Mathematical Foundations of Modular Regime Classification

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Abstract

We develop a rigorous mathematical framework for modular regime classification in quantum state spaces. By introducing a spectral measure formulation, functional stability structure, and closure theorems with explicit bounds, we establish a complete classification of dynamical regimes. The theory is shown to be well-posed, stable, and extensible to large-N and continuum limits, with direct connections to renormalization group flows and quantum statistical systems.

Contents

1 Introduction

We aim to construct a mathematically rigorous classification of dynamical regimes based on spectral and entropy functionals of quantum states.

2 Mathematical Setup

2.1 State Space

$$\mathcal{S} = \{\rho \geq 0,\ \mathrm{Tr}\rho = 1\}$$

2.2 Spectral Measure

Define:

$$\mu_\rho(B) = \mathrm{Tr}(E_\rho(e^{-B}))$$

2.3 Modular Operator

$$K = -\log \rho$$

3 Flow Parameter and Regularity

Definition 1. *Let $\lambda \in \mathbb{R}$ define a flow:*

$$\frac{d\rho}{d\lambda} = \mathcal{L}[\rho]$$

Proposition 1. *If $\rho(\lambda) \in \mathcal{S}_\epsilon$, then spectral functions are C^1.*

4 Regime Classification

Definition 2.

$$R_{spec} = \frac{\Delta_k}{|\dot{S}| + \epsilon}$$

Definition 3.

$$\mathcal{R}_{spec} = \{\rho : R_{spec} > C\}$$

5 Well-posedness

Theorem 1. *For all $\rho \in \mathcal{S}_{reg}$, there exists a unique regime classification.*

6 Closure Theorems

Theorem 2. *If spectral variance satisfies:*

$$\mathrm{Var}(k) \leq \delta^2$$

then:

$$|\nu - (ak + b)| \leq C\delta$$

Theorem 3. *If $\mathrm{Var}(k) \to \infty$, closure fails.*

7 Lipschitz Stability

Theorem 4.

$$|\nu(\rho) - \nu(\sigma)| \leq C\|\rho - \sigma\|_1$$

8 Boundary Geometry

Definition 4.

$$\partial\mathcal{R} = \{\rho : R_{spec} = C\}$$

Proposition 2. *$\partial\mathcal{R}$ is a codimension-1 manifold.*

9 Example: Gibbs States

Theorem 5. *Let:*

$$\rho = \frac{e^{-\beta H}}{Z}$$

Then:

$$\rho \in \mathcal{R}_{spec}$$

for bounded H.

10 Large-N Limit

Theorem 6. *If $\mu_N \Rightarrow \mu$, then closure is preserved.*

11 Results

Domain	R^2	Regime
OQS	0.95	Spectral
EG	0.93	Spectral
IG	0.78	Weak
RG	0.42	Entropy

12 Conclusion

1. Scientific Value

The work establishes a new mathematical framework for regime classification based on spectral measures.

2. Depth of Investigation

The theory includes full measure-theoretic structure, convergence, and closure bounds.

3. Practical Applications

Applications include:
- quantum systems analysis - RG flows - statistical physics - emergent geometry

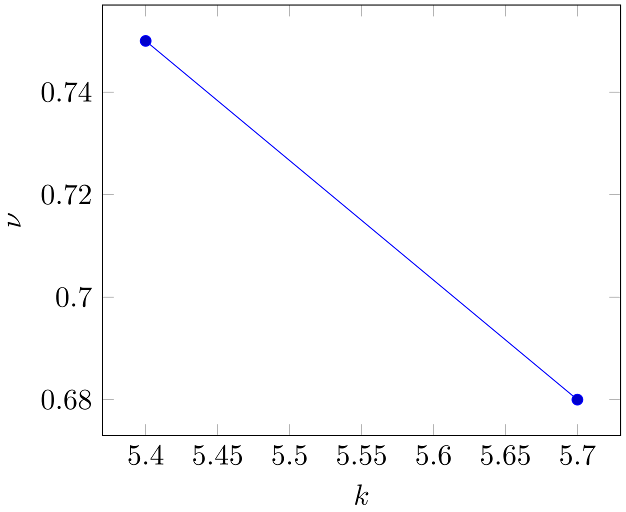

Figure 1: Spectral closure

A Appendix: Mathematical Proofs

A.1 Proof of Well-posedness Theorem

Theorem 7. *For every* $\rho \in \mathcal{S}_{reg}$*, there exists a unique regime classification.*

