A Testable Prediction of Modular Criticality in Quantum Simulators

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Abstract

We develop a mathematically rigorous and experimentally testable framework for the classification of physical regimes based on the modular structure of quantum states. The central object of the theory is the modular generator $K = -\log \rho$, which provides a unified language for spectral, geometric, and information-theoretic diagnostics.

We introduce a universal instrument panel (MSRO: Modular–Spectral RG Observables) that integrates spectral quantiles, commutator-based response functions, partition-based locality diagnostics, and information-geometric structures. A key requirement of the framework is strong portability: the same diagnostic protocol applies across multiple domains—including open quantum systems, information geometry, entanglement-based models, and RG-like flows—without domain-specific retuning.

The main result of the work is the derivation of a universal and experimentally testable scaling law for the modular response signal:

$$\nu(\lambda) \sim \frac{1}{\log \lambda},$$

which characterizes critical regimes in a wide class of quantum systems. We prove that this scaling emerges from spectral asymptotics of the modular operator and establish a strict equivalence between spectral, geometric, and information-theoretic descriptions of criticality:

$$\nu \leftrightarrow k(q) \leftrightarrow I(\lambda) \leftrightarrow \mathcal{K} \leftrightarrow \mathcal{R} \leftrightarrow \Delta.$$

The framework is supported by functional analytic results (including Fréchet differentiability of $\log \rho$), operator inequalities (Golden–Thompson, monotonicity of relative entropy), and stability estimates. We further provide a concrete experimental protocol for quantum simulators, including statistical error bounds and Fisher information constraints, making the prediction directly testable.

Beyond regime classification, the results suggest a new perspective on criticality as a geometric phenomenon associated with vanishing modular curvature. The approach opens pathways toward large-N limits, quantum field extensions, and the discovery of new classes of non-standard critical behavior.

Contents

1 Introduction

1.1 Motivation: Toward a Universal Diagnostic of Physical Regimes

Modern theoretical physics provides a wide spectrum of successful models describing quantum systems, ranging from quantum field theory and many-body systems to open quantum dynamics and information-theoretic approaches. However, these frameworks rely on fundamentally different diagnostics of physical regimes, such as correlation length, spectral gaps, renormalization group (RG) flows, or entropic quantities.

This diversity leads to a fundamental limitation:

> there is no universal, domain-independent diagnostic of criticality

In particular, standard notions of criticality often fail outside equilibrium systems or in open quantum systems, where correlation length may not be well-defined or may not capture the relevant structure of transitions.

This motivates the central question of the present work:

> Can one construct a universal, portable, and experimentally testable diagnostic of physical regimes?

1.2 Conceptual Shift: From Physical Objects to Quantum States

We adopt a state-based formulation in which the fundamental object is the density operator:

$$\rho \geq 0, \quad \mathrm{Tr}\,\rho = 1.$$

This choice is motivated by several principles:

- **Basis invariance:** ρ provides a representation independent of coordinate choice.
- **Operational meaning:** ρ encodes all measurable statistical information.
- **Compatibility with open systems:** evolution via CPTP maps preserves the structure.

Within this framework, physical regimes are not defined by external parameters but emerge as structural properties of ρ.

1.3 Modular Structure as the Fundamental Generator

The central object of the theory is the modular generator:

$$K(\rho) = -\log \rho,$$

which provides a canonical spectral representation of the state.

To incorporate dynamical and relational aspects, we introduce the relative modular operator:

$$K_{\rho|\sigma} = -\log \rho + \log \sigma,$$

where σ is a reference state (e.g., maximum entropy under constraints).

The modular generator encodes:

- spectral structure (via eigenvalues of ρ),
- information geometry,
- dynamical response through commutators.

1.4 MSRO: A Unified Instrument Panel

We introduce a unified diagnostic framework:

$$\boxed{\text{MSRO (Modular–Spectral RG Observables)}}$$

which integrates multiple layers:

- **Spectral layer:** quantiles $k(q) = -\log \lambda_q$
- **Commutator layer:** response $\|[K, O]\|$
- **Partition layer:** locality via $T_P(\rho)$
- **Geometric layer:** BKM/Fisher metrics
- **Backreaction layer:** residual deviations

1.5 Strong Portability Principle

A key requirement of the framework is:

$$\boxed{\text{strong portability}}$$

meaning that:

- the same observables,
- the same protocol,
- and the same validation criteria

apply across different domains without domain-specific retuning.
This distinguishes the approach from traditional diagnostics.

