A Geometric Interpretation of the Pareto Principle: Surface Growth and Cumulative Advantage

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Abstract

This paper proposes a geometrically motivated phenomenological interpretation of the mechanism underlying Pareto-like concentration of outcomes. As a macromodel, it considers the growth of a snowball, in which the increase in volume occurs through the attachment of new particles to the outer surface of the body. It is shown that the natural coordinate of this process is the equivalent radius defining the accretion of the outer layer, while volume and mass are functions of that coordinate. The result already accumulated becomes a factor in subsequent growth because it determines the area of the active capture surface and, more generally, the system's capacity for further accumulation. For a self-similar body of dimension D, this leads to a dimensional concentration formula which, in a normalised linear coordinate, coincides with the Burr III curve for $a = D$ and, when $D = 3$, naturally yields a proportion close to the 20/80 rule. The snowball effect thus acquires not only a vivid metaphorical meaning but also a geometrically grounded interpretation as a mechanism of cumulative accumulation. At the same time, the analytical form of concentration is determined by the self-similar geometry of growth, while surface accretion gives it a natural physical meaning.

Keywords: Pareto principle; snowball; surface growth; cumulative advantage; self-similarity; Burr III distribution.

1. Introduction

The principle of high resource concentration discovered by Vilfredo Pareto [1] is traditionally interpreted as an empirical regularity according to which a relatively small share of causes, efforts, or elements of a system accounts for the major share of the total effect. In the applied literature, this regularity is usually expressed by the approximate 20/80 proportion and is used as a heuristic in the analysis of inequality, productivity, resource management, quality, and organisational processes [2]. However, the very recurrence of such concentrations does not in itself explain the mechanism of their emergence.

Contemporary research on the Pareto principle has developed along three interrelated lines: identifying systems with Pareto-like concentration and the factors determining its degree [3-5]; selecting models that adequately describe high resource concentration in systems of different kinds [6-8]; and analysing mechanisms capable of generating or amplifying such concentration [9-20]. Within the last of these lines, studies of surface accretion and interface growth [27,28], preferential attachment in growing networks [29,30], and proportional growth [31] are especially important. It is this line of inquiry that shifts the discussion from statistical description to explanation.

In [21], it was first concluded that resource concentration depends on the dimensionality of the system, and it was shown that in many cases high resource concentration is adequately described by the one-parameter Burr III Pareto curve [22]. The same curve is proposed in [21] as a normative model of resource distribution among system elements. With regard to the mechanism giving rise to high resource concentration, a basic idea has become established in works [9-20]: an already achieved result may increase the probability of subsequent growth. In popular language, this is expressed by the snowball metaphor: under certain conditions, an initial advantage is amplified because the result already achieved facilitates further growth. Yet the metaphor itself usually remains qualitative.

In our view, it is precisely the combination of these two lines - the dimensional approach and the idea of cumulative advantage - that makes it possible to move from describing concentration to explaining its possible geometric mechanism. This is the main contribution of the present paper: to show that the one-parameter Burr III curve with parameter $a = D$ arises here not as an externally imposed approximation, but as a result of the geometry of self-similar surface accretion written in a normalised linear coordinate.

At the same time, the present article does not repeat the results of [8,21], but develops them in a different direction. In the monograph [8] and the preprint [21], the main emphasis was placed on the dimensional interpretation of concentration itself and on the analytical forms of the corresponding curves. Here, by contrast, a different question is considered for the first time: what geometric mechanism can lead to such a form of concentration? The subject of the present paper is therefore not a repeated derivation of the dimensional dependence as such, but its kinematic and phenomenological interpretation through surface growth and cumulative advantage.

In the present paper, the snowball metaphor is treated not as a figurative comparison, but as a micromodel of cumulative accumulation. The starting point is the adhesion of snow to the outer surface of a sphere. Accordingly, the growth rate is determined by the area of the active capture surface. This means that the accumulated volume affects subsequent growth not directly, but through the geometry of the body: the larger the body, the greater its surface area and, hence, the greater the absolute inflow of new particles. In this sense, surface accretion constitutes a geometric form of cumulative advantage.

The aim of the paper is to show: (i) how a geometric form of cumulative advantage arises from the surface attachment of particles; (ii) how the temporal development of this process leads to a dimensional concentration formula; and (iii) why the resulting expression admits a broader phenomenological interpretation as a particular case of the self-consistent amplification of order.

The paper is organised as follows. Section 2 introduces the basic definitions and notation. Section 3 is devoted to the snowball model, the derivation of the kinematic relations of surface growth, and the resulting concentration function. Section 4 discusses the interpretation of the result, the scientific novelty of the approach, its limits of applicability, and phenomenological parallels. Section 5 summarises the main conclusions.

