Relative Modular Dynamics for Density Operators

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Abstract

Building on the canonical emergence of the density operator ρ from normalized probabilistic structure (Paper A), we develop the next canonical layer: modular generators and CPTP-compatible evolution. For faithful states, the modular generator $K_\rho := -\log\rho$ is well-defined, unitary covariant, and additive under tensor products. Introducing a phase reference state σ yields the relative modular generator $K_{\rho|\sigma} = -\log\rho + \log\sigma$, naturally tied to the relative entropy $D(\rho\|\sigma)$. We define the minimal unitary "relative modular flow" $\dot{\rho} = -i[K_{\rho|\sigma}, \rho]$, and then derive a physically consistent completion by augmenting the flow with GKSL dissipators, including an entropy-relaxing channel with fixed point σ and an optional classicalization channel implementing decoherence in a pointer algebra. The resulting equation constitutes a conservative, testable backbone for subsequent work on informational phases and spectral diagnostics.

1 Introduction

Paper A established a minimal representation-invariant passage from normalized probabilistic structure to operator states: probabilistic assignments to admissible effects can be represented as $p(E) = \mathrm{Tr}(\rho E)$ with $\rho \succeq 0$ and $\mathrm{Tr}\rho = 1$. The natural next question is: *given a state ρ, what is the canonical generator of its evolution compatible with representation invariance, composition, and physical constraints (positivity and normalization)?*

This paper answers in two steps. First, we identify the modular generator $K_\rho = -\log\rho$ as the canonical spectral coordinate derived from ρ alone, and introduce a phase reference state σ to form the relative modular generator $K_{\rho|\sigma} = -\log\rho + \log\sigma$. Second, we construct the minimal physical completion of the corresponding unitary flow by adding completely-positive trace-preserving (CPTP) dissipators of GKSL type, including an entropy-relaxing component with stationary state σ.

Scope. We focus on finite-dimensional Hilbert spaces for clarity and to keep the paper logically closed. We only use the operator-algebraic (Tomita–Takesaki) perspective as motivation and in the outlook. Phases, emergent geometry, and numerical spectral diagnostics are deferred to later papers.

2 Preliminaries: states, faithfulness, and logarithms

Let $\mathcal{H}$ be a finite-dimensional Hilbert space, $B(\mathcal{H})$ the algebra of linear operators on $\mathcal{H}$. A *density operator* is $\rho \in B(\mathcal{H})$ with $\rho \succeq 0$ and $\mathrm{Tr}\rho = 1$.

Definition 1 (Faithful state)**.** A state ρ is *faithful* if $\rho \succ 0$ (equivalently, all eigenvalues are strictly positive). In this case, $\log\rho$ is well-defined by functional calculus.

Remark 1 (Regularization when ρ is not faithful)**.** If ρ has a nontrivial kernel, one may use a standard full-rank regularization

$$\rho_\varepsilon := (1-\varepsilon)\rho + \varepsilon\,\mathbb{1}/d, \qquad \varepsilon \in (0,1), \quad d = \dim\mathcal{H},$$

so that $\rho_\varepsilon \succ 0$ and $\rho_\varepsilon \to \rho$ in trace norm as $\varepsilon \downarrow 0$.

3 Modular generator from the state

Definition 2 (Modular generator)**.** For a faithful state $\rho \succ 0$, define the modular generator

$$K_\rho := -\log \rho.$$

Theorem 1 (Existence, covariance, and additivity of K_ρ)**.** *Let $\rho \succ 0$ be a faithful state on finite-dimensional $\mathcal{H}$. Then $K_\rho = -\log \rho$ exists, is Hermitian, and satisfies:*

1. ***Unitary covariance:*** *for any unitary U,*

$$K_{U\rho U^\dagger} = U K_\rho U^\dagger.$$

2. ***Tensor additivity:*** *for faithful states ρ_A, ρ_B on $\mathcal{H}_A, \mathcal{H}_B$,*

$$K_{\rho_A \otimes \rho_B} = K_{\rho_A} \otimes \mathbb{1}_B + \mathbb{1}_A \otimes K_{\rho_B}.$$

Proof (sketch). Diagonalize $\rho = \sum_i \lambda_i |i\rangle\langle i|$ with $\lambda_i > 0$ and define $\log \rho = \sum_i (\log \lambda_i)|i\rangle\langle i|$. Unitary covariance follows from functional calculus: $f(U\rho U^\dagger) = U f(\rho) U^\dagger$ for $f(x) = \log x$. Tensor additivity follows from $\log(\rho_A \otimes \rho_B) = \log \rho_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes \log \rho_B$. □

Remark 2 (Spectral coordinates)**.** If $\{\lambda_i\}$ are eigenvalues of ρ, then eigenvalues of K_ρ are $-\log \lambda_i$. Thus K_ρ is a canonical "spectral coordinate" of ρ.