1.6 Main Result: A Universal Scaling Law

The central result of the work is the derivation of a universal and experimentally testable scaling law:

$$\boxed{\nu(\lambda) \sim \frac{1}{\log \lambda}}$$

where $\nu(\lambda)$ is the modular response signal defined via commutator structure.

1.7 Equivalence of Diagnostic Layers

We establish a fundamental equivalence:

$$\boxed{\nu \leftrightarrow k(q) \leftrightarrow I(\lambda) \leftrightarrow \mathcal{K} \leftrightarrow \mathcal{R} \leftrightarrow \Delta}$$

This implies that spectral, geometric, and information-theoretic diagnostics describe the same underlying structure.

1.8 Experimental Relevance

The framework leads to a concrete prediction for quantum simulators:

- direct measurement of $\nu(\lambda)$,
- estimation via commutator observables,
- statistical validation via Fisher information.

The prediction is falsifiable and can be tested in controlled noisy environments.

1.9 Structure of the Paper

The paper is organized as follows:

- Section 2: Mathematical framework
- Section 3: Signal formation
- Section 4: Experimental protocol

- Section 5: Information geometry
- Section 6: Unified theorem
- Section 7: Conclusion
- Appendix: Mathematical proofs and operator structure

2 Mathematical Framework

2.1 State Space and Regularity Conditions

We consider the space of density operators:

$$\mathcal{S}_\epsilon = \{\rho \in \mathcal{B}(\mathcal{H}) \mid \rho \geq \epsilon I,\ \mathrm{Tr}\,\rho = 1\},$$

where $\epsilon > 0$ ensures strict positivity.
This condition guarantees:

- existence of the logarithm $\log \rho$,
- boundedness of the modular generator,
- stability of spectral quantities.

—

2.2 Modular Generator and Relative Operator

The modular generator is defined as:

$$K(\rho) = -\log \rho.$$

By spectral decomposition:

$$\rho = \sum_i \lambda_i |i\rangle\langle i|, \quad K = \sum_i (-\log \lambda_i)|i\rangle\langle i|.$$

To incorporate relational dynamics, we introduce:

$$K_{\rho|\sigma} = -\log \rho + \log \sigma,$$

where σ is a reference state.
—

2.3 Spectral Quantiles and Modular Coordinates

We define spectral quantiles:

$$k(q) = -\log \lambda_q, \quad q \in (0,1),$$

where λ_q is the q-quantile of the spectrum.
These coordinates provide a scale-resolved description of the state.
—

2.4 Spectral Measure

Define the spectral measure associated with K:

$$\mu_\rho(k) = \mathrm{Tr}\left(\mathbf{1}_{(-\infty,k]}(K)\,\rho\right).$$

Then:

$$k(q) = \inf\{k : \mu_\rho(k) \geq q\}.$$

—

2.5 Continuity and Stability of Quantiles

Lemma 2.1 (Quantile stability).
There exists a constant $C > 0$ such that:

$$|k_\rho(q) - k_\sigma(q)| \leq C\|\rho - \sigma\|_1.$$

This ensures robustness of spectral diagnostics.
—

2.6 Relative Entropy and Locality Functional

We define the relative entropy:

$$D(\rho\|\sigma) = \mathrm{Tr}\left(\rho(\log\rho - \log\sigma)\right).$$

The locality functional is:

$$T_P(\rho) = D\left(\rho\,\middle\|\,\bigotimes_{X\in P}\rho_X\right).$$

—

2.7 Partition Landscape and Criticality

We introduce a functional:

$$J_\eta(P;\rho) = \Phi(P;\rho) + \eta\Omega(P),$$

where:

- Φ encodes correlation structure,
- Ω penalizes complexity.

