

Modular Criticality Without Correlation Length: A Testable Prediction for Open Quantum Systems

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Abstract

We report the discovery of a new class of critical phenomena in open quantum systems that is not governed by divergence of correlation length. In contrast to the standard paradigm, where criticality is identified by $\xi \rightarrow \infty$, we demonstrate the existence of regimes characterized by:

$$\Delta \rightarrow 0, \quad \Gamma > 0, \quad \xi = O(1),$$

where Δ is the partition gap and Γ is the switching rate of optimal decompositions. We show that such systems exhibit a universal modular scaling law:

$$\nu(\lambda) \sim \frac{1}{\log \lambda},$$

derived from the spectral structure of the modular generator $K = -\log \rho$. This scaling defines a new diagnostic observable independent of spatial correlations.

We formulate a hard, experimentally testable prediction:

$$\Xi(\lambda) = \nu(\lambda) \cdot \log \lambda \rightarrow 1,$$

which provides a falsifiable signature of modular criticality accessible in quantum simulators.

The results establish a new notion of criticality rooted in spectral redistribution, commutator dynamics, and information geometry, rather than correlation length. This reveals a previously unrecognized universality class applicable to open quantum systems, non-equilibrium dynamics, and information-theoretic regimes.

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1 Introduction

1.1 The Standard Paradigm of Criticality

Critical phenomena occupy a central role in modern physics, where they are traditionally associated with divergence of correlation length:

$$\xi \rightarrow \infty.$$

This paradigm underlies the theory of phase transitions, renormalization group (RG) flows, and universality classes. In this framework, criticality is identified through:

- long-range correlations,
- scale invariance,
- divergence of response functions.

1.2 Limitations in Open Quantum Systems

However, this paradigm breaks down in open quantum systems (OQS) and non-equilibrium dynamics.

In such systems:

- correlations often remain short-ranged,
- ξ may remain finite,
- dynamics is dominated by dissipation and mixing,
- no clear spatial geometry may exist.

Nevertheless, these systems may exhibit sharp structural transitions that resemble critical behavior.

This leads to a fundamental inconsistency:

critical-like transitions may occur without $\xi \rightarrow \infty$

1.3 Failure of Correlation-Length Diagnostics

The breakdown of the correlation-length paradigm implies that:

ξ is not a universal diagnostic of criticality

In particular, there exist systems where:

- $\xi = O(1)$ for all parameters,
- yet the system undergoes a qualitative structural change.

Such behavior cannot be captured within the standard framework of phase transitions.

1.4 Conceptual Shift: Modular Description

To resolve this limitation, we adopt a modular perspective based on the density operator ρ . The central object is the modular generator:

$$K = -\log \rho,$$

which encodes the full spectral and informational structure of the system. Rather than analyzing spatial correlations, we focus on:

- spectral redistribution,
- commutator dynamics,
- partition structure,
- information geometry.

1.5 New Definition of Criticality

We introduce a new definition of criticality based on partition structure:

$$\boxed{\Delta \rightarrow 0, \quad \Gamma > 0,}$$

where:

- Δ is the gap in the partition landscape,
- Γ is the switching rate between competing decompositions.

This definition is independent of correlation length.

1.6 Main Result of This Work

We prove that there exists a class of systems satisfying:

$$\boxed{\Delta \rightarrow 0, \quad \Gamma > 0, \quad \xi = O(1),}$$

which defines a new type of criticality.

Moreover, this regime is characterized by a universal scaling law:

$$\boxed{\nu(\lambda) \sim \frac{1}{\log \lambda}.}$$

1.7 Testable Prediction

We define a measurable observable:

$$\Xi(\lambda) = \nu(\lambda) \cdot \log \lambda,$$

and derive the prediction:

$$\boxed{\Xi(\lambda) \rightarrow 1.}$$

This provides a falsifiable experimental signature of modular criticality.

1.8 Scientific Contribution

This work establishes:

- a new class of critical phenomena,
- a universal diagnostic independent of correlation length,
- a modular framework for regime classification,
- a concrete experimental prediction.

1.9 Structure of the Paper

The paper is organized as follows:

- Section 2: Failure of correlation-length diagnostics
- Section 3: Construction of modular critical systems
- Section 4: Existence theorem
- Section 5: Signal formation
- Section 6: Hard prediction
- Section 7: Phase structure
- Section 8: Experimental protocol
- Section 9: Discussion
- Section 10: Conclusion

2 Failure of Correlation-Length Diagnostics

2.1 Correlation Length as a Traditional Diagnostic

In standard statistical physics, criticality is characterized by divergence of correlation length:

$$C(r) \sim e^{-r/\xi}, \quad \xi \rightarrow \infty.$$

This behavior is typically associated with:

- scale invariance,
- universality,
- divergence of susceptibilities.

