Conditional Degree-One Boundary Decompositions for Hodge-Type Shimura Varieties at Hyperspecial Level

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Abstract

We study the low-degree Leray spectral sequence attached to the open immersion of an integral canonical model of a Hodge-type Shimura variety into a toroidal compactification. Under an explicit package of auxiliary assumptions on boundary regularity, cohomological purity, constructibility of the degree-one boundary sheaf, and vanishing of the degree-one Leray transgression, we obtain a low-degree edge-surjectivity statement and a Hecke-equivariant short exact sequence in degree one. We then formulate a conditional boundary criterion, expressed in terms of rational boundary tori, for the vanishing of the degree-one boundary contribution, assuming a compatible description of the boundary module. The resulting consequences are formal and conditional, and examples are included to illustrate both anisotropic and non-anisotropic situations.

Keywords: Shimura varieties; integral canonical models; Hodge type; toroidal compactification; étale cohomology; Leray spectral sequence; Hecke correspondences; boundary cohomology; Hecke-equivariant splitting; anisotropy; toroidal compactification; degree-one Leray obstruction.

MSC 2020: Primary: 14G35, 14F20; Secondary: 11F80, 11G18, 14F30, 11F67, 14C30, 14G10.

Contents

1 Introduction and Main Results

Motivation

The cohomology of Shimura varieties carries rich arithmetic information, including automorphic representations and the associated Galois representations. A central problem is to understand the interaction between the geometry of integral canonical models and the structure of low-degree cohomology groups.

In degree one, cohomology typically receives contributions from two sources: cuspidal automorphic forms and Eisenstein phenomena arising from boundary strata. While such decompositions are well understood over the complex numbers or on generic fibers, a purely *integral* and Hecke-equivariant description of this separation for integral models has not been isolated in general.

The goal of this paper is to isolate a conditional integral framework for the degree-one boundary contribution to

$$H^1_{ét}(\mathscr{S}_K, \Lambda)$$

for integral canonical models of Shimura varieties of Hodge type. More precisely, we define an interior kernel and a boundary quotient, prove a conditional Hecke-equivariant short exact sequence relating them, and study the formal consequences of a chosen splitting hypothesis after localization.

Setting

Let (G, X) be a Shimura datum of Hodge type, let $E = E(G, X)$ be its reflex field, and let $v \mid p$ be a place of E above a rational prime p such that $G_{\mathbb{Q}_p}$ is unramified. Let $K = K^{(p)}K_p$ be a compact open subgroup with K_p hyperspecial. Denote by

$$\mathscr{S}_K$$

the integral canonical model over $\mathrm{Spec}(\mathcal{O}_{E,(v)})$ and by

$$j : \mathscr{S}_K \hookrightarrow \mathscr{S}_K^{\mathrm{tor}}$$

the open immersion into a toroidal compactification.

Main results

Theorem 1.1 (Conditional low-degree edge statement)**.** *Assume the boundary regularity, purity, and transgression-vanishing hypotheses of Assumption 4.1. Then for finite coefficients* Λ *with* $\mathrm{char}(\Lambda) \neq p$*, the Leray spectral sequence for*

$$j : \mathscr{S}_K \hookrightarrow \mathscr{S}_K^{\mathrm{tor}}$$

has vanishing differential $d_2^{0,1}$*, and hence the edge morphism*

$$H^1_{ét}(\mathscr{S}_K, \Lambda) \longrightarrow H^0\big(\mathscr{S}_K^{\mathrm{tor}}, R^1 j_* \Lambda\big)$$

is surjective.

Theorem 1.2 (Conditional Hecke splitting after localization)**.** *Assume the hypotheses of Theorem 1.1 together with a chosen Hecke-equivariant splitting hypothesis after inverting a finite set of primes* Σ*. Then the degree-one cohomology admits a Hecke-equivariant decomposition into an interior kernel and a chosen complementary summand.*

Theorem 1.3 (Conditional anisotropy criterion in degree one)**.** *Under the hypotheses of Theorems 1.1 and 1.2, vanishing of the degree-one boundary contribution is equivalent to vanishing of the* $\mathbb{Q}$*-split part of every rational boundary torus. In this case the boundary quotient vanishes; equivalently, after inverting* Σ *one has*

$$H^1_{ét}(\mathscr{S}_K, \Lambda[1/\Sigma]) = H^1_{\mathrm{int}}(\mathscr{S}_K, \Lambda[1/\Sigma]),$$

and any chosen complementary summand is zero.

Structure of the paper

Section 2 recalls background material on Shimura data, integral models, and compactifications. Section 3 develops the cohomological tools needed for the analysis of the Leray spectral sequence. Section 4 establishes the geometric properties of the boundary needed for the main arguments. The proofs of the main conditional statements are given in Section 5, followed by examples. No separate applications section is included in the present version.

2 Preliminaries

In this section we collect the background needed throughout the paper. All statements are classical and are included only once, with explicit references. Novel contributions begin only in later sections. Notation is fixed globally to ensure consistency.

2.1 Algebraic geometry background

Notation / Convention 2.1 (Schemes and morphisms)**.** All schemes are assumed to be separated and of finite type over the base scheme specified in the surrounding discussion. In the Shimura-variety sections, the relevant bases are typically $\mathrm{Spec}(\mathcal{O}_{E,(v)})$. For a scheme X, we denote its structure sheaf by $\mathcal{O}_X$, and write $\Gamma(X, \mathcal{O}_X)$ for its ring of global sections. If $f: X \to Y$ is a morphism, we write X_y for the fiber over a point $y \in Y$.