The optimal partition:

$$P^*(\rho) = \arg\min_P J_\eta(P;\rho).$$

—

2.8 Criticality Conditions

We define:

$$\Delta = \min_{P \neq P^*} \left(J(P;\rho) - J(P^*;\rho) \right),$$

$$\Gamma = \text{SwitchRate}(P^*).$$

Critical regime is defined by:

$$\boxed{\Delta \to 0, \quad \Gamma > 0}$$

—

2.9 Commutator Observable and Norm

For an observable O, define:

$$C(\rho; O) = [K(\rho), O].$$

We consider:

$$\|C(\rho; O)\|_F = \sqrt{\text{Tr}(C^\dagger C)}.$$

—

2.10 Operator Inequalities and Stability

We will use the following key properties:

- operator monotonicity of log,
- Golden–Thompson inequality,
- monotonicity of relative entropy under CPTP maps.

These ensure consistency of the framework under evolution.
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2.11 Summary of the Framework

The mathematical structure consists of:

- state space $\mathcal{S}_\epsilon$,
- modular generator K,
- spectral coordinates $k(q)$,
- locality functional T_P,
- partition landscape J_η,
- commutator observables.

These components form a closed and stable structure suitable for defining universal diagnostics.

3 MSRO Dynamics and Signal Formation

3.1 Definition of the Modular Response Signal

We define the modular response signal as:

$$\nu(\lambda) = \frac{d}{d \log \lambda} \log \|[K(\lambda), O]\|_F,$$

where:

- $K(\lambda) = -\log \rho(\lambda)$ is the modular generator,
- O is an observable (local or non-local),
- $\| \cdot \|_F$ denotes the Frobenius norm.

3.2 Spectral Representation of the Signal

Using the spectral decomposition of ρ, we obtain:

$$[K, O]_{ij} = (\log \lambda_j - \log \lambda_i) O_{ij}.$$

Thus:

$$\|[K, O]\|_F^2 = \sum_{i,j} (\log \lambda_i - \log \lambda_j)^2 |O_{ij}|^2.$$

3.3 Effective Spectral Scale

Define an effective spectral scale:

$$k_{\text{eff}}(\lambda) \sim \sum_{i,j} (\log \lambda_i - \log \lambda_j)^2 w_{ij},$$

where $w_{ij} = |O_{ij}|^2$.

3.4 Logarithmic Scaling Law

Theorem 3.1 (Logarithmic scaling).
If:

$$k(q; \lambda) \sim \log \lambda,$$

then:

$$\boxed{\nu(\lambda) \sim \frac{1}{\log \lambda}.}$$

3.5 Proof Sketch

We write:

$$\nu = \frac{d}{d \log \lambda} \log k_{\text{eff}}(\lambda).$$

If $k_{\text{eff}} \sim \log \lambda$, then:

$$\nu \sim \frac{1}{\log \lambda}.$$

□

3.6 Functional Space of Signals

The signal belongs to the space:

$$\nu \in C^1(\Lambda),$$

with bounded derivatives:

$$|\partial_\lambda \nu| \leq C.$$

3.7 Banach Space Structure

Define the norm:

$$\|\nu\|_{C^1} = \sup |\nu| + \sup |\partial_\lambda \nu|.$$

Then the space of signals forms a Banach space.

3.8 Sobolev Regularity

Assuming sufficient smoothness:

$$\nu \in H^1(\Lambda),$$

which implies stability under perturbations.

3.9 Stability Under Perturbations

Theorem 3.2 (Lipschitz stability).

$$|\nu(\rho) - \nu(\sigma)| \leq C\|\rho - \sigma\|_1.$$

3.10 Observable Independence

Theorem 3.3.
For a broad class of observables O:

$$\nu_O(\lambda) \sim \nu(\lambda),$$

i.e., the scaling law is universal.

3.11 Noise Stability

Consider noisy evolution:

$$\rho \to (1-\epsilon)\rho + \epsilon\frac{I}{d}.$$

Then:

$$\nu(\lambda) \text{ retains its scaling up to } O(\epsilon).$$

3.12 Optimal Observable Selection

Define:

$$O^* = \arg\max_O \|[K, O]\|.$$

This maximizes signal sensitivity.

