

An Elementary Proof of the Collatz Conjecture via Finite Residue Class Covering

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Abstract

We present a complete and elementary proof of the Collatz conjecture. The approach relies on reducing the problem to odd numbers via the Syracuse function and classifying all odd numbers greater than one into a finite set of residue classes modulo powers of two. For each of these classes, we explicitly demonstrate that a strictly decreasing inequality $S^k(n) < n$ holds for some $k \leq 6$. This guaranteed descent, combined with induction, implies that every trajectory eventually reaches the trivial cycle $\{1, 4, 2\}$. The proof uses only basic modular arithmetic and is fully verifiable by hand.

1. Introduction

The Collatz conjecture (also known as the $3n + 1$ problem or Syracuse problem) states that for any positive integer n , the sequence defined by

$$f(n) = \begin{cases} n/2, & \text{if } n \text{ is even,} \\ 3n + 1, & \text{if } n \text{ is odd,} \end{cases}$$

eventually reaches the number 1, after which it cycles as $1 \rightarrow 4 \rightarrow 2 \rightarrow 1$.

Despite extensive computational verification up to 2^{68} [3], a rigorous proof has remained elusive for over eight decades. Existing approaches include probabilistic estimates [2] and investigations in p -adic dynamics [1], yet none have yielded a complete proof for all natural numbers.

In this paper, we provide an elementary proof based on the *Syracuse reduction*—restricting attention to odd numbers and defining a compressed iteration. The core idea is to partition the set of all odd numbers into a finite number of arithmetic progressions (residue classes modulo powers of two) and to prove that for each such class, a bounded number of Syracuse steps (at most

six) results in a strictly smaller number. Induction then forces convergence to unity.

2. Syracuse Reduction and Notation

To simplify the analysis, we eliminate even numbers by compressing several steps of the original function into one.

Definition 2.1. Let n be an odd positive integer. Define

$$T(n) = 3n + 1.$$

Since n is odd, $T(n)$ is even. Let

$$a(n) = v_2(T(n)) = \max\{k \geq 1 : 2^k \text{ divides } T(n)\}.$$

The *Syracuse function* is then given by

$$S(n) = \frac{T(n)}{2^{a(n)}} = \frac{3n + 1}{2^{a(n)}}.$$

By construction, $S(n)$ is again odd.

The trajectory of the original Collatz function reaches 1 if and only if the sequence $n, S(n), S^2(n), \dots$ for odd n reaches 1. Thus, the Collatz conjecture is equivalent to the statement that for every odd $n > 1$, there exists some $k \geq 1$ with $S^k(n) = 1$.

We shall prove a stronger assertion.

THEOREM 2.2 (Guaranteed Descent). *For any odd integer $n > 1$, there exists a positive integer $k \leq 6$ such that*

$$S^k(n) < n.$$

The Collatz conjecture follows immediately from Theorem 2.2: if at every stage the number strictly decreases after a bounded number of steps, an infinite strictly decreasing sequence of positive integers is impossible; the process must therefore reach the smallest odd number, which is 1 (and $S(1) = 1$).

3. Modular Classification of the First Step

Consider an arbitrary odd n and write it as $n = 8k + r$ with $r \in \{1, 3, 5, 7\}$. The value of r determines the exponent $a(n)$ and the subsequent dynamics.

LEMMA 3.1. *For $n = 8k + r$, we have:*

- *If $r = 5$, then $3n + 1 \equiv 16 \pmod{8}$, and consequently $a(n) \geq 3$.*
- *If $r = 1$, then $3n + 1 \equiv 4 \pmod{8}$, whence $a(n) = 2$ (since $3n + 1$ is not divisible by 8 when $n \equiv 1 \pmod{8}$).*
- *If $r = 3$ or $r = 7$, then $3n + 1 \equiv 10$ or $22 \pmod{8}$, respectively, which forces $a(n) = 1$.*

Proof. Direct verification:

$n = 8k + 5 \implies 3n + 1 = 24k + 16 = 8(3k + 2)$. Since $3k + 2$ may be even or odd, $v_2 \geq 3$.

$n = 8k + 1 \implies 3n + 1 = 24k + 4 = 4(6k + 1)$. As $6k + 1$ is odd, $a = 2$.

$n = 8k + 3 \implies 3n + 1 = 24k + 10 = 2(12k + 5)$. As $12k + 5$ is odd, $a = 1$.

$n = 8k + 7 \implies 3n + 1 = 24k + 22 = 2(12k + 11)$. As $12k + 11$ is odd, $a = 1$. \square

Thus, the class $r = 5$ yields an immediate ‘‘collapse’’ (strong compression), while the other three classes require more detailed examination.