2.2 Implicit Assumptions of the Paradigm

The correlation-length paradigm relies on several implicit assumptions:

1. existence of a spatial metric,
2. locality of interactions,
3. equilibrium or near-equilibrium conditions,
4. well-defined correlation functions.

These assumptions are violated in many physically relevant systems.

2.3 Open Quantum Systems and Finite Correlation Length

Consider a class of open quantum systems governed by CPTP evolution:

$$\rho(\lambda) = \mathcal{E}_\lambda(\rho_0).$$

Let correlations be defined via two-point functions:

$$C_{ij} = \text{Tr}(\rho O_i O_j) - \text{Tr}(\rho O_i) \text{Tr}(\rho O_j).$$

In dissipative systems, one typically observes:

$$C_{ij} \sim e^{-d(i,j)/\xi}, \quad \xi = O(1)$$

for all λ .

2.4 Existence of Structural Transitions Without $\xi \rightarrow \infty$

We now demonstrate that finite ξ does not exclude the existence of critical behavior.

Consider a family of states:

$$\rho(\lambda) = (1 - p(\lambda))\rho_{\text{prod}} + p(\lambda)\rho_{\text{mix}},$$

with:

$$p(\lambda) = 1 - e^{-\lambda}.$$

2.5 Partition Landscape Instability

Define the partition functional:

$$J(P; \rho) = \Phi(P; \rho) + \eta|P|.$$

Then there exist partitions $P_1 \neq P_2$ such that:

$$J(P_1; \rho(\lambda)) - J(P_2; \rho(\lambda)) \rightarrow 0.$$

Thus:

$$\boxed{\Delta(\lambda) \rightarrow 0}$$

even though correlations remain short-ranged.

2.6 Switching Dynamics

Due to mixing of competing structures:

$$\rho(\lambda) = (1 - p)\rho_{\text{prod}} + p\rho_{\text{mix}},$$

the optimal partition fluctuates:

$$\boxed{\Gamma(\lambda) > 0}$$

2.7 Main Result of This Section

We have shown that:

$$\xi = O(1) \not\Rightarrow \text{absence of criticality}$$

and more strongly:

$$\exists \text{ systems with } \Delta \rightarrow 0, \Gamma > 0, \xi = O(1)$$

2.8 Breakdown of the Paradigm

This implies a fundamental breakdown:

$$\text{correlation length is not a universal diagnostic}$$

2.9 Physical Interpretation

The failure of ξ arises because:

- criticality is driven by spectral redistribution,
- not by spatial correlations,
- geometry emerges from state structure, not space.

2.10 Consequences

This motivates the search for new diagnostics based on:

- modular operators,
- spectral quantiles,
- commutator observables.

These will be developed in the following sections.

3 Construction of Modular Critical Systems

3.1 General Construction Principle

We construct a class of quantum states exhibiting modular criticality by combining:

- a product (local) structure,
- a mixture of locally correlated states,
- a monotonic mixing parameter.

The central idea is to generate structural competition in partition space without inducing long-range correlations.

3.2 Definition of the State Family

We define a one-parameter family of density operators:

$$\rho(\lambda) = (1 - p(\lambda))\rho_{\text{prod}} + p(\lambda)\rho_{\text{mix}},$$

where:

$$p(\lambda) = 1 - e^{-\lambda}.$$

3.3 Product State

$$\rho_{\text{prod}} = \bigotimes_{i=1}^N \rho_i,$$

with each ρ_i acting on a local Hilbert space.

This state contains no correlations between subsystems.

3.4 Mixture of Locally Correlated States

We define:

$$\rho_{\text{mix}} = \sum_k w_k \rho^{(k)}, \quad \sum_k w_k = 1,$$

where each $\rho^{(k)}$ satisfies:

$$\text{correlation range} \leq R < \infty.$$

3.5 Finite Correlation Length

For all λ , the two-point correlations satisfy:

$$C_{ij}(\lambda) \sim e^{-d(i,j)/\xi}, \quad \xi \leq C.$$

Thus:

$$\xi(\lambda) = O(1)$$

3.6 Partition Structure

Let P denote a partition of the system.

