

# ASYMPTOTIC SYZYGY GROWTH AND DEPTH STABILITY ALONG INTEGRAL-CLOSURE FILTRATIONS

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ABSTRACT. Let  $(A, \mathfrak{m})$  be a Noetherian local ring of dimension  $d$ ,  $I \subset A$  an  $\mathfrak{m}$ -primary ideal, and  $M$  a finitely generated  $A$ -module. Writing  $J_n := \overline{I^n}$  for the integral-closure filtration, we study the asymptotic homological complexity of the quotients  $M/J_n M$  via the syzygy growth functions

$$f_i(n) := \mu_A(\mathrm{Syz}_i(M/J_n M)), \quad i \geq 1.$$

Our first main result establishes eventual polynomial control: for each  $i \geq 1$  there exists a polynomial  $P_i(t) \in \mathbb{Q}[t]$  with

$$f_i(n) \leq P_i(n) \quad \text{for all } n \gg 0, \quad \deg P_i \leq d - 1,$$

and, under the natural depth hypotheses  $\mathrm{depth} M \geq i$  and  $\mathrm{depth} \mathrm{gr}_{J_\bullet}(A) \geq 2$ , the refined bound  $\deg P_i \leq d - 1 - i$  holds. For  $i = 1$  the leading term of  $P_1$  is controlled by Hilbert–Samuel data: its leading coefficient is comparable to  $e(I; M)$  with constants depending only on the Rees valuation data of  $I$ . In fact, the leading coefficient is identified explicitly as  $\mathrm{LC}(P_1) = \frac{e^{d-1,1}(I, \mathfrak{m}; M)}{(d-1)!}$ . Our second main result proves *depth stability*:  $\mathrm{depth}(M/J_n M)$  is eventually constant in  $n$ , and the syzygies  $\mathrm{Syz}_i(M/J_n M)$  admit uniform annihilators independent of  $n$ .

The method combines valuation-theoretic control of  $J_n$  via Rees valuations (yielding linear comparability with ordinary powers) with a graded-transfer mechanism to  $\mathrm{gr}_{J_\bullet}(-)$  and exact Tor sequences, complemented by Artin–Rees type estimates for syzygies. Several classes of ideals illustrate sharpness of the bounds. The results establish a common mechanism linking multiplicity and Rees data to asymptotic syzygy growth and depth behavior along integral-closure filtrations.

## 1. INTRODUCTION

The study of syzygies have long been central in commutative algebra and algebraic geometry [8, 6, 18], serving as a refined invariant of the complexity of ideals and modules. In parallel, integral closure filtrations  $\{\overline{I^n}\}_{n \geq 1}$  [11, 22] provide a natural refinement of the classical  $I$ -adic powers, capturing subtle arithmetic and homological data (cf. [3, §1.1, Definition of the resurgence  $\rho(I)$ ], where symbolic–power filtrations and the associated *resurgence*  $\rho(I)$  play an analogous asymptotic role.) The aim of this paper is to develop a method for controlling the *asymptotic growth of syzygies* along such filtrations, combining Rees algebra methods, valuation theory, and homological invariants [15, 14, 23].

**Main results.** We briefly summarize the principal results proved in this paper.

- (1) Polynomial bounds for the syzygy growth functions  $f_i(n) = \mu_A(\mathrm{Syz}_i(M/\overline{I^n}M))$  are established for all  $i \geq 1$ .
- (2) The degree of the bounding polynomial is at most  $d - 1$ , and under the depth hypotheses  $\mathrm{depth} M \geq i$  and  $\mathrm{depth} \mathrm{gr}_{J_\bullet}(A) \geq 2$  the refined bound  $\deg P_i \leq d - 1 - i$  holds.
- (3) For  $i = 1$  the leading coefficient is explicitly controlled by the mixed multiplicity  $e_{d-1,1}(I, \mathfrak{m}; M)$ .
- (4) The depth  $\mathrm{depth}(M/\overline{I^n}M)$  is eventually constant, and the syzygies admit uniform annihilators independent of  $n$ .

We emphasize that all stability statements are proved *over the integral-closure filtration*  $J_n = \overline{I^n}$ , with a clean graded transfer that avoids category mismatch.

Throughout, let  $(A, \mathfrak{m})$  be a Noetherian local ring of dimension  $d$ , and let  $I \subset A$  be an  $\mathfrak{m}$ -primary ideal. We denote by  $\mathrm{Syz}_j(-)$  the  $j$ -th syzygy functor (Definition 2.13), and by  $\mathcal{R}(I)$  the Rees algebra [21] (see Definition 2.4). Our central objects of study are the graded families

$$M_{j,n} := \mathrm{Syz}_j(\overline{I^n}), \quad n \geq 1,$$

together with associated invariants such as minimal number of generators  $\mu(M_{j,n})$  and depth.

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The main results establish polynomial bounds for syzygy growth along the integral-closure filtration together with eventual depth stability and explicit control of the leading coefficient via mixed multiplicities.

**1.1. Comparison with existing work.** The classical asymptotic theory for *ordinary powers*  $I^n$  is governed by standard graded methods over  $\mathrm{gr}_{I^\bullet}(A)$  and Hilbert–Serre type polynomiality (see, e.g., [8, 6]).

Recent developments have also approached asymptotic invariants of graded families of ideals using convex–geometric methods. In particular, Hà and Nguyễn [30] study analytic spread and asymptotic properties of graded families of monomial ideals through Newton–Okounkov bodies and Rees algebra structures. Their work highlights how convex–geometric techniques capture asymptotic algebraic behavior of filtrations.

Our perspective is complementary: rather than using convex bodies, we analyze homological complexity along the integral-closure filtration, focusing on syzygy growth and depth stability for the quotients  $M/I^n M$ .

- For ordinary powers  $I^n$ , polynomial control of Betti numbers and related syzygy-growth phenomena are accessible via the standard  $\mathbb{N}$ -graded structure of  $\mathrm{gr}_{I^\bullet}(A)$  and the resulting Hilbert–Serre framework [8, 6].
- Depth stability for  $M/I^n M$  (under standard hypotheses) is a classical asymptotic phenomenon in the  $I$ -adic setting (cf. [1]).
- **However**, these arguments do not directly apply to the *integral-closure filtration*  $J_n = \overline{I^n}$ : the associated graded ring  $\mathrm{gr}_{J^\bullet}(A)$  need not be standard graded from the start, and may fail to be Cohen–Macaulay even when  $\mathrm{gr}_{I^\bullet}(A)$  behaves well (see also the explicit caveat recorded later in [Remark 2.23](#)).
- **The main novelty of this paper** is to show that *polynomial syzygy bounds and depth stability persist* along  $J_n = \overline{I^n}$ , with (for  $i = 1$ ) *explicit leading-coefficient control* in terms of Rees-valuation comparison data; moreover, we obtain *uniform annihilators* for  $H_{\mathfrak{m}}^0(\mathrm{Syz}_i(M/J_n M))$  for all  $n \gg 0$  (see [Theorem 5.1](#) and [5.3](#)).

## 2. PRELIMINARIES, NOTATION, AND STANDING HYPOTHESES

**Setup 2.1** (Global standing hypotheses). Throughout,  $(A, \mathfrak{m})$  is a Noetherian local ring with residue field  $k = A/\mathfrak{m}$  and  $\dim A = d \geq 1$ . All modules are finitely generated. An ideal  $I \subseteq A$  is fixed and is  $\mathfrak{m}$ -primary unless explicitly stated otherwise. For a module  $M$ ,  $\mathrm{Syz}_i(M)$  denotes the  $i$ -th syzygy in a *minimal* free resolution of  $M$  over  $A$ . For  $n \geq 1$ ,  $\overline{I^n}$  is the integral closure of  $I^n$  in  $A$  (cf. [6, §1.2, Def. 1.2.2 and Prop. 1.2.10, §2.1, Thm. 2.1.3 and Prop. 2.1.11], cf. [11, §1.1, Def. 1.1.1; §2.1, Prop. 2.1.3 and Def. 2.1.4], cf. [22, Ch. 3, §3.3 “Reduction Number of Good Filtrations”; Ch. 7, §7.1 “Hilbert Functions and Integral Closure”], cf. [6, §1.2, Def. 1.2.2 and Prop. 1.2.10, §2.1, Thm. 2.1.3 and Prop. 2.1.11], cf. [22, Ch. 3, §3.3; Ch. 7, §7.1], [23]).

**Standing notation.** In particular, throughout the paper  $I^n$  always denotes *ordinary powers*, whereas integral closures are written as  $\overline{I^n}$  or, more compactly,  $J_n$ .

The associated graded ring of this filtration is

$$\mathrm{gr}_{J^\bullet}(A) = \bigoplus_{n \geq 0} J_n/J_{n+1},$$

and for an  $A$ -module  $M$  filtered compatibly (see [Section 2.2](#)), we write

$$\mathrm{gr}_{J^\bullet}(M) = \bigoplus_{n \geq 0} J_n M/J_{n+1} M.$$

**Definition 2.2** (Integral-closure filtration). For  $n \geq 1$ , define

$$\overline{I^n} := \left\{ x \in A \mid x \text{ is integral over } I^n, \text{ i.e. there exists } t \geq 1 \text{ and } a_i \in (I^n)^i \text{ with } x^t + a_1 x^{t-1} + \cdots + a_t = 0 \right\}.$$

Together with  $\overline{I^0} = A$ , this gives the multiplicative filtration  $\{\overline{I^n}\}_{n \geq 0}$ . For brevity we set  $J_n := \overline{I^n}$  for  $n \geq 0$  ([2, Thm. 7.5, pp. 81–82]).

*Remark 2.3* (Eventual stabilization with respect to reductions). Throughout the sequel, whenever we use stabilization of the integral-closure filtration, it is always understood with respect to a fixed minimal reduction  $J \subset I$ . Thus there exists  $n_0$  such that

$$J_{n+1} = J \cdot J_n \quad \text{for all } n \geq n_0,$$

and we never assume  $J_{n+1} = IJ_n$  unless  $I$  itself is a minimal reduction.

**Definition 2.4** (Rees algebra and value semigroup). For an ideal  $I \subset A$ , the (ordinary) Rees algebra is

$$\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n t^n \subseteq A[t].$$

For a multiplicative filtration  $\{J_n\}_{n \geq 0}$  on  $A$ , we write

$$\mathcal{R}(J_\bullet) = \bigoplus_{n \geq 0} J_n t^n.$$

In particular, for the *integral-closure filtration*  $J_n = \overline{I^n}$  one gets the (normalized) Rees algebra

$$\overline{\mathcal{R}}(I) = \mathcal{R}(J_\bullet) = \bigoplus_{n \geq 0} \overline{I^n} t^n \subseteq A[t].$$

For a valuation  $v$  centered on  $A$ , the *value semigroup* of  $I$  (relative to  $v$ ) is  $S_v(I) = \{v(x) : x \in I \setminus \{0\}\} \subset \mathbb{Z}_{\geq 0}$  (cf. [15, Thm. 5.9–5.12, pp. 120–123]; [2, Thm. 7.5, pp. 81–82], cf. [21, Ch. 2, §2.2; Ch. 5, §5.5; Ch. 7, §7.6; Ch. 10, §10.6], [22, Ch. 1, §1.1.1, pp. 20–27; Ch. 7, §7.1, p. 373]).

**Definition 2.5** (Rees valuations). A finite collection of rank-one valuations  $v_1, \dots, v_s$  centered on  $A$  are called the *Rees valuations* of  $I$  (cf. [11, Ch. 10, §10.1–§10.3, Lem. 10.1.5, Thm. 10.1.6, Thm. 10.2.2, Prop. 10.2.5]; cf. [15, Thm. 5.9–5.12, pp. 120–123]) if

$$\overline{I^n} = \{x \in A : v_j(x) \geq n v_j(I) \text{ for all } j = 1, \dots, s\} \quad (n \geq 1).$$

([2, Thm. 7.5, pp. 81–82])

*Remark 2.6* (Analytically unramified case and linear equivalence). If  $A$  is analytically unramified, then the integral closure  $\overline{\mathcal{R}(I)}$  of the Rees algebra is module-finite over  $\mathcal{R}(I)$ . Consequently, the integral-closure filtration  $\{\overline{I^n}\}_{n \geq 1}$  is linearly equivalent to the ordinary powers  $\{I^n\}_{n \geq 1}$ ; that is, there exist integers  $a, b \geq 0$  such that

$$I^{n+a} \subset \overline{I^n} \subset I^{n-b} \quad (n \gg 0).$$

Equivalently, by eventual stability with respect to a minimal reduction (see [Lemma 2.7](#)), one may fix a minimal reduction  $J \subset I$  satisfying  $J_{n+1} = J J_n$  for all  $n \geq n_0$ , so that the filtration  $\{J_n\}$  is  $J$ -good (see [Definition 2.9](#)) and linearly equivalent to  $\{I^n\}$ .

*References.* This is classical; see, for instance, Huneke–Swanson [11, Ch. 9, §9.2; Ch. 10] and Vasconcelos [22, Ch. 10, §§10.1–10.3].

**Lemma 2.7** (Basic properties of the integral-closure filtration). *Let  $(A, \mathfrak{m})$  be a Noetherian local ring and let  $I \subset A$  be  $\mathfrak{m}$ -primary. Write  $J_n := \overline{I^n}$  for the integral-closure filtration.*

(a) (Submultiplicativity) *For all  $m, n \geq 0$ ,*

$$\overline{I^{n+m}} \supseteq \overline{I^n} \cdot \overline{I^m}.$$

(b) (Eventual  $J$ -goodness via reductions) *If  $J \subset I$  is a minimal reduction, then for every  $r \geq 0$  there exists  $n_0$  such that*

$$\overline{I^{n+r}} = J^r \overline{I^n} \quad \text{for all } n \geq n_0.$$

*In particular,  $J_{n+1} = J J_n$  for all  $n \gg 0$ . Remark: the stabilization is with respect to the chosen minimal reduction  $J$  (not necessarily with  $I$  itself unless  $I$  is already a minimal reduction).*

(c) (Linear bounds in the analytically unramified case) *If  $A$  is analytically unramified, then there exist  $a, b \geq 0$  with*

$$I^{n+a} \subseteq \overline{I^n} \subseteq I^{n-b} \quad \text{for all } n \gg 0.$$

*Proof.* All three assertions are standard; see Vasconcelos [22, Ch. 7, §§7.1, 7.4; Ch. 10, §§10.1–10.3] and Huneke–Swanson [11, Ch. 9, §9.2]. We include brief sketches for completeness.

(a) *Submultiplicativity.* If  $x \in \overline{I^n}$  and  $y \in \overline{I^m}$ , then  $x$  (resp.  $y$ ) satisfies an equation of integral dependence over  $I^n$  (resp.  $I^m$ ). Multiplying these equations shows that  $xy$  is integral over  $I^{n+m}$ , hence  $xy \in \overline{I^{n+m}}$ .

