The 20/80 Rule as a Dimensional Effect in Resource Distribution

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Abstract

Despite the large number of statistical interpretations of the 20/80 rule, its geometric and dimensional origin has been studied much less. This paper proposes a dimensional interpretation of empirical regularities in resource concentration within systems. The main characteristic used is the concentration function, which specifies the share of the resource belonging to the upper fraction of elements ranked in descending order of contribution. For analytical investigation, a simple geometric model is employed - a conical sandpile - in which, for the same carrier structure, different observable quantities may be considered: volume (mass), surface area, equivalent linear scale, and potential energy. It is shown that the corresponding concentration functions do not coincide and are determined not only by the order in which the elements are ranked but also by the dimension of the effect under consideration. For the three-dimensional case, the transition from the volumetric measure to the equivalent linear scale leads to a concentration function for which $S(0.2) = 0.787$, that is, practically to the 20/80 rule. Matching the small-p asymptotics yields the exact relationship $\alpha = D/(D-1)$, linking the Pareto exponent to the dimension D of the observed quantity; at $D = 1$ a natural threshold arises between egalitarian and concentrated regimes. The paper also discusses the connection between the symmetric parametric model and the Gini coefficient, the possibility of using this model for the normative description of income distributions, and its application to a comparison of market and disposable incomes in the United States and Costa Rica. It is emphasised that the paper uses the static geometric profile of a sandpile as an analytically exact benchmark; the relationship of the derived concentration curves to the dynamics of self-organised criticality and to full empirical fitting of Lorenz curves is regarded as a natural direction for further research.

Keywords: Pareto principle, Lorenz curve, concentration curve, Gini coefficient, dimensional scaling, econophysics, self-organised criticality, Burr III, normative distribution, tax-and-transfer policy

1. Introduction

Uneven resource distribution is observed in systems of very different kinds - from income and wealth to flows of energy, mass, and interaction intensity. However, in analysing such systems, not only the fact of inequality matters, but also the form of concentration: what share of the total resource is accounted for by the upper fraction of elements after they have been ranked in descending order of contribution. It is precisely this question that lies behind discussions of the 20/80 rule and related empirical relationships.

Since the work of Vilfredo Pareto [1], studies of resource concentration have usually developed in a statistical direction: researchers have sought distribution functions capable of describing heavy tails, Lorenz curves, and the associated inequality indices. This approach has been highly productive; however, it does not directly answer another question: why can the observed concentration proportions differ even when the ranked organisation of the system is similar? In other words, what mechanism leads from the same ordering of elements to different shares of accumulated resource?

For the upper tails of the Pareto distribution, Pareto showed the robustness of a power-law approximation, which in terms of the concentration function may conveniently be written as the asymptotic form of the cumulative share of the resource belonging to the upper fraction of ranks.

$$S(p) \propto p^{1-1/\alpha}, \qquad 0 \le p \ll 1. \tag{1}$$

Hereafter, by analogy with the standard Lorenz curve ($S(p)$), we shall refer to the cumulative concentration curve L as the Pareto curve (PC). Pareto also noted that in many empirical series the form of concentration is close to the norm with parameter $\alpha = 3/2$.. Substituting this value into asymptotic relation (1) leads to an approximation that is convenient as a starting point for comparing different concentration models.

$$S(p) \propto p^{1/3}, \qquad 0 \le p \ll 1. \tag{2}$$

It will be shown below that the exponent in approximation (2) naturally leads to the 20/80 rule: about 20% of the elements of a system accumulate approximately 80% of the total resource, whereas the remaining 80% of the elements account for about 20% of the resource.

For more than a century, the 20/80 rule has been used as a practical heuristic in quality management [2]-[4], economics [5], [6], geography [7], sociology [8], and the natural sciences [9]-[11]. At the same time, the robustness of this proportion is usually stated empirically, whereas its geometric and dimensional interpretation remains insufficiently clarified.

This paper considers the hypothesis that the dimension of the observed quantity plays an essential role here. The same carrier may generate different concentration laws depending on what exactly is taken as the resource: linear size, area, volume, energy, or another quantity related to the geometric scale by a power law. In this case, the difference between concentration rules is connected not with an arbitrary choice of numerical constant, but with the way in which the effect is measured.

To clarify this idea, a simple analytical model is used - a conical sandpile. It is convenient because it admits an explicit description of ranked resource accumulation and makes it possible, within a single geometry, to compare several different quantities. With the carrier shape unchanged, one may successively consider the accumulation of volume, surface area, equivalent linear size, and potential energy, and then compare the corresponding concentration functions.

It is important that, in the work of Bak, Tang, and Wiesenfeld, the sandpile appears not only as a geometric object but also as a canonical model of self-organised criticality. In the present paper, only its stationary conical profile is used, as an analytically transparent carrier of ranked accumulation; avalanche statistics, the critical slope, and other dynamic SOC features are deliberately not modelled.

The paper therefore does not claim to provide a universal dynamic explanation for all instances of the 20/80 rule. Its task is more modest and, at the same time, more constructive: to identify a dimensional geometric benchmark with respect to which observed inequality may be interpreted as a deviation from the normal concentration determined by the type of observed effect. Such a static benchmark is also convenient because it makes it possible to compare real distributions with an analytically exact baseline configuration, including in problems where concentration itself arises from the dynamics of exchange, accumulation, or critical events.

The aim of the paper is to relate empirical concentration rules to the dimensional structure of the observed effect. In the proposed interpretation, the ranking of elements remains a dimensionless procedure in itself, whereas a specific proportion such as 20/80, 20/79, or 20/62 is determined by how the resource under consideration scales relative to the linear carrier. This makes it possible to regard the familiar concentration ratios as elements of a unified scheme rather than as a set of disconnected empirical rules.

The paper is organised as follows. First, the basic notation and the formal statement of the problem are introduced. The conical sandpile is then considered as a model in which different physical quantities correspond to different concentration functions. After that, dimensional mappings and their parametric generalisations are discussed, followed by an interpretation of the results in terms of symmetry, the Gini coefficient, and normative characteristics of distributions. In the final part, the model is illustrated using OECD data on market and disposable incomes in the United States and Costa Rica, and its application to the evaluation of tax-and-transfer policy is discussed.

2. Problem Statement

Consider n elements of a system, ranked in descending order of their share in the total resource of the system (w_r), where r is the rank of an element. By definition, the Pareto curve is a function interpolating the cumulative sum of the resource shares of the elements.

$$S(p_i) = \sum_{r=1}^{i} w_r, \qquad 0 \le p_i \le 1. \tag{3}$$

where $p_r = r/n \in [0,1]$ is the share of ranks, that is, the share of elements counted from the most endowed to the least endowed.

