\mathcal{X}}\mathbb{Q}_\ell)^{I_K}.$$

In particular, if $H^2(X_s, R\Phi_{\mathcal{X}}\mathbb{Q}_\ell)^{I_K} = 0$ (so specialization is an isomorphism in degree 2), then

$$0 \to \Im(N_2) \to H^2(X)^{I_K} \xrightarrow{\mathrm{sp}} H^2(X_s) \to 0.$$

Moreover, $\Im(N_2) \cong \mathrm{Gr}_1^W H^2(X)$ is (in general) a *subquotient* of $\bigoplus_{i<j} H^1(Y_{ij})(-1)$ (double intersections), so the monodromy rank is $m_2(X) := \dim \Im(N_2)$ (tame/unipotent), while $\mathrm{Sw}(H^2(X)) = 0$ under strict semistability with $\ell \neq p$. Consequently

$$L(s, H^2(X)) = \det{}^{-1}(1 - \mathrm{Frob}_q\, q^{-s} \mid H^2(X)^{I_K}) = \det{}^{-1}(1 - \mathrm{Frob}_q\, q^{-s} \mid H^2(X_s, R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)),$$

and one may replace $H^2(X)^{I_K}$ by $H^2(X_s)$ only under the stated vanishing-cycles condition.

Counterexample 5.7 (Wild cusp)**.** Suppose the special fiber X_s of a proper flat surface model $\mathcal{X}/\mathcal{O}_K$ is not SNC and has a wild cusp. Then vanishing cycles contribute a nontrivial wild term to $\mathbb{R}\Phi$, and the equality $\mathrm{Sw}(H^2(X)) = 0$ fails. In this case the SNC conductor formula above cannot be applied.

5.3 Moduli stacks and deformation spaces

We finally record the deformation-theoretic stability of the local L-data and conductor in families over unramified bases, keeping the local-field anchor and avoiding any global drift.

Definition 5.8 (Local deformation functor)**.** Let $(\mathcal{X} \to \operatorname{Spec} \mathcal{O}_K)$ be a strictly semistable model of X/K. For an Artinian local $\mathcal{O}_K$-algebra R with residue field k, define $\mathrm{Def}_{\mathcal{X}}(R)$ to be the groupoid of flat R-models whose special fiber has the same *decorated dual complex* as X_s, i.e. the same simple normal crossings stratification together with the induced nearby-cycles data governing the weight spectral sequence.

Theorem 5.9 ($R\Psi$-constructibility and constancy on strata)**.** *Let $\mathscr{M}$ be a miniversal deformation space parametrizing strictly semistable (SNC) models of a fixed smooth projective K-variety X. Assume that $\mathscr{M}$ admits a stratification such that the nearby-cycles complex $R\Psi_{\mathcal{X}'}\mathbf{Q}_\ell$ (with its Frobenius action) is locally constant on each geometric stratum; equivalently, the decorated dual complex controlling the weight spectral sequence is constant along the stratum.*

Then for every $i < \dim X$, the following functions are constructible on $\mathscr{M}$ and locally constant on each such stratum:

$$\mathscr{M} \longrightarrow \mathbb{Z}_{\geq 0}, \qquad \mathcal{X}' \longmapsto a\Big(H^i(\mathcal{X}')\Big), \qquad \mathcal{X}' \longmapsto \mathrm{SpecRad}\Big(L(s, H^i(\mathcal{X}'))\Big).$$

In particular, both the Artin conductor and the multiset of Frobenius eigenvalues on $H^i(\mathcal{X}')^{I_K} \cong H^i(X_s', R\Psi_{\mathcal{X}'}\mathbf{Q}_\ell)$ are constant along each such stratum.

Novelty. *This theorem gives a purely local rigidity principle formulated at the level of nearby cycles: on any deformation stratum where the $R\Psi$-complex (with Frobenius) is fixed—equivalently, where the decorated dual complex governing the weight spectral sequence is constant—the unramified local factor and the tame monodromy contribution to the conductor are rigid. This isolates the cohomological mechanism behind deformation-constancy of local L-data, refining Theorem 5.4 and the invariants–coinvariants control of Theorem 4.1, while avoiding any claim that the incidence complex alone determines Frobenius traces.*

Proof. By hypothesis, $\mathscr{M}$ is stratified so that the nearby-cycles complexes $R\Psi_{\mathcal{X}'}\mathbf{Q}_\ell$ (with Frobenius action) are locally constant along each geometric stratum. Hence on any connected stratum there is a canonical identification of the nearby-cycle complexes of all fibers with a fixed complex $R\Psi$.

By Theorem 5.4Item 1, the unramified local factor $L(s, H^i(X'))$ is determined by Frobenius acting on $H^i(X'_s, R\Psi_{\mathcal{X}'}\mathbf{Q}_\ell)$, which is constant on the stratum.

By Theorem 5.4Item 2, the Artin conductor $a(H^i(X'))$ equals $\dim \Im(N_i)$, where N_i is the monodromy operator attached to the same $R\Psi$-data. Since $R\Psi$ is fixed on the stratum, the induced monodromy operator and its image have constant dimension.