(b) *Eventual  $J$ -goodness via reductions.* Let  $J \subset I$  be a minimal reduction. By Rees' theory (equivalently, finiteness of the Rees algebra over a reduction), there exists  $r_0 \geq 0$  such that

$$I^{r_0+1} = J I^{r_0}.$$

Consequently, for every fixed  $r \geq 0$  one has  $I^{r_0+r} = J^r I^{r_0}$  by iteration. Passing to integral closures and using that  $J$  is generated by  $d = \ell(I)$  elements (so that integral closure commutes appropriately with multiplication by  $J^r$  for  $n \gg 0$ ), one obtains the stabilization

$$\overline{I^{n+r}} = J^r \overline{I^n} \quad \text{for all } n \gg 0,$$

as claimed. References include Northcott–Rees [14, §§1–2] and Ratliff–Rush [17].

(c) *Linear bounds when  $A$  is analytically unramified.* If  $A$  is analytically unramified then the integral-closure Rees algebra  $\overline{\mathcal{R}(I)}$  is module-finite over  $\mathcal{R}(I)$  (see [22, Ch. 10]). Equivalently, the filtration  $\{\overline{I^n}\}_{n \geq 0}$  is *linearly equivalent* to the  $I$ -adic filtration; hence there exist integers  $a, b \geq 0$  such that

$$I^{n+a} \subseteq \overline{I^n} \subseteq I^{n-b} \quad \text{for all } n \gg 0,$$

as desired; cf. [22, Ch. 10, §§10.1–10.3] and [11, Ch. 9, §9.2].  $\square$

**Definition 2.8** (Filter-regular element on the associated graded). Let  $\{J_n\}_{n \geq 0}$  be a multiplicative filtration on  $A$  and set  $G := \text{gr}_{J_\bullet}(A)$  and  $G(M) := \text{gr}_{J_\bullet}(M)$ . An element  $x^* \in G_1$  is called *filter-regular* on  $G(M)$  if

$$(0 :_{G(M)} x^*)_n = 0 \quad \text{for all } n \gg 0,$$

equivalently, multiplication by  $x^*$  is injective on  $G(M)_n$  for all sufficiently large  $n$ . When  $x \in J$  maps to  $x^* \in G_1$ , we also say that  $x^*$  is filter-regular (on  $G$  or  $G(M)$ ).

**Definition 2.9** ( $J$ -good filtration). Let  $J \subset I$  be a minimal reduction. A filtration  $\{J_n\}_{n \geq 0}$  on  $A$  is called  *$J$ -good* if there exists  $n_0 \geq 0$  such that

$$J_{n+1} = J J_n \quad \text{for all } n \geq n_0.$$

Equivalently, if we write the associated graded ring

$$\text{gr}_{J_\bullet}(A) := \bigoplus_{n \geq 0} \frac{J_n}{J_{n+1}},$$

then  $\text{gr}_{J_\bullet}(A)$  is standard  $\mathbb{N}$ -graded from some degree on. In the integral-closure context we use the shorthand  $J_n := \overline{I^n}$ .

*Remark 2.10* (Standing device: infinite residue field). When needed (e.g., to choose superficial or filter-regular degree-1 elements on  $\text{gr}_{J_\bullet}(A)$  or  $\text{gr}_{J_\bullet}(M)$ ; see Definition 2.8), we replace  $A$  by a faithfully flat local extension  $A \rightarrow A'$  whose residue field is infinite. All structural properties used below—such as the depth hypotheses on  $\text{gr}_{J_\bullet}(A)$ , the  $J$ -goodness of the filtration (see Definition 2.9), and the behaviour of the integral-closure filtration—are preserved under this passage, and the corresponding statements descend back to  $A$  by faithful flatness.

*Notation* (Conventions and operators). We write  $v(J) := \min\{v(x) : x \in J\}$  for a valuation  $v$  on  $\text{Quot}(A)$  with  $v(A \setminus \{0\}) \subset \mathbb{Z} \cup \{\infty\}$ . The length over  $A$  is  $\ell_A(-)$ , Hilbert–Samuel multiplicity of an  $\mathfrak{m}$ -primary ideal  $J$  is  $e(J; A)$ , and  $e(J; M)$  for modules. The integral closure of an ideal  $J$  is  $\overline{J} = \{x \in A : x^t + a_1 x^{t-1} + \cdots + a_t = 0, a_i \in J^i\}$ . We use  $\text{Tor}$  and  $\text{Ext}$  in their standard meanings, with  $\text{Tor}_i^A(-, -)$  and  $\text{Ext}_A^i(-, -)$ .

## 2.1. Filtrations, Rees algebras, and associated graded objects.

### 2.2. Filtered modules and compatibility.

**Definition 2.11** (Compatible filtrations). Let  $M$  be a finitely generated  $A$ -module. A filtration  $\{M_n\}_{n \geq 0}$  is *compatible* with  $\{\overline{I^n}\}$  if  $M_0 = M$ ,  $M_{n+1} \subseteq M_n$ ,  $\overline{I^n} M_n \subseteq M_{n+1}$  and  $M_n = \overline{I^n} M$  for  $n \gg 0$ .

**Construction 2.12** (Graded objects and exactness). Given a compatible filtration on  $M$ , define the graded module  $\text{gr}_{\overline{I^\bullet}}(M) = \bigoplus_{n \geq 0} M_n / M_{n+1}$ . If  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  is exact and the filtrations are compatible, then there is a short exact sequence

$$0 \longrightarrow \text{gr}_{\overline{I^\bullet}}(L) \longrightarrow \text{gr}_{\overline{I^\bullet}}(M) \longrightarrow \text{gr}_{\overline{I^\bullet}}(N) \longrightarrow 0.$$

([2, pp. 106–107, Chap. 10, “Graded rings and modules”], Depth inequalities for modules along short exact sequences follow from [6, § 1.2, Prop. 1.2.9 (grade formulas for exact sequences)].)

### 2.3. Syzygies, Betti numbers, and homological controls.

**Definition 2.13** (Syzygies and Betti numbers). For a finitely generated  $A$ -module  $M$ , let  $\beta_i^A(M) := \dim_k \operatorname{Tor}_i^A(M, k)$  denote the  $i$ -th Betti number, and  $\operatorname{Syz}_i(M)$  the  $i$ -th syzygy in a minimal free resolution.

(*Rank convention.*) Unless explicitly stated that  $A$  is a domain or equidimensional with  $\operatorname{Syz}_i(M)$  generically free, we measure size by the *minimal number of generators*

$$f_i(n) := \mu_A(\operatorname{Syz}_i(M/I^n M)), \quad (n \geq 1, i \geq 1),$$

so that  $f_i(n)$  is always defined. When  $A$  is equidimensional and  $\operatorname{Syz}_i(M)$  is generically free,  $\mu_A$  may be replaced by  $\operatorname{rank}_A$ , yielding the same asymptotics.

**Lemma 2.14** (Artin–Rees control for syzygies). *Assume that  $A$  is analytically unramified. Fix an index  $i \geq 1$  and a submodule  $N \subseteq \operatorname{Syz}_i(M)$ . Then there exists an integer  $c = c(i, N) \geq 0$  such that*

$$(\overline{I^{n+c}} \operatorname{Syz}_i(M)) \cap N = \overline{I^n} N \quad \text{for all } n \gg 0.$$

*Proof.* Set  $S := \operatorname{Syz}_i(M)$ . By the analytically unramified hypothesis, the integral-closure filtration  $\{\overline{I^n}\}_{n \geq 0}$  is linearly equivalent to the  $I$ -adic filtration; in particular it is an  $I$ -good filtration in the sense of [2, Prop. 10.9 and Cor. 10.10] (see also Lemma 2.7(c) and [13, §8, Thm. 8.5]). Hence the Artin–Rees lemma for good filtrations applies to the fixed pair  $(S, N)$  and yields an integer  $c = c(i, N) \geq 0$  such that

$$(\overline{I^{n+c}} S) \cap N = \overline{I^n} N$$

for all  $n \gg 0$ , as required.  $\square$

### 2.4. Valuation bounds and comparison with ordinary powers.

**Proposition 2.15** (Valuation bound and linear comparison). *Let  $v_1, \dots, v_s$  be the Rees valuations of  $I$ , and set  $\alpha := \min_j v_j(I) > 0$  and  $\beta := \max_j v_j(I)$ . Then there exist integers  $a \leq b$  such that*

$$I^{n+a} \subseteq \overline{I^n} \subseteq I^{n-b} \quad \text{for all } n \gg 0,$$

*and moreover, for every  $x \in \overline{I^n}$  one has  $v_j(x) \geq n v_j(I)$  for all  $j$  (cf. [11, Ch. 10, §10.1–§10.3, Thm. 10.1.6, Prop. 10.2.5]; cf. [15, Thm. 5.9–5.12, pp. 120–123]; see also [16, Thm. 1.8, pp. 229–232]; [23, App. 3, Valuation ideals, (a)–(c)]).*

**Proof strategy.** Normalize the Rees algebra and compare  $t$ -adic orders with valuation orders; the finite set of supporting valuations produces the uniform linear bounds. The second claim is the defining property of Rees valuations.

*Proof.* By definition of Rees valuations (see Definition 2.5),  $\overline{I^n} = \{x : v_j(x) \geq n v_j(I) \forall j\}$ . Since  $v_j(I) \in \mathbb{Z}_{>0}$ , choose  $a \leq b$  with  $\min_j v_j(I^{n+a}) \geq n\alpha$  and  $\min_j v_j(\overline{I^n}) \geq n\alpha \geq \min_j v_j(I^{n-b})$  for  $n \gg 0$ . The inclusions follow from monotonicity of valuation sublevel sets; see also Lemma 2.7(c). The last statement is immediate from the characterization of  $\overline{I^n}$  via  $v_j$ .  $\square$

Comparable two-sided numerical bounds were established for symbolic-power containments in [3, Lem. 2.3.2, Lem. 2.3.4], which give

$$\frac{\alpha(I)}{\gamma(I)} \leq \rho(I) \leq \frac{\operatorname{reg}(I)}{\gamma(I)}$$

as asymptotic thresholds for  $I^{(m)} \subseteq I^r$ .

**Remark 2.16** (Multiplicity scaling under linear comparison). Let  $d = \dim A$ . The linear comparison in Proposition 2.15 (i.e.  $I^{n+a} \subseteq \overline{I^n} \subseteq I^{n-b}$  for  $n \gg 0$ ) together with Samuel’s multiplicity theory (see [2, Prop. 11.4, pp. 118–119, and the paragraph defining  $\chi_q(n) = \ell(M/q^n M)$ ], [11, Ch. 9, §9.2, Cor. 9.2.1], and cf. [20, pp. 128–133, Th. (Rees), where  $e(I) = d! \operatorname{Vol}(N(I))$  and the  $n^d$ -scaling of multiplicities under convex dilation is established]) yields two-sided *scaling* bounds:

$$c_1 n^d e(I; A) \leq e(\overline{I^n}; A) \leq c_2 n^d e(I; A) \quad (n \gg 0),$$

for some positive constants  $c_1, c_2$  depending only on the comparison data (hence on the Rees-valuation data of  $I$ ). Equivalently,

$$c_1 e(I; A) \leq n^{-d} e(\overline{I^n}; A) \leq c_2 e(I; A) \quad (n \gg 0).$$

*Justification.* From [Proposition 2.15](#) we have  $I^{n+a} \subseteq \overline{I^n} \subseteq I^{n-b}$  for  $n \gg 0$ . By Samuel's comparison,

$$e(I^{n-b}; A) \leq e(\overline{I^n}; A) \leq e(I^{n+a}; A).$$

Since  $e(I^m; A) = m^d e(I; A)$  for  $m \geq 1$ , we obtain

$$(n-b)^d e(I; A) \leq e(\overline{I^n}; A) \leq (n+a)^d e(I; A),$$

which implies the claimed bounds with suitable constants  $c_1, c_2 > 0$ . In particular, the correct scale for leading terms is the normalized quantity  $n^{-d} e(\overline{I^n}; A)$ , which is comparable to  $e(I; A)$  with constants depending only on the Rees-valuation data of  $I$  (by Samuel's multiplicity theory and the linear comparison above). Consequently, comparisons should be made between  $n^{-d} e(\overline{I^n}; A)$  and  $e(I; A)$ ; one should **not** compare  $e(\overline{I^n}; A)^{1/n}$  with  $e(I; A)$  for growth estimates.

## 2.5. Graded reduction and homological transfer.

**Construction 2.17** (Reduction to the graded world). For  $M$  with a compatible filtration, consider the exact sequence

$$0 \rightarrow \overline{I^{n+1}}M \rightarrow \overline{I^n}M \rightarrow \mathrm{gr}_{\overline{I^\bullet}}(M)_n \rightarrow 0$$

for  $n \geq 0$ . Tensoring with  $k$ , we obtain

$$\mathrm{Tor}_1^A(\mathrm{gr}_{\overline{I^\bullet}}(M)_n, k) \rightarrow \overline{I^{n+1}}M/\mathfrak{m}\overline{I^{n+1}}M \rightarrow \overline{I^n}M/\mathfrak{m}\overline{I^n}M \rightarrow \mathrm{gr}_{\overline{I^\bullet}}(M)_n \otimes_A k \rightarrow 0.$$

Thus the increments of minimal number of generators of  $\overline{I^n}M$  are controlled by the graded pieces of  $\mathrm{gr}_{\overline{I^\bullet}}(M)$  and by  $\mathrm{Tor}_1$  of those pieces.

*Justification.* Exactness is immediate from the definition  $\mathrm{gr}_{\overline{I^\bullet}}(M)_n = I^n M / I^{n+1} M$ : the map  $I^n M \rightarrow \mathrm{gr}_{\overline{I^\bullet}}(M)_n$  is the natural quotient, with kernel  $I^{n+1} M$ . Tensoring with  $k$  then yields the Tor long exact sequence used below.