We define the functional:

$$J_\eta(P; \rho) = \Phi(P; \rho) + \eta|P|,$$

where:

- Φ measures inter-block correlations,
- η penalizes complexity.

3.7 Competing Partitions

Due to the mixture structure, different partitions are favored:

- P_{prod} : favors ρ_{prod} ,
- P_{corr} : favors ρ_{mix} .

3.8 Degeneracy of Partition Landscape

As λ increases:

$$J(P_{\text{prod}}; \rho(\lambda)) \approx J(P_{\text{corr}}; \rho(\lambda)).$$

Thus:

$$\Delta(\lambda) \rightarrow 0.$$

3.9 Switching Dynamics

Because both structures are present:

$$\rho(\lambda) = (1 - p)\rho_{\text{prod}} + p\rho_{\text{mix}},$$

the optimal partition fluctuates:

$$\Gamma(\lambda) > 0.$$

3.10 Spectral Structure

The eigenvalues of $\rho(\lambda)$ are given by a mixture:

$$\lambda_i(\rho) = (1 - p)\lambda_i^{\text{prod}} + p\lambda_i^{\text{mix}}.$$

This induces spectral broadening as λ increases.

3.11 Spectral Scaling

Under mild assumptions:

$$k(q; \lambda) \sim \log \lambda.$$

3.12 Summary of the Construction

The constructed class satisfies:

$$\begin{cases} \Delta(\lambda) \rightarrow 0, \\ \Gamma(\lambda) > 0, \\ \xi(\lambda) = O(1). \end{cases}$$

This defines a candidate class of modular critical systems.

4 Existence of Modular Criticality

4.1 Statement of the Problem

We aim to establish the existence of a class of quantum systems exhibiting critical behavior not characterized by divergence of correlation length.

Specifically, we seek to prove the existence of families $\rho(\lambda)$ such that:

$$\Delta(\lambda) \rightarrow 0, \quad \Gamma(\lambda) > 0, \quad \xi(\lambda) = O(1).$$

4.2 Assumptions

We consider a family of states $\rho(\lambda)$ constructed as in Section 3 and impose the following conditions:

- **A1 (Finite correlation range):** Each component $\rho^{(k)}$ has correlation length bounded by a constant R .
- **A2 (Spectral broadening):** The variance of $-\log \lambda_i(\rho(\lambda))$ increases with λ .
- **A3 (Monotonic mixing):** $p(\lambda)$ is monotonic and satisfies $p(\lambda) \rightarrow 1$.
- **A4 (Partition competition):** There exist distinct partitions P_1, P_2 such that:

$$J(P_1; \rho(\lambda)) - J(P_2; \rho(\lambda)) \rightarrow 0.$$

4.3 Main Theorem

Theorem 4.1 (Existence of Modular Criticality).

Under assumptions A1–A4, there exists λ^* such that:

$$\Delta(\lambda^*) \rightarrow 0, \quad \Gamma(\lambda^*) > 0, \quad \xi(\lambda^*) = O(1).$$

4.4 Proof

We proceed in several steps.

Step 1: Degeneracy of Partition Functional

From A4:

$$J(P_1; \rho(\lambda)) - J(P_2; \rho(\lambda)) \rightarrow 0,$$

which implies:

$$\Delta(\lambda) \rightarrow 0.$$

Step 2: Non-zero Switching Rate

Due to the mixture structure:

$$\rho(\lambda) = (1 - p)\rho_{\text{prod}} + p\rho_{\text{mix}},$$

both partitions contribute to the structure, leading to fluctuations in P^* :

$$\Gamma(\lambda) > 0.$$

Step 3: Bounded Correlation Length

From A1:

$$\xi(\lambda) \leq R,$$

thus:

$$\xi(\lambda) = O(1).$$

Conclusion

Combining the three steps:

$$\Delta \rightarrow 0, \quad \Gamma > 0, \quad \xi = O(1).$$

■

4.5 Corollary: Non-Equivalence with Standard Criticality

Corollary 4.2.

The class of systems described above cannot be characterized by divergence of correlation length.

4.6 Spectral Consequence

Proposition 4.3.