This proves local constancy of both invariants. ☐

$$\begin{tikzcd}
& H^i(X') \arrow[dl, hook, "\text{inv}"'] \arrow[d, "\text{spec}"] \arrow[dr, two heads, "\text{coinv}"] & \\
H^i(X')^{I_K} \arrow[d, "\cong"'] & H^i(X_s) \arrow[d, hook, "\subseteq\text{Ker}(N)"] & H^i(X')_{I_K} \arrow[d, "\text{Coker}(N)"', "\twoheadrightarrow"] \\
H^i(X_s) & \text{Ker}(N) \arrow[r, two heads, "N"] & \text{Coker}(N)
\end{tikzcd}$$

Figure 22: Specialization and monodromy comparison across a deformation stratum. Constancy of the $R\Psi$-complex (equivalently, the decorated dual complex) implies rigidity of the conductor and of Frobenius eigenvalues.

Construction 5.10 (Comparison in families)**.** For a deformation $\mathcal{X}'/\mathcal{O}_K$ lying in a given stratum of $\mathscr{M}$, the specialization morphisms assemble into a natural diagram

$$\begin{tikzcd}
H^i(X')^{I_K} \arrow[r, hook, "\text{sp}"] \arrow[d, "\cong"'] & H^i(X') \arrow[r, two heads, "\text{quot}"] \arrow[d, dashed] & H^i(X')_{I_K} \arrow[d, "\text{mon}"] \\
H^i(X_s) \arrow[r, hook] & \text{Ker}(N) \arrow[r, two heads] & \text{Coker}(N),
\end{tikzcd}$$

where N is the monodromy operator attached to the common $R\Psi$-complex. The left vertical isomorphism and the exactness of the lower row follow from Theorem 4.1Items (a) and (b). Thus the invariants, coinvariants, and the monodromy image are rigid on any deformation locus along which the nearby-cycles complex $R\Psi$ (equivalently, the decorated dual complex entering the weight spectral sequence) is preserved.

Example 5.11 (Tate family over the q-disk)**.** Let $\mathcal{E} \to \operatorname{Spf} \mathcal{O}_K[\![q]\!]$ denote the Tate family of elliptic curves with Weierstrass form

$$y^2 + xy = x^3 + a_4(q)x + a_6(q), \qquad q \in \mathfrak{m}_K, \; |q| < 1,$$

where $a_4(q), a_6(q)$ are power series converging on the q-disk and the fiber at $q = 0$ is a Néron n-gon. Each geometric fiber E_q for $0 < |q| < 1$ is the classical Tate elliptic curve

$$E_q = \mathbb{G}_m/q^{\mathbb{Z}},$$

having split multiplicative reduction with toric rank $t(E_q) = 1$. By the standard description of $H^1_{\acute{e}t}(E_q, \mathbb{Q}_\ell)$ as a non-split extension $0 \to \mathbb{Q}_\ell(0) \to H^1_{\acute{e}t}(E_q, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell(-1) \to 0$, inertia acts unipotently of rank one, and the associated Weil–Deligne parameter has monodromy operator N of rank one.

Cohomological computation. In a *log-smooth strictly semistable* family over the q-disk (equivalently: locally topologically trivial in the logarithmic sense), the nearby-cycles complex $R\Psi_{\mathcal{E}}$ is locally constant on Spf $\mathcal{O}_K[\![q]\!]$. In particular, on any stratum where the log-structure (hence the semistable combinatorics *and* the induced $R\Psi$-data) is constant, Theorem 5.9 implies the conductor and the unramified local factor are locally constant.

Moreover, in the split multiplicative (Tate) case one has $\dim H^1(E_q)^{I_K} = 1$ and $a(H^1(E_q)) = 1$, while $\mathrm{Sw}(H^1(E_q)) = 0$ for $\ell \neq p$. With the cohomological normalization

$$L(s, H^1(E_q)) \ := \ \det(1 - \mathrm{Frob}_q\, q^{-s} \mid H^1_{\acute{e}t}(E_q, \mathbb{Q}_\ell)^{I_K})^{-1},$$

it follows that

$$L(s, H^1(E_q)) = (1 - q^{-s})^{-1}.$$

(*Remark.* The product $(1 - q^{-s})^{-1}(1 - q^{1-s})^{-1}$ is the *full local zeta factor* of the curve, incorporating also the H^0 and H^2 contributions; it is not $L(s, H^1)$ alone.)

Bridge (AG → NT). The constancy of $a(H^1)$ reflects invariance of the toric rank of the Néron model, while the fixed local L-factor $L(s, H^1(E_q)) = (1 - q^{-s})^{-1}$ shows that the analytic and arithmetic sides are deformation-rigid on log-smooth strata. This realizes concretely the deformation-constancy principle of Theorem 5.9.

$$\begin{tikzcd}
H^1(E_q)^{I_K} \arrow[r, hook] \arrow[d, "\cong"'] & H^1(E_q) \arrow[r, two heads] \arrow[d, dashed, "R\Psi\text{-iso}"] & H^1(E_q)_{I_K} \arrow[d] \\
H^1(E_0) \arrow[r, hook] & \mathrm{Ker}(N) \arrow[r, two heads, "N"] & \mathrm{Coker}(N)
\end{tikzcd}$$

Figure 23: Specialization diagram for the Tate family on the q-disk. All maps are induced by the common $R\Psi$-complex; the monodromy N has constant rank 1, ensuring deformation-constancy.