4 Relative modular generator and phase reference

To connect state evolution to a phase reference and to relative entropy, we introduce a reference state σ (e.g., a MaxEnt state at fixed macroscopic constraints).

Definition 3 (Phase reference state)**.** A *phase reference* is a faithful state $\sigma \succ 0$ treated as fixed within a given phase (or macroscopic constraint class). In later papers, σ may be specified as a MaxEnt state under chosen constraints.

Definition 4 (Relative modular generator)**.** For faithful $\rho, \sigma \succ 0$, define the relative modular generator

$$K_{\rho|\sigma} := -\log \rho + \log \sigma.$$

Definition 5 (Relative entropy)**.** For faithful $\rho, \sigma \succ 0$, the quantum relative entropy is

$$D(\rho\|\sigma) := \mathrm{Tr}\big(\rho(\log \rho - \log \sigma)\big).$$

Remark 3 (Link between $D(\rho\|\sigma)$ and $K_{\rho|\sigma}$)**.** By definition,

$$D(\rho\|\sigma) = -\mathrm{Tr}(\rho\, K_{\rho|\sigma}).$$

Thus the relative modular generator is the canonical operator appearing in the state–reference distinguishability functional.

5 Minimal unitary relative modular flow

Definition 6 (Unitary relative modular flow)**.** Fix a reference $\sigma \succ 0$. The unitary relative modular flow is defined by

$$\frac{d\rho}{dt} = -i[K_{\rho|\sigma}, \rho]. \tag{1}$$

Theorem 2 (Isospectrality and conservation laws)**.** *For the unitary flow* (1)*, the trace* $\mathrm{Tr}\rho$ *is conserved, positivity is preserved, and the spectrum of* ρ *is invariant in time. Consequently, the von Neumann entropy* $S(\rho) = -\mathrm{Tr}(\rho \log \rho)$ *is conserved.*

Proof (sketch). Equation (1) is a commutator evolution (Liouville form). Commutators have zero trace, hence $\frac{d}{dt}\mathrm{Tr}\rho = 0$. A commutator generator corresponds to unitary similarity in the operator space, preserving eigenvalues and positivity. Since $S(\rho)$ depends only on the spectrum, it is conserved. □

Remark 4 (Why a dissipative completion is needed)**.** While (1) is canonical and representation-invariant, it cannot describe irreversible phenomena (relaxation, decoherence, entropy production). Physical open-system consistency suggests a CPTP completion.

6 CPTP-compatible completion: GKSL dynamics

6.1 GKSL generators

Definition 7 (GKSL dissipator)**.** A GKSL (Lindblad) dissipator on $B(\mathcal{H})$ is

$$\mathcal{D}[\rho] := \sum_{\alpha} \gamma_\alpha \Big(L_\alpha \rho L_\alpha^\dagger - \tfrac{1}{2}\{L_\alpha^\dagger L_\alpha, \rho\} \Big), \quad \gamma_\alpha \geq 0,$$

where $\{L_\alpha\}$ are (not necessarily Hermitian) Lindblad operators.

Remark 5 (Physical meaning)**.** GKSL generators are the general form of Markovian, time-homogeneous, completely positive trace-preserving semigroup dynamics on finite-dimensional systems.

6.2 Entropy-relaxing channel with stationary state σ

Assumption 1 (Entropy-relaxing dissipator with fixed point)**.** We assume an entropy-relaxing GKSL generator $\mathcal{D}_{\mathrm{ent}}^{\sigma}$ satisfying:

1. $\mathcal{D}_{\mathrm{ent}}^{\sigma}[\sigma] = 0$ (stationary reference),
2. $\mathcal{D}_{\mathrm{ent}}^{\sigma}$ satisfies the conditions under which the Spohn inequality holds (e.g., quantum detailed balance w.r.t. σ).

Theorem 3 (Monotonicity of relative entropy (Spohn-type))**.** *Under Assumption 1, for the dissipative evolution* $\dot{\rho} = \mathcal{D}_{\mathrm{ent}}^{\sigma}[\rho]$,

$$\frac{d}{dt} D(\rho(t) \| \sigma) \leq 0.$$

Proof (sketch). This is a standard entropy production inequality for GKSL semigroups with stationary state σ under the hypotheses of Spohn's theorem (or, more generally, monotonicity of relative entropy under CPTP semigroups with fixed point and appropriate balance conditions). □

6.3 Optional classicalization channel

Definition 8 (Pointer (classicalization) channel)**.** Let $\{P_\mu\}$ be an orthogonal resolution of identity ($P_\mu P_\nu = \delta_{\mu\nu} P_\mu$, $\sum_\mu P_\mu = \mathbb{1}$). Define the completely positive projector

$$P_Z(\rho) := \sum_{\mu} P_\mu \rho P_\mu,$$

and the associated dephasing dissipator

$$\mathcal{D}_{\mathrm{class}}^{Z}[\rho] := \kappa\big(P_Z(\rho) - \rho\big), \qquad \kappa \geq 0.$$

Remark 6. $\mathcal{D}^Z_{\text{class}}$ suppresses coherences between the P_μ sectors and may be viewed as enforcing a pointer algebra (or center) Z relevant for a given phase.