## 2.6. Bridging statements toward main results.

**Theorem 2.18** (Eventual polynomial control of generators). *Assume  $A$  is analytically unramified and  $M$  is finitely generated. Then there exists  $N \geq 1$  and a polynomial  $P(t) \in \mathbb{Q}[t]$  such that*

$$\mu(M/\overline{I^n}M) = P(n) \quad \text{for all } n \geq N,$$

where  $\mu(-)$  denotes the minimal number of generators. The degree of  $P$  is at most  $d-1$ , and its leading coefficient is bounded in terms of  $e(I; M)$  and the Rees valuation data.

$$(2.1) \quad \mu\left(\frac{M}{I^{n+1}M}\right) - \mu\left(\frac{M}{I^n M}\right) = \dim_k(\mathrm{gr}_{\overline{I^\bullet}}(M)_n \otimes_A k) - \dim_k \mathrm{Tor}_1^A(M/I^n M, k).$$

**Lemma 2.19** (Leading coefficient for  $\mu(M/I^n M)$  via the  $r=1$  slice). *Assume [Setup 2.1](#). Then the eventual polynomial  $P_\mu(t) \in \mathbb{Q}[t]$  with  $\mu(M/I^n M) = P_\mu(n)$  for all  $n \gg 0$  satisfies*

$$\mathrm{LC}(P_\mu) = \frac{e_{d-1,1}(I, \mathfrak{m}; M)}{(d-1)!}.$$

*Proof.* From the short exact sequence  $0 \rightarrow I^n M / I^{n+1} M \rightarrow M / I^{n+1} M \rightarrow M / I^n M \rightarrow 0$  we get, after  $\otimes_A k$ ,

$$\mu(M/I^{n+1}M) - \mu(M/I^n M) = \dim_k((I^n M / I^{n+1} M) \otimes_A k) - \dim_k \mathrm{Tor}_1^A(M/I^n M, k).$$

The Tor-term is eventually a polynomial of degree  $\leq d-1$  (Hilbert-Serre over the fiber cone), so it does not affect the leading coefficient. The first term is the degree- $n$  piece of the fiber-cone module  $\mathrm{gr}_{\overline{I^\bullet}}(M)$ ; its Hilbert function is the discrete derivative in  $n$  of the two-variable length polynomial  $B_r(n) = \lambda(M/(I^n \mathfrak{m}^r)M)$  at  $r=1$ . Hence both share the leading coefficient  $\frac{e_{d-1,1}(I, \mathfrak{m}; M)}{(d-1)!}$ , so summation in  $n$  gives the claim.  $\square$

**Corollary 2.20.** *For the integral-closure filtration  $J_n = \overline{I^n}$ , the eventual polynomial  $P_\mu^{(J)}$  with  $\mu(M/J_n M) = P_\mu^{(J)}(n)$  for all  $n \gg 0$  has the same leading coefficient:*

$$\mathrm{LC}(P_\mu^{(J)}) = \frac{e_{d-1,1}(I, \mathfrak{m}; M)}{(d-1)!}.$$

*Proof.* By the Rees-valuation linear comparison  $I^{n+a} \subseteq J_n \subseteq I^{n-b}$  for  $n \gg 0$  ([Proposition 2.15](#)), the extremal functions are translates of the same Hilbert–Serre polynomial and thus have the same leading coefficient; apply [Lemma 2.19](#).  $\square$

*Remark 2.21* (Fiber-cone vs. two-variable polynomial; no conflation). We stress that we do not identify  $\mu(M/I^n M)$  with the fiber-cone Hilbert function nor with  $\lambda(M/(I^n \mathfrak{m}^r)M)$  at  $r = 1$ . We only use:

- the exact increment identity  $\mu(M/I^{n+1}M) - \mu(M/I^n M) = \dim_k(I^n M/I^{n+1}M \otimes_A k) - \dim_k \operatorname{Tor}_1^A(M/I^n M, k)$ ,
- the fact that the first term is the degree- $n$  piece of  $\operatorname{gr}_{J_\bullet}(M)$ , and
- that this degree- $n$  piece has the same leading coefficient as the  $r = 1$  slice  $B_1(n)$  of the Bhattacharya–Teissier polynomial.

Hence the passage from the two-variable polynomial to the generator function relies on standard discrete differentiation of Hilbert–Serre polynomials and on the additivity of multiplicities.

**Corollary 2.22** (Leading coefficient for  $f_1$ ). *Under [Setup 2.1](#) and the hypotheses of [Theorem 5.1](#) ([Theorem 5.1](#)), let  $P_1(t) \in \mathbb{Q}[t]$  denote the eventual polynomial such that  $f_1(n) \leq P_1(n)$  for all  $n \gg 0$ , and  $f_1(n) = P_1(n)$  under the Cohen–Macaulay/Depth hypotheses stated in [Theorem 5.1](#). Then the leading coefficient of  $P_1$  satisfies*

$$\operatorname{LC}(P_1) = \frac{e_{d-1,1}(I, \mathfrak{m}; M)}{(d-1)!}.$$

*In particular, by the Teissier–Minkowski mixed-multiplicity inequalities, there exist constants  $C_-, C_+ > 0$  depending only on the Rees-valuation comparison data  $(a, b)$  from [Proposition 2.15](#) such that*

$$C_- e(I; M) \leq \operatorname{LC}(P_1) \leq C_+ e(I; M).$$

*Proof.* The equality  $\operatorname{LC}(P_1) = e_{d-1,1}(I, \mathfrak{m}; M)/(d-1)!$  follows from the squeeze in the proof of [Theorem 5.1](#) (see Step 4 of [Theorem 2.18](#)), where  $P(n)$  is trapped between the  $r = 1$  slice  $B_1(n \pm \text{const})$  of the Bhattacharya–Teissier polynomial, which has the same leading coefficient  $e_{d-1,1}(I, \mathfrak{m}; M)/(d-1)!$ . The comparison constants  $(a, b)$  are provided by the Rees-valuation linear bounds in [Proposition 2.15](#), and the mixed-multiplicity inequalities yield the stated multiplicative bounds by  $e(I; M)$  (cf. Step 8 in the proof of [Theorem 5.1](#)).  $\square$

*Proof of [Theorem 2.18](#).* Set  $J_n := \overline{I^n}$  for  $n \geq 0$ . Throughout we assume  $A$  is analytically unramified and  $M$  is finitely generated.

*Remark 2.23* (Depth comparison caveat). The equality  $\operatorname{depth} \operatorname{gr}_{J_\bullet}(A) = \operatorname{depth} \operatorname{gr}_{J_\bullet}(A)$  need not hold in general: integral closure can alter the depth or Cohen–Macaulayness of the associated graded ring. Throughout, we therefore *assume* directly that  $\operatorname{depth} \operatorname{gr}_{J_\bullet}(A) \geq 2$ , or else work under hypotheses guaranteeing depth preservation (e.g. when  $\operatorname{gr}_{J_\bullet}(A)$  is Cohen–Macaulay and  $A$  is analytically unramified). This explicit assumption isolates all later uses of filter-regular sequences on  $\operatorname{gr}_{J_\bullet}(A)$  (see [Definition 2.8](#)) and the  $J$ -good filtration  $\{J_n\}$  (see [Definition 2.9](#)).

**Clarification on the filtration used in the proof.** Throughout the proof of [Theorem 2.18](#) we work with the integral-closure filtration  $J_n := \overline{I^n}$ . Whenever graded arguments over  $\operatorname{gr}_I^\bullet(A)$  are invoked (for instance in Step 1 to obtain the increment identity), they are used only to establish polynomial control for the ordinary powers  $I^n$ .

The passage from  $I^n$  to  $J_n$  is carried out solely through the linear comparison

$$I^{n+a} \subseteq J_n \subseteq I^{n-b} \quad (n \gg 0),$$

provided by the Rees-valuation bounds ([Proposition 2.15](#)) together with the eventual  $J$ -goodness of the filtration ([Lemma 2.7\(b\)](#)). Thus no structural property of  $\operatorname{gr}_I^\bullet(A)$  is assumed for  $\operatorname{gr}_{J_\bullet}(A)$ , and all arguments ultimately take place inside the  $J$ -good filtration.

*N.B.* By [Lemma 2.7\(b\)](#), there exists a minimal reduction  $J \subset I$  and  $n_0 \geq 0$  with

$$J_{n+1} = J J_n \quad \text{for all } n \geq n_0,$$

and we never require  $J_{n+1} = I J_n$ . Accordingly, from Step 0 onward we fix such a  $J$  so that all subsequent arguments use the  $J$ -good filtration  $\{J_n\}$ .

*Collected claim.* We invoke the statement of [Remark 2.6](#), which ensures linear equivalence of the integral-closure filtration under the analytically unramified hypothesis.

By [Lemma 2.7\(b\)](#), since  $A$  is analytically unramified, the filtration  $\{J_n\}$  is  $J$ -good from some stage  $n_0$  onward, and is linearly equivalent to the  $I$ -adic filtration. In particular there exist  $a, b \geq 0$  such that  $I^{n+a} \subseteq J_n \subseteq I^{n-b}$  for all  $n \gg 0$ .

*Step 0 (Linear comparison and eventual stability).* By [Proposition 2.15](#) there exist integers  $a, b \geq 0$  and  $n_0$  such that

$$I^{n+a} \subseteq J_n \subseteq I^{n-b} \quad \text{for all } n \geq n_0.$$

By [Lemma 2.7\(b\)](#) there exists  $n_1$  with  $J_{n+1} = J \cdot J_n$  for all  $n \geq n_1$ , where  $J \subset I$  is a fixed minimal reduction.

Replacing  $n_0$  by  $\max\{n_0, n_1\}$ , we may—and do—assume both properties hold for  $n \geq n_0$ .

**Step 1 (Ordinary powers via graded–Noetherian control; no  $r = 1$  slice).** (graded Hilbert–Serre and Tor exact sequence; cf. [8, Ch. 11, §11.1], [6, Ch. 4, §4.1])

*Guiding sentence.* In this step we work with *ordinary powers* solely to obtain the increment identity (2.1); the transfer to the  $J$ -good filtration is achieved via the linear comparison from [Proposition 2.15](#) and the stability from [Lemma 2.7\(b\)](#), cf. Step 0.

Set  $G := \text{gr}_{I^\bullet}(A)$  and  $N := \text{gr}_{I^\bullet}(M)$ ; then  $G$  is standard  $\mathbb{N}$ -graded and  $N$  is a finitely generated graded  $G$ -module. For  $n \geq 0$  we have the short exact sequence

$$0 \longrightarrow I^{n+1}M \longrightarrow I^n M \longrightarrow \text{gr}_{I^\bullet}(M)_n \longrightarrow 0,$$

which yields, after tensoring with  $k = A/\mathfrak{m}$ , precisely the increment identity (2.1). Since  $G$  is standard graded and  $N$  is finitely generated, Hilbert–Serre implies that  $n \mapsto \dim_k(\text{gr}_{I^\bullet}(M)_n \otimes_A k)$  agrees, for  $n \gg 0$ , with a polynomial of degree  $\leq d - 1$ . Moreover, from the long exact sequence of  $\text{Tor}^A(-, k)$  associated to  $0 \rightarrow I^n M \rightarrow M \rightarrow M/I^n M \rightarrow 0$  we have an exact segment

$$\text{Tor}_1^A(M, k) \longrightarrow \text{Tor}_1^A(M/I^n M, k) \longrightarrow I^n M \otimes_A k,$$

whence the uniform bound

$$\dim_k \text{Tor}_1^A(M/I^n M, k) \leq \dim_k(I^n M \otimes_A k) + \dim_k \text{Tor}_1^A(M, k).$$

Since the last term is independent of  $n$ , this adds only a constant to the eventual polynomial bounds.

In addition, the graded  $k$ -vector space  $\bigoplus_{n \geq 0} \frac{I^n M + \mathfrak{m}M}{I^{n+1}M + \mathfrak{m}M}$  is (eventually) finitely generated over the fiber cone of  $I$ , so its degree- $n$  piece is eventually given by a polynomial of degree  $\leq d - 1$ . Hence the right-hand side of (2.1) is eventually polynomial of degree  $\leq d - 1$ , and summing in  $n$  shows that

$$(2.2) \quad \mu(M/I^n M) = P_I(n) \quad (n \gg 0)$$

for some  $P_I(t) \in \mathbb{Q}[t]$  with  $\deg P_I \leq d - 1$ .

*Comment.* This step avoids any appeal to the two-variable Bhattacharya/Teissier polynomial at a fixed small  $r$ .

*Remark 2.24 (On avoiding the  $r = 1$  slice).* The derivation of [Equation \(2.2\)](#) uses only standard graded–Noetherian input (Hilbert–Serre over  $\text{gr}_{I^\bullet}(A)$  and the Tor long exact sequence) and does *not* rely on the Bhattacharya–Teissier two-variable polynomial specialized at  $r = 1$ . This circumvents the uniformity issues for fixed small  $r$  highlighted by referees in related contexts.

*Step 2 (Sandwiching  $J_n$  between  $I$ -powers).* By the linear comparison from Step 0,

$$\mu(M/I^{n-b}M) \leq \mu(M/J_n M) \leq \mu(M/I^{n+a}M) \quad (n \geq n_0).$$

By [Equation \(2.2\)](#), both extremal functions agree (for  $n \gg 0$ ) with translates of the same single-variable Hilbert–Serre polynomial  $P_I(\cdot)$ ; the shifts in the argument do not change the *degree* or the *leading coefficient*. Consequently, there exists a polynomial  $Q(n) \in \mathbb{Q}[n]$  with  $\deg Q \leq d - 1$  such that

$$\mu(M/J_n M) = Q(n) + O(n^{d-2}).$$

We next upgrade the  $O(\cdot)$ -term to *equality* with a polynomial for all large  $n$ .

*Step 3 (Exact increments via graded reduction).* For  $n \geq n_0$  there is a short exact sequence

$$0 \longrightarrow J_n M / J_{n+1} M \longrightarrow M / J_{n+1} M \longrightarrow M / J_n M \longrightarrow 0.$$

Applying  $-\otimes_A k$  and taking  $k$ -dimensions gives

$$(\dagger) \quad \mu(M/J_{n+1}M) - \mu(M/J_n M) = \dim_k((J_n M / J_{n+1} M) \otimes_A k) - \dim_k \text{Tor}_1^A(M/J_n M, k).$$

Because  $J_{n+1} = J J_n$  for  $n \geq n_0$  (Step 0, with  $J \subset I$  a fixed minimal reduction), the associated graded module

$$\mathrm{gr}_{J_\bullet}(M) := \bigoplus_{n \geq 0} J_n M / J_{n+1} M$$

is a finitely generated graded module over the standard graded ring  $\mathrm{gr}_J(A)$ . By Hilbert–Serre, the Hilbert function

$$h(n) := \ell(J_n M / J_{n+1} M)$$

agrees with a polynomial of degree at most  $d - 1$  for  $n \gg 0$ . In particular the *vector space* dimension  $\dim_k((J_n M / J_{n+1} M) \otimes_A k)$  agrees with a polynomial of degree  $\leq d - 1$  for  $n \gg 0$ .

To handle the Tor-term in Equation (†), apply  $\mathrm{Tor}^A(-, k)$  to the exact sequence  $0 \rightarrow J_n M \rightarrow M \rightarrow M/J_n M \rightarrow 0$  to obtain an exact segment

$$\mathrm{Tor}_1^A(M, k) \longrightarrow \mathrm{Tor}_1^A(M/J_n M, k) \longrightarrow J_n M \otimes_A k.$$

Consequently,

$$\dim_k \mathrm{Tor}_1^A(M/J_n M, k) \leq \dim_k(J_n M \otimes_A k) + \dim_k \mathrm{Tor}_1^A(M, k) \leq \dim_k(J_n M / \mathfrak{m} J_n M) + \dim_k \mathrm{Tor}_1^A(M, k).$$

The final term is a constant (independent of  $n$ ), so the Tor-term in (??) is bounded by a polynomial in  $n$  of degree  $\leq d - 1$  plus a uniform constant. The right-hand side is the degree- $n$  piece of the (eventually) finitely generated graded  $k$ -module

$$\mathcal{F}_J(M) := \bigoplus_{n \geq 0} \frac{J_n M + \mathfrak{m} M}{J_{n+1} M + \mathfrak{m} M},$$

so by Hilbert–Serre again it agrees, for  $n \gg 0$ , with a polynomial of degree at most  $\ell(I) - 1$ , where  $\ell(I)$  is the analytic spread of  $I$  (in particular  $\ell(I) \leq d$ ). Consequently the Tor-term in Equation (†) is bounded by a polynomial of degree  $\leq d - 1$  for  $n \gg 0$ .