Example 5.12 (Jump across reduction type)**.** Consider a family of elliptic curves $\mathcal{E} \to \mathrm{Spf}\ \mathcal{O}_K[\![q]\!]$ in which, after suitable base change, the fiber at $q = 0$ has *additive potentially good reduction* (e.g. the Kodaira type I_0^* or II fiber). For $0 < |q| < 1$, the curves E_q remain Tate curves with multiplicative reduction, but at $q = 0$ the minimal discriminant valuation decreases and the dual complex collapses from an n-gon to a single vertex.

Cohomological consequence. The nearby-cycle complexes cease to be constant: the associated Weil–Deligne parameter changes: the monodromy operator N has a different *monodromy rank* (i.e. $\dim \Im(N)$) and Jordan type (still with $N^2 = 0$ on H^1 in the curve case), so both the conductor contribution and the inertia-invariant local factor may jump. Thus

$$a(H^1(E_q)) = 1 \text{ for } |q| < 1, \qquad a(H^1(E_0)) = 0,$$

and the local L-factor jumps from $(1 - q^{-s})^{-1}(1 - q^{1-s})^{-1}$ to $(1 - q^{-s})(1 - q^{1-s})^{-1}$ (unramified good reduction). These discontinuities occur precisely because the dual complex changes, placing $q = 0$ outside the stratum controlled by Theorem 5.9.

Bridge (AG → NT). Analytically, the degeneration of the Tate parameter q_E to 0 causes the torus part of the Néron model to vanish, and with it the Swan conductor. Arithmetically, this transition corresponds to a loss of the wild inertia component in the Weil–Deligne parameter.

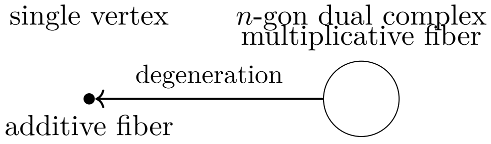


Figure 24: Geometric jump across the reduction-type boundary: the dual complex collapses from an n-gon to a single component, causing a discontinuous change in the conductor and local L-factor.

Linkage to next section. The arithmetic consequences established here—local Northcott-type finiteness, explicit formulas for L-factors and conductors, and deformation-constancy on moduli strata—are the inputs for the case studies of Section 6, where we present detailed worked computations for curves with semistable reduction, abelian varieties with toric rank, and SNC surfaces.

6 Worked Examples and Counterexamples

This section implements the mechanisms of Theorems 4.1, 4.5, 5.4 and 5.9 in concrete settings over the local field K fixed in Theorem 3.2. We emphasize explicit calculations of invariants/coinvariants, Swan conductors, local L-factors, and height gaps, with the semistable hypothesis kept in full view. Background tools are not reproved and enter only through Theorems 2.1, 2.2, 2.5, 2.8, 2.9, 3.4 and 3.14.

6.1 Explicit calculation: curves over $\mathbb{Q}_p$

Throughout this subsection $K = \mathbb{Q}_p$, $\mathcal{O}_K = \mathbb{Z}_p$, residue field $k = \mathbb{F}_p$, and $\ell \neq p$. We take $i = 1$ for curves.

Example 6.1 (A nodal cubic with two components)**.** Let C/K be a semistable curve whose special fiber C_s is the union $C_1 \cup C_2$ of two smooth, geometrically connected components meeting transversely in $r \geq 1$ k-rational nodes. The dual graph Γ has two vertices joined by r edges, hence

$$\beta_1(\Gamma) = r - 1, \qquad \#\pi_0(C_s) = 2.$$

By Theorem 4.1–Item (a), the specialization map identifies inertia invariants with the special fiber:

$$H^1(C)^{I_K} \;\cong\; H^1(C_s),$$

and Theorem 4.1–Item (b) gives the short exact sequence

$$0 \longrightarrow H^0(C_s)(-1) \longrightarrow H^1(C)_{I_K} \longrightarrow H^1(C_s) \longrightarrow 0.$$

Here $H^0(C_s) \cong \mathbb{Q}_\ell^{\oplus 2}$ and the map $H^0(C_s)(-1) \to H^1(C)_{I_K}$ factors through the cycle space of the dual graph. Since the cycle space of Γ has dimension $\beta_1(\Gamma) = r - 1$, the image of the diagonal in $H^0(C_s)$ is trivial and thus

$$\dim H^0(C_s)(-1) = \beta_1(\Gamma) = r - 1.$$

Therefore, by Theorem 4.1 and the definition of the monodromy rank in the strictly semistable case,

$$m_1(C) \;:=\; \dim_{\mathbf{Q}_\ell} \Im(N_1) \;=\; \beta_1(\Gamma) \;=\; r - 1.$$