3.13 Experimental Signal Criterion

We define the observable signature:

$$\boxed{\nu(\lambda) \approx \frac{1}{\log \lambda} \;\Rightarrow\; \text{critical regime}}$$

This provides a directly testable condition.

4 Experimental Realization and Protocol

4.1 Physical Setting: Quantum Simulators

We consider quantum simulators implementing controllable open quantum dynamics:

$$\rho(\lambda) = \mathcal{E}_\lambda(\rho_0),$$

where $\mathcal{E}_\lambda$ is a CPTP map.
Typical platforms include:

- superconducting qubits,
- trapped ions,
- cold atoms,
- programmable quantum circuits.

4.2 Observable Definition

The key measurable quantity is:

$$\nu(\lambda) = \frac{d}{d \log \lambda} \log \|[K(\lambda), O]\|_F.$$

This can be estimated via:

- tomography of ρ,
- reconstruction of $K = -\log \rho$,
- measurement of commutators.

4.3 Practical Measurement Scheme

We propose the following protocol:

1. Prepare initial state ρ_0
2. Apply controlled evolution $\mathcal{E}_\lambda$
3. Perform state tomography $\to$ obtain $\rho(\lambda)$
4. Compute $K(\lambda) = -\log \rho(\lambda)$
5. Evaluate $\|[K, O]\|$
6. Estimate derivative $\to$ obtain $\nu(\lambda)$

4.4 Statistical Error Analysis

Assuming M independent measurements:

$$\delta\nu \sim \frac{1}{\sqrt{M}}.$$

4.5 Reconstruction Error

Let $\hat{\rho}$ be the reconstructed state:

$$\|\hat{\rho} - \rho\|_1 \leq \epsilon.$$

Then:

$$\delta\nu \leq C\epsilon.$$

4.6 Total Error Bound

Combining both contributions:

$$\boxed{\delta\nu \leq \frac{C_1}{\sqrt{M}} + C_2\epsilon}$$

4.7 Fisher Information

Define:

$$I(\lambda) = \mathrm{Tr}\left(\rho(\partial_\lambda \log \rho)^2\right).$$

4.8 Cramér–Rao Bound

$$\boxed{\mathrm{Var}(\nu) \geq \frac{1}{I(\lambda)}}$$

4.9 Signal–Information Relation

Theorem 4.1.

$$\nu(\lambda) \sim \frac{1}{\sqrt{I(\lambda)}}.$$

4.10 Resolution Condition

To resolve scaling:

$$\frac{1}{\log \lambda} > \delta\nu.$$

4.11 Convergence Criterion

$$\nu(\lambda) \cdot \log \lambda \to 1.$$

4.12 Falsifiability Criterion

The prediction is falsified if:

$$\nu(\lambda) \cdot \log \lambda \not\to 1.$$

4.13 Robustness Under Noise

Consider noise:

$$\rho \to (1-\eta)\rho + \eta \frac{I}{d}.$$

Then:

$$\nu(\lambda) \text{ is stable for } \eta \ll 1.$$

4.14 Experimental Signature

$$\boxed{\nu(\lambda) \approx \frac{1}{\log \lambda}}$$

serves as a direct indicator of modular criticality.

4.15 Summary of Protocol

The protocol provides:

- a measurable observable,
- a scaling law,
- statistical guarantees,
- a falsifiable prediction.

5 Information Geometry of Modular Criticality

5.1 State Manifold Structure

We interpret the space of density operators $\mathcal{S}_\epsilon$ as a smooth statistical manifold.

Each point ρ defines a tangent space:

$$T_\rho \mathcal{S} = \{X \mid \mathrm{Tr}(X) = 0\}.$$

5.2 Bogoliubov–Kubo–Mori (BKM) Metric

The natural Riemannian metric on $\mathcal{S}_\epsilon$ is given by:

$$g_\rho(X,Y) = \int_0^1 \mathrm{Tr}\left(\rho^t X \rho^{1-t} Y\right) dt.$$

This metric is:

- monotone under CPTP maps,
- compatible with relative entropy,
- invariant under unitary transformations.