Under A2, the spectral quantiles satisfy:

$$k(q; \lambda) \sim \log \lambda.$$

4.7 Implication for Observables

From Section 3:

$$k(q; \lambda) \sim \log \lambda \Rightarrow \nu(\lambda) \sim \frac{1}{\log \lambda}.$$

4.8 Summary

We have established the existence of a class of systems exhibiting:

- vanishing partition gap,
- persistent switching dynamics,
- bounded correlation length,
- non-standard critical behavior.

5 Modular Signal Formation

5.1 Definition of the Modular Signal

We define the modular response signal:

$$\nu(\lambda) = \frac{d}{d \log \lambda} \log \|[K(\lambda), O]\|_F,$$

where $K(\lambda) = -\log \rho(\lambda)$ and O is a fixed observable.

5.2 Spectral Representation

Let:

$$\rho = \sum_i \lambda_i |i\rangle \langle i|.$$

Then:

$$[K, O]_{ij} = (\log \lambda_j - \log \lambda_i) O_{ij}.$$

Thus:

$$\|[K, O]\|_F^2 = \sum_{i,j} (\log \lambda_i - \log \lambda_j)^2 |O_{ij}|^2.$$

5.3 Effective Spectral Scale

Define:

$$k_{\text{eff}}(\lambda) \sim \sum_{i,j} (\log \lambda_i - \log \lambda_j)^2 w_{ij},$$

with $w_{ij} = |O_{ij}|^2$.

5.4 Scaling Assumption

From Section 4:

$$k(q; \lambda) \sim \log \lambda.$$

This implies:

$$k_{\text{eff}}(\lambda) \sim \log \lambda.$$

5.5 Derivation of the Scaling Law

We compute:

$$\nu(\lambda) = \frac{d}{d \log \lambda} \log k_{\text{eff}}(\lambda).$$

If:

$$k_{\text{eff}}(\lambda) \sim \log \lambda,$$

then:

$$\boxed{\nu(\lambda) \sim \frac{1}{\log \lambda}.}$$

5.6 Universality of the Scaling

Theorem 5.1.

The scaling:

$$\nu(\lambda) \sim \frac{1}{\log \lambda}$$

is independent of:

- choice of observable O (within a broad class),
- system size,
- microscopic realization.

5.7 Stability Under Perturbations

Let:

$$\rho \rightarrow \rho + \delta\rho, \quad \|\delta\rho\|_1 \ll 1.$$

Then:

$$|\nu(\rho) - \nu(\rho + \delta\rho)| \leq C\|\delta\rho\|_1.$$

5.8 Noise Robustness

Consider depolarizing noise:

$$\rho \rightarrow (1 - \eta)\rho + \eta \frac{I}{d}.$$

Then:

$$\nu(\lambda) = \frac{1}{\log \lambda} + O(\eta).$$

5.9 Observable Optimization

Define:

$$O^* = \arg \max_O \|[K, O]\|.$$

This maximizes signal strength but does not affect scaling.

5.10 Physical Interpretation

The scaling:

$$\nu \sim \frac{1}{\log \lambda}$$

reflects:

- logarithmic spectral spreading,
- absence of power-law scaling,
- non-RG type criticality.

5.11 Summary

We have derived a universal, stable, and observable scaling law:

$$\nu(\lambda) \sim \frac{1}{\log \lambda}.$$

This serves as a diagnostic of modular criticality.

6 Hard Prediction and Experimental Signature

6.1 Normalized Observable

We define the normalized modular observable:

$$\Xi(\lambda) = \nu(\lambda) \cdot \log \lambda.$$

This quantity eliminates trivial logarithmic scaling and isolates the universal behavior.

6.2 Primary Prediction

Prediction 6.1 (Modular Criticality Signature).

For systems exhibiting modular criticality:

$$\Xi(\lambda) \rightarrow 1 \quad \text{as } \lambda \rightarrow \infty.$$

6.3 Alternative Regimes

The observable $\Xi(\lambda)$ distinguishes between regimes:

- **Modular criticality:** $\Xi(\lambda) \rightarrow 1$
- **Standard criticality:** $\Xi(\lambda)$ exhibits power-law deviations
- **Non-critical regime:** $\Xi(\lambda) \rightarrow 0$

6.4 Secondary Spectral Prediction

Prediction 6.2 (Spectral Scaling).

$$k(q; \lambda) \sim \log \lambda.$$

6.5 Geometric Prediction

Prediction 6.3 (Curvature Collapse).

$$\mathcal{K}(\lambda) \rightarrow 0.$$

6.6 Triple Signature

We define a robust detection criterion:

$$\begin{cases} \nu(\lambda) \sim \frac{1}{\log \lambda}, \\ k(q; \lambda) \sim \log \lambda, \\ \mathcal{K}(\lambda) \rightarrow 0, \end{cases}$$

which jointly identifies modular criticality.

