

Exact-Affine Rigidity for Log-Concave Measures and Structural Advances Toward the KLS Conjecture

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Abstract

Let $\nu = e^{-W} dz$ be a log-concave probability measure on a Euclidean space E with $W \in C^2(E)$ convex and Legendre-type, and let $m = e^{-\phi} dy$ be log-concave on an interval $I \subset \mathbb{R}$. We study the rigidity of *exact-affine bundles* $T(z, y) = (A(y)z + b(y), y)$ whose pushforward $T_{\#}(\nu \otimes m)$ is log-concave, and develop a systematic structural theory with applications toward the Kannan–Lovász–Simonovits (KLS) conjecture.

Our main unconditional results are as follows.

The unconditional exact-affine layer proves the optimized tangent inequality, the exact Schur-complement formula, the averaged motion estimate

$$\mathcal{J}_{\nu}(B(y)) + \text{Tr}(B(y)^2) \leq \phi''(y),$$

the universal reverse inequality

$$\mathcal{J}_{\nu}(L) + \text{Tr}(L^2) \geq 0,$$

the affine-base exact-affine gauge-collapse theorem, nonradial scalar and fixed-axis full-line collapse, even-fiber positive-part collapse, fixed-direction real-spectrum collapse, fixed real-eigenline full-line rigidity, one-parameter subgroup collapse with a real spectral mode, and complete radial full-line collapse modulo orthogonal gauge. These results preserve essentially the full exact-affine content of the April 2026 manuscript while separating it cleanly from the still-conditional transport layer.

The present revision adds twelve further structural advances. First, after isotropic normalization of the reference fiber, we identify the kernel of the quadratic form

$$\mathcal{Q}_{\nu}(L) := \mathcal{J}_{\nu}(L) + \text{Tr}(L^2)$$

with the infinitesimal orthogonal symmetry algebra of the fiber, prove coercivity on its Hilbert–Schmidt orthogonal complement, and derive a quantitative orthogonal-gauge rigidity theorem on curved bases. In particular, after removing the optimal time-dependent infinitesimal symmetry, the residual linear velocity satisfies

$$\|\tilde{B}(y)\|_{\text{HS}} \leq \lambda_{\nu}^{-1/2} \sqrt{\phi''(y)},$$

where $\lambda_{\nu} > 0$ is the symmetry-gap constant of the isotropic fiber. This yields explicit curvature-budget lower bounds for nontrivial linear distortion, perturbative affine-base rigidity on compact intervals, finite-curvature-tail asymptotic rigidity, and in the Gaussian case the exact identity

$$\mathcal{Q}_{\gamma_E}(L) = 2 \|\text{Sym} L\|_{\text{HS}}^2.$$

Second, we identify the full linear symmetry group G_{ν}^{lin} of the reference fiber and prove a new *symmetry-valued gauge-collapse theorem*: if the exact-affine velocity $B(y) = A(y)^{-1}A'(y)$ lies in the infinitesimal linear symmetry algebra of ν on an arbitrary interval, then $b''(y) \equiv 0$ and the whole linear motion is pure gauge. Thus exact symmetry-valued motion is eliminated with no affine-base or full-line assumption. As a concrete consequence, pure rotational radial exact-affine motion collapses modulo orthogonal gauge on every interval, not only on the full line.

Third, for even reference fibers we prove a curved-base *translation-budget theorem*. Writing

$$c(y) := A(y)^{-1}b''(y) \quad \text{and} \quad \mathcal{T}_{\nu}(v) := \int_E |\langle \nabla W(z), v \rangle| \nu(dz),$$

we show that every even exact-affine bundle satisfies the pointwise budget

$$\mathcal{T}_{\nu}(c(y)) + \mathcal{Q}_{\nu}(B(y)) \leq \phi''(y).$$

After isotropic normalization this combines with the symmetry-gap theorem to give

$$\tau_{\nu} |A(y)^{-1}b''(y)| + \lambda_{\nu} \text{dist}_{\text{HS}}(A(y)^{-1}A'(y), \mathfrak{k}_{\nu})^2 \leq \phi''(y),$$

where $\tau_\nu := \inf_{|u|=1} \mathcal{T}_\nu(u)$ and, for isotropic even fibers, $\tau_\nu \geq 1/\sqrt{3}$ universally. Hence in the even-isotropic world,

$$|A(y)^{-1}b''(y)| \leq \sqrt{3}\phi''(y),$$

and at any flat base point $\phi''(y_0) = 0$ both the normalized translation acceleration and the non-symmetry linear velocity vanish instantaneously.

Fourth, combining the orthogonal-gauge estimate with the even-fiber translation budget, we obtain a quantitative compact-interval translation-slope bound and a new one-sided asymptotic affine-rigidity theorem. On every interval one has explicit control of the translation slope by the integrated base-curvature budget. On a tail $[y_0, \infty)$, if

$$\int_{y_0}^{\infty} \sqrt{\phi''(s)} \, ds < \infty, \quad \int_{y_0}^{\infty} \phi''(s) \, ds < \infty,$$

then after orthogonal gauge the linear part converges in $\mathrm{GL}(E)$ and the translation slope $b'(y)$ converges in E . If in addition

$$\int_{y_0}^{\infty} (s - y_0)\phi''(s) \, ds < \infty,$$

then the bundle is asymptotic on the tail to an affine product with an explicit remainder estimate.

Fifth, for multiplicative one-parameter subgroup motion

$$A(y) = A_0 e^{s(y)G},$$

the averaged-motion and even-fiber translation-budget inequalities collapse to explicit scalar subgroup budgets. Without any evenness assumption one has

$$\mathcal{Q}_\nu(G) |s'(y)|^2 \leq \phi''(y),$$

and after isotropic normalization,

$$\lambda_\nu \, \mathrm{dist}_{\mathrm{HS}}(G, \mathfrak{k}_\nu)^2 |s'(y)|^2 \leq \phi''(y).$$

In the even-fiber world this strengthens to

$$\mathcal{T}_\nu(A(y)^{-1}b''(y)) + \mathcal{Q}_\nu(G) |s'(y)|^2 \leq \phi''(y),$$

hence, after isotropic normalization,

$$\tau_\nu |A(y)^{-1}b''(y)| + \lambda_\nu \, \mathrm{dist}_{\mathrm{HS}}(G, \mathfrak{k}_\nu)^2 |s'(y)|^2 \leq \phi''(y).$$

Consequently subgroup generators lying in the symmetry algebra are pure gauge, while every genuinely non-symmetry subgroup branch—including the spectrally real-free nonselfadjoint branch—must spend explicit base-curvature budget on any interval and freezes asymptotically on one-sided tails with finite $\int \sqrt{\phi''}$.

Sixth, for additive fixed-direction motion

$$A(y) = A_0(I + s(y)N),$$

we prove the companion curved-base budget

$$\mathcal{Q}_\nu(\Gamma_N(s(y))) |s'(y)|^2 \leq \phi''(y), \quad \Gamma_N(t) := (I + tN)^{-1}N.$$

After isotropic normalization this becomes

$$\lambda_\nu \, \mathrm{dist}_{\mathrm{HS}}(\Gamma_N(s(y)), \mathfrak{k}_\nu)^2 |s'(y)|^2 \leq \phi''(y).$$

We then classify the exceptional parameter values at which $\Gamma_N(t)$ can enter the infinitesimal orthogonal symmetry algebra: for nonzero N , such a parameter is either absent or unique, and it exists if and only if

$$N = K(I - t_* K)^{-1}$$

for some $K \in \mathfrak{k}_\nu$. Hence any additive branch that can touch symmetry is, after a fixed affine normalization, again generated by an actual symmetry direction. In the even-fiber world this sharpens to

$$\mathcal{T}_\nu(A(y)^{-1}b''(y)) + \mathcal{Q}_\nu(\Gamma_N(s(y))) |s'(y)|^2 \leq \phi''(y),$$

and on one-sided tails with compact parameter range and finite $\int \sqrt{\phi''}$, the additive parameter necessarily freezes to a finite limit.

Seventh, we prove a local affine-product stability theorem on compact intervals. After orthogonal gauge and normalization at one base point, the integrated curvature budgets

$$\int_J \sqrt{\phi''(s)} \, ds \quad \text{and} \quad \int_J \phi''(s) \, ds$$

control the deviation of the linear coefficient from its base-point value, the deviation of the translation slope from its base-point value, and the affine remainder of the translation itself on the whole interval J . Consequently any normalized nontrivial exact-affine motion carries a definite curvature quantum on every compact scale, and any sequence of exact-affine bundles with uniformly positive symmetry-gap and translation-gap constants and vanishing local curvature budgets converges on compact intervals to an affine product. This supplies the exact-affine compactness half of the desired blow-up bridge.

Eighth, in the full-line even-isotropic setting we globalize that compactness package into a *curvature-density dichotomy*. For every fixed scale R and threshold ε , either there exists a window of length $2R$ on which, after orthogonal gauge and local affine normalization at its center, the exact-affine coefficients are ε -close to an affine product, or else the combined curvature budget

$$\lambda_\nu^{-1/2} \int \sqrt{\phi''} + \tau_\nu^{-1} \int \phi''$$

accumulates at least linearly with rate $\gtrsim_\varepsilon R^{-1}$. Consequently every full-line exact-affine bundle with sublinear combined curvature growth produces arbitrarily long almost-flat windows. This gives a theorem-level production mechanism for the canonical scales sought by the blow-up strategy.

Ninth, we strengthen the full-line blow-up mechanism from mere existence of one good window to a quantitative *bad-center density theorem*. For every fixed scale R and threshold ε , the set of centers whose length- $2R$ windows remain ε -nontrivial has one-dimensional measure at most

$$\frac{4R}{c_\varepsilon} \left(\lambda_\nu^{-1/2} \int \sqrt{\phi''} + \tau_\nu^{-1} \int \phi'' \right), \quad c_\varepsilon := \min\{1, \varepsilon/e\},$$

on every compact base interval. Consequently, whenever the combined curvature density tends to zero on long intervals, almost-flat windows of any fixed scale occur with asymptotic density one. Thus the exact-affine blow-up supply is no longer a single-window statement but a quantitative prevalence theorem.

Tenth, we prove an exact *dual-conjugate characterization* of exact-affine log-concavity. Writing

$$V_y(x) := W(A(y)^{-1}(x - b(y))) + \log \det A(y) + \phi(y),$$

we show that $T_\#(\nu \otimes m)$ is log-concave if and only if, for every covector $\xi \in E$, the partial Legendre conjugate

$$\Psi_\xi(y) := W^*(A(y)^T \xi) + \langle b(y), \xi \rangle - \log \det A(y) - \phi(y)$$

is concave in y . Differentiating recovers the optimized tangent inequality exactly. This recasts the remaining exact-affine problem as a concave-orbit rigidity problem in the dual potential space,

which appears to be the right global formulation of the genuinely anisotropic nonselfadjoint branch.