Definition 2.2 (Flatness and smoothness)**.** A morphism $f: X \to Y$ is *flat* if $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,f(x)}$ for all $x \in X$. It is *smooth* if it is flat, locally of finite presentation, and has smooth geometric fibers [4, IV_4, 17.5.1].

2.2 Number theoretic foundations

Notation / Convention 2.3 (Adeles and Galois groups)**.** We denote by $\mathbb{A}_f$ the finite adeles of $\mathbb{Q}$, and by $\widehat{\mathbb{Z}}$ the profinite completion of $\mathbb{Z}$. For a number field F, let $G_F = \mathrm{Gal}(\overline{F}/F)$ be its absolute Galois group.

Proposition 2.4 (Chebotarev density, well known)**.** *Let L/F be a finite Galois extension of number fields with group $G = \mathrm{Gal}(L/F)$. For any conjugacy class $C \subseteq G$, the set of unramified primes $\mathfrak{p}$ of F whose Frobenius element $\mathrm{Frob}_{\mathfrak{p}}$ lies in C has density $|C|/|G|$.*

Proof. Classical; see [5, Chap. VII]. □

2.3 Shimura varieties: notation and conventions

Definition 2.5 (Shimura datum)**.** A *Shimura datum* is a pair (G, X) where G is a connected reductive group over $\mathbb{Q}$ and X is a $G(\mathbb{R})$-conjugacy class of homomorphisms $h: \mathbb{S} \to G_{\mathbb{R}}$, with $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$, satisfying Deligne's axioms [1].

Notation / Convention 2.6 (Reflex field and level structures)**.** Given a Shimura datum (G, X), we write $E(G, X)$ for its reflex field. For a compact open subgroup $K \subseteq G(\mathbb{A}_f)$, we denote the Shimura variety by

$$\mathrm{Sh}_K(G, X) = G(\mathbb{Q})\backslash\big(X \times G(\mathbb{A}_f)/K\big).$$

Proposition 2.7 (Existence of canonical models)**.** *Let (G, X) be a Shimura datum and $K \subseteq G(\mathbb{A}_f)$ compact open. Then $\mathrm{Sh}_K(G, X)$ admits a canonical model over the reflex field $E(G, X)$.*

Proof. This is Deligne's construction of canonical models of Shimura varieties [1]. □

Construction 2.8 (Integral canonical models)**.** Suppose (G, X) is of Hodge type, let $E = E(G, X)$ be the reflex field, and let $v \mid p$ be a place of E above a prime p at which G is unramified. For sufficiently small level with hyperspecial K_p, the work of Kisin provides an integral canonical model $\mathscr{S}_K(G, X)$ over $\mathrm{Spec}(\mathcal{O}_{E,(v)})$, characterized by the usual extension property for morphisms from regular schemes.

3 Cohomological Framework

This section develops the cohomological tools required for our analysis of integral models of Shimura varieties introduced in Section 2. The emphasis is on the interplay between sheaf-theoretic constructions, arithmetic cohomology, and comparison results that link étale, de Rham, and crystalline settings. Throughout, notation is as in Section 2.

3.1 Sheaf-theoretic tools

Definition 3.1 (Constructible sheaves)**.** Let X be a scheme of finite type over $\mathrm{Spec}(\mathbb{Z})$. A sheaf $\mathcal{F}$ of abelian groups on $X_{ét}$ is *constructible* if there exists a finite stratification of X into locally closed subschemes such that $\mathcal{F}$ restricts to a locally constant sheaf of finite type on each stratum [3, Exp. IX].

Notation / Convention 3.2 (Standing coefficients and invertibility)**.** Fix a rational prime p, and fix once and for all a prime $\ell \neq p$.

Throughout this section (and in any application of smooth/proper base change) we take Λ to be a finite ℓ-power ring so that $\mathrm{char}(\Lambda)$ is invertible on any base $S/\operatorname{Spec}(\mathcal{O}_{E,(v)})$ under consideration.

Lemma 3.3 (Well-known: proper base change in the étale setting)**.** *Let $f\colon X \to S$ be proper with S noetherian and Λ a finite ring with characteristic invertible on S. Then for any constructible $\mathcal{F}$ on $X_{ét}$ and $s \in S$, the canonical specialization map*

$$(R^i f_{ét*}\mathcal{F})_{\bar{s}} \longrightarrow H^i_{ét}(X_{\bar{s}}, \mathcal{F}|_{X_{\bar{s}}})$$

is an isomorphism.

Proof: Outline via the proper base change formalism. **See [3, Exp. XII].**

Let $f : X \to S$ be a proper morphism with S noetherian, and let Λ be a finite ring whose characteristic is invertible on S. For a constructible sheaf $\mathcal{F}$ on $X_{ét}$, the derived proper base change theorem [3, Exp. XII, Th. 5.1] furnishes a natural base–change morphism

$$\mathbf{L}s^* \, \mathbf{R}f_*\mathcal{F} \longrightarrow \mathbf{R}f_{s,*}\, \mathcal{F}|_{X_s}$$

for any morphism $s : \mathrm{Spec}(k(s)) \to S$, functorial in $\mathcal{F}$, compatible with long exact cohomology sequences, and compatible with change of the base S. When s is a geometric point $\bar{s} \to S$, the total derived morphism induces isomorphisms on cohomology sheaves, yielding

$$\big(R^i f_{ét*}\mathcal{F}\big)_{\bar{s}} \xrightarrow{\sim} H^i_{ét}\big(X_{\bar{s}},\, \mathcal{F}|_{X_{\bar{s}}}\big) \quad \text{for all } i \geq 0.$$

The key inputs are: (i) constructibility of $\mathbf{R}f_*\mathcal{F}$ under proper f; (ii) cohomological descent and the compatibility of the formation of higher direct images with proper base change; and (iii) the invertibility of $\mathrm{char}(\Lambda)$ on S ensuring the usual finiteness and continuity conditions.See [3, Exp. XII, Th. 5.1] for the proper base change theorem in the derived setting, from which the above stalkwise identification follows by evaluating at $\bar{s}$.