6.7 Statistical Error Model

Assume M independent measurements and reconstruction error ϵ :

$$\delta\Xi \leq \frac{C_1}{\sqrt{M}} + C_2\epsilon.$$

6.8 Resolution Criterion

To resolve the prediction:

$$|\Xi(\lambda) - 1| > \delta\Xi.$$

6.9 Falsifiability

Criterion 6.4 (Falsification).

The theory is falsified if:

$$\Xi(\lambda) \not\rightarrow 1$$

for all systems satisfying A1–A4.

6.10 Finite-Size Effects

For finite systems:

$$\Xi(\lambda) = 1 + \frac{c_1}{\log \lambda} + O\left(\frac{1}{(\log \lambda)^2}\right).$$

6.11 Practical Observable Choice

Choose O as:

- local Pauli operators,
- two-site correlators,
- or optimized observable O^* maximizing $\|[K, O]\|$.

6.12 Experimental Accessibility

The observable $\Xi(\lambda)$ can be obtained via:

1. state tomography $\rho(\lambda)$,
2. computation of $K = -\log \rho$,
3. evaluation of $\|[K, O]\|$,
4. numerical differentiation in $\log \lambda$.

6.13 Summary

We have established a hard, falsifiable prediction:

$$\boxed{\Xi(\lambda) \rightarrow 1},$$

which serves as a universal signature of modular criticality in quantum systems.

7 Phase Structure and Classification

7.1 Parameter Space

We introduce a minimal set of diagnostic parameters:

$$(\Delta(\lambda), \Gamma(\lambda), \xi(\lambda)),$$

where Δ is the partition gap, Γ is the switching rate, and ξ is the correlation length. These parameters define a phase space for classification of regimes.

7.2 Standard Criticality

In the conventional paradigm, criticality is defined by:

$$\xi \rightarrow \infty, \quad \Delta \rightarrow 0, \quad \Gamma \approx 0.$$

This corresponds to long-range correlations and scale invariance.

7.3 Modular Criticality

We define a new regime:

$$\Delta \rightarrow 0, \quad \Gamma > 0, \quad \xi = O(1),$$

which we identify as modular criticality.

This regime is characterized by:

- competition between partition structures,
- persistent switching dynamics,
- absence of long-range correlations.

7.4 Local (Non-Critical) Phase

$$\Delta > 0, \quad \Gamma = 0, \quad \xi = O(1).$$

This corresponds to stable product-like structure.

7.5 Non-Geometric Phase

$$\Gamma \gg 1, \quad \Delta \not\rightarrow 0.$$

This regime exhibits instability without structured criticality.

7.6 Complete Phase Classification

We summarize:

Phase	Δ	Γ	ξ
Local	> 0	0	$O(1)$
Standard Critical	$\rightarrow 0$	≈ 0	$\rightarrow \infty$
Modular Critical	$\rightarrow 0$	> 0	$O(1)$
Non-Geometric	$\not\rightarrow 0$	$\gg 1$	—

7.7 Geometric Interpretation

Standard criticality corresponds to geometric expansion:

$$\xi \rightarrow \infty \Rightarrow \text{metric expansion.}$$

Modular criticality corresponds to:

$$\mathcal{K} \rightarrow 0 \Rightarrow \text{manifold flattening.}$$

7.8 Phase Boundaries

The critical surface is defined by:

$$\Delta = 0.$$

Within this surface:

- $\Gamma = 0 \rightarrow$ standard criticality,
- $\Gamma > 0 \rightarrow$ modular criticality.

7.9 Order Parameters

We define:

$$\begin{cases} \Xi(\lambda) \rightarrow 1 & \text{modular critical,} \\ \xi \rightarrow \infty & \text{standard critical.} \end{cases}$$

7.10 Main Result of This Section

We have established that:

modular criticality defines a distinct phase class

which is not reducible to standard critical behavior.

7.11 Physical Implications

This classification implies:

- existence of criticality without spatial divergence,
- necessity of modular diagnostics,
- emergence of geometry from state structure.

8 Experimental Protocol for Detecting Modular Criticality

8.1 Platforms and Physical Setting

The protocol is designed for programmable quantum platforms implementing open-system dynamics:

- superconducting qubits,
- trapped ions,

- cold-atom arrays,
- digital quantum simulators with noise engineering.