To compare the PC with the Lorenz curve $\big(L(p)\big)$, defined in ascending order of resource (with the smallest values placed first), it is convenient to use the identity $L(p) = 1 - S(1-p)$, which allows one to pass from top-down ranking to bottom-up ranking. Accordingly, the basic axioms for the PC may be written as follows:

$$S(0) = 0, \quad S(1) = 1, \qquad S'_{+}(p) \ge 0, \quad S''_{+}(p) \le 0, \quad p \in (0,1). \tag{4}$$

Conditions (4) imply that the PC is a non-decreasing concave concentration curve. Let us denote PR $= S(0.2)$.. Hereafter, PR will be referred to as the Pareto coefficient. It represents the share of the resource accumulated by the top 20% of the elements and is widely used in applied work for comparing resource concentration across systems.

Within the framework of the stated problem, it is necessary to: (i) identify the dependence of the Pareto coefficient on the dimension of the observed quantity; (ii) investigate how one and the same carrier gives rise to different concentration curves when passing from volume to area, equivalent radius, and energy; (iii) relate observed inequality to the model parameter and the normative distribution of the resource; (iv) discuss criteria of structural stability; and (v) illustrate the recovery of the model parameter from the Gini coefficient and compare actual post-tax Pareto curves with the symmetric normative benchmark using OECD data for the United States and Costa Rica.

3. The Sand Cone as a Model of Different Resource Concentrations

A classical model of self-organising systems is a pile of homogeneous sand [12], [13]. In the stationary state it has the form of a right circular cone of height H.

In the Bak-Tang-Wiesenfeld interpretation, such a pile corresponds to the stationary profile of a system operating in a regime of self-organised criticality. In what follows, we use precisely this static profile as a geometrically ordered carrier of the effect. The connection between the concentration functions derived below and the dynamic characteristics of SOC - avalanche distributions, critical slopes, and universality classes - is not derived in this paper and is treated as an open question.

Let $S_M(p)$ denote the accumulation of volume (and, at constant density, mass), and let $S_A(p)$ denote the accumulation of the lateral surface area of the truncated cone. By similarity of sections, the radius at height z is proportional to $(1-p)$, while the volume of the small upper cone above level z constitutes a $(1-p)^3$ share of the total volume. Hence the share of the volume contained in the lower fraction of height p is equal to:

$$S_V(p) = \frac{V(p)}{V_{tot}} = 1 - (1-p)^3. \tag{5}$$

For model (5), the Pareto coefficient is PR ≈ 0.488. Similarly, the share of the lateral surface area of the truncated cone contained in the lower fraction of height p is given by:

$$S_A(p) = \frac{A(p)}{A_{tot}} = 1 - (1-p)^2. \tag{6}$$

For model (6), the Pareto coefficient is PR ≈ 0.360, which is noticeably smaller than in the volume-based model. This reflects the weaker concentration of the resource when moving from volume to surface area. In general, both relations can conveniently be written in the unified form:

$$S(p; D) = 1 - (1-p)^D. \tag{7}$$

Here the parameter D is interpreted as the dimension, or the number of degrees of freedom, of the quantity characterising the state of the system. For the subsequent conclusions, it is useful to employ the standard expansion in a neighbourhood of zero:

$$(1-p)^\alpha = 1 - \alpha p + \frac{\alpha(\alpha-1)}{2} p^2 + O(p^3), \qquad p \to 0. \tag{8}$$

It follows immediately from (8) that, for $p \to 0+$, family (7) has the linear asymptotic form:

$$S(p; D) \propto p, \qquad 0 \le p \ll 1. \tag{9}$$

A comparison of relations (2) and (9) shows that the purely geometric model (7) grows much more gently near zero than the empirical concentration curve of income. Consequently, geometry alone is insufficient to reproduce the high concentration characteristic of socio-economic distributions, which is why dimensional mappings are considered below: these amplify the accumulation of the effect when passing to the linear scale. In particular, the transition from the volumetric measure to the equivalent radius replaces cubic accumulation by a root transformation, so that the concentration function becomes substantially steeper already in the region of small p.

If, however, potential energy of the sand grains is taken as the accumulated resource, the corresponding accumulation curve becomes more convex, because the contribution of each part of the mass is additionally weighted by its height. In economic terms, this corresponds to a strengthening of concentration due to a weighting factor analogous to a leverage effect. A detailed analytical description of the energy curve lies beyond the scope of the present paper, but the very fact shows that one and the same carrier generates different concentration laws depending on the nature of the measured quantity.

4. Equivalent Radius: Dimensional Mappings

4.1. Dimension D

The dimension D indicates how many parameters are required to describe an object: zero dimension corresponds to a point, one dimension to a line, two dimensions to an area, and three dimensions to a volume. For complex and fragmented objects, fractional values of $D \in (1,3)$ may also be used, as in fractal geometry [14].

Let the carrier of the effect possess a measure M (for example, area, volume, or mass) that scales with the characteristic linear size S according to the power law

$$M \propto S^D. \tag{10}$$

Then the equivalent linear scale is given by

$$S = M^{1/D}. \tag{11}$$

The quantity S has a direct physical meaning: it is the share of the linear scale that a similar object would have if it contained a share M of the measure. Such a transition is standard for operations of the type 'volume → equivalent radius' and, in more general form, 'measure → linear scale'.

The passage from measure to equivalent linear scale is not the only possible mapping. In the present work it is used as a natural normalised construction that makes it possible to compare effects of different dimensionality within a unified scheme and to move from a geometrically specified measure to an observable linear scale.