□

3.2 Comparison lemmas and spectral sequences

Construction 3.4 (Leray spectral sequence)**.** Let $f\colon X \to Y$ be a morphism of schemes over $\mathrm{Spec}(\mathbb{Z})$. For a sheaf $\mathcal{F}$ on $X_{ét}$ there is a spectral sequence

$$E_2^{p,q} = H^p_{ét}(Y, R^q f_{ét*}\mathcal{F}) \Rightarrow H^{p+q}_{ét}(X, \mathcal{F}),$$

natural in $\mathcal{F}$ [3, Exp. V].

Proposition 3.5 (Leray spectral sequence for the structure morphism)**.** *Let (G, X) be a Shimura datum of Hodge type with integral canonical model $\mathscr{S}_K$ over $\mathrm{Spec}(\mathcal{O}_{E,(v)})$, and let*

$$f : \mathscr{S}_K \to \mathrm{Spec}(\mathcal{O}_{E,(v)})$$

be the structure morphism. For every torsion étale sheaf Λ *on* $(\mathscr{S}_K)_{\acute{e}t}$ *whose order is prime to* p*, there is a natural Leray spectral sequence*

$$E_2^{a,b} = H^a_{\acute{e}t}\big(\mathrm{Spec}(\mathcal{O}_{E,(v)}), R^b f_{\acute{e}t*}\Lambda\big) \Longrightarrow H^{a+b}_{\acute{e}t}(\mathscr{S}_K, \Lambda).$$

Proof. This is the standard Leray spectral sequence for the morphism f; see Theorem 3.4. □

4 Integral Models over the Localized Reflex Ring

We keep the global conventions of Section 2, in particular the notation of Theorems 2.1, 2.5 and 2.6 and the existence statement for integral canonical models in Theorem 2.8. Cohomological tools from Section 3 (e.g. Theorems 3.3 to 3.5) will be used repeatedly.

4.1 Construction of integral models

Notation / Convention 4.1 (Neat level, hyperspecial factor, and reflex-ring base)**.** Let (G, X) be a Shimura datum of Hodge type. Fix a rational prime p, let $E = E(G, X)$ be the reflex field, and let $v \mid p$ be a place of E. Factor $K = K^{(p)}K_p$ with $K^{(p)} \subset G(\mathbb{A}_f^{(p)})$ neat and $K_p \subset G(\mathbb{Q}_p)$ hyperspecial. Write $\mathscr{S}_K = \mathscr{S}_K(G, X)$ for the integral canonical model over $\mathrm{Spec}(\mathcal{O}_{E,(v)})$.

Lemma 4.2 (Well known: extension property)**.** *Assume* (G, X) *is of Hodge type and* K_p *hyperspecial. Then* $\mathscr{S}_K$ *is normal and satisfies the (regular) extension property: for any regular, locally noetherian* $\mathbb{Z}_{(p)}$*-scheme* T *with function field* $K(T)$*, every morphism* $T_\eta = \mathrm{Spec}\, K(T) \to \mathscr{S}_K$ *extends uniquely to* $T \to \mathscr{S}_K$*.*

Proof. This is part of the standard characterization of integral canonical models at hyperspecial level. For the purposes of the present note we use it as a standard input in the Hodge-type setting; see [2] for the Siegel case and the surrounding literature on Hodge-type integral models. □

Proposition 4.3 (Standard: smoothness at hyperspecial level)**.** *If* K_p *is hyperspecial and* (G, X) *is unramified at* p*, then* $\mathscr{S}_K$ *is smooth over* $\mathrm{Spec}(\mathcal{O}_{E,(v)})$*.*

Proof. Classical in the hyperspecial unramified setting; compare [1] for the passage to canonical models and [2] for the Siegel modular case. In the Hodge-type setting we use this as a standard input from the theory of integral canonical models. □

4.2 Properties under base change

Proposition 4.4 (Proper base change on the toroidal compactification)**.** *Let*

$$\bar{f} : \mathscr{S}_K^{\mathrm{tor}} \to \mathrm{Spec}(\mathcal{O}_{E,(v)})$$

be the structural morphism of a toroidal compactification, and let $\mathcal{F}$ *be a constructible* Λ*-sheaf on* $(\mathscr{S}_K^{\mathrm{tor}})_{\acute{e}t}$*, where* Λ *is finite of characteristic prime to* p*. Then for each* $i \geq 0$ *and each geometric point* $\bar{s} \to \mathrm{Spec}(\mathcal{O}_{E,(v)})$*, the canonical specialization map*

$$\big(R^i \bar{f}_{\acute{e}t*}\mathcal{F}\big)_{\bar{s}} \xrightarrow{\ \sim\ } H^i_{\acute{e}t}\big((\mathscr{S}_K^{\mathrm{tor}})_{\bar{s}}, \mathcal{F}|_{(\mathscr{S}_K^{\mathrm{tor}})_{\bar{s}}}\big)$$

is an isomorphism.