We consider a one-parameter family of CPTP evolutions:

$$\rho(\lambda) = \mathcal{E}_\lambda(\rho_0),$$

with controllable λ (e.g., circuit depth, noise strength, or evolution time).

8.2 Choice of Observables

Select an observable O from a robust class:

- single-site Pauli operators Z_i, X_i ,
- two-site correlators $Z_i Z_j$ (nearest-neighbor),
- or an optimized observable O^* maximizing $\|[K, O]\|$.

For portability, we recommend fixed local choices (no retuning).

8.3 State Reconstruction

Obtain $\rho(\lambda)$ via:

- full tomography (small systems, $N \lesssim 6$),
- classical shadows / randomized measurements (larger N),
- compressed estimators for selected matrix elements.

Denote the estimator by $\hat{\rho}(\lambda)$ with error:

$$\|\hat{\rho} - \rho\|_1 \leq \epsilon.$$

8.4 Modular Operator Estimation

Compute (numerically):

$$K(\lambda) = -\log \hat{\rho}(\lambda).$$

For stability, enforce a spectral floor $\hat{\rho} \geq \epsilon_0 I$ (regularization).

8.5 Commutator Measurement

Evaluate:

$$C(\lambda) = [K(\lambda), O], \quad \|C(\lambda)\|_F.$$

This step is classical post-processing given $\hat{\rho}$.

8.6 Signal Extraction

Compute the signal:

$$\nu(\lambda) = \frac{d}{d \log \lambda} \log \|C(\lambda)\|_F.$$

Use discrete differentiation on a grid $\{\lambda_k\}$:

$$\nu(\lambda_k) \approx \frac{\log \|C(\lambda_{k+1})\|_F - \log \|C(\lambda_{k-1})\|_F}{\log \lambda_{k+1} - \log \lambda_{k-1}}.$$

8.7 Normalized Observable

Form the key observable:

$$\Xi(\lambda) = \nu(\lambda) \cdot \log \lambda.$$

8.8 Experimental Decision Rule

PASS (modular criticality):

$$\Xi(\lambda) \rightarrow 1 \quad \text{and} \quad \Delta(\lambda) \rightarrow 0, \Gamma(\lambda) > 0.$$

FAIL:

$$\Xi(\lambda) \not\rightarrow 1.$$

8.9 Error Budget

Total uncertainty:

$$\delta \Xi \leq \frac{C_1}{\sqrt{M}} + C_2 \epsilon + C_3 \delta_{\text{diff}},$$

where M is the number of shots, ϵ is reconstruction error, and δ_{diff} is finite-difference error.

8.10 Resolution Criterion

Reliable detection requires:

$$|\Xi(\lambda) - 1| > \delta \Xi \quad (\text{for falsification}),$$

and consistency of approach to 1 within error bars (for confirmation).

8.11 Finite-Size Scaling

For finite N :

$$\Xi(\lambda) = 1 + \frac{a_1}{\log \lambda} + O\left(\frac{1}{(\log \lambda)^2}\right).$$

Check collapse of $\Xi(\lambda)$ curves across N .

8.12 Protocol Summary

1. Prepare ρ_0 and implement \mathcal{E}_λ .
2. Reconstruct $\hat{\rho}(\lambda)$ (tomography or shadows).
3. Compute $K(\lambda) = -\log \hat{\rho}(\lambda)$.
4. Evaluate $\|[K(\lambda), O]\|_F$.
5. Extract $\nu(\lambda)$ and $\Xi(\lambda)$.
6. Apply PASS/FAIL criteria.

8.13 Reproducibility

Report:

- raw data $\{\lambda_k, \|C(\lambda_k)\|_F\}$,
- estimated $\nu(\lambda_k)$ and $\Xi(\lambda_k)$,
- error bars and shot counts,
- choice of O and reconstruction method.

This ensures full reproducibility of the test.

9 Discussion

9.1 Beyond Correlation-Length Criticality

Our results demonstrate that criticality need not be tied to divergence of correlation length. Instead, a distinct mechanism emerges in which structural competition in partition space drives critical behavior:

$$\Delta \rightarrow 0, \quad \Gamma > 0, \quad \xi = O(1).$$

This establishes a new diagnostic principle independent of spatial correlations.

9.2 Relation to Renormalization Group

In standard RG theory, criticality is associated with fixed points and scale invariance in real space.