Eleventh, we prove a *toric Möbius-symmetry collapse theorem* for the exceptional additive branch. Assume that the symmetry generator in the classification

$$N = K(I - t_*K)^{-1}$$

admits an orthogonal block decomposition

$$K|_{E_j} = \omega_j J_j, \quad J_j^2 = -\text{Id}_{E_j}, \quad J_j \in \mathfrak{k}_\nu,$$

with commuting blockwise complex structures on an orthogonal splitting $E = E_0 \oplus E_1 \oplus \cdots \oplus E_m$. Then the additive family admits an explicit blockwise polar factorization

$$I + uK = R(u)P(u),$$

where $R(u) \in G_\nu^{\text{lin}}$ is a genuine symmetry and $P(u) \in \text{Sym}^+(E)$ is blockwise positive. After symmetry gauge and a fixed ambient affine normalization, the branch therefore enters the self-adjoint positive cone. The full-line even-Legendre positive-cone collapse theorem then forces the additive parameter to be constant and the translation to be affine. In particular, all single-frequency complex-structure Möbius-symmetry branches collapse on the full line.

Twelfth, we show that the compact symmetry generator behind the exceptional additive Möbius branch carries an intrinsic integer weight-lattice decomposition, so that the nonresonant case is automatically torically split. We then prove a dual large-ray rigidity theorem for arbitrary compact symmetry generators: after reducing to a branch of the form $A(y) = I + u(y)K$ with $K \in \mathfrak{k}_\nu$, the block singular radii $\sqrt{1 + \omega_j^2 u(y)^2}$ on the nonzero skew-spectral blocks are concave on every interval. On the full line they are therefore constant, forcing the additive parameter to be constant and the translation to be affine. Consequently the entire additive symmetry-touching full-line branch collapses, not only its toric or nonresonant subclasses.

Conditional on a rank-one transport-edge reduction retained from earlier work, these exact-affine rigidity results eliminate large families of potential terminal-model obstructions to the KLS conjecture, including the entire scalar, self-adjoint positive-definite, radial, fixed-direction real-spectrum, and affine-base classes. The remaining exact-affine frontier is precisely delineated: any surviving obstruction must be genuinely nonselfadjoint and spectrally real-free after every fixed affine normalization; in the additive fixed-direction full-line world it must in addition stay uniformly away from the symmetry algebra on every compact parameter range; and in the even bounded-interval world it must also spend base-curvature budget simultaneously on translation acceleration and genuinely non-symmetry linear motion.

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Contents

1 Introduction

Background and the KLS conjecture

A probability measure μ on $\mathbb{R}^n$ is *log-concave* if for every pair of Borel sets $A, B \subset \mathbb{R}^n$ and every $\theta \in [0, 1]$,

$$\mu(\theta A + (1 - \theta)B) \geq \mu(A)^\theta \mu(B)^{1-\theta}.$$

Equivalently, μ has a density of the form e^{-V} where $V : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex. Log-concave measures form the natural setting for high-dimensional probabilistic and geometric analysis; they include uniform measures on convex bodies, Gaussian measures, and products thereof.

The measure μ is *isotropic* if it is centered

$$\int_{\mathbb{R}^n} x_i \, d\mu(x) = 0 \quad (1 \leq i \leq n)$$

and has identity covariance

$$\int_{\mathbb{R}^n} x_i x_j \, d\mu(x) = \delta_{ij} \quad (1 \leq i, j \leq n).$$

Every full-dimensional log-concave probability measure on $\mathbb{R}^n$ can be brought into isotropic position by an affine transformation.

The *Poincaré constant* $C_P(\mu)$ is the smallest constant $C > 0$ such that

$$\mathrm{Var}_\mu(f) \leq C \int |\nabla f|^2 \, d\mu$$

for every smooth compactly supported $f : \mathbb{R}^n \rightarrow \mathbb{R}$, where $\mathrm{Var}_\mu(f) = \int f^2 \, d\mu - (\int f \, d\mu)^2$. Equivalently, $C_P(\mu)^{-1}$ is the spectral gap of the Langevin operator $L = \Delta - \nabla V \cdot \nabla$ on $L^2(\mu)$.

The Kannan–Lovász–Simonovits conjecture [7] predicts that

$$C_P(\mu) \leq C$$

with a universal constant $C > 0$, independent of the dimension n and of the specific log-concave measure μ . By now Bourgain’s slicing problem has been solved in the affirmative [11], and there is also a recent preprint proof of the thin-shell conjecture [12]. What remains open is the full dimension-free KLS estimate itself. The strongest published general bound currently available for KLS is the $\sqrt{\log n}$ -isoperimetric, equivalently $\log n$ -Poincaré, estimate of Klartag [8].

The transport–soft-edge program and main results

The transport–soft-edge program developed in the earlier memoranda attacked KLS from a different angle. The retained mechanism is the transport operator $\mathcal{H}_\mu$, whose spectrum controls quadratic observables exactly. Combined with the stretched-exponential decay theorem retained from the previous stage, this produces a canonical low-dimensional transport edge. After trimming to the final rank-one edge, curvature disappears and the problem becomes a one-dimensional moving-frame problem over a high-dimensional reference fiber. The weighted rank-one product theorem shows that the residual task is geometric rather than purely spectral: one needs a universal Poincaré bound for the isotropized reference fiber and a universal affine-normalized distortion bound for the best μ_0 -preserving moving frame.

The purpose of the present article is to continue this reduction by eliminating additional exact-affine terminal classes, by strengthening the structural inequalities that govern them, by extracting quantitative curved-base information from the affine-base collapse mechanism, by proving that exact symmetry-valued linear motion is already pure gauge on arbitrary intervals, by deriving a genuine translation-control theorem on curved bases in the even-fiber world, by isolating a new explicit curvature budget for multiplicative one-parameter subgroup motion, by upgrading almost-flat-window production to a density theorem on long intervals, and by recasting exact-affine log-concavity as a pointwise concavity problem for partial Legendre conjugates. The paper now has three layers.

Programmatic layer. We retain from the earlier memoranda the gauge-invariant two-scale theorem, the rank-one trivialization, the exact weighted product reduction, the affine-normalized closure principle, and the conditional rank-one KLS reduction.

Unconditional exact-affine layer. Independently of the transport operator, we study rank-one bundles

$$T(z, y) = (A(y)z + b(y), y)$$

whose pushforward $T_{\#}(\nu \otimes m)$ is log-concave. The exact-affine tangent inequality from the April note already implied several rigidity theorems. The present article pushes this branch further in twenty-four distinct directions.

- (1) We prove an *optimized* tangent inequality by differentiating along affine reference curves $z(y) = z + (y - y_0)u$. This inserts a new positive term depending on $A^{-1}A'$ and the inverse Hessian of the reference potential.
- (2) We identify that optimized inequality with the exact Schur complement of the full Hessian of the global potential.
- (3) By integrating the Schur-complement inequality over the reference fiber, we derive the averaged motion estimate $\mathcal{J}_{\nu}(B) + \text{Tr}(B^2) \leq \phi''$ that ties the linear moving-frame velocity directly to the base curvature.
- (4) We prove the universal reverse inequality $\mathcal{J}_{\nu}(L) + \text{Tr}(L^2) \geq 0$ for every linear field L on the reference fiber.
- (5) We identify the equality case in that lower bound with the infinitesimal affine symmetry algebra of the reference fiber.
- (6) We combine the upper and lower bounds to prove an affine-base exact-affine gauge-collapse theorem on arbitrary intervals: if the base potential is affine, then after composing with a smooth affine ν -preserving moving frame the bundle already has constant linear part and affine translation. Hence the entire affine-base exact-affine terminal class is eliminated.
- (7) We record gauge-free consequences of the affine-base theorem, including actual collapse for trivial affine-symmetry fibers and for globally affine-conjugate positive-cone branches.
- (8) We remove the radial hypothesis from the scalar-dilation full-line theorem: for any Legendre-type fiber potential, exact scalar full-line motion collapses to an affine product.
- (9) We remove the radial hypothesis from the fixed-axis full-line theorem, at the price of assuming unconditional symmetry of the reference fiber.

- (10) We prove a new general structural exclusion over even fibers: for any exact-affine bundle, every real eigenvalue of $A(y)^{-1}A''(y)$ is nonpositive. Thus positive real acceleration modes are impossible even before any symmetry or commutativity assumptions on the linear part. We then close the entire self-adjoint positive-definite branch over even Legendre fibers.

- (11) We show that fixed-direction full-line motion over an even Legendre fiber already collapses as soon as, after a fixed affine normalization, the direction matrix carries a nonzero real eigenvalue. In particular, every invertible odd-dimensional fixed-direction branch is eliminated. We also prove a fixed real-eigenline full-line rigidity theorem and deduce a new one-parameter subgroup collapse theorem: if

$$A(y) = A_0 e^{s(y)M}$$

and $A_0^{-1}MA_0$ has a nonzero real eigenvalue, then A is constant and the translation is affine. We also prove a new local flat-point rigidity statement: if $A''(y_0) = 0$, then $A(y_0)^{-1}A'(y_0)$ cannot carry a nonzero real eigenvalue.

- (12) We integrate the orthogonal-gauge and Lyapunov-order argument from the sequel note and obtain complete full-line collapse of *all* radial exact-affine bundles, not only commuting or nilpotent subclasses.
- (13) We identify the linear symmetry group of the reference fiber and prove that whenever the exact-affine velocity $A(y)^{-1}A'(y)$ lies in its infinitesimal symmetry algebra on an arbitrary interval, the linear motion is pure gauge and the translation is automatically affine. In particular, pure rotational radial motion is eliminated on bounded intervals as well.
- (14) We prove, for even fibers, a new curved-base translation-budget theorem:

$$\mathcal{T}_\nu(A(y)^{-1}b''(y)) + \mathcal{Q}_\nu(A(y)^{-1}A'(y)) \leq \phi''(y),$$

where $\mathcal{T}_\nu(v) = \int |\langle \nabla W(z), v \rangle| \nu(dz)$. After isotropic normalization this yields the simultaneous pointwise budget

$$\tau_\nu |A(y)^{-1}b''(y)| + \lambda_\nu \text{dist}_{\text{HS}}(A(y)^{-1}A'(y), \mathfrak{k}_\nu)^2 \leq \phi''(y),$$

with a universal lower bound $\tau_\nu \geq 1/\sqrt{3}$ in the even-isotropic case. Thus flat base points are instantaneously rigid: when $\phi''(y_0) = 0$, both normalized translation acceleration and non-symmetry linear motion vanish at y_0 .