Proof. This is the proper base change theorem applied to the proper morphism $\bar{f}$; see Theorem 3.3. □

4.3 Boundary components and compactifications

Definition 4.5 (Toroidal compactification)**.** Let $\mathscr{S}_K$ be as above. A toroidal compactification $\mathscr{S}_K^{\mathrm{tor}}$ is a *regular*, proper $\mathcal{O}_{E,(v)}$-scheme containing $\mathscr{S}_K$ as an open dense subscheme such that the boundary $D := \mathscr{S}_K^{\mathrm{tor}} \setminus \mathscr{S}_K$ is a *relative strict normal crossings Cartier divisor* over $\mathrm{Spec}(\mathcal{O}_{E,(v)})$, obtained from an admissible rational polyhedral cone decomposition (refined if necessary) [2].

Notation / Convention 4.6 (Standing boundary regularity)**.** We henceforth work with a choice of S_K^{tor} as in Theorem 4.5; in particular S_K^{tor} is regular and D is a relative SNC Cartier divisor.

Proposition 4.7 (Standard: extension of Hecke correspondences)**.** *For neat level, Hecke correspondences at primes $\ell \neq p$ extend to finite correspondences on $\mathscr{S}_K^{\mathrm{tor}}$ compatible with the open immersion $\mathscr{S}_K \hookrightarrow \mathscr{S}_K^{\mathrm{tor}}$.*

Proof. This is standard for neat level using the moduli interpretation and normality of $\mathscr{S}_K^{\mathrm{tor}}$; see [2]. □

Lemma 4.8 (Boundary purity in codimension one; well known)**.** *Let $X = \mathscr{S}_K^{\mathrm{tor}}$ be regular, and let $i : D \hookrightarrow X$ be the boundary divisor, assumed to be a relative strict normal crossings Cartier divisor. Let Λ be a finite coefficient ring of order prime to p. Then absolute cohomological purity gives*

$$i^!\Lambda \simeq \Lambda(-1)[-2].$$

In particular,

$$H^q_D(X, \Lambda) = 0 \quad \textit{for } q \neq 2, \qquad H^2_D(X, \Lambda) \cong i_*\Lambda(-1).$$

Proof. This is the codimension-one case of absolute cohomological purity for a regular immersion; see [6, Thm. 1.1] and also [7, Exp. XVI]. Since D is a Cartier divisor on the regular scheme X, the immersion $i : D \hookrightarrow X$ has codimension one, so

$$i^!\Lambda \simeq \Lambda(-1)[-2].$$

Applying local cohomology with support in D yields

$$H^q_D(X, \Lambda) = 0 \quad \text{for } q \neq 2, \qquad H^2_D(X, \Lambda) \cong i_*\Lambda(-1).$$

□

Assumption 4.1 (Purity–Kummer fence for the open immersion j)**.** *Let $j : U = \mathscr{S}_K \hookrightarrow X = \mathscr{S}_K^{\mathrm{tor}}$ be the open immersion with boundary $i : D = X \setminus U \hookrightarrow X$. Assume:*

1. *X is regular and D is a* relative Cartier divisor with (strict) normal crossings *over* $\mathrm{Spec}(\mathcal{O}_{E,(v)})$ *(as in Theorem 4.5).*
2. *Coefficients Λ are finite of order prime to p (equivalently, every prime dividing $|\Lambda|$ is $\neq p$).*
3. ***Absolute cohomological purity in codimension** 1: for $i : D \hookrightarrow X$ one has $H^q_D(X, \Lambda) = 0$ for $q \neq 2$ and $H^2_D(X, \Lambda) \cong i_*\Lambda(-1)$.*
4. ***Low-degree boundary control.** For the chosen coefficients Λ, the degree-one boundary sheaf $R^1 j_*\Lambda$ is constructible, and the transgression*

   $$d_2^{0,1} : H^0_{\acute{e}t}\big(X, R^1 j_*\Lambda\big) \longrightarrow H^2_{\acute{e}t}(X, \Lambda)$$

   vanishes.

Remark 4.9. The low-degree results below are conditional on an explicit vanishing hypothesis for the relevant boundary transgression. We do not claim that such a vanishing is automatic in general Hodge-type compactifications; rather, it isolates the precise obstruction that must be controlled in any future unconditional treatment.

Theorem 4.10 (Low-degree edge surjectivity under vanishing of the transgression). *Assume Assumption 4.1. Let (G, X) be of Hodge type with $\mathscr{S}_K$ smooth over* $\mathrm{Spec}(\mathcal{O}_{E,(v)})$*, and let*

$$j : \mathscr{S}_K \hookrightarrow \mathscr{S}_K^{\mathrm{tor}}$$

be a toroidal compactification as above. Then the edge morphism

$$H^1_{\acute{e}t}(\mathscr{S}_K, \Lambda) \longrightarrow H^0_{\acute{e}t}\big(\mathscr{S}_K^{\mathrm{tor}}, R^1 j_{\acute{e}t*}\Lambda\big)$$

is surjective.

Proof. This is the low-degree exactness coming from the Leray spectral sequence of j once the transgression $d_2^{0,1}$ is assumed to vanish. □

Remark 4.11. We do not assert any general vanishing of the higher direct images $R^b j_*\Lambda$ for $b > 1$. The only input used later is the vanishing of the specific transgression $d_2^{0,1}$ in the Leray spectral sequence.

5 Main Results

We retain the global setup and notation from Sections 2 to 4. Fix a Shimura datum (G, X) of Hodge type, a neat compact open $K = K^{(p)}K_p$ with K_p hyperspecial, and write $f : \mathscr{S}_K \to \mathrm{Spec}(\mathcal{O}_{E,(v)})$ for the integral canonical model (Theorems 2.8 and 4.1) and $j : \mathscr{S}_K \hookrightarrow \mathscr{S}_K^{\mathrm{tor}}$ for a toroidal compactification (Theorem 4.5). Throughout Λ denotes a finite coefficient ring with $\mathrm{char}(\Lambda) \neq p$. Hecke correspondences at primes $\ell \neq p$ extend to $\mathscr{S}_K^{\mathrm{tor}}$ (Theorem 4.7).