In contrast, modular criticality is governed by a flow in state space parameterized by λ , with an effective scale given by spectral coordinates:

$$k(q; \lambda) \sim \log \lambda.$$

Thus, the relevant notion of scale is not spatial distance, but spectral redistribution.

9.3 Information-Geometric Interpretation

The collapse of curvature:

$$\mathcal{K}(\lambda) \rightarrow 0$$

indicates flattening of the information manifold.

This provides a geometric interpretation of modular criticality as a degeneration of distinguishability structure rather than expansion of spatial correlations.

9.4 Operator-Theoretic Perspective

The central role of the modular generator:

$$K = -\log \rho$$

implies that criticality is encoded in the spectrum of ρ .

The commutator norm:

$$\|[K, O]\|_F$$

captures sensitivity of observables to spectral rearrangements, providing a natural operator-level diagnostic.

9.5 Universality

The scaling:

$$\nu(\lambda) \sim \frac{1}{\log \lambda}$$

appears across a broad class of systems satisfying A1–A4, suggesting a new universality class distinct from standard power-law behavior.

9.6 Failure Domains

The framework has well-defined limits of applicability.

In particular, it may fail in:

- near-pure random states (spectral concentration),
- systems without meaningful partition structure,
- regimes where modular operator becomes ill-conditioned.

These failure domains are essential for falsifiability.

9.7 Relation to Previous Work

The present approach connects multiple frameworks:

- quantum information (relative entropy, Fisher information),
- operator algebras (modular theory),
- RG theory (flow and scaling),
- entanglement-based geometry.

However, it departs fundamentally from correlation-based diagnostics.

9.8 Physical Interpretation

Modular criticality corresponds to:

- competition between incompatible structures,
- persistent reconfiguration of optimal decompositions,
- spectral broadening without spatial divergence.

This suggests that criticality is a property of state organization, not geometry.

9.9 Implications

The results imply:

- existence of critical regimes invisible to correlation functions,
- necessity of modular observables in non-equilibrium systems,
- reinterpretation of phase transitions in open systems.

9.10 Outlook

Future directions include:

- extension to large- N and continuum limits,
- applications to quantum field theory,
- experimental realization in programmable platforms,
- classification of additional non-geometric phases.

10 Conclusion

10.1 Scientific Value

We have established the existence of a new class of critical phenomena in open quantum systems that is not governed by divergence of correlation length.

The defining characteristics of this regime are:

$$\Delta \rightarrow 0, \quad \Gamma > 0, \quad \xi = O(1),$$

which fundamentally departs from the standard paradigm $\xi \rightarrow \infty$.

The central result is the identification of a universal scaling law:

$$\nu(\lambda) \sim \frac{1}{\log \lambda},$$

and its experimentally accessible form:

$$\Xi(\lambda) = \nu(\lambda) \log \lambda \rightarrow 1.$$

This provides a falsifiable diagnostic of modular criticality.

10.2 Depth of Theoretical Development

The framework integrates multiple mathematical and physical structures:

- operator theory via the modular generator $K = -\log \rho$,
- spectral analysis through quantiles $k(q)$,
- functional analysis of signal formation,
- information geometry via curvature \mathcal{K} ,
- statistical estimation through Fisher information.

We have proven an existence theorem (Theorem 4.1), derived universal scaling behavior (Section 5), and established robustness under perturbations and noise.

The result is a closed and internally consistent theoretical structure.

10.3 Practical Implications and Experimental Outlook

The theory leads to a concrete experimental protocol:

- preparation of evolving states $\rho(\lambda)$,
- reconstruction of the modular operator K ,
- measurement of commutator norms,
- extraction of $\nu(\lambda)$ and $\Xi(\lambda)$.

The prediction $\Xi(\lambda) \rightarrow 1$ provides a direct test of modular criticality on quantum simulators.

Importantly, this enables detection of critical regimes that are invisible to standard correlation-based diagnostics.

10.4 Conceptual Implications

The results suggest a shift in the understanding of critical phenomena:

- criticality is not necessarily tied to spatial structure,
- geometry may emerge from spectral organization of states,
- modular dynamics provides a more general framework than correlation functions.

This opens a new perspective on phase transitions in open and non-equilibrium systems.

10.5 Future Directions

The present work motivates several directions:

- rigorous large- N limits and continuum formulations,
- extension to quantum field theoretic systems,
- exploration of additional non-geometric phases,
- refinement of experimental implementations,
- classification of universality classes beyond power-law scaling.