- (15) We combine the orthogonal-gauge theorem with the even-fiber translation budget to obtain a quantitative compact-interval translation-slope estimate. Thus, after optimal orthogonal gauge, both the linear part and the translation slope are controlled directly by explicit curvature budgets.
- (16) We prove a one-sided asymptotic affine-rigidity theorem: under finite tail budgets for $\sqrt{\phi''}$ and ϕ'' , the gauge-reduced linear part converges and the translation slope converges; under a first-moment tail budget for ϕ'' , the entire bundle is asymptotic to an affine product on the tail.
- (17) For multiplicative one-parameter subgroup motion

$$A(y) = A_0 e^{s(y)G},$$

we prove a new curved-base budget:

$$\mathcal{Q}_\nu(G) |s'(y)|^2 \leq \phi''(y),$$

and, in the even-isotropic setting,

$$\tau_\nu |A(y)^{-1} b''(y)| + \lambda_\nu \operatorname{dist}_{\text{HS}}(G, \mathfrak{k}_\nu)^2 |s'(y)|^2 \leq \phi''(y).$$

Hence subgroup generators inside the symmetry algebra are pure gauge, while every genuinely non-symmetry subgroup branch pays explicit base-curvature budget and becomes asymptotically constant on one-sided tails with finite $\int \sqrt{\phi''}$.

- (18) For additive one-parameter motion

$$A(y) = A_0(I + s(y)N),$$

we prove the companion curved-base budget

$$\mathcal{Q}_\nu(\Gamma_N(s(y))) |s'(y)|^2 \leq \phi''(y), \quad \Gamma_N(t) := (I + tN)^{-1}N,$$

and, in the even-isotropic setting,

$$\tau_\nu |A(y)^{-1} b''(y)| + \lambda_\nu \operatorname{dist}_{\text{HS}}(\Gamma_N(s(y)), \mathfrak{k}_\nu)^2 |s'(y)|^2 \leq \phi''(y).$$

We then classify the possible symmetry-alignment parameters: for nonzero N , the set of t for which $\Gamma_N(t) \in \mathfrak{k}_\nu$ is either empty or a singleton, and this singleton occurs exactly when N is a Möbius transform $K(I - t_* K)^{-1}$ of a genuine infinitesimal symmetry $K \in \mathfrak{k}_\nu$. Thus any additive branch that can touch symmetry is, after a fixed affine normalization, again generated by a symmetry direction; and on compact one-sided tails with finite $\int \sqrt{\phi''}$, the additive parameter necessarily freezes.

- (19) We prove a local affine-product stability theorem on compact intervals: after orthogonal gauge and normalization at a base point, the linear coefficient, the translation slope, and the affine remainder of the translation are all controlled explicitly by the integrated curvature budgets $\int \sqrt{\phi''}$ and $\int \phi''$. As a consequence, every normalized nontrivial exact-affine motion carries a definite curvature quantum on each compact scale, and any family with uniformly positive symmetry-gap and translation-gap constants and vanishing local curvature budgets converges on compact intervals to an affine product.
- (20) We globalize this compactness mechanism on the full line by proving a curvature-density dichotomy: for every fixed window length $2R$ and threshold ε , either some window is ε -close to an affine product after orthogonal gauge and local normalization, or else the combined curvature budget

$$\lambda_\nu^{-1/2} \int \sqrt{\phi''} + \tau_\nu^{-1} \int \phi''$$

must grow at least linearly with rate $\gtrsim_\varepsilon R^{-1}$. Consequently every full-line exact-affine bundle with sublinear combined curvature growth produces arbitrarily long almost-flat windows.

- (21) We sharpen the full-line window-production theorem into a bad-center density estimate: at every fixed scale R and threshold ε , the one-dimensional measure of centers whose windows remain ε -nontrivial is bounded by $4R c_\varepsilon^{-1}$ times the combined curvature budget on a slightly enlarged interval. In particular, vanishing combined curvature density forces almost-flat windows to occur with density one, not merely along a sparse subsequence.

- (22) We prove an exact dual reformulation of exact-affine log-concavity. The global potential is jointly convex if and only if, for every covector ξ , the partial Legendre conjugate $y \mapsto W^*(A(y)^T \xi) + \langle b(y), \xi \rangle - \log \det A(y) - \phi(y)$ is concave. Differentiating this dual criterion recovers the optimized tangent inequality exactly, and thus recasts the unresolved anisotropic nonselfadjoint branch as a concave-orbit rigidity problem in the dual potential space.
- (23) We prove a toric Möbius-symmetry collapse theorem for the exceptional additive branch. If the symmetry generator in the classification $N = K(I - t_* K)^{-1}$ splits orthogonally as

$$K|_{E_j} = \omega_j J_j, \quad J_j^2 = -\text{Id}_{E_j}, \quad J_j \in \mathfrak{k}_\nu,$$

on a decomposition $E = E_0 \oplus E_1 \oplus \cdots \oplus E_m$, then $I + uK$ factorizes blockwise as a genuine symmetry $R(u) \in G_\nu^{\text{lin}}$ times a self-adjoint positive-definite part $P(u)$. After symmetry gauge and a fixed affine normalization, the branch therefore falls into the self-adjoint positive-definite full-line class already eliminated in Section 16. In particular, all single-frequency complex-structure Möbius-symmetry branches collapse on the full line.

- (24) We show that a compact symmetry generator behind the exceptional additive Möbius branch carries an intrinsic integer weight-lattice decomposition, so that the full-line even-fiber Möbius branch collapses in every nonresonant case. We then prove a dual large-ray rigidity theorem for arbitrary compact symmetry generators: after reducing to a branch of the form $A(y) = I + u(y)K$ with $K \in \mathfrak{k}_\nu$, the block singular radii $\sqrt{1 + \omega_j^2 u(y)^2}$ on the nonzero skew-spectral blocks are concave on every interval. On the full line these radii are constant, forcing the additive parameter to be constant and the translation to be affine. Thus the entire additive symmetry-touching full-line branch is eliminated, not only its toric or nonresonant subclasses.

The net effect is a sharper and more useful frontier. Inside the exact-affine full-line world, the remaining unresolved branch is now forced into genuinely nonselfadjoint motion, or else into non-even or nonexact-affine geometry, after optimal moving-frame normalization. More precisely, within fixed-direction motion over an even fiber, any surviving obstruction must be spectrally real-free after every fixed affine normalization; and within fixed one-parameter subgroup motion

$$A(y) = A_0 e^{s(y)M},$$

any surviving full-line obstruction must likewise avoid every nonzero real spectral mode of the generator after fixed affine normalization. On bounded or one-sided bases, the new subgroup-budget and additive-budget theorems show more. For multiplicative subgroup motion, either the generator lies in the symmetry algebra and the motion is pure gauge, or every genuinely non-symmetry subgroup branch pays explicit curvature budget proportional to its distance from the symmetry algebra. For additive fixed-direction motion, the normalized velocity $(I + tN)^{-1}N$ can meet the orthogonal symmetry algebra at at most one parameter value, and this happens only for Möbius-symmetry directions $N = K(I - t_* K)^{-1}$ generated by $K \in \mathfrak{k}_\nu$; otherwise the branch stays uniformly away from symmetry on every compact parameter range and is quantitatively visible to the same base-curvature budget throughout, while compact one-sided tails with finite $\int \sqrt{\phi''}$ still force freezing of the additive parameter. The new local affine-product stability theorem adds a complementary compactness principle: once the local curvature budgets $\int \sqrt{\phi''}$ and $\int \phi''$ are small after normalization at one base point, the exact-affine coefficients are already close to an affine product on the entire interval. Equivalently, any normalized nontrivial exact-affine motion must

carry a definite curvature quantum on every scale. The full-line curvature-density theorem pushes this one step further: if such almost-flat windows fail to appear on a prescribed scale, then the combined budget

$$\lambda_\nu^{-1/2} \int \sqrt{\phi''} + \tau_\nu^{-1} \int \phi''$$

must already accumulate linearly in the window count; conversely, sublinear combined curvature growth automatically produces arbitrarily long almost-flat windows. Thus the exact-affine compactness half of the blow-up strategy is now in place together with a first theorem-level production mechanism for canonical large scales. Within the bounded or one-sided exact-affine world, the entire affine-base subbranch $\phi'' \equiv 0$ is now removed modulo affine ν -symmetry gauge, the exact symmetry-valued branch $A^{-1}A' \in \mathfrak{g}_\nu$ is removed on arbitrary intervals, and in the even-isotropic setting any residual normalized translation acceleration or genuinely non-symmetry linear motion must pay a pointwise base-curvature budget. Outside the exact-affine ansatz, the global affine-normalized distortion problem, the production of canonical almost-flat blow-up scales, and the reference-fiber spectral gap remain the central open tasks of the program.

Organization of the paper

The paper is organized as follows. Sections 2–12 retain and reorganize the gauge-invariant rank-one reduction and the earlier terminal-model eliminations. Section 13 proves the optimized tangent inequality. Sections 14 and 15 contain the nonradial full-line scalar and unconditional fixed-axis collapse theorems. Section 16 proves the new real-eigenmode exclusion and the complete self-adjoint positive-definite collapse theorem over even Legendre fibers. Section 17 proves the fixed-direction real-spectrum collapse theorem over even fibers, the fixed real-eigenline full-line rigidity theorem, and the one-parameter subgroup collapse theorem with a real spectral mode. Sections 18–19 prove complete radial exact-affine full-line collapse modulo orthogonal gauge and record strong fixed-direction corollaries, including the elimination of arbitrary fixed nilpotent one-parameter families. Section 20 develops exact Schur-complement consequences, including the averaged motion inequality, the universal lower bound $\mathcal{J}_\nu(L) + \text{Tr}(L^2) \geq 0$, affine-base gauge collapse, gauge-free affine-base corollaries, and flat-point rigidity. Section 21 proves the new quantitative symmetry-gap coercivity and orthogonal-gauge rigidity theorems on curved bases. Section 22 proves symmetry-valued gauge collapse on arbitrary intervals. Section 23 proves the even-fiber translation-budget theorem on curved bases. Section 24 proves compact-interval translation-slope control and one-sided asymptotic affine rigidity. Section 25 proves the new curved-base subgroup-budget theorems for multiplicative one-parameter motion. Section 26 develops the additive one-parameter budget, the unique-symmetry-parameter theorem, and the Möbius-symmetry reduction for fixed directions. Section 27 proves local affine-product stability, curvature quantization, and vanishing-curvature blow-up compactness. Section 28 proves the full-line curvature-density dichotomy and the basic production theorem for almost-flat windows. Section 29 upgrades this to bad-center density bounds and density-one prevalence of good windows. Section 30 proves the dual-conjugate characterization of exact-affine log-concavity and reformulates the remaining exact-affine difficulty as a concave-orbit problem in the dual potential space. Section 31 proves the toric and nonresonant Möbius-symmetry reductions and then closes the full compact-symmetry additive branch on the full line by a dual large-ray argument. Section 32 summarizes the remaining frontier and formulates the next exact-affine conjectures. Section 33 records the retained theorems from the earlier memoranda, and Section 34 contains the bibliographic note.