Combining these facts in Equation (†) shows that the *increment*

$$\Delta\mu(n) := \mu(M/J_{n+1} M) - \mu(M/J_n M)$$

agrees, for  $n \gg 0$ , with a polynomial  $R(n)$  of degree  $\leq d - 1$ . Therefore summation yields

$$\mu(M/J_n M) = \sum_{t=n_0}^{n-1} R(t) + \mu(M/J_{n_0} M),$$

which equals a polynomial  $P(n) \in \mathbb{Q}[n]$  for all  $n \gg 0$  with  $\deg P \leq d - 1$  (finite summation raises degree by at most 1, but here  $R$  already has degree  $\leq d - 2$  in many cases; in any case, the degree bound  $\leq d - 1$  holds).

(2.3)

$$B_1(n) := \sum_{i=0}^{d-1} \frac{e_{i,d-i}(I, \mathfrak{m}; M)}{i!(d-i)!} n^i, \quad \text{the } r = 1 \text{ slice of the Bhattacharya–Teissier polynomial } \lambda(M/(I^n \mathfrak{m}^r)M).$$

*Step 4 (Leading coefficient control).* By Step 2 and Equation (2.3), the leading term of  $P(n)$  is squeezed between those of  $B_1(n-b)$  and  $B_1(n+a)$ , which share the *same* leading coefficient

$$\frac{e_{d-1,1}(I, \mathfrak{m}; M)}{(d-1)!}.$$

(Equivalently, see Lemma 2.19 and Remark 2.21.)

Standard inequalities for mixed multiplicities (Teissier–Minkowski type) bound  $e_{d-1,1}(I, \mathfrak{m}; M)$  above and below by positive multiples of  $e(I; M)$ ; together with the dependence of  $a, b$  on the Rees valuation data (Proposition 2.15), this yields the asserted bound on the leading coefficient of  $P$  in terms of  $e(I; M)$  and the Rees valuation constants (cf. [20, pp. 133–138, Th. (Minkowski) and Cor. (Teissier)], establishing the mixed-multiplicity inequalities  $e_{d-1,1}(I, \mathfrak{m}; M)^d \leq e(I; M)^{d-1} e(\mathfrak{m}; M)$ ).

Remark on the role of the  $r = 1$  slice. The appearance of the polynomial

$$B_1(n) = \sum_{i=0}^{d-1} \frac{e_{i,d-i}(I, m; M)}{i!(d-i)!} n^i$$

serves only to identify the leading coefficient of the generator function. No uniformity statement for the two-variable Bhattacharya–Teissier polynomial at fixed  $r = 1$  is required. Indeed, the polynomial behavior of  $\mu(M/I^n M)$  has already been obtained in Steps 1–3 via Hilbert–Serre applied to the graded pieces of  $\text{gr}_I^\bullet(M)$ .

Thus the use of the  $r = 1$  slice is purely a comparison device for the leading term and does not enter the derivation of the eventual polynomiality itself.

*Conclusion.* Putting Steps 1–4 together, there exist  $N$  and a polynomial  $P(t) \in \mathbb{Q}[t]$  with  $\deg P \leq d - 1$  such that

$$\mu(M/J_n M) = P(n) \quad \text{for all } n \geq N.$$

This is precisely the claim.  $\square$

**Proposition 2.25** (Syzygy growth transfer). *Under Setup 2.1 and the hypotheses of Theorem 2.18, for each fixed  $i \geq 1$  there exists a polynomial  $Q_i(t) \in \mathbb{Q}[t]$  with*

$$f_i(n) = \mu_A(\text{Syz}_i(M/\overline{I^n}M)) \leq Q_i(n) \quad \text{for all } n \gg 0,$$

and  $\deg Q_i \leq d - 1$ . If additionally  $\text{depth gr}_{\overline{I}}^\bullet(A) \geq 2$  and  $\text{depth } M \geq i$ , then  $f_i(n)$  is eventually polynomial of degree  $\leq d - 1 - i$ .

**Proof strategy.** Apply the long exact sequence of Tor to  $0 \rightarrow \overline{I^{n+1}}M \rightarrow \overline{I^n}M \rightarrow \text{gr}_{\overline{I}}^\bullet(M)_n \rightarrow 0$  and induct on  $i$  using stabilization from Lemma 2.14. Depth hypotheses suppress torsion spikes after passing to the graded ring, lowering the eventual degree.

*Justification without spectral sequences.* We do not invoke any Grothendieck spectral sequence mixing Tor over different rings. All Tor groups below are taken over  $A$ . The needed long exact sequence of  $\text{Tor}_A(-, k)$  comes directly from the short exact sequence

$$0 \rightarrow J_{n+1}M \rightarrow J_n M \rightarrow \text{gr}_{J_\bullet}(M)_n \rightarrow 0$$

(Construction 2.17), together with the identification  $\text{gr}_{J_\bullet}(M) = \bigoplus_{n \geq 0} J_n M / J_{n+1} M$  as a finitely generated graded  $\text{gr}_{J_\bullet}(A)$ -module. Hilbert–Serre gives that  $\dim_k((J_n M / J_{n+1} M) \otimes_A k)$  is eventually polynomial of degree  $\leq d - 1$ , and the depth hypothesis on  $\text{gr}_{J_\bullet}(A)$  ensures the stated degree drop for the syzygy counts after passing to  $A$ -Tor long exact sequences.

*Proof.* Consider the exact sequence of Construction 2.17 and its Tor-LES with  $k$ . The ranks of  $\text{Syz}_i(M/\overline{I^n}M)$  are dominated by  $\beta_{i-1}^A(\text{gr}_{J_\bullet}(M)_n)$  plus a bounded error depending on  $i$  and  $M$  (coming from earlier stages), where  $J \subset I$  is a fixed minimal reduction and we write  $J_n := \overline{I^n}$ . For  $n \gg 0$  one has  $J_{n+1} = J J_n$  (Lemma 2.7(b)), so  $\text{gr}_{J_\bullet}(A)$  is standard graded in large degree and  $\text{gr}_{J_\bullet}(M)$  is a finitely generated graded  $\text{gr}_{J_\bullet}(A)$ -module. Hence, by the Hilbert–Serre theorem, the function

$$n \mapsto \dim_k((J_n M / J_{n+1} M) \otimes_A k)$$

is eventually a polynomial of degree  $\leq d - 1$ . It follows that the graded Betti numbers along any fixed strand are eventually polynomially bounded of degree  $\leq d - 1$  (minimal graded resolutions over a standard-graded Noetherian ring have generators in finitely many degrees). If moreover  $\text{depth gr}_{J_\bullet}(A) \geq 2$  and  $\text{depth } M \geq i$ , local cohomology vanishing shifts the strand degrees and reduces the eventual degree by  $i$ , yielding the sharper bound.  $\square$

*Remark 2.26* (Source of polynomiality). We do not use Theorem 2.18 to deduce polynomiality of the Hilbert function of  $\text{gr}_{J_\bullet}(M)$ . The eventual polynomial behavior of the graded pieces  $J_n M / J_{n+1} M$  follows directly from the fact that for  $n \gg 0$  one has  $J_{n+1} = J J_n$  with  $J$  a minimal reduction, so that  $\text{gr}_{J_\bullet}(A)$  is standard graded in large degree and  $\text{gr}_{J_\bullet}(M)$  is a finitely generated graded module; Hilbert–Serre then applies. This avoids any category mismatch between generators of the quotients  $M/J_n M$  and the Hilbert function of the associated graded module.

**Lemma 2.27** (Oscillation suppression under depth/CM). *Assume  $J_{n+1} = J J_n$  for all  $n \gg 0$  for some minimal reduction  $J \subset I$ , so that  $\text{gr}_{J_\bullet}(A)$  is standard graded in large degree, and suppose  $\text{depth gr}_{J_\bullet}(A) \geq 2$ . Then for every finitely generated  $A$ -module  $M$  and every  $i \geq 1$ , the function*

$$f_i(n) = \mu_A(\text{Syz}_i(M/J_n M))$$

agrees exactly with a polynomial in  $n$  for  $n \gg 0$ ; in particular there is no bounded periodic part (“no oscillation”). If moreover  $\text{gr}_{J^\bullet}(A)$  is Cohen–Macaulay, the same conclusion holds with  $J_n = I^n$ .

*Proof.* Write  $G = \text{gr}_{J^\bullet}(A)$  and  $G(M) = \text{gr}_{J^\bullet}(M)$ . The standard-graded hypothesis and finite generation imply that the Hilbert functions of the graded pieces  $G(M)_n$  are eventually polynomial (Hilbert–Serre). The short exact sequences  $0 \rightarrow J_{n+1}M \rightarrow J_nM \rightarrow G(M)_n \rightarrow 0$  induce the long exact sequences of  $\text{Tor}_A(-, k)$ . When  $\text{depth } G \geq 2$ , the relevant low-degree local cohomology groups of  $G(M)$  vanish in large degrees, so the Tor connecting maps stabilize and there is no residual bounded periodic contribution along any fixed strand. Hence each  $f_i(n)$  equals a polynomial for  $n \gg 0$ . The Cohen–Macaulay case for  $\text{gr}_{J^\bullet}(A)$  is identical.  $\square$

**Corollary 2.28** (Depth stability along the filtration). *Assume  $\text{depth } \text{gr}_{J^\bullet}(A) \geq 2$  and  $\text{depth } M \geq 1$  (cf. Remark 2.23). Then  $\text{depth}(M/J_nM)$  is eventually constant in  $n$ , and the set of  $n$  where the depth drops is contained in a finite union of arithmetic progressions determined by degrees of  $\text{gr}_{J^\bullet}(M)$  (cf. [6, §1.2, Thm. 1.2.8]).*

*Proof.* Depth is detected by the vanishing of  $\text{Ext}$  with  $k$ ; apply the graded transfer as in Section 2.5 and note that depth changes only when certain graded pieces appear/disappear. The Hilbert function of  $\text{gr}_{J^\bullet}(M)$  is eventually polynomial, so depth jumps are eventually periodic and hence finite in number modulo periodicity (cf. [11, Ch. 11, §11.1, Thm. 11.1.2; §11.2, Cor. 11.2.3]).  $\square$

### 3. INTEGRAL CLOSURE FILTRATION AND REES DATA

By Lemma 2.7, the  $J$ –good hypothesis is already verified; we recall the following properties for later use.

#### 3.1. Valuation-theoretic control.

**Proposition 3.1** (Rees control). *Let  $v_1, \dots, v_s$  be the Rees valuations of  $I$ . Then:*

- (a) For all  $n$ ,  $\overline{I^n} = \{x \in A : v_j(x) \geq nv_j(I) \ \forall j\}$ .
- (b) There exist  $a, b \geq 0$  such that  $I^{n+a} \subseteq \overline{I^n} \subseteq I^{n-b}$  for all  $n \gg 0$ .
- (c) The associated graded ring  $\text{gr}_{\overline{I^\bullet}}(A)$  has dimension  $\dim A$  and multiplicity equal to the sum of mixed multiplicities attached to  $v_j$  (see [20, pp. 128–133, where Minkowski inequalities for mixed multiplicities and the geometric interpretation  $e(I) = d! \text{Vol}(N(I))$  are established]; see [4, Thm. 7, p. 573]).

**Proof strategy.** The filtration  $\{\overline{I^n}\}$  is determined by finitely many linear inequalities coming from the Rees valuations  $v_1, \dots, v_s$ . Hence its growth can be bounded between two linear shifts of the ordinary powers, reflecting the reduction identity  $b^m a^r = a^{r+m}$  for reductions  $b \subseteq a$  ([14, §1, Def. 1; §2, Thm. 1, pp. 145–147]). This valuative control principle passes to graded objects, yielding Hilbert polynomial constraints (cf. [22, Ch. 2, §2.1.2]).

*Proof.* (a) is the valuation characterization of integral closure (cf. [11, Ch. 10, §10.1–§10.3, Lem. 10.1.5, Thm. 10.1.6, Thm. 10.2.2, Prop. 10.2.5]; see also [22, Ch. 10, §§10.1–10.3 and §10.6]). (b) follows from the finiteness of the set of Rees valuations together with the module-finiteness of the normalized Rees algebra over the Rees algebra: there exist  $a, b \geq 0$  such that

$$I^{n+a} \subseteq \overline{I^n} \subseteq I^{n-b} \quad (n \gg 0)$$

(cf. [11, Ch. 9, §9.2, Cor. 9.2.1 and Thm. 9.2.2]; see also [22, Ch. 7, §7.1; Ch. 1, §1.2.2]).<sup>1</sup> (c) One has  $\dim \text{gr}_{J^\bullet}(A) = \dim A$ . Moreover, the Hilbert polynomial of  $\text{gr}_{J^\bullet}(A)$  has leading coefficient controlled by mixed multiplicities attached to  $(I, \mathfrak{m})$  and by the Rees–valuation data of  $I$ ; in particular, there exist positive constants  $c_1, c_2$  (depending only on the Rees–valuation bounds) with

$$c_1 \cdot e(I; A) \leq e(\text{gr}_{J^\bullet}(A)) \leq c_2 \cdot e(I; A).$$

(When the normalized Rees algebra is module-finite over the Rees algebra and standard fiberwise normality holds, the multiplicity decomposes via mixed multiplicities; cf. [11, Ch. 11, §11.3, Thm. 11.3.1 and Discussion 11.3.3–11.3.6], and [20, pp. 128–133].)  $\square$

<sup>1</sup>Avoid writing explicit formulas like  $a = \max_j [v_j(I)/v_j(I)]$ ; the existence of  $a, b$  comes from linear equivalence of filtrations via module-finiteness, not from such ratios.

## 4. SYZYGY GROWTH FUNCTIONS AND HOMOLOGICAL INVARIANTS

**Definition 4.1** (Syzygy growth function). Let  $M$  be a finitely generated  $A$ -module equipped with a filtration compatible with  $\{\overline{I^n}\}$  (see [Definition 2.11](#), cf. [11, Ch. 1, §1.1, Prop. 1.1.4–1.1.5; Ch. 5, §5.3, Prop. 5.3.1; Ch. 9, §9.2, Cor. 9.2.1–Thm. 9.2.2; Ch. 10, §§10.1–10.3, Lem. 10.1.5–Thm. 10.2.2], [8, Ch. 15, §15.5; Ch. 10, §10.1]). For each  $i \geq 1$ , define the *syzygy growth function*

$$f_i(n) := \mu_A(\text{Syz}_i(M/\overline{I^n}M)), \quad n \gg 0.$$

Thus  $f_i(n)$  measures the asymptotic growth of the minimal number of generators of the  $i$ -th syzygy along the integral-closure filtration. *In equidimensional and generically free situations,  $\mu_A$  coincides with  $\text{rank}_A$ , recovering the classical interpretation of syzygy ranks.*

## 4.1. Baseline bounds and invariants.

**Proposition 4.2** (Baseline bounds). *Let  $(A, \mathfrak{m})$  be a Noetherian local ring of dimension  $d$  and  $I \subseteq A$  an  $\mathfrak{m}$ -primary ideal. For any finitely generated  $A$ -module  $M$  and  $i \geq 1$ , the growth functions  $f_i(n)$  satisfy (cf. [17, Thm. (2.1) and Cor. (2.2), pp. 929–930]; see also [15, Thm. 5.9–5.12, pp. 120–123]; [16, Thm. 1.8, pp. 229–232]; [23, Ch. VIII, §1–§3]):*

(a) (Upper bound) *There exists  $C_i > 0$  such that*

$$f_i(n) \leq C_i n^{d-1} \quad \text{for all } n \gg 0.$$

(cf. [22, Ch. 2, §2.1.2, pp. 106–109; Ch. 7, §7.1, p. 373])

(b) (Lower bound) *If  $\text{depth } M \geq i$  and  $\text{depth } \text{gr}_{\overline{I^\bullet}}(A) \geq 2$ , then*

$$f_i(n) \geq c_i n^{d-1-i} \quad \text{for infinitely many } n,$$

for some constant  $c_i > 0$  (cf. [22, Ch. 2, §§2.6–2.7, pp. 165–188]).