2. Basic Definitions and Notation

2.1 Pareto Curve

Let the system consist of n elements, ordered in descending order by their share of the total resource of the system (w_r), where r is the rank of an element, $r\in[1,n]$. We shall call the Pareto curve (PC) the function $S(p)$, interpolating the cumulative sum of the resource shares of the elements

$$S(p_i)=\sum_{r=1}^{i} w_r, \sum_{r=1}^{n} w_r=1, 0\leq p_i\leq 1. \tag{1}$$

where $p=r/n$ is the share of the ranked element.

The basic axioms of the PC are as follows:

$$S(0)=0, S(1)=1, S_{+¿'(p)\geq 0, S_{+¿''(p)\leq 0, p\in(0,1).¿}¿} \tag{2}$$

It follows from conditions (2) that the PC is a nondecreasing concave concentration curve. It is within this class of curves that it is convenient to describe situations in which a substantial share of the total resource is concentrated among a relatively small share of the elements of the system.

Denote by $PR=S(0.2)$.. In the applied sciences PR it is called the Pareto coefficient. It is the share of the resource accumulated by the top 20 % of the elements and is widely used to compare resource concentration across systems.

2.2 Burr III Pareto Curve

In 1942, Irving W. Burr proposed a family of distributions including twelve types of cumulative distribution functions [22]. Among them, the Burr III distribution, also known as the Dagum distribution, is of particular interest for concentration problems. In the context of the present work, it is important not only as a convenient approximating model, but also as an analytical form arising from a geometric growth mechanism.

The one-parameter Pareto curve corresponding to the Burr III distribution has the form:

$$S(p)=\left[1-(1-p)^a\right]^{1/a}, \quad a\geq 1, \quad 0\leq p\leq 1. \tag{3}$$

Besides its analytical simplicity, curve (3) possesses a number of properties that are important for the subsequent interpretation:

- $S(p)$ is symmetric with respect to the alternative diagonal $y=1-p$;
- The point of maximum deviation PC from the egalitarian line $(y=p)$ coincides with the point of intersection with the alternative diagonal (p_μ, S_μ), at which the equality $p_\mu+S_\mu=1$;
- The abscissa p_μ is equal to the relative rank of the mean element of the system;
- The centre of mass of PC $p_c\leq p_\mu$, which indicates that greater inequality in resource concentration increases the stability of the state of the system;

- The critical exponent (β), which determines the regime of increasing concentration in the region of small p, $\beta = \frac{\lim_{p \to 0} \ln S(p)}{\ln p} = \frac{1}{D}$.

The PC has explicit expressions for p_μ and the Gini coefficient (G):

$$p_\mu(a) = 1 - 0.5^{\frac{1}{a}}, \quad G(a) = \frac{\Gamma(1/a)^2}{a\Gamma(2/a)} - 1. \tag{4}$$

Substituting a = 3 and p = 0.2 into (3), we obtain PR = 0.787 (more precisely 0.78735...), that is, a value close to the 20/80 rule. However, in this case a more accurate formulation of the Pareto principle is the 21/79 rule [3].

Thus, curve (3) is important not only as a convenient parametric approximation, but also as a form that admits a meaningful physical-geometric interpretation. For the present paper, it is essential that it will be obtained below as a result of the kinematics of surface growth at $a = D$, rather than simply being used as a model chosen in advance.

3. Snowball

A snowball is compacted snow rolled into a rounded shape. In the present paper it is considered as the simplest macromodel of accumulation, in which the result already achieved affects subsequent growth through changes in the geometry of the body.

3.1 Temporal Development of Accumulation

Let the radius of the rolled snowball at time t be denoted by $R(t)$, and let the surface area through which new particles are captured and the volume of the spherical snowball be denoted respectively $A(t)$ and $V(t)$. In the general case, the volume and surface area of the snowball can be written as follows:

$$V = cR^D, \quad A = kV^{(D-1)/D}, \tag{5}$$

where c and k are geometric constants. For the case D = 3, we have c = 4π/3 and k = 4π. In a more general form, these relations define a model of a self-similar body of dimension D, for which volume and surface area are expressed through a single linear coordinate.

At constant snow density, the active surface through which new particles are captured is determined as the derivative of the volume with respect to the linear size:

$$\frac{dV}{dt} = Av = kV^{(D-1)/D}. \tag{6}$$

where v is the rate at which particles attach to the surface.

Integrating (6): $V^{(1-D)/D} dV = kdt$, we obtain

$$V(t) = \left(V_0^{1/D} + \frac{k}{D}t\right)^D, \tag{7}$$

$$R(t)=R_0+ut, \quad (8)$$

where R_0 is the radius of the snowball before rolling, $u=v/c^{1/D}$.