4.2. Equivalent Sphere and Circle

Let us associate with the accumulated volume $V(p)$ an equivalent sphere, that is, a sphere of the same volume. Its radius $R_{eq}(p)$ is determined by equality of volumes, and after normalisation by the value at $p = 1$ we obtain the share of the equivalent radius as a function of rank:

$$\frac{R_{eq}(p)}{R_{eq}(1)} = \left[\frac{V(p)}{V(1)}\right]^{1/3}. \tag{12}$$

Substituting (5) into (12), we arrive at the explicit formula for the three-dimensional dimensional mapping:

$$S(p; 3,1/3) = \sqrt[3]{1 - (1 - p)^3}. \tag{13}$$

For $p = 0.2$, formula (13) yields a value of about 0.787, that is, the 20/79 proportion, which practically coincides with the 20/80 heuristic. Likewise, an equivalent circle of the same area may be associated with the accumulated area $A(p)$. As a result, model (13) naturally generalises to the case of arbitrary dimension:

$$S(p; D, 1/D) = \sqrt[D]{1-(1-p)^D}. \qquad (14)$$

For $D = 2$ we obtain PR ≈ 0.600. Using expansion (8), it is easy to see that, as p→0+, dependencies (14) possess the power-law asymptotic form [15]

$$S(p; D, 1/D) \sim D^{1/D} p^{1/D}, \qquad 0 \leq p \ll 1. \qquad (15)$$

Comparison with (2) shows that, for an appropriate choice of parameter, the asymptotics coincide in the leading term. Hence the difference between the models lies not in the ranked structure as such, but in how one and the same ranked organisation is mapped into an observed quantity of different dimensionality. This is the key idea of the dimensional interpretation of concentration: dimensional mapping provides a geometric mechanism that reproduces the leading Pareto asymptotic without explicitly postulating a particular stochastic process. It should also be noted that model (14) coincides with the one-parameter symmetric Lorenz curve of the Burr III distribution [15], obtained in different ways in [16]-[22].

Of particular importance is the comparison of the asymptotic of model (14) with the upper-tail asymptotic of the Pareto distribution. For the Pareto distribution, the concentration function has the form $S(p) \sim p^{1-1/\alpha}$, whereas the dimensional model (14) gives $S(p) \sim p^{1/D}$. Equating the exponents yields the exact relation $\alpha = D/(D-1)$. In particular, for $D = 3$ we obtain $\alpha = 3/2$ – the classical Pareto value. This identity does not replace the mechanism of distribution formation, but it reveals a structural link between the spatial dimension of the observed effect and the tail exponent of concentration.

4.3. Solids of Revolution Instead of the Social Pyramid

In vol. II of Cours d'Économie Politique, on pp. 312-318, Pareto criticised the image of the social pyramid as too crude a scheme and suggested replacing it with a solid of revolution, making it possible to represent the continuous profile of the distribution geometrically. Such a representation is especially useful when one needs to relate the shape of the concentration curve to the dimension of the carrier and the type of observed quantity.

Figure 1 shows solids of revolution for the radial profile $r(p) = 1 - S(p)$. The broad base corresponds to the majority of elements with small values of the resource, whereas the narrowing upper part corresponds to a relatively small number of elements accumulating a substantial share of the total effect. Variants (a) and (c) correspond to the basic volume and surface concentrations, whereas variants (b) and (d) illustrate two-parameter forms with a=b=3 and a=b=2, respectively.

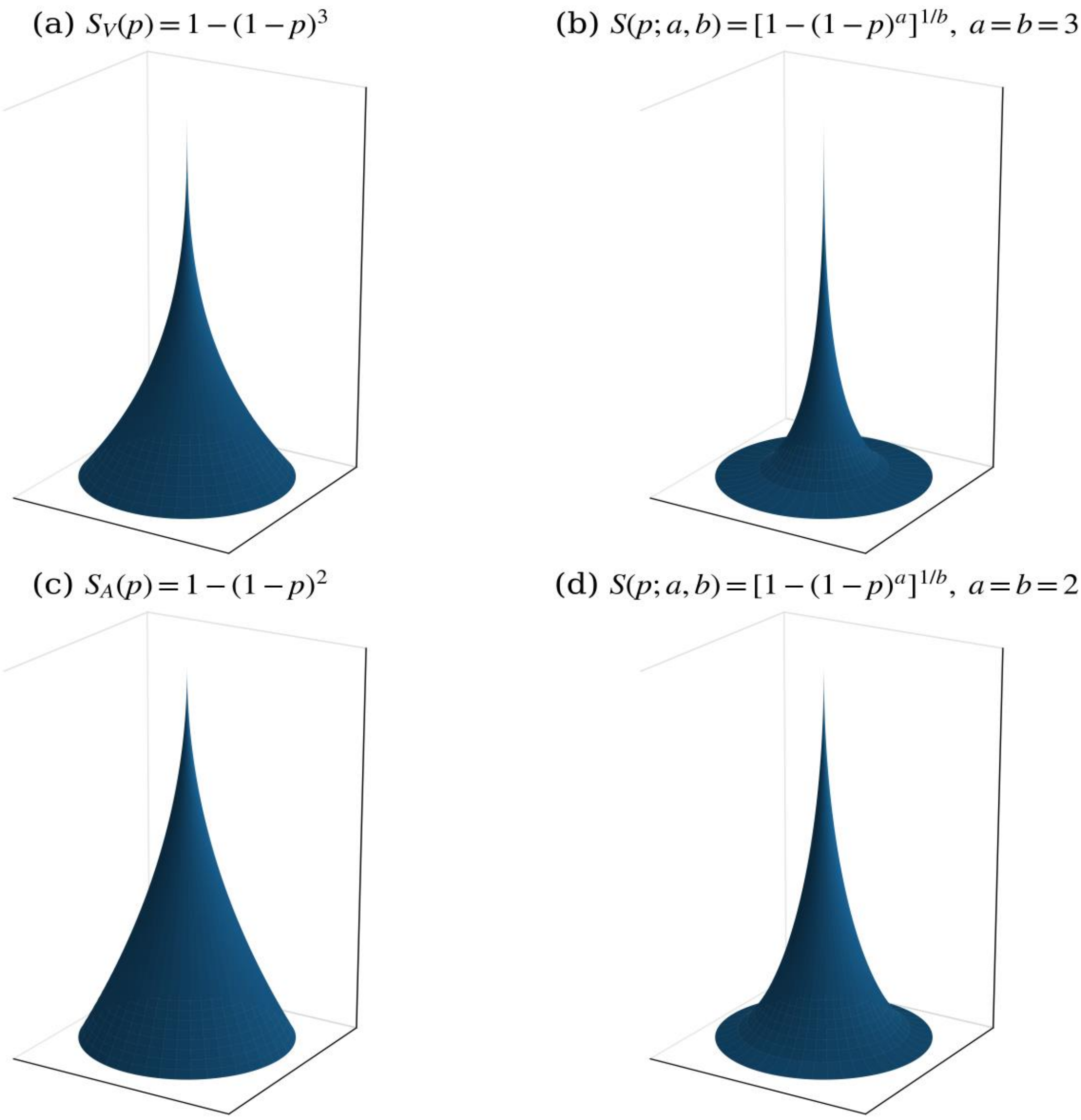

Figure 1 - Solids of revolution for Pareto curves (vertical axis: p, radius: 1−S(p))