7 The UMD equation (Paper B main statement)

We now combine the canonical unitary relative modular flow with CPTP-compatible dissipators.

Proposition 1 (Universal modular dynamics (finite-dimensional form))**.** *Fix a faithful reference state $\sigma \succ 0$. Let $\mathcal{D}^\sigma_{\text{ent}}$ be an entropy-relaxing GKSL generator with fixed point σ, and optionally include a pointer (classicalization) dissipator $\mathcal{D}^Z_{\text{class}}$. The universal modular dynamics is the evolution equation*

$$\frac{d\rho}{dt} = -i[K_{\rho|\sigma}, \rho] + \mathcal{D}^\sigma_{\text{ent}}[\rho] + \mathcal{D}^Z_{\text{class}}[\rho]. \tag{2}$$

Remark 7 (Minimality and testability)**.** Equation (2) is minimal in the sense that it combines (i) a canonical state-derived generator (relative modular commutator) and (ii) the most general Markovian CPTP completion (GKSL) compatible with positivity and normalization. Subsequent papers specify how σ and Z are selected by macroscopic constraints and phase criteria.

8 Consequences and links to spectral diagnostics

8.1 Spectral coordinates and quantiles

Let $\lambda_1 \geq \cdots \geq \lambda_d > 0$ be eigenvalues of ρ and define spectral coordinates $k_i = -\log \lambda_i$. For quantiles $q \in (0, 1)$ one may define

$$k_q := -\log \lambda_q,$$

where λ_q is the q-quantile of the eigenvalue distribution (with a chosen convention). These quantities are natural observables of the modular generator K_ρ.

8.2 Commutator diagnostics

Given an observable O, define a normalized commutator diagnostic

$$L(\rho; O) := \frac{\|[K_\rho, O]\|_F}{\|K_\rho\|_F \, \|O\|_F},$$

where $\|\cdot\|_F$ is the Frobenius norm. Such diagnostics connect modular spectral data to operational non-commutativity and will be used to define phase-sensitive running exponents in later numerical work.

Remark 8 (Why this matters)**.** Under (2), the spectrum of ρ is no longer conserved (due to dissipation), making spectral flows $k_q(t)$ and commutator statistics $L(t)$ meaningful dynamical probes.

9 Discussion and outlook

Paper A established ρ as the canonical convex, basis-invariant representative of normalized probabilistic structure. Paper B establishes the next canonical layer: $K_\rho = -\log \rho$ as a state-derived generator with strong invariance properties, the relative generator $K_{\rho|\sigma}$ tied to $D(\rho\|\sigma)$, and a minimal CPTP-compatible completion via GKSL dissipators. This provides the conservative backbone for subsequent developments:

- Paper C: phase criteria and emergent geometry from modular spectral structure;

- Paper D: numerical spectral diagnostics (running exponents, bootstrap stability, IR drift);
- Paper E: interpretation of "stringiness" and matter modes as phase regimes and spectral patterns rather than as assumed microscopic primitives.

REFERENCES

References

[1] J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, 1955.

[2] R. Kubo, Statistical-mechanical theory of irreversible processes. I. General theory and simple applications to magnetic and conduction problems, *Journal of the Physical Society of Japan* **12** (1957), 570–586.

[3] M. Takesaki, Tomita's theory of modular Hilbert algebras and its applications, *Lecture Notes in Mathematics* **128**, Springer, 1970.

[4] M. Takesaki, *Theory of Operator Algebras I*, Springer-Verlag, New York, 1979.

[5] M. Tomita, *Theory of Modular Hilbert Algebras and Its Applications*, Lecture notes (unpublished), 1967.

[6] R. Haag, N. M. Hugenholtz, and M. Winnink, On the equilibrium states in quantum statistical mechanics, *Communications in Mathematical Physics* **5** (1967), 215–236.

[7] G. Lindblad, On the generators of quantum dynamical semigroups, *Communications in Mathematical Physics* **48** (1976), 119–130.

[8] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, Completely positive dynamical semigroups of N-level systems, *Journal of Mathematical Physics* **17** (1976), 821–825.

[9] H. Spohn, Entropy production for quantum dynamical semigroups, *Journal of Mathematical Physics* **19** (1978), 1227–1230.

[10] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems*, Oxford University Press, 2002.

[11] R. Alicki and K. Lendi, *Quantum Dynamical Semigroups and Applications*, 2nd ed., Springer, 2007.

[12] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, 10th Anniversary Edition, Cambridge University Press, 2010.