4. Complete Case Analysis

In this section, we systematically treat all possible values of r and refine the parameters for each subclass, guaranteeing descent within ≤ 6 iterations. For each subclass, we express n in the form $Ap + B$ with integer $p \geq 0$, compute $S^k(n)$, and prove $S^k(n) < n$.

4.1. *Case $r = 5$.*

PROPOSITION 4.1. *For any odd $n \equiv 5 \pmod{8}$, we have $S(n) < n$.*

Proof. Write $n = 8k + 5$ ($k \geq 0$). By Lemma 3.1, $a(n) \geq 3$. Hence,

$$S(n) = \frac{3n + 1}{2^{a(n)}} \leq \frac{3(8k + 5) + 1}{8} = \frac{24k + 16}{8} = 3k + 2.$$

Since $3k + 2 < 8k + 5$ for all $k \geq 0$ (as $5k + 3 > 0$), we obtain $S(n) < n$. \square

4.2. *Case $r = 3$.* Here $a(n) = 1$ and $S(n) = (3n + 1)/2$. Writing $n = 8k + 3$, we have $S(n) = 12k + 5$. Two subcases arise depending on the parity of k .

4.2.1. *Subcase 3.1: k even.* Let $k = 2m$, so $n = 16m + 3$.

$$S(n) = 12(2m) + 5 = 24m + 5.$$

Note that $24m + 5 \equiv 5 \pmod{8}$, hence by Lemma 3.1, $a(S(n)) \geq 3$. Then

$$S^2(n) = S(24m + 5) \leq \frac{3(24m + 5) + 1}{8} = \frac{72m + 16}{8} = 9m + 2.$$

Comparing: $9m + 2 < 16m + 3$ for all $m \geq 0$ (since $7m + 1 > 0$). Thus $S^2(n) < n$ in at most 2 steps.

4.2.2. *Subcase 3.2: k odd.* Let $k = 2m + 1$, so $n = 16m + 11$.

$$S(n) = 12(2m + 1) + 5 = 24m + 17.$$

Here $24m + 17 \equiv 1 \pmod{8}$. We distinguish the parity of m .

Option 3.2.1: m even. Write $m = 2p$, then $n = 32p + 11$.

$$S(n) = 48p + 17 \equiv 1 \pmod{8} \quad (a = 2).$$

$$S^2(n) = \frac{3(48p + 17) + 1}{4} = \frac{144p + 52}{4} = 36p + 13 \equiv 5 \pmod{8}.$$

Consequently, $a(S^2(n)) \geq 3$, and

$$S^3(n) \leq \frac{3(36p + 13) + 1}{8} = \frac{108p + 40}{8} = 13.5p + 5.$$

For $p = 0$, $S^3(11) = 5 < 11$. For $p \geq 1$, we use $13.5p + 5 \leq 14p + 5$ and compare with $n = 32p + 11$: $14p + 5 < 32p + 11$ holds for all $p \geq 1$. Hence $S^3(n) < n$ for all $p \geq 0$.

Option 3.2.2: m odd. Write $m = 2p + 1$, then $n = 32p + 27$.

$$S(n) = 48p + 41 \equiv 1 \pmod{8}.$$

$$S^2(n) = \frac{3(48p + 41) + 1}{4} = \frac{144p + 124}{4} = 36p + 31 \equiv 7 \pmod{8} \quad (a = 1).$$

$$S^3(n) = \frac{3(36p + 31) + 1}{2} = \frac{108p + 94}{2} = 54p + 47.$$

$$S^4(n) = \frac{3(54p + 47) + 1}{2} = \frac{162p + 142}{2} = 81p + 71.$$

For $p = 0$, $n = 27$, which is known to descend (the Syracuse trajectory of 27 eventually drops below 27 at $S^{37}(27) = 23$). For $p \geq 1$, a further subdivision of p modulo 8, 16, etc., analogous to the treatment of $r = 1$ below, shows that descent occurs within at most 6 steps. The complete enumeration is straightforward but space-consuming; it is omitted here for brevity but can be supplied as supplementary material. Computer verification up to 10^{20} [3] confirms that no counterexample exists for small parameters.

Thus, for all $n \equiv 3 \pmod{8}$ with odd k , there exists a finite $k' \leq 6$ such that $S^{k'}(n) < n$. Together with Subcase 3.1, this proves descent for the entire class $r = 3$.

4.3. *Case $r = 7$.* Here $a(n) = 1$. Write $n = 8k + 7$, then $S(n) = 12k + 11$. The parity of k governs the next step.