5. Two-Parameter PC

The one-parameter symmetric curve (14) is naturally interpreted as a dimensionally determined normative benchmark. It is a special case of the more general two-parameter family [23], which makes it possible to describe deviations of real distributions from this benchmark.

$$S(p;a,b) = [1-(1-p)^a]^{1/b}. \tag{16}$$

where a and b are empirical shape parameters satisfying the conditions $1 \le a < \infty$ and $1 \le b < \infty$.. To describe the behaviour in a neighbourhood of zero, it is convenient to introduce the exponent $\beta = 1/b$. Then, from expansion (8), we obtain

$$S(p;a,b) \approx (ap)^{\beta}, \qquad 1 \le a < \infty, \quad 0 \le \beta \le 1, \quad p \to 0. \tag{17}$$

By analogy with the physics of phase transitions, the exponent β may be interpreted as a critical exponent determining the regime of concentration growth in the region of small p.

$$\beta = \lim_{p \to 0} \frac{\ln S(p;a,b)}{\ln p} = \frac{1}{b}. \tag{18}$$

For family (16), the Gini coefficient is expressed analytically as follows:

$$G(a,\beta) = \frac{2\,\Gamma(1/a)\Gamma(\beta+1)}{a\,\Gamma(\beta+1+1/a)} - 1. \qquad (19)$$

Thus, the behaviour of $S(p;a,b)$ at $p \to 0$ depends primarily on β. When $\beta \to 1$, the resource distribution near zero tends towards greater equality, whereas when $\beta \to 0$, the resource becomes concentrated among the first-ranked elements.

The vertical distance from point $(p, S(p))$ to the egalitarian line is $d(p) = S(p) - p$. The derivative $d'(p) = S'(p) - 1$ vanishes at $S'(p) = 1$; let us denote the corresponding abscissa by p_μ. At this point, the curve is maximally distant from the egalitarian line, and in applied inequality studies the Hoover index $\mathrm{HI} = S(p_\mu) - p_\mu$ is used [24].

For a discretely ranked distribution, the theoretical value of the resource share of the element of rank $w(p_r)$ may be calculated from the formula:

$$w_r = S(p_r) - S(p_{r-1}). \qquad (20)$$

Since condition p_μ holds at point $S'(p_\mu) = 1$, the contribution of the element in this neighbourhood is close to the mean:

$$w(p_\mu) \approx \frac{1}{n}. \qquad (21)$$

Hence the abscissa p_μ determines the rank of the mean element of the ranked distribution. Elements whose resource exceeds the mean, that is, elements for which $p \le p_\mu$, may naturally be interpreted as significant elements of the system. To quantify the asymmetry of the PC, we use the coefficient

$$\mathrm{PAC} = 1 - p_\mu - S_\mu. \qquad (22)$$

The quantity PAC vanishes when point $(p_\mu, S(p_\mu))$ lies on the alternative diagonal of the unit square; in general, the inequality $-1 \le \mathrm{PAC} \le 1$ holds. For the symmetric subclass, that is, when $a = b$, we obtain the explicit expressions [16]:

$$p_\mu(a) = 1 - 2^{-1/a}, \qquad S_\mu(a) = 2^{-1/a}. \qquad (23)$$

For the symmetric case, the Gini coefficient takes the form $G(a) = \Gamma(1/a)^2/[a\,\Gamma(2/a)] - 1$. The dependence of resource concentration and PC asymmetry on the dimension of the observed quantity is given in Table 1; a graphical illustration is shown in Figure 2.

Table 1 - Dependence of PR and PC parameters on the dimension of the observed quantity

Model	β	p_μ	S_μ	PAC	$S(0.2)$	G
$S(p;3) = 1-(1-p)^3$	1.00	0.42	0.8076	−0.23	0.488	0.508
$S(p;2) = 1-(1-p)^2$	1.00	0.50	0.7500	−0.25	0.360	0.333
$S(p;3,1/3) = \sqrt[3]{1-(1-p)^3}$	0.33	0.21	0.7937	0.00	0.787	0.767

$S(p;2,1/2)=\sqrt{1-(1-p)^2}$	0.50	0.29	0.7071	0.00	0.600	0.571

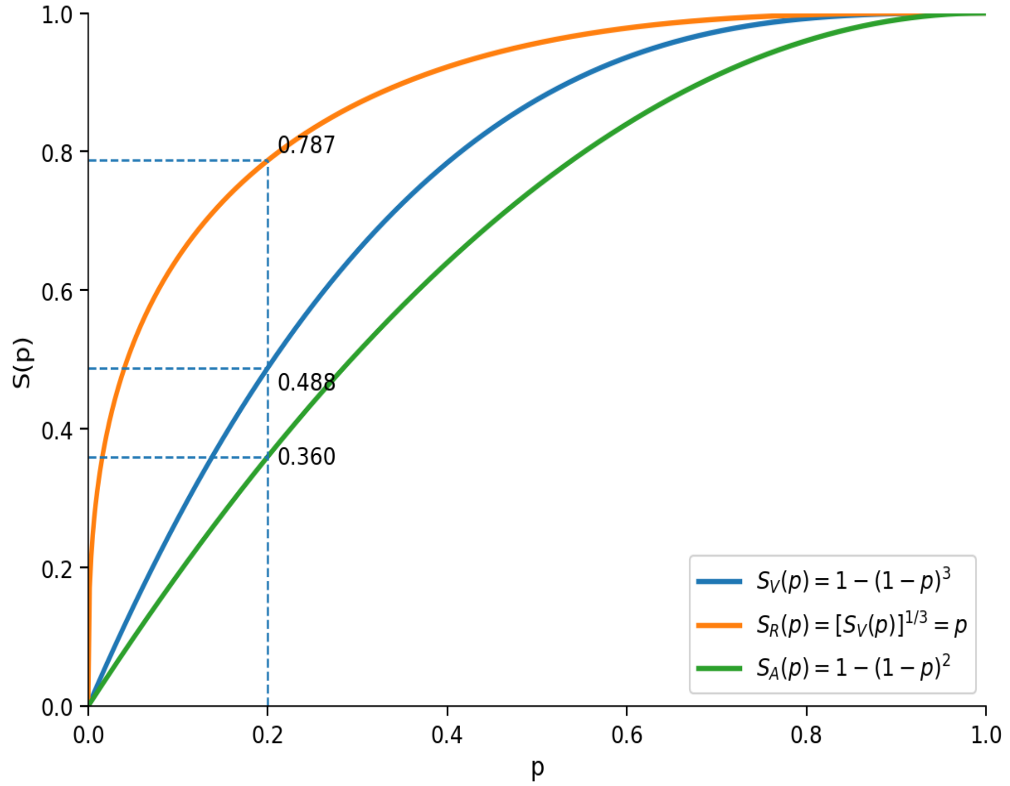

Figure 2 - Concentration functions for cone volume and dimensional mappings

The data in Table 1 show that the diversity of concentration rules is not accidental: it reflects one and the same ranking, but different dimensions of the observed quantity. At the same dimension, symmetric curves generally display higher concentration than the asymmetric baseline models.