5.1 First main theorem: structural statement

Definition 5.1 (Boundary quotient and interior kernel in degree one). Let

$$\mathrm{edge} : \ H^1_{\acute{e}t}(\mathscr{S}_K, \Lambda) \longrightarrow H^0_{\acute{e}t}\big(\mathscr{S}_K^{\mathrm{tor}}, R^1 j_{\acute{e}t*}\Lambda\big)$$

be the edge morphism of Theorem 4.10. Define the degree-one interior kernel by

$$H^1_{\mathrm{int}}(\mathscr{S}_K, \Lambda) := \ker(\mathrm{edge}),$$

and the degree-one boundary quotient by

$$H^1_{\mathrm{bdry}}(\mathscr{S}_K, \Lambda) := H^0_{\acute{e}t}\big(\mathscr{S}_K^{\mathrm{tor}}, R^1 j_{\acute{e}t*}\Lambda\big).$$

If a Hecke-equivariant splitting of the edge sequence is chosen, we write $H^1_{\mathrm{comp}}(\mathscr{S}_K, \Lambda)$ for the corresponding chosen complement to H^1_{int} inside $H^1_{\acute{e}t}(\mathscr{S}_K, \Lambda)$.

Theorem 5.2 (Hecke-equivariant splitting after a chosen localization hypothesis). *Assume (G, X) is of Hodge type, K is neat with K_p hyperspecial, and Λ as above. Then:*

(i) There is a Hecke-equivariant short exact sequence

$$0 \longrightarrow H^1_{\mathrm{int}}(\mathscr{S}_K, \Lambda) \longrightarrow H^1_{\acute{e}t}(\mathscr{S}_K, \Lambda) \xrightarrow{\mathrm{edge}} H^1_{\mathrm{bdry}}(\mathscr{S}_K, \Lambda) \longrightarrow 0,$$

functorial under level change away from p.

Here $H^1_{\mathrm{int}}(\mathscr{S}_K, \Lambda) := \ker(\mathrm{edge})$ and

$$H^1_{\mathrm{bdry}}(\mathscr{S}_K, \Lambda) := H^0_{\acute{e}t}\big(\mathscr{S}_K^{\mathrm{tor}}, R^1 j_{\acute{e}t*}\Lambda\big)$$

are the interior kernel and boundary quotient introduced in Theorem 5.1. We do not identify H^1_{int} a priori with automorphic interior cohomology or with an Eisenstein/cuspidal decomposition in the classical sense.

(ii) *Assume in addition that, after inverting a finite set* Σ *of rational primes disjoint from* $\{p\}$*, the short exact sequence in* (i)

$$0 \longrightarrow H^1_{\mathrm{int}}(\mathscr{S}_K, \Lambda[1/\Sigma]) \longrightarrow H^1_{\acute{e}t}(\mathscr{S}_K, \Lambda[1/\Sigma]) \xrightarrow{\text{edge}} H^1_{\mathrm{bdry}}(\mathscr{S}_K, \Lambda[1/\Sigma]) \longrightarrow 0$$

splits in the category of $\mathbb{T}^{(\Sigma)}$*-modules. Then there exists a* $\mathbb{T}^{(\Sigma)}$*-equivariant idempotent endomorphism of* $H^1_{\acute{e}t}(\mathscr{S}_K, \Lambda[1/\Sigma])$ *whose image is a chosen Hecke-stable complement to* $H^1_{\mathrm{int}}(\mathscr{S}_K, \Lambda[1/\Sigma])$.

Proof. **Step 1: Hecke action on the Leray package.** Let $j : \mathscr{S}_K \hookrightarrow \mathscr{S}_K^{\mathrm{tor}}$ be the open immersion and $i : D \hookrightarrow \mathscr{S}_K^{\mathrm{tor}}$ the boundary. By Theorem 4.7, for each prime $\ell \neq p$ and each double coset $[K_\ell g K_\ell]$, there is a finite correspondence

$$\mathscr{S}_K^{\mathrm{tor}} \xleftarrow{\ p_1\ } \mathcal{H}_g^{\mathrm{tor}} \xrightarrow{\ p_2\ } \mathscr{S}_K^{\mathrm{tor}}$$

extending the Hecke correspondence on $\mathscr{S}_K$, with p_1, p_2 finite and étale over the open. The induced functors $(p_i)_{\acute{e}t*}$ act on $R^b j_{\acute{e}t*}\Lambda$ and commute with the differentials of the Leray spectral sequence. Hence both $H^1_{\acute{e}t}(\mathscr{S}_K, \Lambda)$ and $H^0_{\acute{e}t}(\mathscr{S}_K^{\mathrm{tor}}, R^1 j_{\acute{e}t*}\Lambda)$ are $\Lambda[\mathrm{Hecke}^{(p)}]$-modules and the edge map is Hecke-equivariant.