Guide to the main results

For the reader's convenience we collect a high-level summary of the principal theorems proved unconditionally in this article.

Result	Class eliminated	Ref.
Optimized tangent inequality	(structural tool)	Prop. 13.1
Exact Schur complement	(structural identification)	Prop. 20.1
Averaged motion inequality	$\mathcal{Q}_\nu(B) \leq \phi''$	Prop. 20.3
Universal lower bound $\mathcal{Q}_\nu \geq 0$	(Brascamp–Lieb)	Prop. 20.5
Affine-base gauge collapse	$\phi'' \equiv 0$ on any interval	Thm. 20.7
Full-line scalar dilation	$A(y) = a(y)I$	Thm. 14.2
Full-line unconditional fixed-axis	unconditional fibers	Thm. 15.3
Self-adj. pos.-def. full-line (even)	$A(y) \in \text{Sym}^+(E)$	Thm. 16.7
Fixed-dir. real-spectrum (even)	$A = A_0(I + sN)$, real mode	Thm. 17.1
Radial full-line collapse	all radial A	Thm. 18.7
Symmetry-gap coercivity	quantitative budget	Thm. 21.3
Symmetry-valued gauge collapse	$B \in \mathfrak{k}_\nu$ on any interval	Thm. 22.3
Even translation budget	$\mathcal{T}_\nu(c) + \mathcal{Q}_\nu(B) \leq \phi''$	Thm. 23.3
Local affine-product stability	compactness theorem	Thm. 27.1
Curvature-density dichotomy	almost-flat window production	Thm. 28.5
Dual conjugate characterization	concave-orbit reformulation	Thm. 30.1
Toric Möbius collapse	additive toric branches	Thm. 31.3
Nonresonant Möbius collapse	additive nonresonant symmetry branches	Thm. 31.8
Full Möbius collapse	additive symmetry-touching full-line branches	Thm. 31.11

Notation and conventions

Throughout, E denotes a finite-dimensional Euclidean space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $|\cdot|$. For $A \in \text{End}(E)$, $\|A\|_{\text{HS}} = (\text{Tr}(A^T A))^{1/2}$ is the Hilbert–Schmidt norm, $\|A\|_{\text{op}} = \sup_{|v|=1} |Av|$ is the operator norm, and $\text{Sym}(A) := \frac{1}{2}(A + A^T)$ denotes the symmetric part. We write $\text{Sym}^+(E)$ for the cone of symmetric positive-definite endomorphisms, $\mathfrak{so}(E)$ for the Lie algebra of skew-symmetric endomorphisms, and $\text{GL}(E)$, $\text{O}(E)$ for the general linear and orthogonal groups.

A convex function $W \in C^2(E)$ is *Legendre-type* if $\nabla^2 W(z)$ is positive-definite for every $z \in E$ and $\nabla W(E) = E$. The Legendre conjugate is $W^*(p) = \sup_{z \in E} (\langle p, z \rangle - W(z))$.

For a log-concave probability measure $\nu = e^{-W} dz$ on E , we write Var_ν and Cov_ν for variance and covariance with respect to ν . The measure ν is *isotropic* if $\int z \nu(dz) = 0$ and $\int z \otimes z \nu(dz) = \text{Id}$.

2 Retained verified inputs

We briefly record the statements from the earlier memoranda that remain valid and that will be used explicitly below.

2.1 The transport operator and exact quadratic transference

For an isotropic log-concave measure μ on $\mathbb{R}^n$, the transport operator $\mathcal{H}_\mu$ is the positive operator defined by

$$\langle \mathcal{H}_\mu v, v \rangle = \|\langle x, v \rangle\|_{H^{-1}(\mu)}^2.$$

Theorem 2.1 (Exact quadratic transference, retained). *Let ν be an isotropic log-concave probability measure on a Euclidean space E , and let $B : E \rightarrow E$ be symmetric. Then*

$$\text{Var}_\nu(\langle By, y \rangle - \text{Tr} B) \leq 4 \text{Tr}(B^2 \mathcal{H}_\nu).$$

In particular, if $\eta_1(\nu) \leq K$, then every quadratic form on E obeys

$$\text{Var}_\nu(\langle By, y \rangle - \text{Tr} B) \leq 4K \text{Tr}(B^2).$$

2.2 Transport-spectrum decay

Write

$$\eta_1(\mu) \geq \dots \geq \eta_n(\mu) \geq 0$$

for the eigenvalues of $\mathcal{H}_\mu$, and let

$$d_\mu(K) = \#\{i : \eta_i(\mu) > K\}.$$

Theorem 2.2 (Stretched-exponential transport spectrum, retained). *There exist universal constants $C, c, \beta > 0$ such that for every isotropic log-concave probability measure μ on $\mathbb{R}^n$,*

$$\eta_k(\mu) \leq C(1 + \log(n/k))^\beta \quad (1 \leq k \leq n),$$

and consequently

$$d_\mu(K) \leq Cn e^{-cK^{1/\beta}} \quad (K \geq 1).$$

With the current thin-shell proof one may take $\beta = 16$.

Corollary 2.3 (Canonical rank-one transport trimming). *There exists $C_0 > 0$, depending only on the constants in [theorem 2.2](#), such that for*

$$K_1(n) := C_0(1 + \log n)^\beta$$

one has

$$d_\mu(K_1(n)) \leq 1.$$

Hence the canonical transport edge

$$F_1 := \text{span}\{u_i : \eta_i(\mu) > K_1(n)\}$$

has dimension at most one.

Proof. Choose C_0 large enough so that

$$Cn e^{-c(C_0(1+\log n)^\beta)^{1/\beta}} = Cn e^{-cC_0^{1/\beta}(1+\log n)} \leq 1$$

for every $n \geq 1$. Then [theorem 2.2](#) yields $d_\mu(K_1(n)) \leq 1$. □

2.3 Why the previous obstruction was not final

Let $\mathbb{R}^n = E \oplus F$ be an orthogonal splitting, write $\mu(\mathrm{d}x, \mathrm{d}y) = \mu_y(\mathrm{d}x)m(\mathrm{d}y)$, and let $\mathcal{U}$ denote the Poisson gauge from the previous memorandum. The old interaction matrix was

$$\mathcal{K}(y) = \int_E \mathcal{U}(x, y)^* \mathcal{U}(x, y) \mu_y(\mathrm{d}x).$$

The previously proved obstruction showed that on the canonical transport edge F_K , the quantity $\sup_y \|\mathcal{K}(y)\|_{\mathrm{op}}$ must be large if $P_{F_K} \mathcal{H}_\mu P_{F_K}$ is large. The point of the present note is that $\mathcal{K}$ is attached to the L^2 -minimal gauge; it is not a gauge-invariant invariant of the fiber family.

3 Admissible gauges and gauge invariance

Fix an orthogonal decomposition

$$\mathbb{R}^n = E \oplus F$$

with $\dim E = m$, $\dim F = r$, and assume for the main results of this section that

$$\mu(\mathrm{d}x, \mathrm{d}y) = e^{-V(x, y)} \mathrm{d}x \mathrm{d}y$$

is regular and strictly positive. Write

$$\mu(\mathrm{d}x, \mathrm{d}y) = \mu_y(\mathrm{d}x) m(\mathrm{d}y)$$

for the disintegration over F . For $v \in F$, define

$$\Xi_v(x, y) := \partial_v V(x, y) - \int_E \partial_v V(x', y) \mu_y(\mathrm{d}x').$$

Then $\int_E \Xi_v \mathrm{d}\mu_y = 0$.

Definition 3.1 (Admissible gauge). A smooth map

$$W : E \times F \rightarrow \mathcal{L}(F, E)$$

is called an *admissible gauge* for the disintegration $(\mu_y)_{y \in F}$ if for every $y \in F$, every $v \in F$, and every smooth compactly supported $\phi : E \rightarrow \mathbb{R}$,

$$\partial_v \int_E \phi(x) \mu_y(\mathrm{d}x) = - \int_E \langle \nabla_x \phi(x), W(x, y)v \rangle \mu_y(\mathrm{d}x).$$

Equivalently, for every $v \in F$,

$$-\mathrm{div}_{\mu_y}(W_v) = \Xi_v, \quad W_v(\cdot, y) := W(\cdot, y)v.$$

The Poisson gauge of the previous memorandum is one admissible gauge: for each $v \in F$, solve

$$-L_y u_v = \Xi_v$$

and set $U_v = \nabla_x u_v$.

Proposition 3.2 (Gauge-invariant derivative of the fiber average). *Let W be any admissible gauge and let*

$$\bar{f}(y) := \int_E f(x, y) \mu_y(\mathrm{d}x).$$

Then for every smooth $f : E \times F \rightarrow \mathbb{R}$ and every $v \in F$,

$$\partial_v \bar{f}(y) = \int_E (\partial_v f(x, y) - \langle W(x, y)v, \nabla_x f(x, y) \rangle) \mu_y(\mathrm{d}x).$$

Proof. Differentiate $\bar{f}(y) = \int_E f(\cdot, y) \mathrm{d}\mu_y$ in direction v :

$$\partial_v \bar{f}(y) = \int_E \partial_v f \mathrm{d}\mu_y + \partial_v \left(\int_E f(\cdot, y) \mathrm{d}\mu_y \right).$$

Apply [theorem 3.1](#) with $\phi(x) = f(x, y)$. This gives

$$\partial_v \left(\int_E f(\cdot, y) \mathrm{d}\mu_y \right) = - \int_E \langle \nabla_x f(x, y), W(x, y)v \rangle \mathrm{d}\mu_y,$$

which is the stated formula. □

Definition 3.3 (Covariant derivative associated with W). For an admissible gauge W , define

$$D_v^W f := \partial_v f - \langle Wv, \nabla_x f \rangle.$$

If $(e_j)_{j=1}^r$ is an orthonormal basis of F , we write

$$D_F^W f := (D_{e_1}^W f, \dots, D_{e_r}^W f) \in F.$$

Theorem 3.4 (Exact gauge-invariant two-scale Poincaré theorem). *Assume that*

$$C_P(\mu_y) \leq A \quad \text{for all } y \in F$$

and

$$C_P(m) \leq B.$$

Then for every admissible gauge W and every smooth $f : E \times F \rightarrow \mathbb{R}$,

$$\mathrm{Var}_\mu(f) \leq A \int |\nabla_x f|^2 \mathrm{d}\mu + B \int |D_F^W f|^2 \mathrm{d}\mu.$$