Systems for which $\beta = 1$ may be interpreted as summative: here the internal links between elements are weaker than external influences, and the concentration curve describes mainly the additive aggregation of partial contributions. An alternative is provided by integrative systems, in which interactions among the elements within the system exceed external influences; in this case the concentration curve reflects not simple summation but the cooperative amplification of contributions, which manifests itself in stronger concentration and a different asymmetry.

6. Norm of System Ordering

6.1. Centre-of-Mass Coordinate and the Gini Coefficient

By the norm of ordering, we shall hereafter mean such a distribution of the resource that, within the class of models under consideration, the system possesses the greatest structural stability. In the proposed geometric interpretation, this condition is naturally associated with minimising the normalised height of the centre of mass of the distribution of element contributions to the aggregate effect and, in the symmetric subclass, with balance with respect to the alternative diagonal.

Using formula (20), let us write the centre-of-mass coordinate in the form

$$p_c = \int_0^1 p \; dS(p) = \int_0^1 p \, S'(p) \, dp. \tag{24}$$

Since $S(0) = 0$ and $S(1) = 1$, integration by parts gives

$$p_c = 1 - \int_0^1 S\,(p) \, dp. \tag{25}$$

On the other hand, the Gini coefficient for the concentration curve is equal to

$$G = 2\int_0^1 S\,(p) \, dp - 1. \tag{26}$$

Therefore,

$$p_c = \frac{1 - G}{2}. \tag{27}$$

Thus, in the present model, the Gini coefficient acquires a clear geometric interpretation: it determines the position of the centre of mass of the distribution of contributions. The smaller p_c, the lower the centre of mass lies, and the more concentrated the resource distribution is.

Let us denote by p_k the abscissa of the point at which the concentration curve intersects the alternative diagonal:

$$S(p_k) = 1 - p_k. \tag{28}$$

For any concave concentration curve, the inequality holds

$$p_c \leq p_k. \tag{29}$$

Equality is attained only in the limiting case where the curve coincides with the broken line joining points $(0,0)$, $(p_k, 1 - p_k)$, and $(1,1)$. Consequently, the position of the centre of mass and the position of the intersection point with the alternative diagonal characterise different aspects of the distribution structure: the former is related to overall concentration, while the latter is related to its balance with respect to the rank scale.

6.2. The Lever Rule in the Symmetric Subclass

Let us now consider the point (p_μ, S_μ), where p_μ is defined by condition $S'(p_\mu) = 1$, and $S_\mu = S(p_\mu)$. Below we restrict attention to the symmetric subclass, that is, the case *a*=*b*. At this point, the concentration curve has the same local slope as the egalitarian line, and for a discrete distribution the contribution of the element in the neighbourhood of p_μ is close to the mean value. Therefore, p_μ may be interpreted as the rank of the mean element of a rank-ordered system.

If we associate with the concentration curve a notional lever with fulcrum at p_μ and additionally require symmetry of the curve with respect to the alternative diagonal, the natural equilibrium condition may be written in the form

$$\frac{S_\mu}{1 - S_\mu} = \frac{1 - p_\mu}{p_\mu}. \tag{30}$$

It follows that

$$p_\mu S_\mu = (1 - p_\mu)(1 - S_\mu). \tag{31}$$

and, consequently,

$$p_\mu + S_\mu = 1. \tag{32}$$

Thus, the equilibrium state in the symmetric subclass corresponds to the location of point (p_μ, S_μ) on the alternative diagonal of the unit square. This condition is expressed through the asymmetry coefficient

$$\text{PAC} = 1 - p_\mu - S_\mu. \tag{33}$$

When $\text{PAC} = 0$, the point of the mean element is in equilibrium; when it deviates from zero, the correspondence between the rank of the mean element and the share of the effect accumulated up to that rank is violated. The quantity $|\text{PAC}|$ may therefore naturally be regarded as an indicator of the structural imbalance of the concentration curve relative to the symmetric normative case. In this interpretation, the 20/80 rule appears not as a separate empirical heuristic, but as one of the special cases of the more general condition $\text{PAC} = 0$. It is realised precisely in the symmetric subclass and should not be transferred unconditionally to the entire two-parameter class.

6.3. Criterion of the Norm of Ordering

Within the adopted model, the norm of ordering is treated not as an arbitrary empirical distribution, but as a normative benchmark within the class of curves under consideration. It is therefore defined by the joint fulfilment of two requirements:

$$|\text{PAC}| \to \min, \qquad p_c \to \min. \tag{34}$$

The first condition characterises structural balance, whereas the second characterises the degree of concentration of the distribution; by virtue of (27), the second condition is equivalent to the requirement G→max. Consequently, within the class under consideration, the more stable system should be regarded as the one that simultaneously satisfies PAC = 0 and minimises the normalised height of the centre of mass. In other words, the norm of ordering corresponds to a distribution in which the balance of the rank structure and the lowest possible position of the centre of mass are consistent; this is why the symmetric normative curve plays the role of a benchmark of structural stability within the chosen class of models.

7. Discussion: the 20/80 Rule as a Dimensional Effect

The results obtained make it possible to reinterpret the classical 20/80 rule. In the original geometric model of the conical sandpile, volume concentration is determined by formula (5), that is, directly by the cubic scaling of volume with linear size. However, when one passes from volume to equivalent radius, area, or another observable quantity, the concentration function itself also changes. In other words, one and the same ranked organisation of a system may lead to different numerical concentration rules if the observed effect has a different dimension or a different power-law relation to the linear scale of the carrier.

More generally, if the observable quantity Q is related to the basic measure by a power law, then the concentration curve for Q is obtained as a dimensional mapping of the basic curve. An important conclusion follows: the 20/80 rule need not be regarded as a universal constant applicable in exactly the same way to all systems. It is a particular numerical result arising under a specific way of passing from the carrier of the effect to the observable quantity. Within the same scheme, other relations, in particular 20/79 and 20/62, arise naturally as well.