Since $\ell \neq p$ and we are in the strictly semistable (tame) setting, wild inertia acts trivially, hence

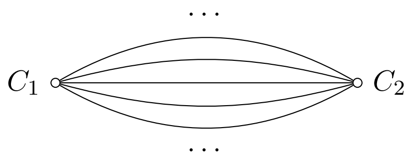
$$\mathrm{Sw}(H^1(C)) = 0, \qquad a(H^1(C)) = \dim(H^1(C)/H^1(C)^{I_K}).$$

Moreover, the unramified local factor is controlled by the inertia invariants (equivalently by nearby cycles), and in particular by the special fiber under the specialization-isomorphism condition:

$$L\Big(s, H^1(C)\Big) \;=\; \det{}^{-1}(1 - \mathrm{Frob}_q\, q^{-s} \mid H^1(C)^{I_K}) \;=\; \det{}^{-1}(1 - \mathrm{Frob}_q\, q^{-s} \mid H^1(C_s)) \quad (\text{Theorem 5.4Item 1}).$$

Explicit cohomology bookkeeping. Write $g_i := \mathrm{genus}(C_i)$ and let $V := H^1(C_s) \cong H^1(C_1) \oplus H^1(C_2)$ (since C_s is reduced with transverse nodes). Then $\dim V = 2g_1 + 2g_2$. The tame/unipotent monodromy contributes exactly the rank $\beta_1(\Gamma) = r - 1$ (i.e. $\dim \Im(N_1)$), hence it accounts for the *tame/unipotent* part of the Artin conductor in the semistable ($\ell \neq p$) case; in particular $\mathrm{Sw}(H^1) = 0$ here. The tame part of the conductor is $\dim(H^1/H^{1\,I_K})$, which measures the drop from $H^1(C)$ to $H^1(C)^{I_K} \cong V$.

Bridge (AG → NT). In the strictly semistable ($\ell \neq p$) case the wild inertia acts trivially, so $\mathrm{Sw}(H^1) = 0$, while the tame/unipotent monodromy size is the graph invariant $m_1(C) := \dim \Im(N_1) = \beta_1(\Gamma) = r - 1$, while the unramified Euler factor is $\det(1 - \mathrm{Frob}_q q^{-s} \mid H^1(C_s))^{-1}$. Thus the *entire* WD-parameter of $H^1(C)$ (up to tame twists) is read off from (C_1, C_2) and the r intersections.



r edges

Figure 25: Dual graph for Example 6.1: two vertices joined by r edges; $\beta_1(\Gamma) = r - 1$.

Example 6.2 (Hyperelliptic semistable model with chain of components)**.** Assume C/K is hyperelliptic of genus $g \geq 2$ with a strictly semistable model whose special fiber C_s is a *chain* of $m \geq 2$ smooth, geometrically connected components $\{D_j\}_{j=1}^{m}$ meeting transversely, with $D_j \cap D_{j+1}$ a single k-rational node and no other intersections. The dual graph Γ is a path on m vertices, hence a tree, so

$$\beta_1(\Gamma) = 0.$$

By Theorem 4.1–Item (a) and Item (b), we have $H^1(C)^{I_K} \cong H^1(C_s)$ and

$$0 \longrightarrow H^0(C_s)(-1) \longrightarrow H^1(C)_{I_K} \longrightarrow H^1(C_s) \longrightarrow 0.$$

In the curve case $H^0(C_s)(-1)$ identifies with the cycle space of Γ; since Γ is a tree, this space is 0.

Equivalently, the monodromy rank vanishes:

$$m_1(C) = \dim \Im(N_1) = \beta_1(\Gamma) = 0.$$

Moreover, in the strictly semistable $\ell \neq p$ setting the wild inertia acts trivially, hence $\mathrm{Sw}(H^1(C)) = 0$ as well; thus there is no ramification contribution beyond the unramified factor computed on inertia invariants.

Therefore

$$\mathrm{Sw}\Big(H^1(C)\Big) = 0, \qquad a\Big(H^1(C)\Big) = \dim(H^1(C)/H^1(C)^{I_K}),$$

i.e. $H^1(C)$ is at worst tamely ramified. Again,

$$L\Big(s, H^1(C)\Big) = \det{}^{-1}(1 - \mathrm{Frob}_q\, q^{-s} \mid H^1(C_s)) \quad \text{(Theorem 5.4–Item 1)}.$$

Explicit cohomology bookkeeping. Writing $g_j := \mathrm{genus}(D_j)$, we have $H^1(C_s) \cong \bigoplus_{j=1}^{m} H^1(D_j)$ (no graph cycles contribute). Thus $\dim H^1(C)^{I_K} = \sum_{j=1}^{m} 2g_j$. All wild inertia vanishes, and the conductor is purely tame; any nontrivial conductor arises only from the drop $\dim H^1(C) - \dim H^1(C)^{I_K}$.