Proof. Disintegrate the variance:

$$\mathrm{Var}_\mu(f) = \int_F \mathrm{Var}_{\mu_y}(f(\cdot, y)) m(\mathrm{d}y) + \mathrm{Var}_m(\bar{f}).$$

The first term is bounded by the fiber Poincaré inequality:

$$\int_F \mathrm{Var}_{\mu_y}(f(\cdot, y)) m(\mathrm{d}y) \leq A \int |\nabla_x f|^2 \mathrm{d}\mu.$$

For the second term, apply the Poincaré inequality on m :

$$\mathrm{Var}_m(\bar{f}) \leq B \int_F |\nabla \bar{f}(y)|^2 m(\mathrm{d}y).$$

Choose an orthonormal basis $(e_j)_{j=1}^r$ of F . By [theorem 3.2](#),

$$\partial_{e_j} \bar{f}(y) = \int_E D_{e_j}^W f(x, y) \mu_y(\mathrm{d}x).$$

Hence, by Jensen,

$$|\partial_{e_j} \bar{f}(y)|^2 \leq \int_E |D_{e_j}^W f(x, y)|^2 \mu_y(\mathrm{d}x).$$

Summing over j and integrating in y yields

$$\mathrm{Var}_m(\bar{f}) \leq B \int |D_F^W f|^2 \mathrm{d}\mu.$$

Combining the bounds proves the theorem. $\square$

Lemma 3.5 (Gauge freedom). *If W and W' are admissible gauges, then $Z := W' - W$ satisfies*

$$\mathrm{div}_{\mu_y}(Z_v) = 0 \quad \text{for every } y \in F, v \in F.$$

Conversely, if W is admissible and Z is a smooth field with

$$\mathrm{div}_{\mu_y}(Z_v) = 0 \quad (y \in F, v \in F),$$

then $W + Z$ is admissible.

Proof. Subtract the weighted divergence equations. The converse is immediate by adding a divergence-free field to an admissible one. $\square$

Proposition 3.6 (The Poisson gauge is the L^2 -minimal gauge). *Fix $y \in F$ and $v \in F$, and let $U_v(\cdot, y) = \nabla_x u_v(\cdot, y)$ be the Poisson gauge, i.e.*

$$-L_y u_v = \Xi_v.$$

If $W_v(\cdot, y)$ is any other admissible gauge in the same direction v , then

$$\int_E |W_v|^2 \mathrm{d}\mu_y = \int_E |U_v|^2 \mathrm{d}\mu_y + \int_E |W_v - U_v|^2 \mathrm{d}\mu_y.$$

In particular, U_v is the unique $L^2(\mu_y)$ -minimizer among all admissible gauges with divergence Ξ_v .

Proof. By [theorem 3.5](#), $Z_v := W_v - U_v$ is weighted divergence-free:

$$\mathrm{div}_{\mu_y}(Z_v) = 0.$$

Since $U_v = \nabla_x u_v$, an integration by parts gives

$$\int_E \langle U_v, Z_v \rangle \mathrm{d}\mu_y = \int_E \langle \nabla_x u_v, Z_v \rangle \mathrm{d}\mu_y = - \int_E u_v \mathrm{div}_{\mu_y}(Z_v) \mathrm{d}\mu_y = 0.$$

Therefore

$$\int_E |W_v|^2 \mathrm{d}\mu_y = \int_E |U_v + Z_v|^2 \mathrm{d}\mu_y = \int_E |U_v|^2 \mathrm{d}\mu_y + \int_E |Z_v|^2 \mathrm{d}\mu_y.$$

$\square$

Remark 3.7. [theorem 3.6](#) explains the status of the old interaction matrix

$$\mathcal{K}(y) = \int_E \mathbf{U}(x, y)^* \mathbf{U}(x, y) \mu_y(\mathrm{d}x).$$

It records the minimal *kinetic* cost of lifting the derivative of the conditional family in Wasserstein tangent space. It does *not* record the best gauge for metric distortion or affine-normalized geometry.

4 Curvature, trivialization, and the rank-one case

For a general base F , it is natural to define the gauge curvature in the standard way. If $(e_j)_{j=1}^r$ is an orthonormal basis of F , write

$$W_j(x, y) := W(x, y)e_j.$$

Definition 4.1 (Gauge curvature).

$$\mathcal{R}_{jk}^W := \partial_{y_j} W_k - \partial_{y_k} W_j + [W_j, W_k],$$

where

$$[W_j, W_k] = (W_j \cdot \nabla_x) W_k - (W_k \cdot \nabla_x) W_j.$$

Proposition 4.2 (Commutator identity). *Let*

$$D_j^W := \partial_{y_j} - W_j \cdot \nabla_x.$$

Then

$$[D_j^W, D_k^W] = -\mathcal{R}_{jk}^W \cdot \nabla_x.$$

Proof. This is the standard computation for first-order transport operators:

$$[D_j^W, D_k^W] = -(\partial_{y_j} W_k - \partial_{y_k} W_j + [W_j, W_k]) \cdot \nabla_x.$$

□

In the previous memorandum we proved a general flatness theorem for the Poisson gauge on simply connected F . Exactly the same proof works for any admissible gauge. For the present note, however, the decisive simplification is that the canonical transport edge can be chosen rank one. In that case no curvature tensor survives.

Theorem 4.3 (Rank-one trivialization). *Assume $F = \mathbb{R}e$ is one-dimensional, and let W be an admissible gauge. Suppose that $W(\cdot, y)$ is globally Lipschitz in x , uniformly on compact y -intervals. Let $T_y : E \rightarrow E$ solve*

$$\partial_y T_y(z) = -W(T_y(z), y), \quad T_0(z) = z.$$

Then for every $y \in \mathbb{R}$, T_y is a diffeomorphism and

$$\mu_y = (T_y)_\# \mu_0.$$

Hence every admissible rank-one gauge globally trivializes the family of fibers.

Proof. Global well-posedness of the ODE follows from the Lipschitz assumption in x . Standard ODE theory gives that each T_y is a diffeomorphism.

Define

$$\nu_y := (T_y)_\# \mu_0.$$

For every smooth compactly supported $\phi : E \rightarrow \mathbb{R}$,

$$\frac{d}{dy} \int_E \phi(x) \nu_y(dx) = \frac{d}{dy} \int_E \phi(T_y(z)) \mu_0(dz) = - \int_E \langle \nabla_x \phi(T_y(z)), W(T_y(z), y) \rangle \mu_0(dz).$$

Equivalently,

$$\frac{d}{dy} \int_E \phi \, d\nu_y = - \int_E \langle \nabla_x \phi(x), W(x, y) \rangle \nu_y(dx).$$

On the other hand, admissibility of W gives

$$\frac{d}{dy} \int_E \phi \, d\mu_y = - \int_E \langle \nabla_x \phi(x), W(x, y) \rangle \mu_y(dx).$$

Thus $(\nu_y)_{y \in \mathbb{R}}$ and $(\mu_y)_{y \in \mathbb{R}}$ solve the same linear continuity equation with the same initial datum $\nu_0 = \mu_0$. Uniqueness for the transport equation with globally Lipschitz velocity yields $\nu_y = \mu_y$. $\square$

Proposition 4.4 (Gauge freedom as moving-frame freedom). *Assume $F = \mathbb{R}$, and let T_y and T'_y be two rank-one trivializations satisfying*

$$\mu_y = (T_y)_\# \mu_0 = (T'_y)_\# \mu_0 \quad \text{for all } y.$$

Then

$$S_y := T_y^{-1} \circ T'_y$$

preserves the reference fiber μ_0 :

$$(S_y)_\# \mu_0 = \mu_0.$$

Conversely, if T_y is any rank-one trivialization and $(S_y)_{y \in \mathbb{R}}$ is a smooth family of diffeomorphisms with

$$(S_y)_\# \mu_0 = \mu_0,$$

then $T'_y := T_y \circ S_y$ is another trivialization of the same family (μ_y) .

Proof. If $T'_y = T_y \circ S_y$, then

$$\mu_0 = (T_y^{-1})_\# \mu_y = (T_y^{-1} \circ T'_y)_\# \mu_0 = (S_y)_\# \mu_0.$$

Conversely,

$$(T_y \circ S_y)_\# \mu_0 = (T_y)_\# \mu_0 = \mu_y.$$

$\square$

Remark 4.5. [theorem 4.4](#) is the geometric meaning of gauge freedom in rank one. The true optimization problem is not “choose a vector field W ”. It is: starting from one trivialization T_y , compose it with a μ_0 -preserving moving frame S_y so as to optimize the resulting metric distortion. This is the rank-one counterpart of gauge fixing in geometric analysis.

5 Exact weighted product reduction in rank one

Assume from now on that $F = \mathbb{R}$. Let W be an admissible rank-one gauge, and let T_y be the corresponding trivialization from [theorem 4.3](#). Set

$$\nu := \mu_0.$$

Define the triangular map

$$T : E \times \mathbb{R} \rightarrow E \times \mathbb{R}, \quad T(z, y) = (T_y(z), y).$$

Then

$$\mu = T_\#(\nu \otimes m).$$

5.1 Chain rules

Write

$$J_y(z) := D_z T_y(z) \in \mathcal{L}(E, E).$$

Proposition 5.1 (Exact pullback formulas). *Let $g = f \circ T$. Then*

$$\nabla_z g(z, y) = J_y(z)^T \nabla_x f(T_y(z), y),$$

and

$$\partial_y g(z, y) = (D^W f)(T_y(z), y).$$

Consequently,

$$|\nabla g(z, y)|^2 = |J_y(z)^T \nabla_x f(T_y(z), y)|^2 + |D^W f(T_y(z), y)|^2.$$

Proof. The first identity is the ordinary chain rule in the z -variable. For the second,

$$\partial_y g(z, y) = \partial_y f(T_y(z), y) + \langle \nabla_x f(T_y(z), y), \partial_y T_y(z) \rangle.$$

Since $\partial_y T_y(z) = -W(T_y(z), y)$, this becomes

$$\partial_y g(z, y) = \partial_y f(T_y(z), y) - \langle W(T_y(z), y), \nabla_x f(T_y(z), y) \rangle = (D^W f)(T_y(z), y).$$

The final identity follows by summing the squares of the z - and y -derivatives. $\square$

5.2 Metric tensors induced by the trivialization

Definition 5.2 (Fiber distortion and full distortion tensors). Define, on $E \times \mathbb{R}$,

$$Q(T_y(z), y) := J_y(z) J_y(z)^T$$

and

$$G(T_y(z), y) := DT(z, y) DT(z, y)^T.$$

Equivalently, Q records the fiber metric distortion and G records the full triangular distortion.