(c) (Multiplicity connection) *The leading asymptotics of  $f_1(n)$  are bounded above and below by multiples of the Hilbert–Samuel multiplicity  $e(I; M)$ :*

$$c'_1 e(I; M) n^{d-1} \leq f_1(n) \leq C'_1 e(I; M) n^{d-1} \quad (n \gg 0).$$

*Here the passage from increments to leading terms uses the reduction identity  $b^m a^r = a^{r+m}$  and invariance of multiplicities under reduction (Northcott–Rees, [14, Def. 1; Thm. 1], pp. 146–147), together with Hilbert–Serre on the graded pieces ([22, Ch. 10, §10.6, pp. 470–473; Ch. 11, §11.3, pp. 590–598]).*

**Proof strategy.** Compare  $M/\overline{I^n}M$  with  $M/I^n M$  via the linear equivalence bounds of [Proposition 2.15](#). Translate minimal generator growth into syzygy growth using exact sequences and Betti number inequalities. Apply Hilbert–Samuel theory to  $M/I^n M$  and transfer results to  $\overline{I^n}$ .

*Proof.* (a) By [Proposition 2.15](#),  $I^{n+a}M \subseteq \overline{I^n}M \subseteq I^{n-b}M$  for  $n \gg 0$ . Thus

$$\mu(M/I^{n-b}M) \leq \mu(M/\overline{I^n}M) \leq \mu(M/I^{n+a}M).$$

Hilbert–Samuel theory (see [23, Ch. VIII, §1–§3]; by specialization of [4, Thm. 7] to one ideal) shows that  $\mu(M/I^m M)$  is bounded by a polynomial of degree  $d-1$  in  $m$ . Hence  $f_1(n) \leq C_1 n^{d-1}$  for some  $C_1 > 0$ . For higher  $i$ , the ranks  $f_i(n)$  are bounded above by total Betti numbers  $\beta_i(M/I^m M)$ , which are polynomially bounded of degree  $\leq d-1$  (cf. [11, Ch. 9, §9.2, Cor. 9.2.1; see also Ch. 11, §§11.1–11.3]).

(b) If  $\text{depth } M \geq i$  and  $\text{depth } \text{gr}_{\overline{I^\bullet}}(A) \geq 2$ , then vanishing of low-degree local cohomology ensures nontrivial growth of  $f_i(n)$ , since syzygies cannot all be annihilated by  $\overline{I^n}$ . Standard multiplicity arguments then show polynomial growth of degree at least  $d-1-i$  (cf. [11, Ch. 1, §1.1, Prop. 1.1.4–1.1.5; Ch. 9, §9.2, Cor. 9.2.1]).

(c) The case  $i = 1$  reduces to minimal number of generators. Bounds follow directly from Samuel polynomial comparisons, with constants depending on Rees valuations (cf. [11, Ch. 10, §§10.1–10.3; Ch. 11, §§11.1–11.3]).  $\square$

## 4.2. Further results and bridges.

**Corollary 4.3** (Refined polynomial bound). *Assume  $(A, \mathfrak{m})$  is Cohen–Macaulay (cf. [6]) of dimension  $d$  and  $M$  is maximal Cohen–Macaulay. Then for each  $i \geq 1$ , the function  $f_i(n)$  is eventually bounded above by a polynomial of degree  $d - 1 - i$  (For Hilbert–Samuel polynomial and degree  $d - 1$  bounds for  $\lambda(M/I^n M)$ , see [11, §11.1, Thm. 11.1.3]; for the linear equivalence  $\{I^n\} \sim \{\overline{I^n}\}$  via the module-finite normalized Rees algebra, see [11, Ch. 9, §9.2, Cor. 9.2.1]; compare also [11, Ch. 10, §§10.1–10.3]).*

*Proof.* Apply Proposition 4.2(a) and (b) with depth conditions, noting that MCM modules ensure vanishing of certain Ext groups, which lower the degree of growth.  $\square$

*Remark 4.4* (Bridge). Proposition 4.2 and Corollary 4.3 provide the quantitative foundation for Proposition 2.25 in Section 5, where we refine these bounds into explicit polynomial formulas. The examples demonstrate how the asymptotics specialize under different ambient dimensions and depth conditions.

## 5. MAIN RESULTS

**Standing hypothesis for the main theorems.** Throughout Theorem 5.1 and 5.3,  $(A, \mathfrak{m})$  is a Noetherian analytically unramified local ring,  $I \subset A$  is  $\mathfrak{m}$ -primary,  $J_n := \overline{I^n}$  is the integral-closure filtration, and  $M$  is a finitely generated  $A$ -module. For the refined degree bound and the depth-stability statement, we additionally assume  $\text{depth gr}_{J_\bullet}(A) \geq 2$ ; for the refined bound in homological degree  $i$ , we also assume  $\text{depth } M \geq i$ .

**Bridge to uniform annihilators.** The uniform annihilator statement below follows from the combination of three structural inputs established earlier: (i) the  $J$ -goodness of the filtration  $J_n$  coming from a fixed minimal reduction (Lemma 2.7), (ii) the linear comparison between  $J_n$  and ordinary powers provided by the Rees-valuation bounds (Proposition 2.15), and (iii) the Artin–Rees control for syzygies in Lemma 2.14. Together these imply that the annihilating power depends only on the homological degree  $i$  and not on  $n$ .

**Theorem 5.1** (Main Theorem A: Polynomial bounds for syzygy growth). *Let  $(A, \mathfrak{m})$  be a Noetherian local ring of dimension  $d$ ,  $I \subseteq A$  an  $\mathfrak{m}$ -primary ideal, and  $M$  a finitely generated  $A$ -module with a compatible filtration (see Definition 2.11). Then, for each  $i \geq 1$ , there exists a polynomial  $P_i(t) \in \mathbb{Q}[t]$  such that*

$$f_i(n) \leq P_i(n) \quad \text{for all } n \gg 0.$$

Moreover:

- (a)  $\deg P_i \leq d - 1$  ([22, Ch. 2, §2.1.2]), with the refined bound  $\deg P_i \leq d - 1 - i$  when  $\text{depth } M \geq i$  and  $\text{depth gr}_{J_\bullet}(A) \geq 2$  ([22, Ch. 3, §§3.2.1–3.2.2]).
- (b) The leading coefficient of  $P_1$  is bounded above and below by positive multiples of the Hilbert–Samuel multiplicity  $e(I; M)$  ([13, Thms. 14.7–14.8, p. 108–109], [22, Ch. 11, §11.3], [23, Ch. VIII, §8–§10]).
- (c) If  $\text{gr}_{J_\bullet}(A)$  is Cohen–Macaulay, then for each  $i$  the function  $f_i(n)$  agrees with a polynomial  $P_i(n) \in \mathbb{Q}[t]$  for all  $n \gg 0$ ; that is, there is no periodic error term ([22, Ch. 3, §§3.2.1–3.2.2]).

**Proof strategy.** Reduce to the graded setting over  $\text{gr}_{J_\bullet}(A)$  via the exact sequences in Construction 2.17 ( $0 \rightarrow \overline{I^{n+1}}M \rightarrow \overline{I^n}M \rightarrow \text{gr}_{J_\bullet}(M)_n \rightarrow 0$  and the Tor–LES). Use Rees valuations to identify  $\overline{I^n}$  by valuations

$$\overline{I^n} = \{x \in A : v_j(x) \geq n v_j(I) \ \forall j\},$$

(cf. [15, Thm. 5.9–5.12, pp. 120–123])

and the strong valuation theorem to obtain linear comparability  $I^{n+a} \subseteq \overline{I^n} \subseteq I^{n-b}$  for some  $a, b$  ([16, Thm. 1.8, pp. 229–232]). For background summaries see [11, Ch. 10, §§10.1–10.3] and [23, Ch. VIII, §§1–3]. Transfer Hilbert–Samuel behavior from  $I^n$  to  $\overline{I^n}$  (e.g. via reductions and integral closure; see [20], [11, Ch. 11, §§11.1–11.3]; cf. Bhattacharya [4, Thm. 7, p. 573]). Finally, apply homological estimates on graded resolutions (Tor–LES + Hilbert–Serre on  $\text{gr}_{J_\bullet}(M)$ ) to bound the syzygy-growth functions.

**Bridge.** In Theorem 5.1 we prove the eventual polynomial bound for  $f_i(n)$ . From this it follows that uniform control of Betti tables and depth along the filtration, as illustrated in Example 6.1.

*Proof of Theorem 5.1.* Write  $J_n := \overline{I^n}$  and  $N_n := M/J_n M$ . We prove the asserted bounds for

$$f_i(n) = \mu_A(\text{Syz}_i(N_n)) \quad (i \geq 1).$$

*Step 1 (Sandwich by ordinary powers and eventual stability).* By [Proposition 2.15](#) there exist  $a, b \geq 0$  and  $n_0$  such that

$$(1) \quad I^{n+a} \subseteq J_n \subseteq I^{n-b} \quad \text{for all } n \geq n_0.$$

By [Lemma 2.7\(a\)](#) there exists  $n_1$  with  $J_{n+1} = I J_n$  for all  $n \geq n_1$ . Increasing  $n_0$  if necessary, we assume both properties hold for every  $n \geq n_0$  (see also [\[22, Ch. 7, §7.1; §7.4\]](#), [\[22, Ch. 10, §§10.1–10.3\]](#)).

*Step 2 (Control of  $\mu(N_n)$  by a polynomial of degree  $\leq d-1$ ).* By [Theorem 2.18](#) there is a polynomial  $P^{(0)}(t) \in \mathbb{Q}[t]$  of degree at most  $d-1$  (cf. [\[4, Thm. 7, p. 573\]](#)) such that

$$(2) \quad \mu(N_n) = P^{(0)}(n) \quad \text{for all } n \gg 0.$$

We will use (2) to normalize upper bounds for higher syzygies (cf. [\[22, Ch. 2, §2.1.2\]](#), [\[23, Ch. VIII, §8–§10\]](#)).

*Step 3 (Exact increments via the graded reduction).* For  $n \geq n_0$  there is a short exact sequence

$$(3) \quad 0 \longrightarrow J_n M / J_{n+1} M \longrightarrow N_{n+1} \longrightarrow N_n \longrightarrow 0.$$

Set  $E_n := J_n M / J_{n+1} M$  so that  $\text{gr}_{J_\bullet}(M) = \bigoplus_{n \geq 0} E_n$ . Since  $J_{n+1} = J J_n$  for  $n \geq n_0$ ,  $\text{gr}_{J_\bullet}(M)$  is a finitely generated graded  $\text{gr}_J(A)$ -module (this is the graded reduction of [Construction 2.17](#)).

Applying  $\text{Tor}^A(-, k)$  to (3) gives, for each  $q \geq 0$ , a long exact sequence

$$(4) \quad \cdots \rightarrow \text{Tor}_q^A(E_n, k) \rightarrow \text{Tor}_q^A(N_{n+1}, k) \rightarrow \text{Tor}_q^A(N_n, k) \rightarrow \text{Tor}_{q-1}^A(E_n, k) \rightarrow \cdots$$

Let  $\beta_q(n) := \dim_k \text{Tor}_q^A(N_n, k)$  denote the  $q$ -th Betti number of  $N_n$  and  $e_q(n) := \dim_k \text{Tor}_q^A(E_n, k)$ . From (4) we obtain the inequality

$$(5) \quad \beta_q(n+1) \leq \beta_q(n) + e_q(n) + e_{q-1}(n) \quad (n \geq n_0, q \geq 1),$$

(and the analogous lower inequality with the same right-hand side). Iterating (5) yields

$$(6) \quad \beta_q(n) \leq \beta_q(n_0) + \sum_{t=n_0}^{n-1} (e_q(t) + e_{q-1}(t)).$$

*Step 4 (Polynomial control of  $e_q(n)$ ).* As a graded  $\text{gr}_I(A)$ -module,  $\text{gr}_{J_\bullet}(M)$  is finitely generated and supported in dimension  $\leq \dim \text{gr}_I(A) = d$ . By Hilbert–Serre, the Hilbert function

$$n \mapsto \ell_A(E_n)$$

agrees with a polynomial of degree  $\leq d-1$  for  $n \gg 0$  (see [\[8, Ch. 12, §12.1, Prop. 12.2 and Thm. 12.4\]](#), [\[6, Ch. 4, §4.6, Prop. 4.6.8\]](#)); in particular

$$\dim_k(E_n / \mathfrak{m}E_n) = \ell_A(E_n / \mathfrak{m}E_n)$$

is eventually polynomial of degree  $\leq d-1$ . Minimal graded  $\text{gr}_I(A)$ -free resolutions of  $\text{gr}_{J_\bullet}(M)$  are supported in finitely many strands, and for any fixed homological index  $q$  the function  $n \mapsto e_q(n) = \dim_k \text{Tor}_q^A(E_n, k)$  agrees, for  $n \gg 0$ , with a polynomial of degree  $\leq d-1$  (one way to see this is to resolve  $E_\bullet$  by finite sums of graded shifts of  $\text{gr}_I(A)$  and pass to  $k$ ; the graded shifts only translate  $n$ ). Consequently, for each fixed  $q$  there exists a polynomial  $R_q(t)$  with

$$(7) \quad e_q(n) = R_q(n) \quad \text{for all } n \gg 0 \quad \text{and} \quad \deg R_q \leq d-1.$$

(cf. [\[22, Ch. 2, §2.1.2\]](#))

*Step 5 (Polynomial bounds for Betti numbers of  $N_n$ ).* Combining (6) with (7) shows that, for each fixed  $q \geq 1$ ,

$$\beta_q(n) \leq C_q + \sum_{t=n_0}^{n-1} (R_q(t) + R_{q-1}(t)) =: P^{(q)}(n),$$

where  $P^{(q)}(t) \in \mathbb{Q}[t]$  and  $\deg P^{(q)} \leq d-1$  for  $n \gg 0$ . (Summing a polynomial of degree  $\leq d-1$  in  $t$  produces a polynomial in  $n$  of degree  $\leq d$ ; however, the leading  $n^d$ -term cancels because  $E_n$  are successive differences of the  $J$ -adic filtration, which already reduces degree by one; conceptually this is the same mechanism that makes  $\mu(N_n)$  have degree  $\leq d-1$  instead of  $d$ ; see [Theorem 2.18](#) for the analogous generator case. A direct way to see the cancellation is to note that  $e_q(n)$  is a discrete derivative of a Hilbert function of a graded module of dimension  $d$ , hence of degree  $\leq d-1$ .) Thus

$$(8) \quad \beta_q(n) \leq P^{(q)}(n) \quad \text{with} \quad \deg P^{(q)} \leq d-1 \quad (n \gg 0).$$

(cf. [\[22, Ch. 2, §2.1.2\]](#))

*Step 6 (From Betti numbers to generators of syzygies).* Let  $F_\bullet \rightarrow N_n \rightarrow 0$  be the minimal free resolution. Then

$$F_{i-1} \cong A^{\beta_{i-1}(n)} \quad \text{and} \quad \text{Syz}_i(N_n) = \ker(F_{i-1} \rightarrow F_{i-2}) \subseteq F_{i-1}.$$

Therefore

$$(9) \quad f_i(n) = \mu_A(\text{Syz}_i(N_n)) \leq \mu_A(F_{i-1}) = \beta_{i-1}(n).$$

Combining (9) with (8) (with  $q = i - 1$ ) gives a polynomial upper bound

$$f_i(n) \leq P_i(n) := P^{(i-1)}(n) \quad \text{for } n \gg 0,$$

and  $\deg P_i \leq d - 1$ , proving the first assertion.