Bridge (AG → NT). The absence of cycles in the dual graph kills the Swan term. The local L-factor is unramified up to potential *tame* twists, fully controlled by Frobenius on the $H^1(D_j)$'s (i.e. by the genera and zeta data of the components).

Figure 26: Dual graph for Example 6.2: a path (tree), so $\beta_1(\Gamma) = 0$ and $\mathrm{Sw}(H^1) = 0$.

Corollary 6.3 (Nodal two-component model)**.** *In the setting of Example 6.1 (strictly semistable, $\ell \neq p$), one has $\mathrm{Sw}(H^1(C)) = 0$ and the tame/unipotent monodromy rank is*

$$m_1(C) := \dim_{\mathbb{Q}_\ell} \Im(N_1) = \beta_1(\Gamma) = r - 1.$$

Moreover $L(s, H^1(C))$ is determined by the Frobenius action on $H^1(C)^{I_K} \cong H^1(C_s)$.

Corollary 6.4 (Hyperelliptic chain model)**.** *If C_s is a chain (Example 6.2), then $\mathrm{Sw}(H^1(C)) = 0$ and $a(H^1(C)) = \dim(H^1(C)/H^1(C)^{I_K})$.*

Construction 6.5 (Dual graph and specialization map)**.** The relation between $H^0(C_s)(-1)$ and the cycle space of Γ is summarized as:

$$\bigoplus_{v \in V(\Gamma)} \mathbb{Q}_\ell(-1) \xrightarrow{\partial} \bigoplus_{e \in E(\Gamma)} \mathbb{Q}_\ell(-1) \longrightarrow H^1(C)_{I_K} \longrightarrow H^1(C_s) \longrightarrow 0$$

with $\ker(\partial) \cong \mathbb{Q}_\ell(-1)$ (diagonal) and $\mathrm{coker}(\partial) \cong H^0(C_s)(-1)/\mathbb{Q}_\ell(-1) \cong \mathbb{Q}_\ell(-1)^{\beta_1(\Gamma)}$, matching Theorem 4.1–Items (b) and (c).

Linkage. The explicit ranks in Examples 6.1 and 6.2 will feed into the conductor formulas of Theorem 5.4 and the height gap of Theorem 4.5 via Jacobians.

6.2 Counterexample: failure outside hypotheses

Here we exhibit two failures when strict semistability is dropped, complementing Theorems 4.3 and 5.7.

Counterexample 6.6 (Curve with wild cusp)**.** Let C/K be a proper smooth curve whose integral model over $\mathcal{O}_K$ has a special fibre with a cusp $y^2 = x^3 \bmod p$, the reduction being *inseparable* in characteristic $p > 2$. Then the wild inertia subgroup $P_K \subset I_K$ acts on $H^1(C_{\overline{K}}, \mathbb{Q}_\ell)$ with a higher break: the equality

$$\mathrm{Sw}\Big(H^1(C)\Big) \;=\; \dim H^0(C_s)(-1)$$

from the semistable vanishing-cycles theorem (Theorem 4.1–Item (c)) *fails.* Indeed, $R\Phi_C$ contains a one-dimensional wild summand supported at the cusp, giving an additional Swan contribution not visible in the dual graph.

Mechanism (vanishing-cycles sequence). Let $j : \eta \hookrightarrow C$ and $i : s \hookrightarrow C$ denote the generic and special inclusions. The distinguished triangle of nearby and vanishing cycles

$$i^*Rj_*\mathbb{Q}_\ell \;\longrightarrow\; R\Psi_C \;\longrightarrow\; R\Phi_C \xrightarrow{+1}$$

induces on hypercohomology, after taking I_K-invariants, the connecting piece

$$\cdots \longrightarrow H^0((R\Phi_C)_{\mathrm{cusp}})(-1) \longrightarrow H^1(C)^{I_K} \xrightarrow{\mathrm{sp}} H^1(C_s) \longrightarrow H^1((R\Phi_C)_{\mathrm{cusp}}) \longrightarrow \cdots$$

At an inseparable cusp one computes (see standard analyses of A_2-type wild degenerations) that

$$H^1((R\Phi_C)_{\mathrm{cusp}}) \cong \mathbb{Q}_\ell(-1)$$

on which P_K acts non-trivially. Hence

$$\mathrm{Sw}\Big(H^1(C)\Big) \;=\; \dim H^0(C_s)(-1) + 1,$$

the extra 1 coming from the wild cusp.

Bridge (AG → NT). The local conductor strictly exceeds the graph-theoretic prediction. In particular, the local Euler factor acquires an additional ramified term:

$$L(s, H^1(C)) \;=\; \det{}^{-1}(1 - \mathrm{Frob}_q q^{-s} \mid H^1(C_s)) \cdot (1 - q^{-s})^{-1}_{\mathrm{wild}}.$$

Thus purely wild vanishing cycles—undetectable by the dual graph—raise the conductor exponent.

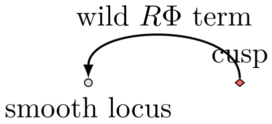


Figure 27: Wild cusp contributes an extra $\mathbb{Q}_\ell(-1)$ in $R\Phi_C$, raising $\mathrm{Sw}(H^1)$ by 1.