Since

$$DT(z, y) = \begin{pmatrix} J_y(z) & \partial_y T_y(z) \\ 0 & 1 \end{pmatrix},$$

one has

$$DT(z, y)^T \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} J_y(z)^T p \\ q + \langle \partial_y T_y(z), p \rangle \end{pmatrix}.$$

In particular, by [theorem 5.1](#),

$$\langle G \nabla f, \nabla f \rangle = \langle Q \nabla_x f, \nabla_x f \rangle + |D^W f|^2.$$

Theorem 5.3 (Exact weighted rank-one product reduction). *Let*

$$A_0 := C_P(\nu), \quad B := C_P(m).$$

Then every smooth $f : E \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies

$$\text{Var}_\mu(f) \leq A_0 \int \langle Q \nabla_x f, \nabla_x f \rangle \, d\mu + B \int |D^W f|^2 \, d\mu.$$

Equivalently,

$$\text{Var}_\mu(f) \leq \max\{A_0, B\} \int \langle G \nabla f, \nabla f \rangle \, d\mu.$$

Proof. Let $g = f \circ T$. Since $\mu = T_{\#}(\nu \otimes m)$,

$$\mathrm{Var}_{\mu}(f) = \mathrm{Var}_{\nu \otimes m}(g).$$

Tensorization of Poincaré gives

$$\mathrm{Var}_{\nu \otimes m}(g) \leq A_0 \int |\nabla_z g|^2 d(\nu \otimes m) + B \int |\partial_y g|^2 d(\nu \otimes m).$$

Use [theorem 5.1](#) and push forward by T :

$$\int |\nabla_z g|^2 d(\nu \otimes m) = \int \langle Q \nabla_x f, \nabla_x f \rangle d\mu,$$

and

$$\int |\partial_y g|^2 d(\nu \otimes m) = \int |D^W f|^2 d\mu.$$

This proves the first inequality. The second follows from

$$\langle G \nabla f, \nabla f \rangle = \langle Q \nabla_x f, \nabla_x f \rangle + |D^W f|^2.$$

□

Remark 5.4. [theorem 5.3](#) is not merely a restatement of the gauge-invariant two-scale theorem. It identifies the actual metric tensor induced by the rank-one trivialization. The old Poisson-connection picture isolated the covariant derivative D^W ; the present theorem shows that the missing complementary object is the fiber distortion tensor Q , and together they form the full distortion tensor G .

5.3 Evolution of the fiber metric

Proposition 5.5 (Evolution equation for the fiber metric). *Along the rank-one trivialization,*

$$\partial_y J_y(z) = -\nabla_x W(T_y(z), y) J_y(z).$$

Consequently,

$$\partial_y (J_y(z) J_y(z)^T) = -A_y(z) Q_y(z) - Q_y(z) A_y(z)^T,$$

where

$$A_y(z) := \nabla_x W(T_y(z), y), \quad Q_y(z) := J_y(z) J_y(z)^T.$$

In particular, if

$$\sigma(y) := \sup_{x \in E} \|\mathrm{Sym} \nabla_x W(x, y)\|_{\mathrm{op}},$$

then

$$e^{-2\Sigma(y)} \mathrm{Id} \preceq Q(\cdot, y) \preceq e^{2\Sigma(y)} \mathrm{Id}, \quad \Sigma(y) := \int_0^{|y|} \sigma(\mathrm{sgn}(y) s) ds.$$

Proof. Differentiate the ODE for T_y with respect to z :

$$\partial_y J_y(z) = -\nabla_x W(T_y(z), y) J_y(z).$$

Therefore

$$\partial_y Q_y = (\partial_y J_y) J_y^T + J_y (\partial_y J_y)^T = -A_y Q_y - Q_y A_y^T.$$

For any $u \in E$,

$$\frac{d}{dy} \langle Q_y u, u \rangle = -2 \langle \text{Sym}(A_y) Q_y u, u \rangle.$$

Hence

$$\left| \frac{d}{dy} \log \langle Q_y u, u \rangle \right| \leq 2 \|\text{Sym}(A_y)\|_{\text{op}} \leq 2 \sigma(y).$$

Integrating from 0 to y and using $Q_0 = \text{Id}$ gives the stated bounds. $\square$

Corollary 5.6 (Strain control). *If*

$$\Lambda := \sup_{y \in \mathbb{R}} \Sigma(y) < \infty,$$

then

$$e^{-2\Lambda} \text{Id} \preceq Q(x, y) \preceq e^{2\Lambda} \text{Id} \quad \text{for all } (x, y).$$

Hence the first term in [theorem 5.3](#) obeys

$$\int \langle Q \nabla_x f, \nabla_x f \rangle \, d\mu \leq e^{2\Lambda} \int |\nabla_x f|^2 \, d\mu.$$

Proof. Immediate from [theorem 5.5](#). $\square$

Remark 5.7. The evolution of Q only sees the symmetric part of $\nabla_x W$. This is the precise sense in which infinitesimal rotations of the fibers are harmless for the *fiber* metric. However the full distortion tensor G still contains the vertical velocity $\partial_y T_y = -W \circ T_y$, and this term may be large. Thus even in rank one, raw L^2 -kinetic size and raw Lipschitz size remain too coordinate-sensitive; affine normalization will be necessary.

6 Closure theorems and affine normalization

The weighted theorem of the previous section yields several immediate closure criteria.

Corollary 6.1 (Direct quasi-isometric closure). *Assume that the triangular trivialization T satisfies*

$$\text{ess sup}_{(z,y)} \|DT(z, y)\|_{\text{op}} \leq L.$$

Then

$$C_P(\mu) \leq L^2 \max\{C_P(\nu), C_P(m)\}.$$

Proof. For $G = DT DT^T \circ T^{-1}$,

$$\langle G \nabla f, \nabla f \rangle \leq L^2 |\nabla f|^2$$

for μ -almost every point. Apply [theorem 5.3](#). $\square$

Remark 6.2. [theorem 6.1](#) is too strong to be the final invariant. A pure affine triangular shear may have arbitrarily large operator norm even though the corresponding isotropic measure satisfies KLS by affine invariance. This is why the correct closure criterion must first remove the affine part.

We now formulate a closure theorem in affine-normalized form.

Theorem 6.3 (Affine-normalized quasi-product closure). *Let ν be a log-concave probability measure on E , let m be a log-concave probability measure on $\mathbb{R}$, and let $T : E \times \mathbb{R} \rightarrow E \times \mathbb{R}$ be a triangular trivialization such that*

$$\mu = T_{\#}(\nu \otimes m)$$

is isotropic. Let $A_E : E \rightarrow E$ and $a : \mathbb{R} \rightarrow \mathbb{R}$ be affine maps such that

$$\tilde{\nu} := (A_E)_{\#}\nu \quad \text{and} \quad \tilde{m} := a_{\#}m$$

are isotropic. Set

$$A := A_E \oplus a \quad \text{and} \quad S := T \circ A^{-1}.$$

Assume that S is L -Lipschitz:

$$\text{ess sup } \|DS\|_{\text{op}} \leq L.$$

Then

$$C_P(\mu) \leq L^2 \max\{C_P(\tilde{\nu}), C_P(\tilde{m})\}.$$

In particular, if $\tilde{\nu}$ has dimension-free KLS and L is universal, then μ satisfies KLS.

Proof. Let

$$\sigma := A_{\#}(\nu \otimes m) = \tilde{\nu} \otimes \tilde{m}.$$

Then

$$\mu = S_{\#}\sigma.$$

Since σ is a product measure,

$$C_P(\sigma) = \max\{C_P(\tilde{\nu}), C_P(\tilde{m})\}.$$

For any smooth f , let $g = f \circ S$. Then

$$\text{Var}_{\mu}(f) = \text{Var}_{\sigma}(g) \leq C_P(\sigma) \int |\nabla g|^2 \, d\sigma.$$

Because S is L -Lipschitz,

$$|\nabla g| \leq L |\nabla f| \circ S.$$

Thus

$$\text{Var}_{\mu}(f) \leq L^2 C_P(\sigma) \int |\nabla f|^2 \, d\mu,$$

which is the claim. □

Corollary 6.4 (Rank-one closure criterion in isotropic form). *Let μ be isotropic, let*

$$\mu(\mathrm{d}x, \mathrm{d}y) = \mu_y(\mathrm{d}x) m(\mathrm{d}y)$$

be a rank-one disintegration, and let T_y be any trivialization of the conditional family. Write $\nu = \mu_0$. If

- (i) *the isotropization $\tilde{\nu}$ of ν satisfies $C_P(\tilde{\nu}) \leq C_f$,*
- (ii) *the isotropization $\tilde{m}$ of m satisfies $C_P(\tilde{m}) \leq C_b$,*
- (iii) *after isotropizing the reference product, the map $S = T \circ A^{-1}$ is L -Lipschitz,*

then

$$C_P(\mu) \leq L^2 \max\{C_f, C_b\}.$$

Proof. This is just [theorem 6.3](#). □

Remark 6.5. For a one-dimensional isotropic log-concave marginal $\tilde{m}$, the quantity $C_P(\tilde{m})$ is universally bounded. Thus, once one reaches rank one, the only genuinely high-dimensional difficulty lies in the isotropized reference fiber and in the affine-normalized distortion of the trivialization.

7 Canonical rank-one transport edge reduction

We now combine the rank-one theory with the retained transport-spectrum theorem.

Let μ be isotropic log-concave on $\mathbb{R}^n$. Choose

$$K_1(n) = C_0(1 + \log n)^\beta$$

as in [theorem 2.3](#), and let

$$F_1 := \text{span}\{u_i : \eta_i(\mu) > K_1(n)\}, \quad E_1 := F_1^\perp.$$

Then $\dim F_1 \leq 1$. Disintegrate μ over the canonical edge:

$$\mu(dx, dy) = \mu_y(dx) m(dy), \quad x \in E_1, y \in F_1.$$

Theorem 7.1 (Canonical rank-one reduction). *For every isotropic log-concave probability measure μ on $\mathbb{R}^n$, the KLS problem reduces to the following rank-one data attached to the canonical transport edge F_1 :*

- (i) *the isotropized reference fiber $\tilde{\nu}$, where $\nu = \mu_0$;*
- (ii) *a choice of ν -preserving moving frame S_y on the reference fiber;*
- (iii) *the affine-normalized triangular trivialization obtained from $T_y \circ S_y$.*

More precisely, if for the canonical rank-one edge one can prove uniformly that

- (a) *$C_P(\tilde{\nu}) \leq C_f$ with a universal constant;*
- (b) *there exists a smooth ν -preserving moving frame S_y such that the corresponding affine-normalized trivialization is L -Lipschitz with a universal constant;*

then

$$C_P(\mu) \leq L^2 \max\{C_f, C_b\},$$

where C_b is the universal one-dimensional isotropic bound.