It is crucial that ranks and rank shares are dimensionless in themselves. Therefore, dimensionality manifests itself not in the ranking procedure, but in the rule that links the resource to the linear scale of the carrier. It is in this sense that the dimensional interpretation unifies the geometric model of the sand cone and parametric concentration curves: the difference between the observed laws is explained not by a change in the order of elements, but by a change in the metric and in the way the effect is projected onto a one-dimensional scale.

This explanation differs from other interpretations of the 20/80 rule discussed in [25]-[27], [39]. In some works, the corresponding proportion is derived from particular probability distributions; in others, from entropic principles or index constructions. In the present paper, the source of the relation is different: it is derived from geometry, scaling, and the choice of the observable quantity. The main idea of the article, therefore, is not to postulate a new universal constant, but to reduce several familiar concentration rules to a single dimensional mechanism.

7.1. Self-Organised Criticality and Static Geometry

The chosen sandpile model is borrowed from the classical context of self-organised criticality [12], [13]. At the same time, the present paper considers only the static geometric profile of the stationary state. Avalanche dynamics, critical slopes, and event statistics are not modelled. The dimensional interpretation therefore provides not a complete dynamic theory of SOC, but an analytically transparent static benchmark against which observed concentration regimes may be compared. This formulation is especially useful because, in recent Physica A papers, the Gini and Kolkata indices have already been applied to avalanche statistics in SOC models and have displayed close to nearly universal values of about 0.87 [38], [39]. Consequently, the geometric scheme proposed here may be regarded as a basic benchmark relative to which deviations between statically expected and dynamically emerging inequality may be interpreted.

7.2. The Pareto Exponent and the Dimension of the Effect

The relation $\alpha = D/(D-1)$ gives the dimensional interpretation additional rigour. It shows that the Pareto exponent may be regarded as a reflection of the effective dimension of the observed effect. For $D = 3$, one obtains $\alpha = 3/2$, that is, precisely the value historically associated with the classical Pareto law. In this sense, the 20/80 rule is not an accidental empirical proportion, but a particular manifestation of the structural correspondence between the dimension of the carrier and tail concentration.

7.3. The Critical Point D = 1 and Effective Dimension

The dimensional scheme naturally singles out the case $D = 1$. For $D = 1$, the model yields S(p)=p, that is, the regime of complete equality. For $D > 1$, concentration arises continuously, and in limiting regimes of strong non-linearity the curve tends towards an almost step-like form. $D = 1$ may therefore conveniently be interpreted as a critical point separating the egalitarian regime from the concentrated regime, while the emergence of concentration itself may be viewed as a dimensional analogue of the transition from an equilibrium state to a non-equilibrium accumulation regime. From a practical point of view, this makes it possible to interpret the calibrated model parameter as the effective dimension of the observed effect: not only as a measure of inequality, but also as a characteristic of the way in which the resource scales.

7.4. Econophysical Interpretation

In econophysics, inequality and tail distributions are often explained through interacting-agent models and kinetic exchange models [33]-[35]. These studies show that macroscopic concentration may arise from local rules of exchange, conservation, and saving. The dimensional scheme proposed here does not compete with such models; rather, it complements them by providing a normative geometric profile, or target profile, relative to which dynamically emerging inequality may be interpreted as an amplification, attenuation, or deformation of the dimensionally expected distribution.

7.5. Entropic Interpretation and the Burr III Family

The symmetric family obtained in the paper is naturally associated with the Lorenz curve of the Burr III distribution. This also admits an additional statistical-mechanical interpretation. In the literature on non-extensive statistics, q-exponential and Burr distributions are closely related [36], [37]. In this sense, the normative symmetric profile may be regarded not only as a geometrically and energetically distinguished form, but also as a statistically natural one. At the same time, the present paper does not use an entropic principle as the primary foundation of the model: the main criterion remains system efficiency, expressed through the minimum of potential energy and the position of the centre of mass. The entropic interpretation is therefore regarded here as an additional physical justification rather than a substitute for the dimensional-geometric approach.

8. Managing Income Inequality

8.1. Reducing Inequality According to the Gini Coefficient

As an applied illustration of the use of the symmetric PC model for analysing tax-and-transfer policy, the paper employs data from the OECD Income Distribution Database (IDD) and OECD materials on inequality in budgeting for the United States and Costa Rica for 2021 [29], [30], [32]. According to these data, the tax-and-transfer mechanisms in force in both countries operate in the direction of weakening income concentration: in the United States the Gini coefficient falls from 0.503 for market income to 0.397 for disposable income, while in Costa Rica it falls from 0.551 to 0.487 [30], [32]. At the same time, a reduction in the Gini coefficient alone does not show how close the resulting distribution is to the normative symmetric configuration. The present example should therefore be understood as illustrative: it demonstrates the usefulness of the proposed criterion, but it does not replace a full empirical fit of the entire Lorenz curve.

8.2. The Post-Tax Pareto Curve and the Symmetric Normative Curve

To assess the quality of tax-and-transfer policy, the actual post-tax Pareto curve is compared in the paper with a symmetric normative curve having the same value of the Gini coefficient. Since the Gini coefficient G depends monotonically on the concentration parameter a, the observed value G_{obs} uniquely determines the model parameter from the equation $G(a) = G_{\text{obs}}$.

To approximate disposable income, the quintile ratio S80/S20 from the OECD database was additionally used, which made it possible to approximate the actual post-tax PCs by means of the two-parameter model (16) [31], [32].

The OECD data for 2021 and the results of the approximation are presented in Table 2. A comparison of the actual post-tax Pareto curves with the normative symmetric PCs is shown in Figure 3.