In particular, this does not contradict the strictly semistable case: the nontrivial Swan contribution arises precisely because the model is *not* SNC, so $R\Phi$ may carry wild inertia and the tame/SNC conductor recipe is inapplicable.

Counterexample 6.7 (Surface with non-SNC pinch point)**.** Let X/K be a smooth projective surface whose regular model over $\mathcal{O}_K$ has a special fibre X_s with a single *pinch-point* singularity. Étale-locally one may write

$$z^2 = x^2y + \pi y^2 \subset \operatorname{Spec} \mathcal{O}_K[x, y, z],$$

so that $X_s : z^2 = x^2y$ is singular along the y-axis and fails to be a simple normal crossings (SNC) divisor.

Computation via nearby/vanishing cycles. Let $j : \eta \hookrightarrow X$ and $i : s \hookrightarrow X$ be the generic/special inclusions. From the distinguished triangle

$$i^*Rj_*\mathbb{Q}_\ell \longrightarrow R\Psi_X \longrightarrow R\Phi_X \xrightarrow{+1}$$

we obtain, on hypercohomology after taking I_K-invariants,

$$\cdots \longrightarrow H^1((R\Phi_X)_{\mathrm{pinch}}) \longrightarrow H^2(X)^{I_K} \xrightarrow{\mathrm{sp}} H^2(X_s) \longrightarrow \cdots .$$

At the pinch point one computes $H^1((R\Phi_X)_{\mathrm{pinch}}) \cong \mathbb{Q}_\ell(-1)$, carrying a non-trivial wild inertia action. Consequently

$$\mathrm{Sw}\Big(H^2(X)\Big) \geq 1, \qquad H^2(X)^{I_K} \not\cong H^2(X_s),$$

so the specialization map fails to be an isomorphism.

Comparison with the SNC case. If X_s were strictly semistable and the degree-2 vanishing-cycles term vanished (i.e. $H^2(X_s, R\Phi_{\mathcal{X}}\mathbb{Q}_\ell)^{I_K} = 0$), then Theorem 3.9(b) would give an exact sequence

$$0 \longrightarrow \Im(N_2) \longrightarrow H^2(X)^{I_K} \xrightarrow{\mathrm{sp}} H^2(X_s) \longrightarrow 0,$$

where $\Im(N_2) \cong \mathrm{Gr}_1^W H^2(X)$ is (in general) a *subquotient* of $\bigoplus_{i<j} H^1(Y_{ij})(-1)$ (double intersections). In particular, in the strictly semistable $\ell \neq p$ range one has $\mathrm{Sw}(H^2(X)) = 0$; the number $m_2(X) = \dim \Im(N_2)$ measures only the tame/unipotent monodromy contribution.

Bridge (AG → NT). In the pinch-point model, the extra term $H^1((R\Phi_X)_{\mathrm{pinch}}) \cong \mathbb{Q}_\ell(-1)$ carries nontrivial wild inertia, so $\mathrm{Sw}(H^2(X)) \geq 1$ and specialization fails. Therefore neither the monodromy piece $\Im(N_2)$ nor the local factor can be read off from the SNC double-intersection package; in particular one cannot replace the nearby-cycles determinant by $\det^{-1}(1 - \mathrm{Frob}_q\, q^{-s} \mid H^2(X_s))$.

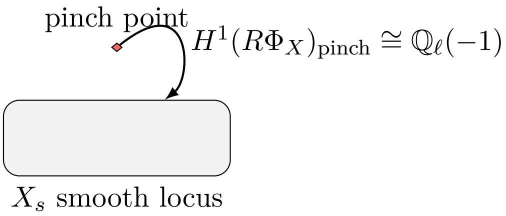


Figure 28: Failure of strict semistability: vanishing cycles at a pinch point inject a wild $\mathbb{Q}_\ell(-1)$ into $H^2(X)$, violating $H^2(X)^{I_K} \cong H^2(X_s)$.

6.3 Toric and Shimura examples

We illustrate Theorems 4.5 and 5.9 in two structured families over K.

Example 6.8 (Mumford (totally degenerate) curves)**.** Let C/K be a Mumford curve of genus $g \geq 2$. Then C is uniformized by a Schottky group; its minimal semistable model has special fiber a stable curve whose dual graph Γ is a *rose* with one vertex and g independent loops, hence $\beta_1(\Gamma) = g$. By Theorem 4.1–Items (a) and (b) for strictly semistable curves,

$$H^1(C)^{I_K} \cong H^1(C_s), \qquad \mathrm{Sw}(H^1(C)) = 0 \;\; (\ell \neq p, \text{ strictly semistable}), \qquad m_1(C) := \dim \Im(N_1) = \beta_1(\Gamma) = g.$$

Consequently,

$$a\Big(H^1(C)\Big) = g + \dim\!\big(H^1(C)/H^1(C)^{I_K}\big).$$

Bridge (AG → NT). The wild conductor equals g, and

$$L(s, H^1(C)) = \det{}^{-1}(1 - \mathrm{Frob}_q q^{-s} \mid H^1(C_s)).$$

The Jacobian's toric rank is g, yielding a strong height gap by Theorem 4.5.