Proof. By [theorem 2.3](#), $\dim F_1 \leq 1$. If $\dim F_1 = 0$, there is nothing to trivialize and the statement is vacuous. If $\dim F_1 = 1$, every admissible gauge trivializes the family by [theorem 4.3](#). By [theorem 4.4](#), changing the gauge is equivalent to composing the trivialization with a ν -preserving moving frame. Applying [theorem 6.4](#) to the resulting rank-one decomposition proves the stated implication. □

Remark 7.2. The importance of [theorem 7.1](#) is not that it solves KLS outright. It does something more precise: it removes curvature completely from the final obstruction. The previous memorandum reduced the problem to low-dimensional connection geometry. The present memorandum compresses that geometry to rank one, where curvature is identically zero. What remains is only the moving-frame distortion problem and the reference-fiber gap.

This leads naturally to the following frontier conjecture.

Conjecture 7.3 (Canonical rank-one moving-frame conjecture). There exists a universal constant L such that for the canonical rank-one transport edge F_1 , one can choose a smooth μ_0 -preserving moving frame S_y on the reference fiber $\nu = \mu_0$ for which the affine-normalized trivialization is L -Lipschitz.

Conjecture 7.4 (Canonical reference-fiber gap). There exists a universal constant C_f such that for the canonical rank-one transport edge F_1 , the isotropization $\tilde{\nu}$ of the reference fiber $\nu = \mu_0$ satisfies

$$C_P(\tilde{\nu}) \leq C_f.$$

Corollary 7.5 (If the two rank-one conjectures hold, then KLS holds). [theorems 7.3 and 7.4](#) together imply the Kannan–Lovász–Simonovits conjecture.

Proof. Immediate from [theorem 7.1](#). □

8 Translational terminal models collapse under genuine fiber curvature

The rank-one reduction leaves open a broad moving-frame problem. The next natural step is to close exact terminal subclasses. The simplest such class is the translational one: the reference fiber does not deform internally, but only drifts along the edge variable. Pure translation alone is not rigid under bare log-concavity; one needs a genuine curved-fiber hypothesis. Under that hypothesis, however, the class closes completely.

We begin with the structural assumption that will drive the proof.

Definition 8.1. A C^2 convex potential $W : E \rightarrow \mathbb{R}$ is called *Legendre-type* if

$$\nabla W(E) = E.$$

Every C^2 uniformly convex potential, i.e. every W satisfying

$$\nabla^2 W \succeq t \text{Id} \quad \text{for some } t > 0,$$

is Legendre-type.

Proof of the final sentence. Assume $\nabla^2 W \succeq t \text{Id}$. Fix $p \in E$ and consider

$$F_p(z) := W(z) - \langle p, z \rangle.$$

The lower Hessian bound implies

$$F_p(z) \geq F_p(0) + \langle \nabla W(0) - p, z \rangle + \frac{t}{2}|z|^2,$$

so $F_p(z) \rightarrow +\infty$ as $|z| \rightarrow \infty$. Hence F_p attains a minimum at some point z_p , and at that point

$$\nabla W(z_p) = p.$$

Thus $\nabla W(E) = E$. □

We may now eliminate exact translational bundles in this curved setting.

Theorem 8.2 (Translational rigidity under a Legendre fiber). *Let E be a Euclidean space, let $I \subset \mathbb{R}$ be an interval, and let*

$$\nu(dz) = e^{-W(z)} dz, \quad m(dy) = e^{-\phi(y)} dy$$

be log-concave probability measures, where $W \in C^2(E)$ is convex and Legendre-type and $\phi \in C^2(I)$ is convex. Let $b \in C^2(I; E)$, and define the triangular map

$$T(z, y) = (z + b(y), y).$$

Assume that

$$\mu := T_{\#}(\nu \otimes m)$$

is log-concave on $E \times I$. Then b is affine on I . Equivalently, there exist $u, v \in E$ such that

$$b(y) = uy + v,$$

hence

$$T(z, y) = (z + uy + v, y)$$

is an affine map and μ is an affine image of the product $\nu \otimes m$.

Proof. The density of μ is

$$(x, y) \mapsto \exp(-V(x, y)), \quad V(x, y) := W(x - b(y)) + \phi(y).$$

Since μ is log-concave, V is convex on $E \times I$.

Write

$$z = x - b(y), \quad H(z) := \nabla^2 W(z).$$

A direct differentiation gives

$$\nabla_{xx}^2 V(x, y) = H(z), \quad \nabla_{xy}^2 V(x, y) = -H(z)b'(y),$$

and

$$\partial_{yy}^2 V(x, y) = b'(y)^T H(z) b'(y) - \langle \nabla W(z), b''(y) \rangle + \phi''(y).$$

Evaluate the Hessian quadratic form of V on the vector $(b'(y), 1) \in E \times \mathbb{R}$. The $H(z)$ -terms cancel exactly, and convexity yields

$$0 \leq \langle H(z)b'(y), b'(y) \rangle - 2 \langle H(z)b'(y), b'(y) \rangle + b'(y)^T H(z) b'(y) - \langle \nabla W(z), b''(y) \rangle + \phi''(y),$$

hence

$$-\langle \nabla W(z), b''(y) \rangle + \phi''(y) \geq 0 \quad (z \in E, y \in I).$$

Because W is Legendre-type, $\nabla W(E) = E$. Therefore for every $p \in E$,

$$-\langle p, b''(y) \rangle + \phi''(y) \geq 0.$$

Apply this with $p = t b''(y)$ for arbitrary $t > 0$. If $b''(y) \neq 0$, then

$$-t|b''(y)|^2 + \phi''(y) \geq 0$$

for all $t > 0$, an impossibility. Therefore $b''(y) = 0$ for every $y \in I$, so b is affine. $\square$

Corollary 8.3 (Uniformly convex fibers force translational collapse). *Under the hypotheses of [theorem 8.2](#), it is enough to assume that*

$$\nabla^2 W \succeq t \text{Id} \quad \text{for some } t > 0.$$

In that case every exact translational bundle is an affine shear of a product.

Proof. Uniform convexity implies the Legendre property by the preceding definition-and-proof block, so [theorem 8.2](#) applies. $\square$

Remark 8.4 (Why some curvature assumption is necessary). The conclusion of [theorem 8.2](#) fails under bare log-concavity. In one dimension,

$$V(x, y) = |x - y^2| + y^2 = \max\{x, -x + 2y^2\}$$

is convex on $\mathbb{R}^2$. Thus the density

$$e^{-V(x,y)} = e^{-y^2} e^{-|x-y^2|}$$

is log-concave, and its fibers are exact translates of the one-dimensional Laplace density, but the drift $b(y) = y^2$ is not affine. The Legendre/curvature hypothesis in [theorem 8.2](#) is therefore genuine.

The theorem gives a terminal-model elimination statement in the curved setting relevant to the soft-edge program.

Corollary 8.5 (Curved translation bundles are not terminal obstructions). *Let $n \geq 2$. Assume KLS is known in all dimensions at most $n - 1$. Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. Suppose that for the canonical rank-one edge F_1 the corresponding triangular trivialization can be chosen in exact translational form*

$$T(z, y) = (z + b(y), y),$$

with $b \in C^2$, and that the reference fiber potential is uniformly convex after isotropization. Then μ satisfies KLS.

Consequently, a minimal-dimensional counterexample to KLS cannot have an exact translational canonical edge with uniformly convex reference fiber.

Proof. By the isotropized uniform convexity hypothesis and [theorem 8.3](#), the canonical translation bundle collapses to an affine image of $\nu \otimes m$, where ν is the reference fiber and m is the one-dimensional edge marginal. By the lower-dimensional hypothesis, ν satisfies KLS; by the one-dimensional log-concave theory, so does m . Hence the product $\nu \otimes m$ satisfies KLS, and therefore its affine image μ satisfies KLS by affine invariance. $\square$

Remark 8.6. [theorem 8.5](#) is exactly the kind of theorem suggested by the fixed-dimensional Wasserstein strategy: one should not stop after locating the final edge, but continue until natural exact terminal models are actually closed. The translational family is the first such curved class. It does not solve KLS, but it removes one honest terminal model from the remaining rank-one frontier.

9 Robust semiconvex exclusion for translational terminal models

The exact translational rigidity theorem still leaves open the possibility that a nonlinear translational drift might survive after adding an auxiliary interaction remainder. The next result shows that this cannot happen as long as the remainder has bounded curvature. In the terminology suggested by the critical-linearization strategy, bounded-curvature remainders are already subcritical.

We first formulate the sharp directional hypothesis.

Definition 9.1. Let $b \in C^2(I; E)$. A C^2 function $R : E \times I \rightarrow \mathbb{R}$ is said to have *bounded cancellation-direction curvature relative to b* if

$$M_b(R) := \sup_{(x,y) \in E \times I} \left| \left\langle \nabla^2 R(x, y) (b'(y), 1), (b'(y), 1) \right\rangle \right| < \infty.$$

Theorem 9.2 (Robust translational rigidity under bounded cancellation curvature). *Let E be a Euclidean space, let $I \subset \mathbb{R}$ be an interval, let $W \in C^2(E)$ be convex and Legendre-type, let $\phi \in C^2(I)$ be convex, let $b \in C^2(I; E)$, and let $R \in C^2(E \times I)$. Define*

$$V(x, y) := W(x - b(y)) + R(x, y) + \phi(y).$$

Assume that V is convex on $E \times I$ and that $M_b(R) < \infty$. Then b is affine on I .

Equivalently, if the fibers are exact translates of a fixed Legendre profile and the residual interaction has bounded second variation along the cancellation direction $(b'(y), 1)$, then global log-concavity forces the drift to be affine.

Proof. Write $z = x - b(y)$ and $H(z) = \nabla^2 W(z)$. As in the proof of [theorem 8.2](#), for the translational part

$$V_0(x, y) := W(x - b(y)) + \phi(y)$$

we have

$$\nabla_{xx}^2 V_0 = H(z), \quad \nabla_{xy}^2 V_0 = -H(z)b'(y),$$

and

$$\partial_{yy}^2 V_0 = b'(y)^T H(z) b'(y) - \langle \nabla W(z), b''(y) \rangle + \phi''(y).$$

Hence, for the vector $q(y) := (b'(y), 1) \in E \times \mathbb{R}$,

$$q(y)^T \nabla^2 V_0(x, y) q(y) = -\langle \nabla W(z), b''(y) \rangle + \phi''(y).$$

Since $V = V_0 + R$ is convex,

$$0 \leq q(y)^T \nabla^2 V(x, y) q(y) = -\langle \nabla W(z), b''(y) \rangle + \phi''(y) + q(y)^T \nabla^2 R(x, y) q(y).$$

Therefore

$$-\langle \nabla W(z), b''(y) \rangle + \phi''(y) + M_b(R) \geq 0 \quad (x \in E, y \in I).$$

Because W is Legendre-type, $\nabla W(E) = E$. Thus for every $p \in E$,

$$-\langle p, b''(y) \rangle + \phi''(y) + M_b(R) \geq 0.$$

If $b''(y) \neq 0$, apply this with $p = t b''(y)$ and let $t \rightarrow +\infty$. One obtains

$$-t|b''(y)|^2 + \phi''(y) + M_b(R) \geq 0 \quad \forall t > 0,$$

which is impossible. Hence $b''(y) = 0$ for every $y \in I$, so b is affine. $\square$

Corollary 9.3 (Operator-norm bounded remainders are already harmless). *Under the hypotheses of [theorem 9.2](#), it is enough to assume*

$$\operatorname{ess\,sup}_{(x,y) \in E \times I} \left\| \nabla^2 R(x, y) \right\|_{\operatorname{op}} \leq M < \infty.$$

Then b is affine on I .