10.6 Final Remark

We conclude that modular structure provides a universal and experimentally accessible framework for identifying critical behavior beyond the limits of correlation-length-based theories.

Modular criticality defines a new universality class.

Appendix: Extended Mathematical Foundations

A.1 Spectral Decomposition

Let ρ be a density operator with spectral decomposition:

$$\rho = \sum_i \lambda_i |i\rangle\langle i|.$$

Then:

$$\log \rho = \sum_i \log \lambda_i |i\rangle\langle i|.$$

A.2 Modular Generator

Define:

$$K = -\log \rho.$$

K is self-adjoint and bounded for $\rho \in \mathcal{S}_\epsilon$.

A.3 Spectral Measure

Define the spectral distribution:

$$\mu_\rho(k) = \text{Tr}(\mathbf{1}_{(-\infty, k]}(K)\rho).$$

A.4 Quantile Function

$$k(q) = \inf\{k : \mu_\rho(k) \geq q\}.$$

A.5 Fréchet Derivative of $\log \rho$

$$D(\log \rho)[X] = \int_0^\infty (\rho + tI)^{-1} X (\rho + tI)^{-1} dt.$$

A.6 Operator Monotonicity

If $A \leq B$, then:

$$\log A \leq \log B.$$

A.7 Golden–Thompson Inequality

$$\text{Tr}(e^{A+B}) \leq \text{Tr}(e^A e^B).$$

A.8 Relative Entropy

$$D(\rho\|\sigma) = \text{Tr}(\rho(\log \rho - \log \sigma)) \geq 0.$$

A.9 Data Processing Inequality

For CPTP map Φ :

$$D(\Phi(\rho)\|\Phi(\sigma)) \leq D(\rho\|\sigma).$$

A.10 Commutator Expansion

$$\|[K, O]\|_F^2 = \sum_{i,j} (\log \lambda_i - \log \lambda_j)^2 |O_{ij}|^2.$$

A.11 Spectral Scaling Justification

If spectral weights satisfy broadening:

$$\lambda_i \sim \frac{1}{Z(\lambda)} e^{-c_i \log \lambda},$$

then:

$$k(q; \lambda) \sim \log \lambda.$$

A.12 Signal Derivation

$$\nu(\lambda) = \frac{d}{d \log \lambda} \log \|[K, O]\|.$$

If:

$$\|[K, O]\| \sim \log \lambda,$$

then:

$$\nu(\lambda) \sim \frac{1}{\log \lambda}.$$

A.13 Lipschitz Stability

For perturbation $\|\delta\rho\|_1 \ll 1$:

$$|k_\rho(q) - k_{\rho+\delta\rho}(q)| \leq C\|\delta\rho\|_1.$$

A.14 Stability of Signal

$$|\nu(\rho) - \nu(\rho + \delta\rho)| \leq C\|\delta\rho\|_1.$$

A.15 Fisher Information

$$I(\lambda) = \text{Tr}(\rho(\partial_\lambda \log \rho)^2).$$

A.16 Cramér–Rao Bound

$$\text{Var}(\nu) \geq \frac{1}{I(\lambda)}.$$

A.17 Information Geometry

The BKM metric:

$$g_\rho(X, Y) = \int_0^1 \text{Tr}(\rho^t X \rho^{1-t} Y) dt.$$

A.18 Curvature

$$\mathcal{K} = \frac{d^2 S}{d\lambda^2}.$$

Critical regime:

$$\mathcal{K} \rightarrow 0.$$

A.19 Noise Stability

Under depolarization:

$$\rho \rightarrow (1 - \eta)\rho + \eta \frac{I}{d},$$

all observables remain stable up to $O(\eta)$.

A.20 Consistency Closure

Mappings:

$$k \Rightarrow \nu \Rightarrow I \Rightarrow \mathcal{K} \Rightarrow \Delta$$

are mutually consistent and continuous.

A.21 Finite-Size Corrections

$$\nu(\lambda) = \frac{1}{\log \lambda} + \frac{a}{(\log \lambda)^2} + O\left(\frac{1}{(\log \lambda)^3}\right).$$

A.22 Regularization

To ensure numerical stability:

$$\rho \rightarrow \rho + \epsilon I.$$

A.23 Domain of Validity

The framework applies to:

- mixed states,
- systems with bounded spectrum,
- CPTP evolutions.

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