Step 2: Proof of (i). Consider the Leray spectral sequence

$$E_2^{a,b} = H^a_{\acute{e}t}(\mathscr{S}_K^{\mathrm{tor}}, R^b j_{\acute{e}t*}\Lambda) \Rightarrow H^{a+b}_{\acute{e}t}(\mathscr{S}_K, \Lambda).$$

By Theorem 4.10, the specific transgression

$$d_2^{0,1} : E_2^{0,1} \longrightarrow E_2^{2,0}$$

vanishes. Therefore the standard low-degree exact sequence attached to Leray yields

$$0 \to E_2^{1,0} \longrightarrow H^1_{\acute{e}t}(\mathscr{S}_K, \Lambda) \xrightarrow{\text{edge}} E_2^{0,1} \to 0.$$

Since $E_2^{1,0} = E_\infty^{1,0}$ and $E_2^{0,1} = E_\infty^{0,1}$ in total degree 1, this is

$$0 \to E_\infty^{1,0} \longrightarrow H^1_{\acute{e}t}(\mathscr{S}_K, \Lambda) \xrightarrow{\text{edge}} E_\infty^{0,1} = H^0_{\acute{e}t}(\mathscr{S}_K^{\mathrm{tor}}, R^1 j_{\acute{e}t*}\Lambda) \to 0.$$

Step 3: Splitting hypothesis after localization. After inverting Σ, assume that the short exact sequence of Item (i) splits in the category of $\mathbb{T}^{(\Sigma)}$-modules. No general semisimplicity claim for $\mathbb{T}^{(\Sigma)}$-modules is used here.

Step 4: Construction of a Hecke-equivariant splitting.
Write

$$M := H^1_{\acute{e}t}(\mathscr{S}_K, \Lambda[1/\Sigma]), \qquad B := H^0_{\acute{e}t}\big(\mathscr{S}_K^{\mathrm{tor}}, R^1 j_*\Lambda\big)[1/\Sigma].$$

By Step 3, the sequence

$$0 \to H^1_{\mathrm{int}}(\mathscr{S}_K, \Lambda)[1/\Sigma] \longrightarrow M \xrightarrow{\text{edge}} B \to 0$$

is exact in the abelian category of $\mathbb{T}^{(\Sigma)}$-modules. By the splitting hypothesis assumed in Item (ii), this exact sequence admits a $\mathbb{T}^{(\Sigma)}$-equivariant section. Choose such a splitting

$$s : B \longrightarrow M$$

of the edge map. Then

$$e := s \circ \mathrm{edge} \in \mathrm{End}_{\mathbb{T}^{(\Sigma)}}(M)$$

is an idempotent endomorphism of M, because

$$e^2 = s \circ \mathrm{edge} \circ s \circ \mathrm{edge} = s \circ \mathrm{id}_B \circ \mathrm{edge} = e.$$

Moreover,

$$\ker(e) = \ker(\mathrm{edge}) = H^1_{\mathrm{int}}(\mathscr{S}_K, \Lambda)[1/\Sigma],$$

and

$$\mathrm{im}(e) = s(B) \cong B.$$

Therefore

$$M = H^1_{\mathrm{int}}(\mathscr{S}_K, \Lambda)[1/\Sigma] \oplus s(B)$$

as $\mathbb{T}^{(\Sigma)}$-modules. The idempotent e is thus the projector onto the chosen complementary summand $s(B)$ along

$$H^1_{\mathrm{int}}(\mathscr{S}_K, \Lambda)[1/\Sigma].$$

Accordingly, define

$$\mathbf{P}_{\mathrm{comp}} := e, \qquad \mathbf{P}_{\mathrm{int}} := 1 - e.$$

Then both $\mathbf{P}_{\mathrm{comp}}$ and $\mathbf{P}_{\mathrm{int}}$ are $\mathbb{T}^{(\Sigma)}$-equivariant idempotents, with

$$\mathrm{im}(\mathbf{P}_{\mathrm{comp}}) = s(B), \qquad \mathrm{im}(\mathbf{P}_{\mathrm{int}}) = H^1_{\mathrm{int}}(\mathscr{S}_K, \Lambda)[1/\Sigma].$$

Equivalently,

$$\ker(\mathbf{P}_{\mathrm{comp}}) = H^1_{\mathrm{int}}(\mathscr{S}_K, \Lambda)[1/\Sigma], \qquad \ker(\mathbf{P}_{\mathrm{int}}) = s(B).$$

Thus the short exact sequence splits after inverting Σ, and one obtains a Hecke-stable complement to

$$H^1_{\mathrm{int}}(\mathscr{S}_K, \Lambda)[1/\Sigma].$$

We emphasize, however, that this complementary summand depends on the choice of $\mathbb{T}^{(\Sigma)}$-equivariant splitting s unless additional hypotheses are imposed to make the splitting canonical. □

Remark 5.3 (Conditional non-boundary localization)**.** Suppose that, in a given arithmetic situation, one knows by independent input that the localized boundary module

$$H^1_{\mathrm{bdry}}(\mathscr{S}_K, \Lambda[1/\Sigma])_{\mathfrak{m}}$$

vanishes for a maximal ideal $\mathfrak{m} \subset \mathbb{T}^{(\Sigma)}$. Then the localized edge map is zero, and the $\mathfrak{m}$-localization of $H^1_{\acute{e}t}(\mathscr{S}_K, \Lambda[1/\Sigma])$ identifies with the localized interior kernel.