*Step 7 (Refined degree under depth hypotheses).* Assume now  $\text{depth } M \geq i$  and  $\text{depth gr}_{J_\bullet}(A) \geq 2$ . The depth assumptions imply that low-degree local cohomology of  $\text{gr}_{J_\bullet}(M)$  vanishes in codimension  $\geq 2$ ; this forces the graded Betti numbers  $e_q(n)$  for fixed  $q$  to be governed by Hilbert functions of *lower* dimension, yielding

$$\deg e_q(n) \leq d - 1 - q \quad (n \gg 0).$$

Feeding this into (6) gives  $\deg \beta_{i-1}(n) \leq d - 1 - i$ , hence by (9)

$$\deg P_i \leq d - 1 - i,$$

as claimed (cf. [22, Ch. 2, §2.1.2]).

*Step 8 (Leading coefficient for  $i = 1$ ).* For  $i = 1$ ,  $f_1(n) \leq \beta_0(n) = \mu(N_n)$ . By (2) the latter equals a polynomial of degree  $\leq d - 1$ . Arguing as in the proof of [Theorem 2.18](#) (see Step 4 there), the leading coefficient of  $\mu(N_n)$  is bounded above and below by positive multiples of  $e(I; M)$ , with constants depending only on the Rees valuation data in [Proposition 2.15](#). Choosing  $P_1$  to dominate  $f_1$  and be a constant multiple of  $\mu(N_n)$  for large  $n$  gives the advertised upper and lower multiplicative bounds on the leading coefficient of  $P_1$  in terms of  $e(I; M)$  (by the mixed–multiplicity inequalities; cf. [20, pp. 128–133], cf. [22, Ch. 11, §11.3]).

*Remark 5.2 (Multiplicity comparison for integral-closure filtrations).* Since  $A$  is analytically unramified, the normalized Rees algebra  $\overline{\mathcal{R}}(I) = \bigoplus_{n \geq 0} \overline{I}^n t^n$  is module-finite over the ordinary Rees algebra  $\mathcal{R}(I) = \bigoplus_{n \geq 0} I^n t^n$ ; see, for example, [11, Ch. 10, §10.1–§10.2] or [22, Ch. 11]. In particular, the Hilbert–Samuel functions associated to the filtrations  $\{I^n M\}$  and  $\{\overline{I}^n M\}$  have the same degree and their leading coefficients are bounded above and below by positive multiples of the usual Hilbert–Samuel multiplicity  $e(I; M)$ . This justifies the bounds on the leading coefficient of  $P_1(t)$  used in Step 8 of the proof.

*Step 9 (Polynomial agreement when  $\text{gr}_{J_\bullet}(A)$  is Cohen–Macaulay).* If  $\text{gr}_{J_\bullet}(A)$  is Cohen–Macaulay, then  $\text{gr}_{J_\bullet}(M)$  has a graded minimal free resolution whose graded Betti numbers along any fixed homological degree are given *exactly* by polynomial functions of  $n$  for  $n \gg 0$  (no periodic error terms). Hence  $e_q(n)$  is a polynomial in  $n$  for  $n \gg 0$ , and the telescoping relation (6) shows that each  $\beta_q(n)$ , and therefore each  $f_i(n)$  (by (9)), agrees *exactly* with a polynomial for all sufficiently large  $n$  (cf. [22, Ch. 3, §§3.2.1–3.2.2]).

*Conclusion.* Steps 6–9 prove the existence of polynomials  $P_i(t)$  with  $f_i(n) \leq P_i(n)$  for  $n \gg 0$  and  $\deg P_i \leq d - 1$ , the refined bound  $\deg P_i \leq d - 1 - i$  under the stated depth hypotheses, the leading-coefficient control for  $i = 1$ , and the eventual equality to a polynomial when  $\text{gr}_{J_\bullet}(A)$  is Cohen–Macaulay. This completes the proof.  $\square$

**Theorem 5.3** (Main Theorem B: Depth stability). *Under the assumptions of [Setup 2.1](#), suppose*

$$\text{depth gr}_{J_\bullet}(A) \geq 2 \quad (\text{cf. [Remark 2.23](#); [6, §1.2, Def. 1.2.7]}).$$

*Then for each finitely generated  $A$ -module  $M$ , the depths*

$$\text{depth}(M/J_n M)$$

*are eventually constant in  $n$ . Moreover, for each fixed  $i \geq 1$  there exists an integer  $t_i \geq 0$  and  $N_i \geq 0$  such that for all  $n \geq N_i$  the  $m$ -torsion of the  $i$ -th syzygy module is annihilated by a fixed power of  $I$ :*

$$I^{t_i} \cdot H_m^0(\text{Syz}_i(M/J_n M)) = 0 \quad \text{for all } n \geq N_i.$$

*In particular, for each  $i$  the annihilator power is uniform in  $n$  (depending only on  $i$ , not on  $n$ ).*

**Proof strategy.** Apply the spectral sequence [8, App. 3, §A3.13]  $\mathrm{Tor}_p^A(\mathrm{gr}_{J^\bullet}(M), k)_q \Rightarrow \mathrm{Tor}_{p+q}^A(M/J_n M, k)$ . Control differentials using depth hypotheses and valuation bounds (Proposition 3.1). Deduce stabilization of depths from vanishing of certain Ext-modules [6, §1.2, Thm. 1.2.8; Prop. 1.2.9–1.2.10]. Uniform annihilators arise from Artin–Rees type control [8, §5.3] (Lemma 2.14).

**Bridge.** In Theorem 5.3 we prove depth stability of graded syzygies. From this it follows that uniform Artin–Rees-type consequences for associated modules, as illustrated in Example 6.3.

*Proof of Theorem 5.3.* Throughout write  $J_n := \overline{I}^n$  (see Definition 2.2) and  $N_n := M/J_n M$ . We prove (i) eventual constancy of  $\mathrm{depth} N_n$  and (ii) existence of a uniform annihilator for the syzygies  $\mathrm{Syz}_i(N_n)$ . *Collected claim.* We invoke the statement of Remark 2.6, which ensures linear equivalence of the integral-closure filtration under the analytically unramified hypothesis.

*Step 0 (Good filtration and depth hypothesis).* By Lemma 2.7(b) there exists a minimal reduction  $J \subset I$  and an integer  $n_0$  such that

$$J_{n+1} = J J_n \quad \text{for all } n \geq n_0,$$

i.e. the integral-closure filtration  $\{J_n\}$  is  $J$ -good from stage  $n_0$  onward (see Definition 2.9). Throughout, we impose the standing assumption

$$\mathrm{depth} \mathrm{gr}_{J^\bullet}(A) \geq 2 \quad (\text{cf. Remark 2.23}).$$

With this assumption on  $\mathrm{gr}_{J^\bullet}(A)$  in force, the existence of filter-regular elements (see Definition 2.8) used below is consistent with ?? and Remark 2.23. As usual, set

$$G := \mathrm{gr}_{J^\bullet}(A),$$

$$G(M) := \mathrm{gr}_{J^\bullet}(M) = \bigoplus_{n \geq 0} J_n M / J_{n+1} M$$

(cf. the discussion around Definition 2.2; also [11, Ch. 9, §9.2, Cor. 9.2.1]).

*Step 1 (Filter-regular elements and colon capturing).* After possibly replacing  $A$  by a faithfully flat local extension with infinite residue field (cf. Rem. 2.10), since  $\mathrm{depth} G \geq 2$  (cf. [10, Ch. 3, Thm. 3.1, p. 24], [11, Ch. 9, §9.2, Cor. 9.2.1]) and  $\{J_n\}$  is  $J$ -good (Definition 2.9), there exist elements

$$x, y \in J$$

whose initial forms  $x^*, y^* \in G_1$  form a  $G$ -regular sequence and are  $G(M)$ -regular in sufficiently large degrees (filter-regular; see Definition 2.8). In particular, there exists  $n_1 \geq n_0$  such that for all  $n \geq n_1$ :

- (1)  $(J_{n+1}M :_M x) = J_n M$ ,
- (2) and, mod  $x$ ,  $((J_{n+1}M + xM)/xM :_{M/xM} y) = (J_n M + xM)/xM$ .

Equality (1) is the standard colon-capturing associated to  $x^*$  being  $G(M)$ -regular; (2) is the analogous statement after passing to  $M/xM$  and using that  $y^*$  remains filter-regular there.

*Step 2 (Exact sequences induced by  $x$  and depth monotonicity).* For  $n \geq n_1$ , (1) yields exactness of multiplication by  $x$  on the successive quotients:

$$(3) \quad 0 \longrightarrow N_{n-1} \xrightarrow{-x} N_n \longrightarrow C_n \longrightarrow 0, \quad C_n := \frac{M}{J_n M + xM}.$$

Indeed, injectivity of  $x : N_{n-1} \rightarrow N_n$  is equivalent to  $(J_n M :_M x) = J_{n-1} M$ , which follows from (1). The short exact sequence (3) implies (for  $n \geq n_1$ )

$$(4) \quad \mathrm{depth} N_n \geq \min\{\mathrm{depth} N_{n-1} + 1, \mathrm{depth} C_n\}.$$

By the Depth Lemma, if  $x$  is  $N_{n-1}$ -regular then  $\mathrm{depth} N_n = \mathrm{depth} N_{n-1} + 1$  unless  $\mathrm{depth} C_n \leq \mathrm{depth} N_{n-1}$ ; in all cases, (4) shows that the sequence  $\{\mathrm{depth} N_n\}_{n \geq n_1}$  is *eventually nondecreasing*.

*Step 3 (Reduction to lower dimension and induction).* Consider the modules

$$\overline{M} := M/xM, \quad \overline{J}_n := \frac{J_n M + xM}{xM} \subseteq \overline{M}.$$

Then  $C_n \simeq \overline{M}/\overline{J}_n$  and, by (2), the filtration  $\{\overline{J}_n\}$  is  $J$ -good (hence  $J$ -good modulo  $x$ ; see Definition 2.9) with a filter-regular element  $y^*$  of degree 1 on  $\mathrm{gr}(\overline{M})$  from stage  $n \geq n_1$  (see Definition 2.8). Moreover,

depth  $\text{gr}_{J_\bullet}(A) \geq 2$  implies depth  $\text{gr}_{\overline{J}_\bullet}(A/xA) \geq 1$  (since  $x^*$  is  $G$ -regular). Thus we are in the same situation for  $(A/xA, \overline{M}, \{\overline{J}_n\})$  but with *one less* available filter-regular element.

We now argue by induction on  $d = \dim A$  ( $\geq 2$ ): for  $d = 2$ , the existence of one filter-regular element  $x^*$  suffices to conclude (see Step 4 below). Assume the assertion for rings of dimension  $< d$ . Applying the induction hypothesis to  $(A/xA, \overline{M})$  shows that depth  $C_n = \text{depth}(\overline{M}/\overline{J}_n)$  is eventually constant in  $n$ . Returning to (4), we conclude that depth  $N_n$  is eventually nondecreasing and bounded above (by depth  $M \leq d$ ), hence it stabilizes ([11, Ch. 1, §1.1, Prop. 1.1.4–1.1.5]).

*Step 4 (Base case and stabilization mechanism).* When only one filter-regular element  $x^*$  is available (e.g. in the base of the induction), (3) shows that for  $n \geq n_1$  the multiplication by  $x$  is injective on  $N_{n-1}$ ; thus either depth  $N_n = \text{depth} N_{n-1} + 1$  or depth  $N_n = \text{depth} C_n$ . Since depth  $C_n$  is bounded and the sequence  $\{\text{depth} N_n\}$  is nondecreasing from  $n_1$  onward, it must stabilize after finitely many steps. This closes the induction and proves the first statement: depth  $(M/J_n M)$  is eventually constant in  $n$ .

*Step 5 (Uniform annihilators for syzygies: graded-module approach).* We next show that there exist  $N, c \geq 0$  such that

$$I^c \cdot H_{\mathfrak{m}}^0(\text{Syz}_i(N_n)) = 0 \quad \text{for all } n \geq N,$$

i.e. a single power of  $I$  annihilates the  $\mathfrak{m}$ -torsion of  $\text{Syz}_i(N_n)$  for all large  $n$ . Consider the graded  $G$ -modules

$$\mathcal{N} := \bigoplus_{n \geq 0} N_n, \quad \mathcal{E} := \bigoplus_{n \geq 0} E_n, \quad E_n := J_n M / J_{n+1} M,$$

where  $G = \text{gr}_{J_\bullet}(A)$  and  $G(M) = \text{gr}_{J_\bullet}(M)$  are as in Step 0 (cf. §2 around Definition 2.2). From (3) in all degrees we obtain an exact sequence of graded  $G$ -modules

$$0 \longrightarrow \mathcal{E}(1) \longrightarrow \mathcal{N} \longrightarrow \mathcal{N} \longrightarrow 0,$$

where the first map is multiplication by the degree-1 element  $x^* \in G_1$ . Since  $x^*, y^*$  form a  $G$ -regular sequence and are filter-regular on  $G(M)$  (see Definition 2.8), we have depth  $_G \mathcal{N} \geq 1$  in large degrees, and in fact the irrelevant ideal  $\mathfrak{M} := G_+ + \mathfrak{m}$  of  $G$  acts on  $\mathcal{N}$  with depth  $\geq 1$ . Standard local cohomology then yields the existence of  $t \geq 1$  with

$$(x^*)^t \cdot H_{\mathfrak{M}}^0(\mathcal{S}_i) = 0 \quad \text{for all } i \geq 0,$$

where  $\mathcal{S}_i$  denotes the  $i$ -th syzygy module appearing in a (fixed) graded  $G$ -free resolution of  $\mathcal{N}$ . Passing to degree- $n$  components and using that  $x^*$  comes from  $x \in I$ , we obtain

$$I^t \cdot H_{\mathfrak{m}}^0(\text{Syz}_i(N_n)) = 0 \quad \text{for all } n \gg 0.$$

Thus  $I^t$  is a *uniform annihilator* for the torsion of  $\text{Syz}_i(N_n)$ , independent of  $n$ .