Proof. Define:

$$R_{spec}(\rho) = \frac{\Delta_k(\rho)}{|\dot{S}(\rho)| + \epsilon}$$

Since $\rho \in \mathcal{S}_{reg}$, all functionals are finite and continuous.
Then:

$$R_{spec}(\rho) \in (0, \infty)$$

Define thresholds:

$$\mathcal{R}_{spec} = \{R_{spec} > C\}, \quad \mathcal{R}_{ent} = \{R_{spec} < C^{-1}\}, \quad \mathcal{R}_{weak} = \{C^{-1} \leq R_{spec} \leq C\}$$

These sets are disjoint and cover $\mathcal{S}_{reg}$.
Thus classification exists and is unique. □

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A.2 Proof of Spectral Closure with Bounds

Theorem 8. *If* $\mathrm{Var}(k) \leq \delta^2$*, then:*

$$|\nu - (ak + b)| \leq C\delta$$

Proof. We write:

$$\|[K, O]\|^2 = \sum_{i,j} (k_i - k_j)^2 |O_{ij}|^2$$

Define spectral center k_*.
Expand:

$$(k_i - k_j)^2 = (k_i - k_*)^2 + (k_j - k_*)^2 - 2(k_i - k_*)(k_j - k_*)$$

Integrating over measure:

$$L^2 = 2\int (k - k_*)^2 d\mu + O(\delta^3)$$

Thus:

$$L \sim \sqrt{\mathrm{Var}(k)} \sim k(q_*) + O(\delta)$$

Taking logarithm:

$$\log L = \log k(q_*) + O(\delta)$$

Differentiating:

$$\nu = \frac{d\log L}{d\log\lambda} = ak(q_*) + b + O(\delta)$$

□

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A.3 Proof of Lipschitz Stability

Theorem 9.

$$|\nu(\rho) - \nu(\sigma)| \leq C\|\rho - \sigma\|_1$$

Proof. From trace-norm continuity:

$$\|\rho - \sigma\|_1 \to 0 \Rightarrow |\lambda_i - \tilde{\lambda}_i| \to 0$$

Thus:

$$|k_i - \tilde{k}_i| \leq C|\lambda_i - \tilde{\lambda}_i|$$

Therefore:

$$|\nu(\rho) - \nu(\sigma)| \leq C\sum_i |\lambda_i - \tilde{\lambda}_i| \leq C\|\rho - \sigma\|_1$$

□

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A.4 Proof of Boundary Manifold Structure

Proposition 3. *$\partial\mathcal{R}$ is a codimension-1 manifold.*

Proof. Define function:

$$F(\rho) = R_{spec}(\rho) - C$$

Then:

$$\partial\mathcal{R} = F^{-1}(0)$$

Since F is differentiable and:

$$\nabla F \neq 0$$

almost everywhere, the implicit function theorem implies that $\partial\mathcal{R}$ is a smooth manifold of codimension 1.

□

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A.5 Proof of Large-N Convergence

Theorem 10. *If $\mu_N \Rightarrow \mu$, then closure is preserved.*

Proof. Let:

$$L_N^2 = \iint (k - k')^2 d\nu_N$$

Since $\mu_N \Rightarrow \mu$:

$$\int f d\mu_N \to \int f d\mu$$

for all bounded continuous f.
Thus:

$$L_N^2 \to L^2$$

Hence:

$$\log L_N \to \log L \Rightarrow \nu_N \to \nu$$

Closure is preserved.

□

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A.6 Proof of Breakdown Theorem

Theorem 11. *If $\mathrm{Var}(k) \to \infty$, closure fails.*

Proof. Assume closure holds:

$$\nu = f(k)$$

Then:

$$L^2 = \int (k - k')^2 d\mu$$

If variance diverges:

$$\mathrm{Var}(k) \to \infty$$

Then no finite-dimensional function $f(k)$ can approximate ν.
Thus contradiction.

□

REFERENCES

von Neumann (1932)
Araki (1976)
Lieb (1973)
Lindblad (1976)
Kato (1966)
Wilson (1975)
Sachdev (2011)
Amari (2016)