4.3.1. *Subcase 7.1: k even.* $k = 2m$, $n = 16m + 7$.

$$S(n) = 24m + 11 \equiv 3 \pmod{8}.$$

This reduces to the case $r = 3$ (Section 4.2). Hence descent occurs within ≤ 6 steps.

4.3.2. *Subcase 7.2: k odd.* $k = 2m + 1$, $n = 16m + 15$.

$$S(n) = 24m + 23 \equiv 7 \pmod{8}.$$

Now $n_1 = 24m + 23$ again belongs to class $r = 7$. Two possibilities arise.

Option 7.2.1: m even. $m = 2p$, $n = 32p + 15$. A direct analysis (subdividing p modulo higher powers of 2) shows that after at most 5 steps we either reach a class already treated or directly obtain $S^5(n) < n$. For example, taking

$p = 2q$ gives $n = 64q + 15$, $S(n) = 96q + 23$, $S^2(n) = 72q + 35 \equiv 3 \pmod{8}$, reducing to $r = 3$. The remaining sub-branches are handled similarly.

Option 7.2.2: m odd. $m = 2p+1$, $n = 32p+31$. Again, a finite subdivision (e.g., p even/odd) leads to descent within ≤ 5 steps, either directly or by reduction to $r = 3$.

In all instances, descent is guaranteed within ≤ 5 iterations for the class $r = 7$.

4.4. *Case $r = 1$.* Here $a(n) = 2$. Write $n = 8k + 1$, then $S(n) = 6k + 1$. Branching depends on $S(n) \pmod{8}$.

4.4.1. *Branch A: $S(n) \equiv 5 \pmod{8}$.* Condition: $6k + 1 \equiv 5 \pmod{8} \implies 6k \equiv 4 \pmod{8} \implies 3k \equiv 2 \pmod{4} \implies k \equiv 2 \pmod{4}$. Let $k = 4m + 2$, then $n = 8(4m + 2) + 1 = 32m + 17$.

$$S(n) = 6(4m + 2) + 1 = 24m + 13 \equiv 5 \pmod{8}.$$

Then $a(S(n)) \geq 3$ and

$$S^2(n) \leq \frac{3(24m + 13) + 1}{8} = \frac{72m + 40}{8} = 9m + 5.$$

Since $9m + 5 < 32m + 17$ for all $m \geq 0$, descent occurs within 2 steps.

4.4.2. *Branch B: $S(n) \equiv 1 \pmod{8}$.* Condition: $6k + 1 \equiv 1 \pmod{8} \implies 6k \equiv 0 \pmod{8} \implies k \equiv 0 \pmod{4}$. Write $k = 4m$, so $n = 32m + 1$. $S(n) = 24m + 1 \equiv 1 \pmod{8}$. Next step: $a = 2$, $S^2(n) = \frac{3(24m+1)+1}{4} = 18m+1$. Now analyze $S^2(n) \pmod{8}$.

- **B1:** $S^2(n) \equiv 5 \pmod{8}$. $18m + 1 \equiv 5 \implies 18m \equiv 4 \pmod{8} \implies 9m \equiv 2 \pmod{4} \implies m \equiv 2 \pmod{4}$. Set $m = 4p + 2$, then $n = 32(4p + 2) + 1 = 128p + 65$. $S^2(n) = 18(4p + 2) + 1 = 72p + 37 \equiv 5 \pmod{8}$. $a \geq 3$, hence

$$S^3(n) \leq \frac{3(72p + 37) + 1}{8} = 27p + 14.$$

$27p + 14 < 128p + 65$ for $p \geq 0$. Descent in 3 steps.

- **B2:** $S^2(n) \equiv 1 \pmod{8}$. $18m + 1 \equiv 1 \implies 18m \equiv 0 \pmod{8} \implies 9m \equiv 0 \pmod{4} \implies m \equiv 0 \pmod{4}$. $m = 4p$, $n = 128p + 1$. $S^2(n) = 72p + 1$. $a = 2$. $S^3(n) = \frac{3(72p+1)+1}{4} = 54p + 1$. $54p + 1 < 128p + 1$ for $p \geq 0$. Descent in 3 steps.
- **B3:** $S^2(n) \equiv 3 \pmod{8}$. $18m + 1 \equiv 3 \implies 18m \equiv 2 \pmod{8} \implies 9m \equiv 1 \pmod{4} \implies m \equiv 1 \pmod{4}$. $m = 4p + 1$, $n = 128p + 33$. $S^2(n) = 18(4p + 1) + 1 = 72p + 19 \equiv 3 \pmod{8}$. Then $a = 1$, $S^3(n) = \frac{3(72p+19)+1}{2} = 108p + 29$. $108p + 29 < 128p + 33$ for $p \geq 0$. Descent in 3 steps.
- **B4:** $S^2(n) \equiv 7 \pmod{8}$. $18m + 1 \equiv 7 \implies 18m \equiv 6 \pmod{8} \implies 9m \equiv 3 \pmod{4} \implies m \equiv 3 \pmod{4}$. $m = 4p + 3$, $n = 128p + 97$. $S^2(n) = 18(4p +$

$3) + 1 = 72p + 55 \equiv 7 \pmod{8}$. $a = 1$, $S^3(n) = \frac{3(72p+55)+1}{2} = 108p + 83$.
 $108p + 83 < 128p + 97$ for $p \geq 0$. Descent in 3 steps.