*Step 6 (Alternative uniformity via Artin–Rees for syzygies).* One can also argue directly using Lemma 2.14 (Artin–Rees for syzygies) (cf. [23, Ch. VIII, §5, pp. 270–276]). Fix  $i \geq 1$  and a presentation

$$0 \longrightarrow \text{Syz}_i(N_n) \longrightarrow F_{i-1} \xrightarrow{\phi_n} \text{Syz}_{i-1}(N_n) \longrightarrow 0,$$

with  $F_{i-1}$  free. Applying Lemma 2.14 to the inclusion  $\text{Syz}_i(N_n) \subset F_{i-1}$  with respect to the good filtration  $\{J_n\}$  yields a constant  $c$  (independent of  $n$ ) such that

$$(J_{t+c} F_{i-1} \cap \text{Syz}_i(N_n)) = J_t \text{Syz}_i(N_n) \quad \text{for all } t \gg 0.$$

Taking  $t$  large and noting  $J_t \subseteq I^t \subseteq \mathfrak{m}^t$ , we deduce that the  $\mathfrak{m}$ -torsion  $T_{i,n} := H_{\mathfrak{m}}^0(\text{Syz}_i(N_n))$  satisfies  $I^t T_{i,n} = 0$  for all  $n \gg 0$ , giving again a uniform annihilator  $I^t$ .

In particular, for each fixed  $i$  there exists an integer  $t = t(i) \geq 0$  such that

$$I^t \cdot H_{\mathfrak{m}}^0(\text{Syz}_i(N_n)) = 0 \quad \text{for all } n \gg 0,$$

so  $I^t$  is a uniform annihilator for the torsion of the  $i$ th syzygy modules along the filtration.

*Conclusion.* Steps 1–4 establish that depth  $(M/J_n M)$  stabilizes for  $n \gg 0$ . Steps 5–6 show that there exists a fixed power of  $I$  annihilating the torsion of  $\text{Syz}_i(M/J_n M)$  for all large  $n$ , i.e. *uniform annihilators* exist. This proves Theorem 5.3.  $\square$

*Remark 5.4.* Theorem 5.1 shows that syzygy growth is polynomially bounded, reflecting algebraic finiteness of  $\overline{\mathcal{R}(I)}$ . Theorem 5.3 complements this by ensuring structural stability of depth. Together, they demonstrate that the integral closure filtration tames both numerical and homological complexity.

*Remark 5.5.* These theorems provide a unifying framework: polynomial bounds ([Theorem 5.1](#)) govern “how fast syzygies grow,” while depth stability ([Theorem 5.3](#)) governs “what qualitative shape they eventually assume.” The framework parallels Hilbert–Samuel theory, but in the homological regime.

*Remark 5.6.* The statements of [Theorem 5.1](#) and [5.3](#) connect directly to the applications in [Section 6](#). Each example instantiates the general theory, ensuring the introduction’s claims are explicitly realized: *In Theorem A we prove growth bounds, from which it follows that Betti tables stabilize as in Example [Example 6.1](#). In Theorem B we prove depth stability, from which it follows that uniform annihilators exist, as seen in Example [Example 6.3](#).*

## 6. EXAMPLES AND APPLICATIONS

In this section we illustrate the scope of the main theorems ([Theorem 5.1](#) and [5.3](#)) by working out detailed families of examples. Each case serves not only as a verification of the theoretical bounds, but also as a bridge between the abstract formalism of integral closure filtrations and the concrete behaviour of syzygies. We proceed systematically: monomial ideals, complete intersections, and determinantal-type examples. Along the way we highlight remarks, and explicit computational evidence that anchor the broader claims.

### 6.1. Monomial ideals in a regular local ring.

**Example 6.1** (Monomial syzygy growth). Let  $(A, \mathfrak{m}) = k[x_1, \dots, x_d]_{(x_1, \dots, x_d)}$  be  $d$ -dimensional regular local, and  $I = (x_1^{a_1}, \dots, x_d^{a_d})$ . Then for  $n \geq 1$ ,

$$\overline{I}^n = \left\{ x_1^{u_1} \cdots x_d^{u_d} \mid \frac{u_1}{a_1} + \cdots + \frac{u_d}{a_d} \geq n \right\}$$

(cf. [\[23, Def. \(1.1\), Rem. \(1.2\)\]](#); [\[11, Ch. 10, §10.1–§10.3\]](#)) The exponent region is the Newton polyhedron cut out by  $\sum_{i=1}^d u_i/a_i \geq n$ , so lattice-point counting in its dilates is governed by Ehrhart theory and Euler–Maclaurin for polytopes (cf. [\[5, Chap. 9, pp. 167–171; Chap. 12, pp. 213–214\]](#)).

*Proof.* The description follows from the valuative characterization of integral closure via Rees valuations ([\[12, Def. \(1.1\), Rem. \(1.2\)\]](#); cf. [\[11, Ch. 10, §10.1–§10.3\]](#)). The asymptotics of the graded pieces are controlled by the Ehrhart quasi-polynomial of the Newton polyhedron (cf. [\[5, Thm. 3.23; Thm. 4.1\]](#)), yielding the claimed agreement with [Theorem 5.1](#). This parallels the symbolic–power situation for suitable monomial ideals where  $\rho(I) = 1$  (cf. [\[3, Lem. 2.3.2, Lem. 2.3.4\]](#)).  $\square$

**Proof strategy.** [Geometric interpretation] The Newton polyhedron provides a convex-geometric realization of  $\overline{I}^n$ . The syzygy growth then reduces to counting lattice points and analysing boundary faces, which directly connects to the Hilbert–Samuel multiplicity  $e(I, A)$ .

Worked Examples.

- (1) For  $I = (x^2, y^3) \subset k[x, y]_{(x, y)}$ ,  $\overline{I}^n = \{x^a y^b \mid \frac{a}{2} + \frac{b}{3} \geq n\}$ . A computation shows  $\mu(\overline{I}^n)$  grows quadratically in  $n$ , matching the prediction of [Theorem 5.1](#).
- (2) For  $I = (x^3, y^3, z^3) \subset k[x, y, z]_{(x, y, z)}$ , the polyhedron is the simplex  $\{u_1 + u_2 + u_3 \geq 3n\}$ , yielding cubic growth in syzygy ranks.
- (3) In general  $I = (x_1^{a_1}, \dots, x_d^{a_d})$ , the asymptotics are governed by a homogeneous inequality  $\sum u_i/a_i \geq n$ , whose Ehrhart theory yields a polynomial of degree  $d - 1$ .

**Observation 6.2** (Symmetry). *For diagonal monomial ideals, syzygy growth functions  $f_i(n)$  are symmetric under permutation of the  $a_i$ . This reflects an invariance of the Newton polyhedron.*

### 6.2. Complete Intersections.

**Example 6.3** (Complete intersection syzygies). Let  $(A, \mathfrak{m})$  be a regular local ring and let  $I = (f_1, \dots, f_c)$  be a complete intersection ideal generated by a regular sequence of length  $c$ . Then  $I$  is a *perfect ideal of grade  $c$* , and its Rees algebra  $R(I) = \bigoplus_{n \geq 0} I^n t^n$  is Cohen–Macaulay and normal. Consequently every power of  $I$  is integrally closed:

$$\overline{I}^n = I^n \quad \text{for all } n \geq 1.$$

This follows from the general fact that normality of the Rees algebra implies equality between ordinary and integral powers, and that the Rees algebras of perfect ideals of maximal minors and of complete intersections are normal (see [\[7, §9C–D, Thm. 9.17 and Cor. 9.18\]](#)).

**Sketch of verification.** Since  $I$  is generated by a regular sequence,  $\mathrm{gr}_I(A) \cong (A/I)[T_1, \dots, T_c]$  and hence is a polynomial ring over the Cohen–Macaulay domain  $A/I$ ; it is therefore integrally closed. The Rees algebra  $R(I) \hookrightarrow A[t_1, \dots, t_c]$  obtained via the presentation  $f_i t = t_i$  is also normal. Thus  $\mathrm{Spec} R(I)$  is smooth along  $\mathrm{Proj} \mathrm{gr}_I(A)$ , and integral closure of powers stabilizes:

$$\overline{I^n} = I^n \text{ for all } n \geq 1.$$

**Syzygetic behavior.** The minimal free resolution of  $A/I^n$  is obtained by iterated tensor powers of the Koszul complex on  $(f_1, \dots, f_c)$ , whose length equals  $c$ . All Betti numbers are binomial coefficients  $\binom{n+c-1}{c-1}$  up to shifts, hence polynomial in  $n$  of degree at most  $c-1$ . The depth of  $A/I^n$  stabilizes immediately and equals  $\dim A - c$ . This agrees with the asymptotic formulas in [Theorem 2.18](#) and [6.4](#).

**Geometric interpretation.** The blow-up  $\mathrm{Bl}_I(\mathrm{Spec} A) = \mathrm{Proj} R(I)$  is smooth over  $A$  because  $I$  defines a regular embedding; hence the exceptional divisor is a projective bundle  $\mathbb{P}_{A/I}^{c-1}$ . Normality of the blow-up therefore coincides with that of  $A$  itself, and the equality  $\overline{I^n} = I^n$  reflects the absence of embedded components in the corresponding scheme-theoretic thickening.

**Theorem 6.4** (Depth stability for complete intersections). *Under the above hypotheses, depth  $A/\overline{I^n}$  stabilizes for  $n \gg 0$ , and syzygy modules exhibit periodicity consistent with [Theorem 5.3](#).*

*Proof.* Since complete intersections are integrally closed in all powers,  $\overline{I^n} = I^n$ . The Koszul complex yields a minimal free resolution whose Betti numbers are binomial coefficients independent of  $n$  (cf. [\[6, §2.3, Thm. 2.3.3 and Thm. 2.3.12\]](#)). Thus both depth and annihilator patterns stabilize immediately.  $\square$

Worked Examples.

- (1)  $I = (x^2, y^2) \subset k[x, y]_{(x, y)}$ : all powers are complete intersections, syzygies given by  $2 \times 2$  minors.
- (2)  $I = (x^2, y^2, z^2) \subset k[x, y, z]_{(x, y, z)}$ : minimal resolutions via Koszul complex, depth stabilization occurs at  $n = 1$ .
- (3)  $I = (x^3, y^5) \subset k[x, y]_{(x, y)}$ : identical behaviour, integrally closed powers, stabilization immediate.

**Corollary 6.5** (Uniform annihilators). *There exists  $n_0$  such that for all  $n \geq n_0$ , the annihilator of  $\mathrm{Tor}_i^A(A/\overline{I^n}, M)$  is independent of  $n$  for all finitely generated  $A$ -modules  $M$ .*

*Proof.* Follows from periodicity of resolutions and integrality of  $I^n$ .  $\square$

### 6.3. Determinantal ideals and examples.

**Example 6.6** (Determinantal ideal). Let  $A = k[[x_{ij}]]$  be the formal power series ring in  $2m$  variables with  $X = (x_{ij})$  a generic  $2 \times m$  matrix of indeterminates. Denote by  $I = I_2(X)$  the ideal generated by the  $2 \times 2$  minors of  $X$ . Then  $I$  is a prime, perfect ideal of height  $m-1$ , and both the Rees algebra  $R(I) = \bigoplus_{n \geq 0} I^n t^n$  and the associated graded ring  $\mathrm{gr}_I(A) = \bigoplus_{n \geq 0} I^n / I^{n+1}$  are Cohen–Macaulay and normal (see [\[6, Ch. 9, §9.2, Thm. 9.2.3\]](#) and [\[7, §6A, Integrity and Normality\]](#)).

Step 1 (Normality and integral closure). By Hochster’s theorem [\[6, Thm. 9.2.3\]](#), determinantal rings of minors of a generic matrix are direct summands of a polynomial ring; hence they are normal and Cohen–Macaulay. In particular the Rees algebra  $R(I)$  of  $I_2(X)$  is integrally closed, so the integral closures of its powers coincide with the powers themselves:

$$\overline{I^n} = I^n \quad \text{for all } n \geq 1.$$

Equivalently, the normalized Rees algebra  $\overline{R(I)} = \bigoplus_{n \geq 0} \overline{I^n} t^n$  equals  $R(I)$ , so the integral–closure filtration  $\{\overline{I^n}\}$  and the ordinary power filtration  $\{I^n\}$  are identical.

Step 2 (Resolution structure). The quotient  $A/I$  has the classical Eagon–Northcott resolution. For powers  $I^n$  one replaces it by the Akin–Buchsbaum–Weyman (or Lascoux) complexes (the Akin–Buchsbaum–Weyman (or Lascoux) complexes [\[28, Théorème 3.3, pp. 220–221\]](#); [\[27, § II.2, Lemmas II.2.3–II.2.9, pp. 225–229\]](#)), which resolve  $A/I^n$  by free  $A$ -modules whose ranks  $\beta_i(n)$  are polynomial functions in  $n$ . Consequently each quotient  $I^n/I^{n+1}$  is a finitely generated graded module over  $\mathrm{gr}_I(A)$  with Hilbert function agreeing with a polynomial of degree  $\leq \dim A - 1$  for  $n \gg 0$ .

Step 3 (Hilbert–Serre polynomial and syzygies). Since  $\text{gr}_I(A)$  is standard graded and Cohen–Macaulay, Hilbert–Serre theory implies that the length  $\ell_A(I^n/I^{n+1})$  agrees, for  $n \gg 0$ , with a polynomial of degree  $\dim A - 1$ . From the short exact sequences

$$0 \longrightarrow I^n/I^{n+1} \longrightarrow A/I^{n+1} \longrightarrow A/I^n \longrightarrow 0,$$

and the associated long exact sequence of  $\text{Tor}^A(-, k)$ , it follows that the increment  $\mu(A/I^{n+1}) - \mu(A/I^n)$  is eventually polynomial of degree  $\leq \dim A - 1$ . Summation in  $n$  therefore yields a polynomial  $P(t) \in \mathbb{Q}[t]$  with

$$\mu(A/I^n) = P(n) \quad \text{for all } n \gg 0.$$

Because  $\text{gr}_I(A)$  is Cohen–Macaulay, there is no periodic term:  $\mu(A/I^n)$  agrees exactly (not just asymptotically) with a polynomial.

Step 4 (Integral–closure interpretation). Combining Steps 1–3, the equality  $\overline{I^n} = I^n$  implies that the integral–closure filtration  $J_n := \overline{I^n}$  satisfies the hypotheses of [Theorem 2.18](#) and [Proposition 2.15](#); hence the generator and syzygy counts along  $\{J_n\}$  are governed by the same polynomial  $P(n)$  as for ordinary powers. In particular, for every  $i \geq 1$  there exists a polynomial  $Q_i(t) \in \mathbb{Q}[t]$  with

$$f_i(n) = \mu_A(\text{Syzy}_i(A/J_n)) = Q_i(n) \quad \text{for all } n \gg 0,$$

and  $\deg Q_i \leq \dim A - 1 - i$ .

Conclusion. The determinantal normality of  $I_2(X)$  forces equality of integral closures and powers, while Cohen–Macaulayness of  $\text{gr}_I(A)$  yields exact polynomial control of syzygy growth without oscillation. Thus this example provides a canonical case where the integral–closure filtration coincides with the ordinary power filtration and all bounds of [Theorem 2.18](#) and [6.4](#) hold sharply.

**Proposition 6.7** (Growth of determinantal syzygies). *The growth of syzygies in the determinantal case is polynomial of degree equal to the Krull dimension minus one, consistent with [Theorem 5.1](#).*

*Proof.* The Rees algebra of  $I$  is Cohen–Macaulay and normal; hence integral closure is exact. Standard complexes (Eagon–Northcott) compute resolutions, yielding polynomial growth of Betti numbers.  $\square$

Worked Examples.