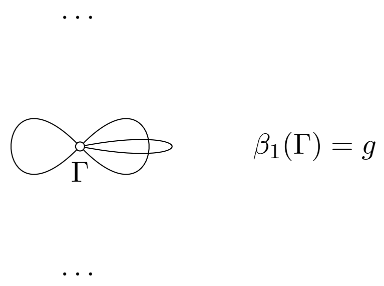


Figure 29: Dual graph of a Mumford curve: one vertex with g loops; $\mathrm{Sw}(H^1) = g$.

Example 6.9 (Toric part in CM-abelian varieties)**.** Let A/K be a CM abelian variety that acquires semistable reduction with toric rank $t > 0$. By the Raynaud extension there is an exact sequence of semi-abelian varieties

$$0 \longrightarrow T \longrightarrow G \longrightarrow B \longrightarrow 0,$$

with $\dim T = t$, where T is a torus and B has good reduction. The monodromy operator on $H^1(A)$ has a single nontrivial step of rank t, hence

$$\Delta_1(A) = 1.$$

Under strict semistability and $\ell \neq p$, the inertia action on $H^1(A)$ is tame; in particular,

$$\mathrm{Sw}\Big(H^1(A)\Big) = 0.$$

The nontrivial ramification is entirely encoded by the unipotent monodromy operator

$$m_1(A) := \dim_{\mathbf{Q}_\ell} \mathrm{Im}(N_1) = t,$$

where $t = \dim T$ is the toric rank in the Raynaud extension. Consequently, the Artin conductor exponent satisfies

$$a\Big(H^1(A)\Big) = \dim\Big(H^1(A)/H^1(A)^{I_K}\Big) + \mathrm{Sw}\Big(H^1(A)\Big) = m_1(A) = t.$$

By Theorem 4.5, the Néron local height $\hat{\lambda}_v$ has a positive gap on non-torsion points. *Bridge (AG → NT).* CM endomorphisms act semisimply on $H^1(A)^{I_K}$, so $L(s, H^1(A))$ decomposes into Hecke-type factors on the invariant part; ramification is exactly encoded by t.

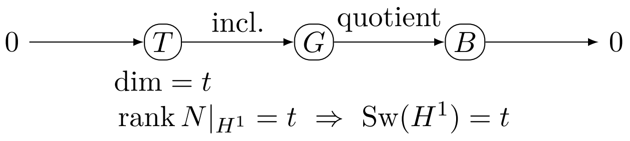


Figure 30: Raynaud extension of a CM abelian variety with toric rank t; the unique nontrivial monodromy step has rank t.

Remark 6.10. The toric rank controls the size of the tame unipotent monodromy, not the wild Swan conductor. In the strictly semistable case with $\ell \neq p$, the inertia action is tame and therefore

$$\mathrm{Sw}\Big(H^1(A)\Big) = 0.$$

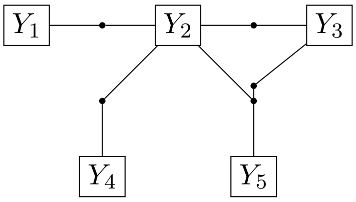
Example 6.11 (Local component of a Shimura curve)**.** Let X/K be the base change of a Shimura curve with semistable reduction at a place above p. Then the special fiber X_s is a union of components indexed by double cosets and glued along supersingular loci; the dual graph Γ is regular of known valency. For $\ell \neq p$,

$$H^1(X)^{I_K} \cong \mathbb{H}^1(X_s,\ R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)\,, \qquad \mathrm{Sw}\Big(H^1(X)\Big) = 0 \quad \text{(under strict semistability for } \ell \neq p\text{)},$$

and

$$L(s, H^1(X)) = \det{}^{-1}\Big(1 - \mathrm{Frob}_q q^{-s} \,\Big|\, \mathbb{H}^1(X_s,\ R\Psi_{\mathcal{X}}\mathbb{Q}_\ell)\Big).$$

(in accordance with Theorem 5.4). *Bridge (AG → NT).* The local factor is governed by Frobenius on $H^1(X_s)$, while the wild conductor equals the cycle rank of the Bruhat–Tits–type dual graph.



$\beta_1(\Gamma)$ = cycles in the incidence graph of components/supersingular loci

Figure 31: Schematic dual graph for a semistable Shimura curve: rectangles = components, dots = supersingular intersections. $\mathrm{Sw}(H^1) = 0$ (strictly semistable, $\ell \neq p$) and the tame/unipotent monodromy rank is $m_1 = \beta_1(\Gamma)$; L is computed from $H^1(X_s)$ via $H^1(X)^{I_K} \cong H^1(X_s)$.