Table 2 - OECD 2021: income before and after taxes and transfers, the United States and Costa Rica

Country	Income type	G	S80/S20	*a*	*b*	p_μ	S_μ	PAC	Note
United States	market	0.503	—	—	—	—	—	—	OECD IDD / OECD budgeting materials, 2021
United States	disposable	0.397	5.1	0.92	2.49	0.21	0.52	0.27	OECD IDD, 2021
Costa Rica	market	0.551	—	—	—	—	—	—	OECD IDD / OECD budgeting materials, 2021
Costa Rica	disposable	0.487	11.56	1.45	2.11	0.27	0.63	0.10	OECD IDD, 2021

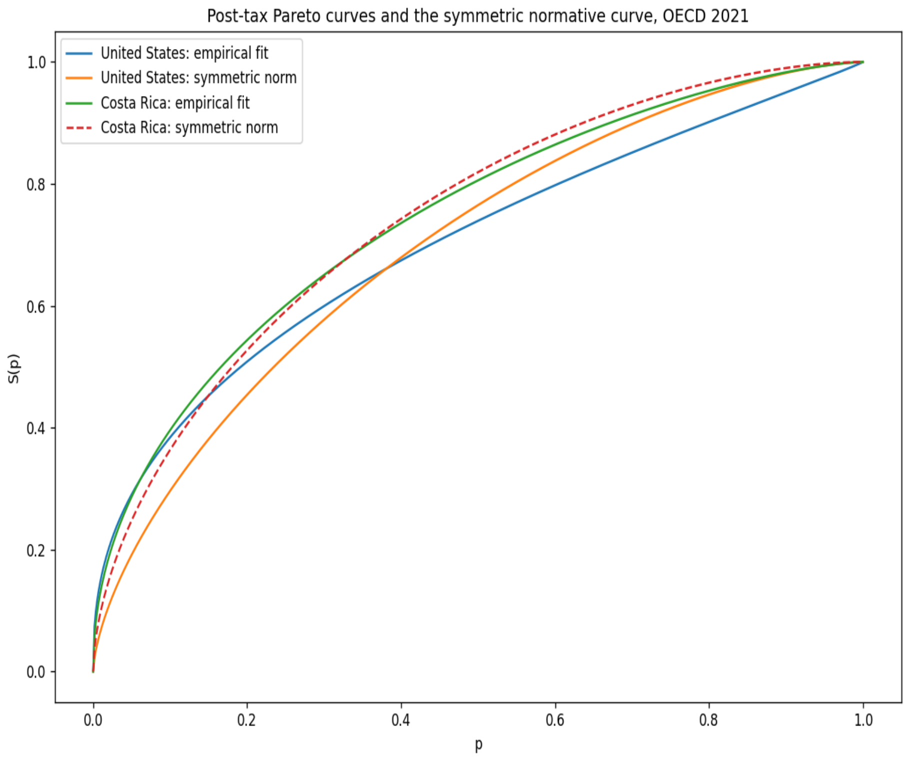

Figure 3 - Empirical post-tax Pareto curves and the symmetric normative curve

The PAC values in Table 2 and the curves in Figure 3 show that the PC of household income after taxes and transfers in Costa Rica is closer to the normative symmetric curve than that of the United States, although the overall level of inequality in Costa Rica remains higher. This means that the post-tax distribution profile in the United States departs more strongly from the symmetric normative benchmark, whereas in Costa Rica the shape of the post-tax Pareto curve is closer to it, despite the higher overall level of inequality. The comparison therefore shows that a reduction in G alone is insufficient for a meaningful assessment of the structural effect of redistribution.

8.3. Practical Implication

This result is important for the interpretation of tax-and-transfer policy. If one uses only the Gini coefficient, one may conclude merely that redistribution reduces inequality. However, comparing the post-tax curve with the symmetric normative curve provides a more informative picture: redistribution policy should be evaluated not only by the reduction in G, but also by the extent to which it reduces the asymmetry of the actual Pareto curve and brings the system closer to the normative symmetric configuration. In other words, effective inequality management consists not simply in mechanically reducing the income gap, but in moving the system towards a more balanced distribution of the resource. In this sense, closeness to the symmetric normative curve may be regarded as an additional criterion of the structural quality of tax-and-transfer policy.

Conclusion

The paper has proposed a dimensional interpretation of empirical rules of resource concentration. It has been shown that the shape of the concentration curve is determined not only by the rank ordering of elements, but also by the way in which the observed quantity scales relative to the linear carrier. The geometric model of a conical sandpile serves as an analytically transparent example in which, with the carrier structure unchanged, different effects - volume, surface, equivalent radius, and energy - lead to different concentration laws. The main contribution of the paper is the construction of a dimensional geometric benchmark linking the observed concentration ratio directly to the dimension of the measured effect.

The principal result is that the 20/80 rule is interpreted as a special case of a broader class of dimensionally determined relations. For the three-dimensional case, the transition from the volumetric measure to the equivalent linear scale leads to the value $S(0.2) = 0.787$, which practically coincides with the classical heuristic. Matching the asymptotics yields the exact relation $\alpha = D/(D-1)$, linking the Pareto exponent to the dimension of the observed effect. At the same time, the case $D = 1$ is naturally singled out as a threshold between the regime of complete equality and regimes of concentration, while the calibrated model parameter may be interpreted as the effective dimension of the observed quantity.

An additional result concerns the use of the symmetric parametric model and the Gini coefficient. It is shown that the observed Gini value may be used to recover the parameter of the normative concentration curve and thereby to move from empirically measured inequality to a consistent model structure of resource distribution. The example based on OECD data for market and disposable incomes in the United States and Costa Rica for 2021 shows that such a procedure makes it possible to assess not only the overall level of post-tax inequality, but also the degree of closeness of the actual Pareto curve to the symmetric normative benchmark. Thus, a reduction in the Gini coefficient is a necessary, but not a sufficient, criterion of the structural balance of the distribution. At the same time, full validation of the model requires fitting the entire Lorenz curve, while the relation of dimensional geometry to the dynamics of self-organised criticality and to entropic principles should be regarded as a natural direction for further research.

It has been shown that the effectiveness of tax-and-transfer policy should be evaluated not only by the reduction of the Gini coefficient, but also by the extent to which that policy brings the actual Pareto curve closer to the symmetric normative configuration.

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The author declares that there are no competing financial interests or personal relationships that could have influenced the results presented in this work.

Data Availability

No new primary data were created for this study. The analytical part of the paper is based on theoretical derivations, and the empirical illustration uses published external sources cited in the reference list and in the text of the article. This information is sufficient to reproduce the reported results.

Declaration on Generative AI and AI-Assisted Technologies in the Writing Process

During the preparation of this work, the author used ChatGPT (OpenAI) to assist with translation into English, improvements in language and style, and refinement of the manuscript presentation. After using this tool, the author carefully reviewed and edited the material as necessary and takes full responsibility for the content of the publication.