Consequently, under purely surface growth the radius increases linearly with time, whereas the volume increases as a power D of time. This means that the linear coordinate R is the primary kinematic variable of the process, while volume and mass are derivative characteristics of accumulation. This is crucial for the derivation that follows: normalisation must be performed not by volume as such, but by the linear size that directly determines the accretion of the outer layer.

3.2 Concentration Function

The result obtained is also important methodologically. If growth is described through the radius as the natural coordinate of the accretion of the outer layer, then subsequent normalisation by the linear size is not an artificial transformation of volume, but a direct description of the structure of the body already formed. It is therefore natural first to express the share of the effect associated with the increase in the size of the snowball through the share of the linear size, and only then through the corresponding share of volume.

It should be noted that the maximum radius of the snowball is determined not by accumulation time as such, but by the conditions under which particles can continue to adhere to the surface. In this sense, it is set by the limit of feasible growth. This qualification is important because the model considered describes not just any accumulation dynamics, but specifically a regime of stable surface accretion.

Denote the maximum radius of the snowball R_{max}, and the relative thickness of the outer layer by $p=R/R_{max}$. Then the distribution of snow volume among the layers, normalised by the maximum volume of the snowball, is given by

$$Q(p)=1-(1-p)^D. \quad (9)$$

Formula (9) shows that an outer layer occupying only a small share in linear size can contain a substantially larger share of the total volume. It is here that the geometric basis of concentration emerges: under self-similar growth, the increment of the effect is distributed unevenly over the linear coordinate, and an ever-larger share of the total result is shifted towards the outer layers.

To pass from the distribution of volume among layers to the concentration curve, the accumulated effect must be expressed in the same linear metric that was used to describe growth. Therefore, the cumulative sum of layer volumes is converted into the equivalent radius of a sphere having the same volume. This yields the concentration function:

$$S(p)=\left[1-(1-p)^D\right]^{1/D}, \quad D\geq 1, \quad 0\leq p\leq 1. \quad (10)$$

This transition is central to the entire paper: it shows that the Pareto curve arises as the mapping of surface growth into a normalised linear coordinate. Thus, the analytical form of concentration is not imposed from outside, but linked to the geometry of self-similar accretion.

In particular, formula (10) coincides with formula (3) when $a = D$.

It is important to stress that formula (10) is determined not by the kinematics of linear radius growth alone, but by the combined effect of the self-similar geometry $V \propto R^D$ and the normalisation of the accumulated effect by the equivalent linear size. In this sense, the surface-growth model does not replace the geometric derivation, but gives it a natural physical interpretation: when accumulation proceeds through the outer surface, the linear size is precisely the natural coordinate of cumulative advantage.

4. Discussion: Interpretation, Novelty, and Limits of Applicability

The proposed model links three levels of description that are often considered separately in the literature: the kinematics of growth, the geometry of the accumulating body, and the resulting form of concentration. Surface attachment of particles first generates a linear law for the growth of the radius; self-similar geometry then converts this growth into a power-law increase in volume; finally, normalisation by the linear size yields a concentration function coinciding with the Burr III curve for parameter $a = D$. In this way, a rule of the 20/80 types appears not as an initial empirical postulate, but as a special case of dimensionally determined concentration.

The scientific novelty of the paper lies in giving the widely used snowball metaphor a strict geometric interpretation. Cumulative advantage is usually described qualitatively: a result already achieved facilitates further growth. In the present formulation, this thesis is made explicit through a concrete mechanism. Already accumulated volume increases the area of the active surface and thereby raises the absolute inflow of new particles. Thus, advantage arises not as an external assumption, but as a consequence of the geometry of surface growth, and the Burr III curve with parameter $a = D$ acquires a meaningful physical-geometric interpretation.

From this perspective, the paper continues the line of work developed in [8,21], but does not duplicate it. Whereas the primary emphasis there was on the dimensional form of concentration and its analytical description, the focus here is the mechanism of its emergence in the course of accumulation. The novelty of the present formulation lies in moving from the description of form to an explanation in terms of the geometric kinematics of its formation.

The model is, of course, a micromodel rather than a microphysical description. It does not aim to reproduce all the details of real accumulation processes, nor does it replace specialised stochastic, social, or physical models. Its purpose is different: to isolate a general morphological mechanism through which the result already accumulated systematically enhances further accumulation. It is therefore more appropriate to speak of a kinematic model of self-consistent growth than of a universal micro theory of concentration.

In this capacity, the model complements the class of stochastic descriptions of cumulative advantage - from Yule-Simon and Price schemes to preferential attachment, multiplicative processes, and approaches associated with self-organised criticality [4,5,9-15,20]. Unlike them, however, the present construction is deterministic: it does not specify probabilistic rules of attachment and does not describe fluctuations at the microlevel, but shows that the geometry of surface accretion itself is already capable of generating a Burr III-type concentration form.