5.2 Second main theorem: cohomological invariants

Definition 5.4 (Degree-one boundary rank)**.** Define the degree-one boundary rank by

$$\delta_\Lambda(\mathscr{S}_K) := \mathrm{rank}_\Lambda\, H^0_{\acute{e}t}\big(\mathscr{S}_K^{\mathrm{tor}}, R^1 j_{\acute{e}t*}\Lambda\big).$$

Lemma 5.5 (Diagrammatic edge package for (S^{tor}, D))**.** *Let $j : U := \mathscr{S}_K \hookrightarrow X := \mathscr{S}_K^{\mathrm{tor}}$ be the open immersion and $i : D := X \setminus U \hookrightarrow X$ the boundary divisor. For any finite Λ with $\mathrm{char}(\Lambda) \neq p$, the localization triangle*

$$j_!\Lambda \longrightarrow \Lambda \longrightarrow i_*i^*\Lambda \xrightarrow{+1}$$

induces the long exact sequence in cohomology

$$\cdots \to H^1(X, \Lambda) \;\to\; H^1(U, \Lambda) \;\xrightarrow{\partial}\; H^2_D(X, \Lambda) \;\to\; H^2(X, \Lambda) \to \cdots,$$

where ∂ is the boundary map. By cohomological purity for the normal crossings divisor D, $H^2_D(X, \Lambda) \cong H^0(X, i_*\Lambda(-1))$, and (étale locally) $R^1 j_*\Lambda \simeq \bigoplus_k i_{k*}\Lambda(-1)$ for the components $D = \bigcup_k D_k$. The connecting map

$$\partial : \; H^1(U, \Lambda) \longrightarrow H^2_D(X, \Lambda)$$

from the localization long exact sequence is related to the transgression in the Leray spectral sequence of j, but we do not identify the Leray edge map itself with a map into $H^2(X, \Lambda)$.

Theorem 5.6 (Conditional boundary-torus criterion). *Assume the setup of Theorem 5.2, and assume in addition that there is an identification*

$$H^1_{\mathrm{bdry}}(\mathscr{S}_K, \Lambda[1/\Sigma]) \;\cong\; \bigoplus_c \mathrm{Hom}_{\mathbb{Z}}\big(X^*(T_c)^{G_{\mathbb{Q}}}, \Lambda[1/\Sigma](-1)\big),$$

where c runs over the relevant rational boundary components. Then the following are equivalent:

(a) $\delta_{\Lambda[1/\Sigma]}(\mathscr{S}_K) = 0$*;*

(b) every rational boundary torus T_c associated with a maximal parabolic is anisotropic over $\mathbb{Q}$;

(c)

$$H^1_{\mathrm{bdry}}(\mathscr{S}_K, \Lambda[1/\Sigma]) = 0.$$

Proof. We recall the *boundary defect* δ_Λ from Theorem 5.4:

$$\delta_\Lambda(\mathscr{S}_K) \;:=\; \mathrm{rank}_\Lambda\; H^0_{\acute{e}t}\big(\mathscr{S}_K^{\mathrm{tor}}, R^1 j_{\acute{e}t*}\Lambda\big), \quad j : \mathscr{S}_K \hookrightarrow \mathscr{S}_K^{\mathrm{tor}}.$$

By Theorem 4.10 there is a Hecke-equivariant surjection (edge map)

$$\mathrm{edge}:\; H^1_{\acute{e}t}(\mathscr{S}_K, \Lambda) \twoheadrightarrow H^0_{\acute{e}t}\big(\mathscr{S}_K^{\mathrm{tor}}, R^1 j_{\acute{e}t*}\Lambda\big).$$

Step 1: Use of the boundary-module hypothesis. The substantive geometric input in the present theorem is the additional hypothesis that

$$H^1_{\mathrm{bdry}}(\mathscr{S}_K, \Lambda[1/\Sigma]) \;\cong\; \bigoplus_c \mathrm{Hom}_{\mathbb{Z}}\big(X^*(T_c)^{G_{\mathbb{Q}}}, \Lambda[1/\Sigma](-1)\big).$$

We do not prove this identification here from local Kummer theory and toroidal charts. Rather, the theorem records the formal consequences that follow once such a boundary-module description is available.

In particular, under this hypothesis one has

$$\delta_{\Lambda[1/\Sigma]}(\mathscr{S}_K) = \sum_c \mathrm{rank}_{\mathbb{Z}}\big(X^*(T_c)^{G_{\mathbb{Q}}}\big),$$

namely the total $\mathbb{Q}$-split rank of the relevant rational boundary tori.

Step 2: Item (a) ⇒ Item (b). If $\delta_{\Lambda[1/\Sigma]}(\mathscr{S}_K) = 0$, then the equality established in Step 1,

$$\delta_{\Lambda[1/\Sigma]}(\mathscr{S}_K) = \sum_c \mathrm{rank}_{\mathbb{Z}}\big(X^*(T_c)^{G_{\mathbb{Q}}}\big),$$

forces

$$\mathrm{rank}_{\mathbb{Z}}\big(X^*(T_c)^{G_{\mathbb{Q}}}\big) = 0$$

for every cusp c. Hence

$$X^*(T_c)^{G_{\mathbb{Q}}} = 0$$

for every c. Equivalently, the $\mathbb{Q}$-split subtorus of T_c is trivial, so T_c is anisotropic over $\mathbb{Q}$.

Step 3: Item (b) ⇒ Item (c). If each T_c is anisotropic, then $X^*(T_c)^{G_{\mathbb{Q}}} = 0$ for every cusp c. By the assumed boundary-torus description, this implies

$$H^1_{\mathrm{bdry}}(\mathscr{S}_K, \Lambda[1/\Sigma]) = 0,$$

which is exactly Item (c).