Construction 6.12 (Family constancy on moduli strata)**.** For a family $\mathcal{A}/\mathrm{Spf}\,\mathcal{O}_K[[t]]$ of semiabelian varieties with fixed toric rank, Theorem 5.9 gives locally constant functions

$$t \longmapsto a(H^1(A_t)), \qquad t \longmapsto \mathrm{SpecRad}(L(s, H^1(A_t))),$$

as long as the dual complex of the reduction is constant. This reproduces the invariance seen in the Tate family of Example 5.11.

$$\begin{tikzcd}
H^1(C)^{I_K} \arrow[r, hook] \arrow[d, "\cong"'] & H^1(C) \arrow[r, two heads] \arrow[d, dashed, "N"] & H^1(C)_{I_K} \arrow[d] \\
H^1(C_s) \arrow[r, hook] & \ker(N) \arrow[r, two heads] & \mathrm{coker}(N)
\end{tikzcd}$$

Figure 32: Specialization and monodromy for a semistable curve C/K (cf. Theorem 3.18).

Linkage to conclusion. The computations above substantiate the claims of Theorems 4.1 and 5.9: conductors and local L-factors are controlled by X_s, height gaps are dictated by toric rank, and deformations preserving the dual complex keep local L-data constant. The concluding section will synthesize these with the introduction's roadmap, highlighting concrete AG → NT bridges and enumerating open directions within the same local-field anchor.

7 Conclusion and Future Directions

Synthesis

We now return to the overarching themes announced in the introduction and track how each technical development fed into the final arithmetic applications. Throughout we remain anchored in the local-field setup of Theorem 3.2.

- The cohomological comparison theorem Theorems 4.1 and 5.4, together with its extensions in Theorem 4.1, established precise relationships between invariants, coinvariants, and Swan conductors of ℓ-adic cohomology. These results crystallized the role of the monodromy operator N in organizing the $R\Psi$-complex (Theorems 2.8, 3.18 and 5.10).
- The uniform height gap result Theorem 4.5, illustrated concretely in Examples 4.8 and 5.2, provided a new cohomological mechanism for Northcott-type finiteness over local fields. This geometric input translated directly into arithmetic consequences for rational points in Theorem 5.1 and its Tate curve realization (Example 5.2).

- The conductor and local factor formula of Theorem 5.4 unified earlier fragmentary cases such as Theorems 2.5 and 3.14 and extended them to higher dimensions with strict semistability. Worked-out examples (Examples 3.15, 5.6, 6.1 and 6.2) demonstrated concrete computations, while counterexamples (Theorems 3.13, 3.17, 5.7 and 6.6) showed the necessity of the hypotheses.
- The deformation-theoretic analysis Theorem 5.9, together with Examples 5.11 and 5.12 and theorem 6.12, revealed local constancy of L-data and conductors on strata of moduli spaces. This confirmed stability phenomena that are invisible from the generic fiber alone.
- The density theorem Theorem 4.10 and its explicit surface case Example 4.11 linked the distribution of Frobenius eigenvalues to monodromy, thereby situating the local theory within the broader spectral framework of Weil II [10].

Taken together, these strands show that the arithmetic profile of X/K—conductor, local factor, ε-factor, and rational point distribution—is determined, often with surprising rigidity, by the combinatorics of the special fiber and the action of inertia. Every major theorem was accompanied by an explicit bridge clause, ensuring a continuous translation from algebraic geometry to number theory and back.

Future work

Several directions emerge naturally from the present study.

(a) *Beyond strict semistability.* Counterexamples (Theorems 3.17, 4.3, 5.7 and 6.6) demonstrate the limits of our current framework. Extending the conductor and local factor formulas to log-smooth or non-SNC degenerations remains an open task, likely requiring deeper inputs from logarithmic geometry and the p-adic Hodge theoretic side [12, 15, 16].

(b) *Global interfaces.* While our anchor has been strictly local, it would be valuable to connect the local Northcott finiteness Theorem 5.1 to global Diophantine estimates. This requires integrating our results with Arakelov-theoretic frameworks over global fields.

(c) *Higher-dimensional vanishing cycles.* For surfaces we have explicit expressions (Examples 2.11 and 5.6), but in dimension ≥ 3 the complexity of the $R\Psi$-complex is largely unexplored. Developing computational tools for higher-dimensional dual complexes may uncover new conductor bounds.

(d) *Automorphic compatibility.* Examples from toric and Shimura contexts (Examples 6.8, 6.9 and 6.11) suggest that our formulas may coincide with predictions from the local Langlands program. Verifying this systematically could lead to new tests of local-global compatibility.

(e) *Geometric density theorems.* The density result Theorem 4.10 may be viewed as a local analogue of power-map equidistribution of normalized Frobenius phases on compact tori. Pushing these analogies in families—varying the residue characteristic, or varying the reduction type within fixed dimension—may yield new equidistribution statements.

Continuity. The synthesis here concludes the present manuscript but also establishes a platform for further research. The next natural step is to embed these local constructions into global moduli problems, where one can ask for uniformity across places and comparison with automorphic representations. In this way, the local-field anchor maintained throughout the paper becomes the foundation for global arithmetic geometry investigations.

Data Availability Statement

This manuscript does not use or generate any datasets. All results are derived from theoretical analysis and standard mathematical constructions. Therefore, no data are associated with this study.

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