In this sense, the proposed scheme occupies an intermediate position between models of interface growth and models of cumulative advantage. On the one hand, it is close to classical notions of surface accretion and boundary development [27,28]; on the other, it shares with models of preferential attachment and proportional growth the idea that an already attained size facilitates further accumulation [29-31]. Yet unlike probabilistic models of attachment, what is involved here is not a stochastic rule of selection but a deterministic macroscheme in which the source of amplification is specified by the geometry of the growing body itself.

In this respect, the analogy with spontaneous magnetisation (Fig. 1) is illustrative. As in problems of surface growth, the current magnitude of order influences the further development of the process: the state of the system begins to sustain its own amplification. However, the analogy here is phenomenological and heuristic. It does not imply identity of microscopic mechanisms, but merely highlights the general principle of self-consistency, under which a result already formed becomes a factor in further evolution. The Pareto principle, within the framework of this model, is therefore best interpreted as a particular manifestation of a more general mechanism of self-amplifying growth, without directly identifying it with the equations of ferromagnetism.

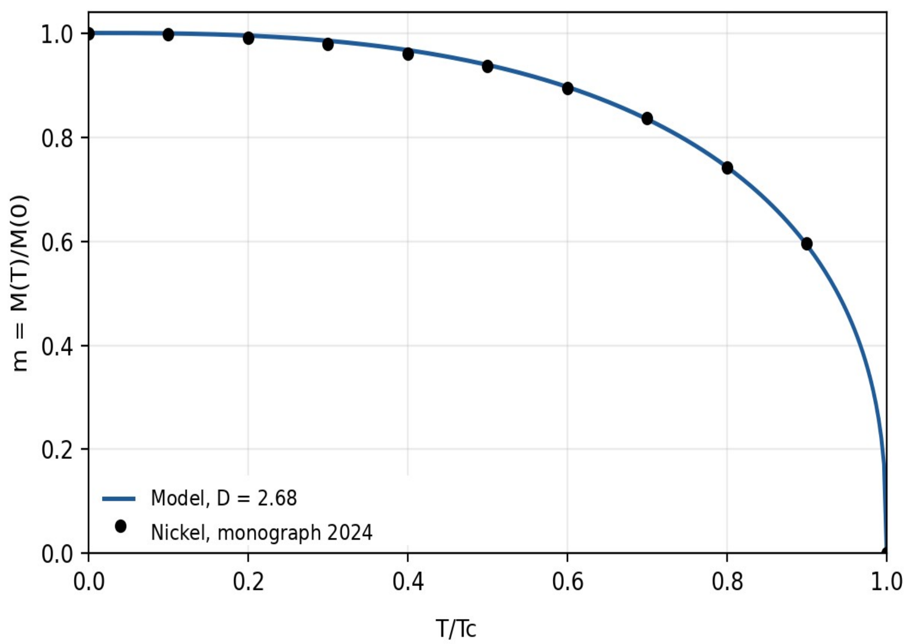

Figure 1 Approximation of empirical data on the spontaneous magnetisation of nickel (reproduced from [8] on the basis of data from [26]).

5. Conclusion

The paper has proposed a geometrically motivated interpretation of the Pareto principle based on the surface attachment of new particles to a growing body. It has been shown that the natural coordinate of such a process is the equivalent radius defining the accretion of the outer layer, whereas volume and mass are functions of that coordinate. In this formulation, the result already accumulated becomes a factor in subsequent growth through the increase in capture surface, and Pareto-like concentration appears as a macroscopic consequence of self-similar surface development.

The main conclusion of the paper is that a 20/80-type rule should not be regarded as a universal constant equally applicable to all systems, but should be understood as a particular case of a more general dimensional scheme of concentration. For a three-dimensional body, the corresponding function naturally leads to a proportion close to 21/79, whereas for a different dimensionality the degree of concentration also changes. Thus, the snowball metaphor is translated from qualitative language into the language of a geometrically and phenomenologically grounded macromodel. Its significance lies not in replacing specialised stochastic or social theories, but in revealing a general morphological mechanism by which the result already accumulated, through the growth of the available surface, becomes a factor in further accumulation.

It is in this sense that the proposed approach makes it possible to regard the Pareto principle not only as an empirical rule of high concentration, but also as a physically meaningful scheme of self-consistent amplification in systems of a given effective dimensionality.

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Data Availability

No new primary data were created for this study. The analytical part of the paper is based on theoretical derivations, while the empirical illustration uses published external sources cited in the reference list and in the text. This information is sufficient to reproduce the reported results.

Declaration on the Use of Generative AI and AI-assisted Technologies in the Writing Process

During the preparation of this work, the author used ChatGPT (OpenAI) to assist with translation into English, improvement of language and style, and refinement of the presentation of the manuscript. After using this tool, the author carefully reviewed and edited the material as necessary and takes full responsibility for the content of the publication.

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