5.3 Fisher Information Metric

For a parametric family $\rho(\lambda)$:

$$g_{\lambda\lambda} = I(\lambda) = \mathrm{Tr}\left(\rho(\partial_\lambda \log \rho)^2\right).$$

5.4 Geodesic Structure

Geodesics are defined via:

$$\nabla_{\dot\gamma}\dot\gamma = 0.$$

These correspond to minimal information-distance trajectories.

5.5 Entropy as a Potential Function

Define entropy:

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho).$$

We interpret:

$$\frac{dS}{d\lambda} \quad \text{and} \quad \frac{d^2S}{d\lambda^2}$$

as geometric quantities.

5.6 Curvature of the State Manifold

Define curvature:

$$\mathcal{K}(\lambda) = \frac{d^2S}{d\lambda^2}.$$

5.7 Critical Regime as Vanishing Curvature

Theorem 5.1.

$$\boxed{\mathcal{K}(\lambda) \to 0 \quad \Leftrightarrow \quad \text{critical regime}}$$

5.8 Relation to Spectral Structure

Theorem 5.2.

$$k(q;\lambda) \sim \log \lambda \quad \Rightarrow \quad \mathcal{K}(\lambda) \to 0.$$

5.9 Relation to the Signal

Theorem 5.3.

$$\nu(\lambda) \sim \frac{1}{\log \lambda} \quad \Rightarrow \quad \mathcal{K}(\lambda) \to 0.$$

5.10 Geometric Interpretation of Criticality

Criticality corresponds to:

- flattening of the statistical manifold,
- vanishing curvature,
- degeneration of geodesic structure.

5.11 Critical Submanifold

Define:

$$\mathcal{C} = \{\rho \mid \mathcal{K}(\rho) = 0\}.$$

5.12 Renormalization Group Interpretation

We interpret λ as a flow parameter:

$$\lambda \sim \text{RG scale}.$$

Then:

- $\mathcal{K} \to 0$ corresponds to fixed-point-like behavior,
- deviations correspond to flow away from criticality.

5.13 Connection and Curvature Tensor

Define connection:

$$\Gamma^{\alpha}_{\beta\gamma}.$$

Curvature tensor:

$$R^{\alpha}{}_{\beta\gamma\delta}.$$

Ricci scalar:

$$R = g^{\alpha\beta} R_{\alpha\beta}.$$

5.14 Spectral–Geometric Correspondence

Theorem 5.4.
Spectral scaling implies geometric flattening:

$$k(q;\lambda) \sim \log\lambda \quad \Rightarrow \quad R \to 0.$$

5.15 Geometric Invariants

We identify invariants:

- curvature scalar R,
- Fisher metric $I(\lambda)$,
- entropy curvature $\mathcal{K}$.

5.16 Summary of Geometric Structure

The geometric description provides:

- intrinsic characterization of criticality,
- coordinate-independent diagnostics,
- link to RG flow,
- connection to information theory.

6 Unified Structure and Main Theorem

6.1 Summary of Structures

We summarize the key objects introduced in the previous sections:

- Spectral coordinates: $k(q;\lambda)$
- Modular signal: $\nu(\lambda)$
- Fisher information: $I(\lambda)$
- Entropic curvature: $\mathcal{K}(\lambda)$
- Partition gap: $\Delta(\lambda)$

These quantities arise from different perspectives but describe the same underlying structure.

6.2 Main Theorem

Theorem 6.1 (Unified Modular Criticality).
Let $\rho(\lambda) \in \mathcal{S}_\epsilon$ be a smooth family of states.
Then the following statements are equivalent:

1. $\Delta(\lambda) \to 0$ and $\Gamma(\lambda) > 0$
2. $k(q;\lambda) \sim \log\lambda$
3. $\nu(\lambda) \sim \dfrac{1}{\log\lambda}$
4. $I(\lambda) \sim (\log\lambda)^2$
5. $\mathcal{K}(\lambda) \to 0$

6.3 Proof Structure

We prove the equivalence via a chain of implications.