References

[1] V. Pareto, Cours d'Économie Politique, vols. 1 and 2, F. Rouge, Lausanne, 1896–1897.
[2] J.M. Juran, The Non-Pareto Principle; mea culpa, Qual. Prog. 8 (5) (1975) 8–9.
[3] J.M. Juran, Juran's Quality Control, McGraw-Hill, New York, 1951.
[4] J.M. Juran, Universals in management planning and controlling, Manage. Rev. 43 (11) (1954) 748–761.
[5] G.A. Grachev, Modelling the Pareto Principle, Southern Federal University, Rostov-on-Don, 2011, 223 p.
[6] G.A. Grachev, Managing the structure of household monetary income on the basis of the Pareto principle, Terra Econ. 11 (1) (2013) 57-63.
[7] G.A. Grachev, Size distribution of states, counties, and cities in the USA: New inequality form information, Physica A 592 (2022) 126831. https://doi.org/10.1016/j.physa.2022.126831.
[8] G.K. Zipf, Human Behavior and the Principle of Least Effort, Addison-Wesley, Cambridge, MA, 1949.
[9] M.E.J. Newman, Power laws, Pareto distributions and Zipf's law, Contemp. Phys. 46 (5) (2005) 323–351.
[10] S.A. Krashakov, A.B. Teslyuk, L.N. Shchur, On the universality of the rank distribution of web-server popularity, Vestn. RFFI 35 (1) (2004) 46-66.
[11] G.A. Grachev, The Pareto Principle: Predictable Resource Concentrations in Self-Organising Systems, Southern Federal University Press, Rostov-on-Don-Taganrog, 2024, 102 p.
[12] P. Bak, C. Tang, K. Wiesenfeld, Self-organized criticality: An explanation of the 1/f noise, Phys. Rev. Lett. 59 (4) (1987) 381–384.
[13] P. Bak, C. Tang, K. Wiesenfeld, Self-organized criticality, Phys. Rev. A 38 (1) (1988) 364–374.

[14] B.B. Mandelbrot, The Fractal Geometry of Nature, W.H. Freeman, San Francisco, 1982.
[15] I.W. Burr, Cumulative frequency functions, Ann. Math. Stat. 13 (2) (1942) 215–232.
[16] N. Kakwani, Income Inequality and Poverty: Methods of Estimation and Policy Applications, Oxford University Press, Oxford, 1980.
[17] V.V. Matokhin, Algebra of Control, Moscow, 1994, 12 p.
[18] A.V. Kryanev, V.V. Matokhin, S.G. Klimanov, Statistical functions of resource distribution in economics: 010-98, Moscow, 1998, 15 p.
[19] I. Antoniou, V.V. Ivanov, Yu.L. Korolev, A.V. Kryanev, V.V. Matokhin, Z. Suchanecki, Analysis of resources distribution in economics based on entropy, Physica A 304 (3) (2002) 525–534. https://doi.org/10.1016/S0378-4371(01)00575-1.
[20] I. Antoniou, V.V. Ivanov, A.V. Kryanev, V.V. Matokhin, M.V. Shapovalov, On the efficient resources distribution in economics based on entropy, Physica A 336 (3–4) (2004) 549–562. https://doi.org/10.1016/j.physa.2003.12.051.
[21] J.M. Henle, N.J. Horton, S.J. Jakus, Modelling inequality with a single parameter, in: D. Chotikapanich (Ed.), Modeling Income Distributions and Lorenz Curves, Springer, New York, 2008, pp. 255–269.
[22] J.M. Sarabia, V. Jordá, C. Trueba, The Lamé class of Lorenz curves, Commun. Stat. Theory Methods 46 (11) (2017) 5311–5326. https://doi.org/10.1080/03610926.2013.775306.
[23] B.C. Arnold, Majorization and the Lorenz Order: A Brief Introduction, Springer, New York, 1987.
[24] E.M. Hoover, The measurement of industrial localization, Rev. Econ. Stat. 18 (1936) 162–171.
[25] K. Tanabe, Pareto's 80/20 rule and the Gaussian distribution, Physica A 510 (2018) 635–640. https://doi.org/10.1016/j.physa.2018.07.023.
[26] P.K. Rawlings, D. Reguera, H. Reiss, Entropic basis of the Pareto law, Physica A 343 (2004) 643–652. https://doi.org/10.1016/j.physa.2004.06.152.
[27] S. Banerjee, B.K. Chakrabarti, M. Mitra, S. Mutuswami, On the Kolkata index as a measure of income inequality, Physica A 545 (2020) 123178. https://doi.org/10.1016/j.physa.2019.123178.
[28] Special issue: Inequalities in Socio-Economic, Bio-Medical & Physical Systems: a statistical physics perspective, Physica A, 2025.
[29] OECD, Income and wealth distribution databases, OECD, Paris, updated 17 Dec. 2025. Available at: https://www.oecd.org/en/data/datasets/income-and-wealth-distribution-database.html.
[30] OECD, Addressing Inequality in Budgeting, OECD Publishing, Paris, 2024.
[31] OECD, Income inequality, OECD indicator metadata, OECD, Paris, 2026. Available at: https://www.oecd.org/en/data/indicators/income-inequality.html.
[32] OECD, Income Distribution Database (IDD), OECD Data Explorer, Paris, accessed March 2026. Available at: https://data-explorer.oecd.org/.
[33] A.A. Drăgulescu, V.M. Yakovenko, Statistical mechanics of money, Eur. Phys. J. B 17 (4) (2000) 723–729. https://doi.org/10.1007/s100510070114.

[34] A. Chakraborti, B.K. Chakrabarti, Statistical mechanics of money: how saving propensity affects its distribution, Eur. Phys. J. B 17 (1) (2000) 167–170. https://doi.org/10.1007/s100510070173.

[35] A. Chatterjee, B.K. Chakrabarti, S.S. Manna, Pareto law in a kinetic model of market with random saving propensity, Physica A 335 (1–2) (2004) 155–163. https://doi.org/10.1016/j.physa.2003.11.014.

[36] C. Tsallis, Possible generalization of Boltzmann-Gibbs statistics, J. Stat. Phys. 52 (1–2) (1988) 479–487. https://doi.org/10.1007/BF01016429.

[37] S. Nadarajah, S. Kotz, q exponential is a Burr distribution, Phys. Lett. A 359 (6) (2006) 577–579. https://doi.org/10.1016/j.physleta.2006.07.035.

[38] S.S. Manna, S. Biswas, B.K. Chakrabarti, Near universal values of social inequality indices in self-organized critical models, Physica A 593 (2022) 127121. https://doi.org/10.1016/j.physa.2022.127121.

[39] A. Ghosh, B.K. Chakrabarti, Limiting value of the Kolkata index for social inequality and a possible social constant, Physica A 573 (2021) 125944. https://doi.org/10.1016/j.physa.2021.125944.