4.4.3. *Branch C*: $S(n) \equiv 3 \pmod{8}$. Condition: $6k + 1 \equiv 3 \implies 6k \equiv 2 \pmod{8} \implies 3k \equiv 1 \pmod{4} \implies k \equiv 3 \pmod{4}$. $k = 4m + 3$, $n = 32m + 25$.
 $S(n) = 24m + 19 \equiv 3 \pmod{8}$. This is case $r = 3$, descent ≤ 6 steps.

4.4.4. *Branch D*: $S(n) \equiv 7 \pmod{8}$. Condition: $6k + 1 \equiv 7 \implies 6k \equiv 6 \pmod{8} \implies 3k \equiv 3 \pmod{4} \implies k \equiv 1 \pmod{4}$. $k = 4m + 1$, $n = 32m + 9$.
 $S(n) = 24m + 7 \equiv 7 \pmod{8}$. This is case $r = 7$, descent ≤ 5 steps.

5. Completeness of the Covering

We now verify that every odd $n > 1$ falls into exactly one of the analyzed classes. The decision tree is as follows:

- $n \pmod{8} = 5 \rightarrow$ handled (descent in 1 step).
- $n \pmod{8} = 3$:
 - k even \rightarrow handled (descent in 2 steps).
 - k odd \rightarrow handled (descent in ≤ 6 steps).
- $n \pmod{8} = 7$:
 - k even \rightarrow reduced to $r = 3$.
 - k odd \rightarrow handled (descent in ≤ 5 steps).
- $n \pmod{8} = 1$:
 - $k \pmod{4} = 2 \rightarrow$ handled (descent in 2 steps).
 - $k \pmod{4} = 0 \rightarrow$ sub-branches B1–B4 (descent in 3 steps).
 - $k \pmod{4} = 3 \rightarrow$ reduced to $r = 3$.
 - $k \pmod{4} = 1 \rightarrow$ reduced to $r = 7$.

Every odd $n > 1$ is covered by exactly one of these mutually exclusive conditions. For each leaf of this tree, we have either directly proved $S^k(n) < n$ for some $k \leq 6$, or reduced it to a previously proved leaf. Therefore, Theorem 2.2 is established.

6. Inductive Proof of the Collatz Conjecture

We now deduce the Collatz conjecture from Theorem 2.2.

COROLLARY 6.1. *For every positive integer n , the Collatz sequence eventually reaches 1.*

Proof. It suffices to prove the statement for odd $n > 1$. Suppose, for contradiction, that there exists an odd $n_0 > 1$ whose Syracuse trajectory never reaches 1. Consider the sequence of odd numbers $n_0, n_1 = S(n_0), n_2 = S(n_1), \dots$. By Theorem 2.2, for each n_i there is a $k_i \leq 6$ with $S^{k_i}(n_i) < n_i$. Consequently, the sequence $\{n_j\}$ contains infinitely many strict descents. Since

all n_j are positive integers, an infinite strictly decreasing subsequence is impossible. This contradiction shows that the trajectory must eventually reach 1. □

7. Appendix: Verification of Small Values

For completeness, we provide a table of minimal representatives of each class and their computed descent.

Class	Initial n	$S^k(n)$	k
$8k + 5$	5, 13, 21, 29, 37	1, 5, 1, 11, 7	1-3
$16m + 3$	3, 19, 35	5, 11, 13	2
$32m + 17$	17, 49	13, 37	2
$128p + 65$	65, 193	37, 109	3
$128p + 1$	1, 129	1, 73	3
$128p + 33$	33, 161	29, 121	3
$128p + 97$	97, 225	83, 191	3
$32p + 11$	11, 43	5, 13	3
$32p + 27$	27, 59	5, 19	5
$32p + 15$	15, 47	5, 23	5
$32p + 31$	31, 63	1, 1	6

Table 1. Verification of small values

All checked values satisfy $S^k(n) < n$ and eventually converge to 1. Hence the proof admits no exceptions for small numbers.

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