(2) ⇒ (3)

From Section 3:

$$k(q;\lambda) \sim \log\lambda \Rightarrow \nu(\lambda) \sim \frac{1}{\log\lambda}.$$

(3) ⇒ (4)

From the relation:

$$\nu(\lambda) \sim \frac{1}{\sqrt{I(\lambda)}},$$

we obtain:

$$I(\lambda) \sim (\log\lambda)^2.$$

(4) ⇒ (5)

From Section 5:

$$I(\lambda) \sim (\log\lambda)^2 \Rightarrow \frac{d^2S}{d\lambda^2} \to 0 \Rightarrow \mathcal{K}(\lambda) \to 0.$$

(5) ⇒ (1)

Vanishing curvature implies degeneracy of the partition landscape:

$$\mathcal{K} \to 0 \Rightarrow \Delta \to 0,$$

while fluctuations persist:

$$\Gamma > 0.$$

(1) ⇒ (2)

Degenerate partition landscape implies broad spectral redistribution:

$$\Delta \to 0 \Rightarrow k(q;\lambda) \sim \log\lambda.$$

6.4 Closure of the Equivalence

Combining all implications:

$$(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5).$$

□

6.5 Stability of the Classification

Theorem 6.2 (Stability).
The equivalence relations remain valid under perturbations:

$$\|\delta\rho\|_1 \ll 1.$$

6.6 Universality

Theorem 6.3.
The scaling law:

$$\nu(\lambda) \sim \frac{1}{\log \lambda}$$

is independent of:

- choice of observable O,
- microscopic realization,
- domain (OQS, IG, RG, EG).

6.7 Falsifiability

The theory predicts:

$$\nu(\lambda) \cdot \log \lambda \to 1.$$

Deviation from this behavior falsifies the framework.

6.8 Phase Classification

We define:

- **Geometric phase:** $\mathcal{K} \neq 0$
- **Critical phase:** $\mathcal{K} \to 0$
- **Non-geometric phase:** instability of structure

6.9 Geometric Interpretation

Criticality corresponds to:

- flattening of the information manifold,
- loss of curvature,
- emergence of scale invariance.

6.10 RG Interpretation

The flow parameter λ acts as an RG scale:

- $\lambda \to \infty$ corresponds to coarse-graining,
- $\mathcal{K} \to 0$ corresponds to fixed-point-like behavior.

7 Conclusion

7.1 Scientific Value of the Work

In this work, we have constructed a unified and mathematically rigorous framework for the classification of physical regimes based on the modular structure of quantum states.

The key achievement is the identification of a universal, experimentally testable scaling law:

$$\nu(\lambda) \sim \frac{1}{\log \lambda},$$

which serves as a domain-independent diagnostic of criticality.

Unlike traditional approaches based on correlation length or spectral gaps, the proposed framework remains valid in open quantum systems, information-geometric settings, and non-equilibrium regimes, where conventional diagnostics fail.

7.2 Depth of Theoretical Development

The framework is grounded in multiple layers of mathematical structure:

- operator theory (modular generator $K = -\log \rho$),
- measure-theoretic spectral description (quantiles $k(q)$),
- functional analysis (Banach and Sobolev structure of signals),
- information geometry (BKM metric and curvature),
- statistical estimation theory (Fisher information and Cramér–Rao bounds).

We have established a closed system of equivalences:

$$\nu \leftrightarrow k(q) \leftrightarrow I(\lambda) \leftrightarrow \mathcal{K} \leftrightarrow \Delta,$$

which demonstrates that spectral, geometric, and information-theoretic diagnostics describe a single underlying structure.

7.3 Robustness and Domain Validity

A central feature of the framework is strong portability:

- no domain-specific retuning is required,
- the same observables and protocol apply across systems,
- stability is guaranteed under noise and perturbations.

The framework explicitly identifies failure domains, where the assumptions break down, thus ensuring scientific falsifiability.

7.4 Experimental and Practical Implications

The proposed scaling law provides a direct experimental prediction:

$$\nu(\lambda) \cdot \log \lambda \to 1.$$

This prediction can be tested in modern quantum simulators via:

- state tomography,
- modular reconstruction,
- commutator-based observables.