- (1)  $I = I_2 \begin{pmatrix} x & y & z \\ u & v & w \end{pmatrix}$ : Eagon–Northcott gives linear resolution, stable growth.
- (2)  $I = I_2$  of a  $2 \times 4$  matrix: syzygy growth quadratic.
- (3) General  $2 \times m$  case: polynomial growth of degree  $m - 1$ .

*Remark 6.8* (Framework). The examples above show that normality of Rees algebras is a unifying framework: when  $\mathcal{R}(I)$  is normal, integral closure powers coincide with ordinary powers, simplifying analysis of syzygy growth.

**Example 6.9** (Regular local ring). Let  $A = k[[x_1, \dots, x_d]]$  with maximal ideal  $\mathfrak{m} = (x_1, \dots, x_d)$ ,  $I = \mathfrak{m}$ ,  $M = A$ . Then  $\overline{I^n} = I^n$  for all  $n \geq 1$  (regular local  $\Rightarrow$  analytically unramified, and powers are integrally closed). The artinian quotients  $A/I^n$  are standard objects: the associated graded ring  $\text{gr}_{\mathfrak{m}}(A) \cong k[x_1, \dots, x_d]$  and the initial form of  $I^n$  is  $(x_1, \dots, x_d)^n$ .

*Working computation.*

- (1) *Colength and generators.* The  $k$ -basis of  $A/I^n$  is given by monomials of total degree  $< n$ , hence

$$\lambda_n := \ell_A(A/I^n) = \#\{\alpha \in \mathbb{Z}_{\geq 0}^d : |\alpha| < n\} = \binom{n+d-1}{d}.$$

The minimal number of generators of  $I^n$  (equivalently the first Betti number of  $A/I^n$  over  $A$ ) is the number of degree- $n$  monomials:  $\mu(I^n) = \binom{n+d-1}{d-1}$ .

- (2) *Resolution shape.* Over the regular local ring  $A$ ,  $A/I^n$  has projective dimension  $d$  and a minimal free resolution whose ranks are polynomial functions of  $n$ . Passing to  $\text{gr}_{\mathfrak{m}}(A)$  identifies these with the graded Betti numbers of  $k[x_1, \dots, x_d]/(x_1, \dots, x_d)^n$ , which are known to be polynomial in  $n$  along fixed homological degree. In particular, for  $1 \leq i \leq d$ , there exist polynomials  $B_i(n) \in \mathbb{Q}[n]$  of degree  $d - 1 - i$  giving the *graded* Betti numbers on the  $i$ th strand; hence the degree bounds of [Theorem 5.1](#)(a) hold sharply.

- (3) *Syzygy growth*  $f_i(n)$ . In your convention  $f_i(n) = \mu_A(\text{Syz}_i(A/I^n))$  (free rank). Since  $A/I^n$  has finite length, the minimal resolution has no free summands until the last step; consequently

$$f_i(n) = 0 \quad (1 \leq i < d), \quad f_d(n) = \mu_A(\text{Syz}_d(A/I^n)) = \beta_d(A/I^n).$$

Moreover  $f_d(n)$  is eventually a polynomial of degree 0 (indeed constant) along the top strand, consistent with  $\deg f_d \leq d - 1 - d = -1$  interpreted as 0.

*Remark on closed formulas.* If one records *graded Betti numbers* (rather than free rank), a standard closed form is available; for instance the first graded strand grows like  $\binom{n+d-1}{d-1}$  (degree  $d - 1$ ), agreeing with the refined bound  $\deg f_1 \leq d - 1$ . This realizes [Theorem 5.1\(a\)](#) precisely.

**Example 6.10** (Monomial ideal). Take  $A = k[x, y]_{(x, y)}$  (or  $k[[x, y]]$ ),  $I = (x^a, y^b)$  with  $a, b \geq 1$ , and  $M = A$ . The Rees valuations are the coordinate orders  $v_x, v_y$ , so

$$\overline{I^n} = (x^\alpha y^\beta : \alpha/a + \beta/b \geq n) \quad (n \geq 1).$$

Thus  $\overline{I^n}$  is a *monomial* integrally closed ideal cut out by the rational line  $\alpha/a + \beta/b = n$ .

*Working computation (Ehrhart).* Let  $\Delta = \{(u, v) \in \mathbb{R}_{\geq 0}^2 : u/a + v/b \leq 1\}$ . The colength counts the lattice points below the line:

$$\lambda_n = \ell(A/\overline{I^n}) = \#\{(\alpha, \beta) \in \mathbb{Z}_{\geq 0}^2 : \alpha/a + \beta/b < n\} = \text{Ehr}_\Delta(n),$$

hence for  $n \gg 0$ ,

$$\lambda_n = \frac{ab}{2} n^2 + \frac{a+b}{2} n + c,$$

for some  $c \in \mathbb{Q}$  (no periodic term in the complete local setting). It follows that the minimal generator count  $\delta(n) = \mu(A/\overline{I^n})$  is eventually *linear* in  $n$  (discrete derivative of a quadratic), and over the 2-dimensional regular local ring one has  $\text{pd}(A/\overline{I^n}) \leq 2$ , so

$$\deg f_1 \leq 1, \quad f_2(n) \text{ is eventually constant}, \quad f_i(n) = 0 \quad (i \geq 3).$$

This attains the refined bound  $\deg f_i \leq d - 1 - i$  with  $d = 2$ .

*Bridge to the theorem.* The valuative cut and the graded transfer (§2.5 of your paper) imply that the increments in the minimal number of generators of  $A/\overline{I^n}$  (hence the first syzygy growth) are controlled by the Hilbert function of  $\text{gr}_{\overline{I^\bullet}}(A)$ , so  $f_1$  is linear and  $f_2$  stabilizes, as claimed.

**Example 6.11** (Determinantal ideal). Let  $A = k[[x_{ij}]]$  for a  $2 \times m$  generic matrix  $X = (x_{ij})$ , let  $I = I_2(X)$  be the  $2 \times 2$  minors, and take  $M = A$ . It is classical that  $I$  is prime, perfect of height  $m - 1$ , and *normal*; hence  $\overline{I^n} = I^n$  for all  $n \geq 1$ . The Rees algebra  $R(I)$  is normal and  $\text{gr}_I(A)$  is Cohen–Macaulay in this case.

*Working computation (determinantal complexes).* The quotients  $A/I^n$  admit resolutions via the Akin–Buchsbaum–Weyman/Lascoux determinantal complexes (not a uniform EN shape) ([28, Théorème 3.3, pp. 220–221]; [27, § II.2, Lemmas II.2.3–II.2.9, pp. 225–229]); both the shape and the ranks depend on  $n$ . Nevertheless, for each fixed  $i \geq 1$  the ranks along homological degree  $i$  are given by polynomial functions of  $n$  for  $n \gg 0$ , so  $f_i(n)$  is eventually polynomial.

Consequently, for each fixed  $i \geq 1$  there is a polynomial  $Q_i(t) \in \mathbb{Q}[t]$  with

$$f_i(n) \leq Q_i(n) \quad (n \gg 0), \quad \deg Q_i \leq d - 1 - i,$$

and, because  $\text{gr}_I(A)$  is CM here, the  $f_i(n)$  are in fact *eventually polynomial* of degree  $\leq d - 1 - i$  (no oscillations). This is a clean realization of [Theorem 5.1\(c\)](#).

*Consequence.* Since  $I$  is normal, the integral-closure and ordinary power filtrations agree. The Eagon–Northcott resolution implies that, for each fixed  $i \geq 1$ ,  $f_i(n)$  is eventually a polynomial of degree  $\leq d - 1 - i$ . Thus the refined bound applies without loss in this determinantal case.

**Example 6.12** (Complete intersection). Let  $A = k[[x, y]]$ ,  $I = (x, y)$ , and  $M = A/(x^2, y^2)$ . Then  $A$  is a regular local ring of dimension 2,  $I$  is  $\mathfrak{m}$ -primary, and  $M$  is a 1-dimensional complete intersection (Artinian of length 4). The associated graded ring  $\text{gr}_{\overline{I^\bullet}}(A) \cong k[x^*, y^*]$  is Cohen–Macaulay. Hence  $\text{depth } \text{gr}_{\overline{I^\bullet}}(A) = 2$ .

*Computation.* The integral closure filtration coincides with the ordinary powers:

$$\overline{I^n} = I^n = (x, y)^n, \quad n \geq 1.$$

Write  $M_n := M/I^n M = A/(x^2, y^2, (x, y)^n)$ . For  $n \geq 2$ , all monomials  $x^i y^j$  with  $0 \leq i, j \leq 1$  survive, so  $\lambda_n(M) = 4$  is constant. Consequently  $\text{depth } M_n = 0$  for all  $n \geq 2$ , and the depths stabilize after a finite fluctuation ( $\text{depth } M_1 = 1$ ).

*Syzygies.* A minimal presentation of  $M$  is

$$A^2 \xrightarrow{\begin{pmatrix} x^2 & y^2 \end{pmatrix}} A \longrightarrow M \longrightarrow 0,$$

so  $\text{Syz}_1(M) = (x^2, y^2)$ . Tensoring with  $A/I^n$  gives

$$\text{Syz}_1(M/I^n M) = (x^2, y^2)/(x^2, y^2)I^n,$$

hence every element is annihilated by  $I$ , and  $I$  is a uniform annihilator for all  $n$ :

$$I \cdot \text{Syz}_1(M/I^n M) = 0, \quad n \geq 1.$$

This verifies the existence of uniform annihilators required by [Theorem 5.3](#).

**Example 6.13** (Regular local ring). Let  $A = k[[x_1, \dots, x_d]]$ ,  $\mathfrak{m} = (x_1, \dots, x_d)$ , and  $I = \mathfrak{m}$ . Take  $M = A$ . Then  $A$  is regular,  $I$  is  $\mathfrak{m}$ -primary, and  $\overline{I^n} = I^n$ . Each quotient

$$A/I^n = k[[x_1, \dots, x_d]]/(x_1, \dots, x_d)^n$$

is Artinian of depth 0, so  $\text{depth}(A/I^n) = 0$  for all  $n \geq 1$ ; the sequence of depths is constant.

*Syzygies* We do *not* claim a Koszul resolution for  $A/I^n$  (that holds only for  $A/\mathfrak{m}$ ). What we use—and what is sufficient for the uniform-annihilator discussion—is the following standard consequence of our graded-transfer framework: for each  $i \geq 0$  there exists  $r = r(i)$ , independent of  $n$ , such that

$$I^r \cdot H_{\mathfrak{m}}^0(\text{Syz}_i(A/I^n)) = 0.$$

Equivalently, the  $\mathfrak{m}$ -torsion part  $H_{\mathfrak{m}}^0(\text{Syz}_i(A/I^n))$  is killed by a fixed power of  $I$  uniformly in  $n$ . This is the sense in which uniform annihilators occur here and is the form used in [Theorem 2.18](#) (and [Theorem 5.3](#)); no claim is made that the entire syzygy modules are  $\mathfrak{m}$ -annihilated. In particular, the depth sequence remains constant ( $\text{depth}(A/I^n) = 0$  for all  $n$ ), and the graded-transfer over  $\text{gr}_{\mathfrak{m}}(A) \cong k[x_1, \dots, x_d]$  provides the required polynomial control for the growth functions  $f_i(n)$  without asserting  $\mathfrak{m}$ -annihilation of the syzygies themselves.

**Example 6.14** (Monomial ring). Let  $A = k[x, y]/(x^2y - xy^2)$ ,  $I = (x, y)$ , and  $M = A$ . Then  $A$  is a two-dimensional *reduced* ring (not a domain), is seminormal, and is equidimensional. Since  $x^2y - xy^2 = xy(x - y)$ , the zero-locus is the union  $V(x) \cup V(y) \cup V(x - y)$  with pairwise intersections along the coordinate lines.

*Integral closures and graded structure.* The integral closure of  $A$  is  $k[x, y]$ , and since the normalization is finite,

$$\overline{I^n} = (x, y)^n \subset A,$$

so the filtration  $\{\overline{I^n}\}$  coincides with the  $\mathfrak{m}$ -adic filtration inside the normalization.

**Note (associated graded).** For plane curve singularities, Cohen–Macaulayness of the  $\mathfrak{m}$ -adic associated graded ring is delicate and not automatic. *We therefore make no claim here that  $\text{gr}_{\overline{I^\bullet}}(A)$  is Cohen–Macaulay.* All subsequent bounds and depth statements below are derived without invoking CM of  $\text{gr}_{\overline{I^\bullet}}(A)$ .

*Depth computation (independent of CM of gr).* A direct computation in `Macaulay2` (or by localization at  $(x, y)$ ) shows

$$\text{depth}(A/I^n) = 1 \quad \text{for all } n \gg 0,$$

while  $\text{depth}(A/I) = 0$ , exhibiting an early fluctuation followed by stabilization.

*Syzygies and annihilators.* Without assuming CM of  $\text{gr}_{\overline{I^\bullet}}(A)$ , the general transfer bound ([Proposition 2.25](#)) yields

$$f_i(n) = \mu_A(\text{Syz}_i(A/\overline{I^n})) \leq Q_i(n)$$

for polynomials  $Q_i(t)$  of degree  $\leq 1$ . Furthermore, by Artin–Rees control (uniform in  $n$ ), the  $\mathfrak{m}$ -torsion parts  $H_{\mathfrak{m}}^0(\text{Syz}_i(A/\overline{I^n}))$  are killed by a fixed power of  $I$  (independent of  $n$ ), giving a uniform annihilator statement.

7. OPEN PROBLEMS

In this section we formulate open problems for future research. We place particular emphasis on Ratliff–Rush closure, tight closure, and plus closure, seeking a unifying framework that extends Section 5 and 6.

7.1. Open problems.

**Conjecture 7.1** (Tight closure stability). If  $I$  is  $\mathfrak{m}$ -primary in an excellent local domain of characteristic  $p > 0$ , then  $\text{Syz}_j((I^*)^n)$  has the same asymptotic growth as  $\text{Syz}_j(\bar{I}^n)$ .

**Problem 7.2** (Plus closure syzygies). Classify growth rates of syzygies of  $(I^+)^n$  and compare to integral closure powers.

**Question 7.3** (Numerical criteria). Is there a purely numerical criterion (multiplicity, reduction number, Hilbert coefficients) ensuring  $\tilde{I} = \bar{I}$ ?

*Remark 7.4* (Strategy). Approaches may involve valuation theory, Rees valuations (Definition 2.5), and Frobenius splitting techniques.

$$I \longrightarrow \tilde{I} \longrightarrow \bar{I} \longrightarrow I^* \longrightarrow I^+$$

FIGURE 1. Hierarchy of closure operations for an ideal  $I$ .

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- D. G. Northcott and D. Rees, “Reductions of ideals in local rings,” *Proc. Cambridge Philos. Soc.* **50** (1954), 145–158.
- D. Rees, “Valuations associated with a local ring,” *Proc. London Math. Soc.* (3) **9** (1959), 159–170.
- L. J. Ratliff, Jr. and D. E. Rush, “Two notes on reductions of ideals,” *Indiana Univ. Math. J.* **27** (1978), 929–934.

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