Step 4: Item (c) ⇒ Item (a). If

$$H^1_{\mathrm{bdry}}(\mathscr{S}_K, \Lambda[1/\Sigma]) = 0,$$

then by definition its rank is zero, so

$$\delta_{\Lambda[1/\Sigma]}(\mathscr{S}_K) = 0.$$

Step 5: Final remark. Any precise comparison with intersection cohomology on the minimal compactification would require additional input from intermediate extension, perverse truncation, and decomposition results that are not developed in the present manuscript. Accordingly, no such identification is claimed here. □

Example 5.7 (Illustrative Hilbert modular surface case under the boundary-module hypothesis)**.** Let $G = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$ for a real quadratic field F and take $K = K^{(p)}K_p$ with K_p hyperspecial and K neat. Then $\mathscr{S}_K = \mathscr{S}_K(G, X)$ is a Hilbert modular surface over $S = \mathrm{Spec}(\mathcal{O}_{E,(v)})$, whose minimal and toroidal compactifications fit into the setup of Theorem 5.6.

(1) Boundary tori and their $\mathbb{Q}$-split rank. Each cusp c of $\mathscr{S}_K$ corresponds to a rational parabolic of G with Levi quotient $M_c \simeq \mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_1 \times \mathrm{GL}_1 / \mathbb{G}_m$. The associated boundary torus is

$$T_c \;\simeq\; \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m / \mathbb{G}_m,$$

whose character lattice is $X^*(T_c) \cong \mathbb{Z}^{[F:\mathbb{Q}]}/\mathbb{Z}$ endowed with a nontrivial $G_{\mathbb{Q}}$–fixed line. Hence T_c has $\mathbb{Q}$–split rank 1 and is *not anisotropic*. In particular,

$$\delta_{\Lambda[1/\Sigma]}(\mathscr{S}_K) = \sum_c \mathrm{rank}_{\mathbb{Z}} X^*(T_c)^{G_{\mathbb{Q}}} = \#\{\text{cusps}\} \neq 0,$$

so Item (a) of Theorem 5.6 fails.

(2) Boundary computation under the hypothesis of Theorem 5.6. Assuming the boundary-module identification appearing in Theorem 5.6, one obtains

$$H^1_{\mathrm{bdry}}(\mathscr{S}_K, \Lambda[1/\Sigma]) \;\cong\; \bigoplus_c \Lambda[1/\Sigma](-1),$$

since in the Hilbert modular surface case each relevant boundary torus contributes a one-dimensional $\mathbb{Q}$-split character space. This boundary term records the failure of anisotropy: each cusp contributes a nonzero boundary summand, and therefore the edge quotient does not vanish.

(3) Interpretation under Theorem 5.6.

- Item (a) fails since $\delta_{\Lambda[1/\Sigma]} \neq 0$.
- Item (b) fails because T_c is $\mathbb{Q}$-split of rank 1.
- Consequently, Item (c) fails: the boundary quotient

  $$H^1_{\mathrm{bdry}}(\mathscr{S}_K, \Lambda[1/\Sigma])$$

  is nonzero, so the edge quotient does not vanish.

Thus the Hilbert modular surface provides a concrete illustration of the *necessity of anisotropy* for the vanishing of the boundary defect and the boundary quotient.

(4) Contrast (compact anisotropic case). If G is replaced by a quaternionic inner form of $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$ that is anisotropic modulo its center, then $\mathscr{S}_K$ is proper, so there is no boundary contribution in degree one. In that case

$$H^1_{\mathrm{bdry}}(\mathscr{S}_K, \Lambda[1/\Sigma]) = 0$$

and the edge sequence identifies

$$H^1_{\acute{e}t}(\mathscr{S}_K, \Lambda[1/\Sigma]) = H^1_{\mathrm{int}}(\mathscr{S}_K, \Lambda[1/\Sigma]).$$

This gives the compact model for the vanishing statement predicted by Theorem 5.6. Any further comparison with automorphic cuspidal cohomology or intersection cohomology lies beyond the scope of the present manuscript.

5.3 Equivalences and classification results

Diagrammatic equivalence. The long exact sequence and edge–purity package used in this subsection are summarized in Theorem 5.5.

Remark 5.8 (Localization heuristic)**.** Let $\mathfrak{m} \subset \mathbb{T}^{(\Sigma)}$ be a maximal ideal. If one knows by independent input that

$$H^1_{\mathrm{bdry}}(\mathscr{S}_K, \Lambda[1/\Sigma])_{\mathfrak{m}} = 0,$$

then Theorem 5.3 shows that the localized edge map vanishes and

$$H^1_{\acute{e}t}(\mathscr{S}_K, \Lambda[1/\Sigma])_{\mathfrak{m}} = H^1_{\mathrm{int}}(\mathscr{S}_K, \Lambda[1/\Sigma])_{\mathfrak{m}}.$$

Thus any genuine Eisenstein/non-Eisenstein classification at $\mathfrak{m}$ requires additional arithmetic input on the localized boundary module and is not proved in the present paper.

Corollary 5.9 (Stability under level change away from p)**.** *Let $K' \subset K$ be neat with the same hyperspecial K_p. The formation of H^1_{int}, H^1_{bdry}, and, after choosing a splitting as in Item (ii), the projector $\mathbf{P}_{\mathrm{comp}}$, is compatible with pullback along finite étale level maps and with Hecke operators away from $\{p\} \cup \Sigma$.*

Conclusion of the present note. The results obtained here isolate the degree-one boundary obstruction, record the formal splitting consequences of a chosen Hecke-equivariant section after localization, and show how an explicit boundary-torus description would force vanishing of the boundary quotient in the anisotropic case. Arithmetic applications require additional comparison input and are deferred to future work.

Declarations

Availability of data and material

No datasets were generated or analyzed during the current study. All arguments and constructions are purely theoretical and contained within the manuscript.

Competing interests

The authors declare that they have no competing interests.

Funding

This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.

Authors' contributions

Rahul Thakurdas Kundnani conceived the main results, developed the proofs, and wrote the manuscript.

Acknowledgements

N/A

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