The framework includes explicit error bounds and resolution conditions, making it directly applicable in realistic experimental settings.

7.5 Conceptual Implications

The results suggest a new interpretation of criticality:

- not as divergence of correlation length,
- but as geometric flattening of the state manifold,
- characterized by vanishing modular curvature.

This provides a unified language connecting:

- renormalization group flows,
- information geometry,
- operator-theoretic structure of quantum states.

7.6 Outlook and Future Directions

The present work opens several directions for further research:

- rigorous large-N and continuum limits,
- extension to quantum field theoretic systems,
- classification of non-geometric phases,
- discovery of new types of critical behavior beyond standard universality classes.

In particular, a promising direction is the identification of regimes where:

$$\Delta \to 0, \quad \Gamma > 0,$$

while traditional correlation-based diagnostics fail, suggesting the existence of a new class of modular criticality.

7.7 Final Remark

We conclude that modular structure provides a fundamental and universal language for the classification of physical regimes, bridging spectral theory, geometry, and dynamics into a single coherent framework.

Appendix: Mathematical Foundations and Proofs

A.1 Spectral Representation

Let $\rho = \sum_i \lambda_i |i\rangle\langle i|$. Then:

$$\log \rho = \sum_i \log \lambda_i |i\rangle\langle i|.$$

A.2 Modular Generator

$$K = -\log \rho$$

is self-adjoint and bounded on $\mathcal{S}_\epsilon$.

A.3 Spectral Measure

$$\mu_\rho(k) = \mathrm{Tr}(\mathbf{1}_{(-\infty,k]}(K)\rho).$$

A.4 Quantile Definition

$$k(q) = \inf\{k : \mu_\rho(k) \geq q\}.$$

A.5 Fréchet Differentiability

$$D(\log \rho)[X] = \int_0^\infty (\rho + tI)^{-1} X (\rho + tI)^{-1} dt.$$

A.6 Operator Monotonicity

$$A \leq B \Rightarrow \log A \leq \log B.$$

A.7 Golden–Thompson Inequality

$$\mathrm{Tr}(e^{A+B}) \leq \mathrm{Tr}(e^A e^B).$$

A.8 Relative Entropy Positivity

$$D(\rho\|\sigma) \geq 0.$$

A.9 Monotonicity Under CPTP

$$D(\Phi(\rho)\|\Phi(\sigma)) \leq D(\rho\|\sigma).$$

A.10 Lipschitz Stability of Quantiles

$$|k_\rho(q) - k_\sigma(q)| \leq C\|\rho - \sigma\|_1.$$

A.11 Commutator Norm Expansion

$$\|[K, O]\|_F^2 = \sum_{i,j} (\log \lambda_i - \log \lambda_j)^2 |O_{ij}|^2.$$

A.12 Signal Regularity

$$\nu \in C^1(\Lambda), \quad |\partial_\lambda \nu| \leq C.$$

A.13 Banach Space Structure

$$\|\nu\|_{C^1} = \sup |\nu| + \sup |\partial_\lambda \nu|.$$

A.14 Sobolev Embedding

$$\nu \in H^1(\Lambda).$$

A.15 Fisher Information Scaling

$$I(\lambda) = \mathrm{Tr}(\rho(\partial_\lambda \log \rho)^2).$$

A.16 Cramér–Rao Bound

$$\mathrm{Var}(\nu) \geq \frac{1}{I(\lambda)}.$$

A.17 Entropy Curvature

$$\mathcal{K} = \frac{d^2 S}{d\lambda^2}.$$

A.18 Stability Under Noise

$$\rho \to (1-\eta)\rho + \eta I/d.$$

A.19 Full Stability Closure

All quantities remain continuous under:

$$\|\delta\rho\|_1 \ll 1.$$

A.20 Consistency of the Framework

All mappings:

$$k \Rightarrow \nu \Rightarrow I \Rightarrow \mathcal{K} \Rightarrow \Delta$$

are mutually consistent and stable.

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