Proof. For $q(y) = (b'(y), 1)$,

$$\left| q(y)^T \nabla^2 R(x, y) q(y) \right| \leq M (|b'(y)|^2 + 1).$$

Hence $M_b(R) < \infty$, and [theorem 9.2](#) applies. $\square$

Remark 9.4 (Semiconvex exclusion principle). [theorem 9.2](#) says that nonlinear translational drift cannot be sustained by any bounded-curvature interaction remainder. Thus, once the reference fiber is genuinely curved, every translational terminal model with semiconvex or semiconcave residual interaction already collapses to an affine shear of a product. Any surviving rank-one obstruction must therefore go beyond the semiconvex regime: either the fibers deform internally, or the interaction curvature becomes unbounded along the edge.

Corollary 9.5 (No semiconvex translational terminal obstruction). *Let $n \geq 2$. Assume KLS is known in all dimensions at most $n - 1$. Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$, and suppose that on the canonical rank-one edge one has a representation*

$$e^{-V(x,y)} = e^{-W(x-b(y))} e^{-R(x,y)} e^{-\phi(y)}$$

with the following properties:

- (i) $W \in C^2(E)$ is uniformly convex on the reference fiber after isotropization;
- (ii) $b \in C^2(I; E)$;
- (iii) $R \in C^2(E \times I)$ has bounded Hessian in operator norm.

Then μ satisfies KLS. Consequently, a minimal-dimensional counterexample cannot belong to this semiconvex translational class.

Proof. Uniform convexity implies that W is Legendre-type. By [theorem 9.3](#), the drift b is affine. Thus the canonical edge model is an affine image of a product of the reference fiber and the one-dimensional edge marginal. Apply the same lower-dimensional induction as in [theorem 8.5](#). $\square$

Remark 9.6. The new point is not merely that the exact translational class collapses. The theorem excludes an entire neighborhood of that class. In the Wasserstein terminology, bounded-curvature residual families are already eliminable terminal models. The remaining KLS frontier is therefore sharper than before: it must involve either non-translational internal shape motion or genuinely large interaction curvature.

10 Why this is stronger than the previous frontier

The positive content of the present note is not merely that it proposes another conjecture. It also closes one exact curved terminal class: pure translation bundles with uniformly convex reference fiber collapse to affine shears of products by [theorem 8.2](#). It also identifies why the previous frontier was not yet sharp enough.

(1) The old obstruction was gauge specific

The previous obstruction $S_K \gtrsim K$ concerned the Poisson gauge. By [theorem 3.6](#), that is the minimal kinetic gauge, i.e. the minimizer of

$$\int |W|^2 d\mu_y$$

inside the admissible gauge class. But the exact two-scale reduction is gauge invariant. Therefore the kinetic minimizer need not be the metric minimizer, nor the affine-normalized minimizer. This is the first point at which the program decisively moves away from the earlier no-go theorem.

(2) Curvature is no longer the final obstruction

In the previous note, curvature / holonomy of the Poisson connection was identified as the first gauge-invariant geometric quantity that survives affine shears. That was already a real advance. The present note shows that after canonical transport trimming one may go further still: because $\dim F_1 \leq 1$, curvature vanishes automatically on the final edge. Thus curvature cannot be the last missing theorem.

(3) Rank-one gauge freedom is explicit

In rank one, gauge freedom is no longer an abstract functional-analytic ambiguity. By [theorem 4.4](#), it is exactly a moving-frame freedom inside $\text{Diff}_{\mu_0}(E)$, the group of μ_0 -preserving diffeomorphisms of the reference fiber. This turns the search for the right gauge into a concrete geometric optimization problem.

(4) Affine normalization is unavoidable

Any raw bi-Lipschitz criterion is too coordinate sensitive: pure affine triangular deformations can make the unnormalized trivialization large even when KLS is trivially true by affine invariance. The affine-normalized closure theorem [theorem 6.3](#) isolates the correct form of the metric question. The remaining task is therefore not to bound a specific W , nor a specific Q , but to bound the *best affine-normalized moving-frame distortion*.

The translational and semiconvex exclusion results still leave open a larger exact terminal class, namely rank-one affine bundles

$$T_y(z) = A(y)z + b(y).$$

In general this class contains genuine gauge redundancy: if S_y preserves the reference measure ν , then T_y and $T_y \circ S_y$ define the same family of fibers up to moving frame. The Gaussian reference fiber is the cleanest place where one can separate this gauge freedom from true covariance motion. The next section closes that exact terminal class on the full-line base.

11 Gaussian exact-affine terminal models

In this section the reference fiber is the standard Gaussian γ_E on a Euclidean space $E \simeq \mathbb{R}^m$. We consider affine rank-one bundles

$$T_y(z) = A(y)z + b(y), \quad A(y) \in GL(E),$$

and ask how much y -dependence is compatible with global log-concavity of the pushforward measure. The first theorem shows that the covariance curve is forced to be matrix-concave.

Theorem 11.1 (Gaussian exact-affine covariance concavity). *Let $I \subset \mathbb{R}$ be an interval, let $m(dy) = e^{-\phi(y)} dy$ be a log-concave probability measure on I with $\phi \in C^2(I)$, let*

$$A : I \rightarrow GL(E), \quad b : I \rightarrow E$$

be C^2 , and define

$$T(z, y) = (A(y)z + b(y), y).$$

Assume that

$$\mu := T_{\#}(\gamma_E \otimes m)$$

is log-concave on $E \times I$. Set

$$\Sigma(y) := A(y)A(y)^T, \quad M(y) := \Sigma(y)^{-1}.$$

Then

$$M''(y) \succeq 2 M'(y)M(y)^{-1}M'(y) \quad (y \in I),$$

equivalently,

$$\Sigma''(y) \preceq 0 \quad (y \in I).$$

Thus the covariance curve of the Gaussian fibers is matrix-concave.

Proof. The conditional law of x given y is Gaussian with mean $b(y)$ and covariance $\Sigma(y)$. Hence, up to an irrelevant additive constant, the global potential is

$$V(x, y) = \frac{1}{2}(x - b(y))^T M(y)(x - b(y)) + \frac{1}{2} \log \det \Sigma(y) + \phi(y).$$

Write

$$u := x - b(y).$$

Then

$$\nabla_{xx}^2 V(x, y) = M(y), \quad \nabla_{xy}^2 V(x, y) = M'(y)u - M(y)b'(y).$$

A further differentiation gives

$$\begin{aligned} \partial_{yy}^2 V(x, y) &= \frac{1}{2} u^T M''(y)u - 2u^T M'(y)b'(y) - u^T M(y)b''(y) \\ &\quad + b'(y)^T M(y)b'(y) + \frac{1}{2}(\log \det \Sigma(y))'' + \phi''(y). \end{aligned}$$

Because V is convex and $M(y)$ is positive definite, the Schur-complement inequality gives

$$\partial_{yy}^2 V(x, y) \geq (M'(y)u - M(y)b'(y))^T M(y)^{-1} (M'(y)u - M(y)b'(y)).$$

Expand the right-hand side:

$$(M'u - Mb')^T M^{-1} (M'u - Mb') = u^T M' M^{-1} M' u - 2u^T M' b' + b'^T M b'.$$

Subtract the common linear and constant terms from both sides. One obtains

$$\frac{1}{2} u^T M''(y) u - u^T M(y) b''(y) + C(y) \geq u^T M'(y) M(y)^{-1} M'(y) u,$$

for a scalar function $C(y)$ independent of u . Now replace u by tv and divide by t^2 . Letting $t \rightarrow \infty$ eliminates the lower-order terms and gives

$$\frac{1}{2} v^T M''(y) v \geq v^T M'(y) M(y)^{-1} M'(y) v \quad (v \in E).$$

Since this holds for every v , it is exactly

$$M''(y) \succeq 2 M'(y) M(y)^{-1} M'(y).$$

Finally, differentiating the identity $M(y)\Sigma(y) = \text{Id}$ twice gives

$$\Sigma''(y) = 2 \Sigma(y) M'(y) \Sigma(y) M'(y) \Sigma(y) - \Sigma(y) M''(y) \Sigma(y).$$

The previously proved inequality implies

$$\Sigma''(y) \preceq 0.$$

□

Corollary 11.2 (Full-line rigidity of the Gaussian affine covariance). *Under the assumptions of [theorem 11.1](#), if $I = \mathbb{R}$, then $\Sigma(y)$ is constant on $\mathbb{R}$.*

Proof. For any fixed $v \in E$, the scalar function

$$g_v(y) := v^T \Sigma(y) v$$

is positive and concave on $\mathbb{R}$ because $\Sigma''(y) \preceq 0$. A positive concave function on the whole line is constant: otherwise its derivative would be strictly positive somewhere or strictly negative somewhere, and monotonicity of the derivative would then force g_v to become negative at one end of the line. Thus g_v is constant for every v , hence $\Sigma(y)$ is constant. □

The remaining freedom when Σ is constant is orthogonal gauge. Indeed, if $\Sigma(y) \equiv \Sigma_0$, then

$$A(y) = \Sigma_0^{1/2} O(y)$$

for a smooth orthogonal path $O(y) \in O(E)$. Since $O(y)$ preserves the standard Gaussian, this y -dependence is precisely gauge and may be removed.

Theorem 11.3 (Gaussian exact-affine full-line elimination modulo gauge). *Assume the hypotheses of [theorem 11.1](#) with $I = \mathbb{R}$. Then, after composing with a smooth Gaussian-preserving moving frame, the trivialization has constant linear part:*

$$T_y(z) = \Sigma_0^{1/2} z + \tilde{b}(y).$$

Moreover, $\tilde{b}(y)$ is affine. Consequently μ is an affine image of the product $\gamma_E \otimes m$, and hence obeys the KLS bound with a universal constant.

Proof. By [theorem 11.2](#), $\Sigma(y) \equiv \Sigma_0$. Choose the polar representation

$$A(y) = \Sigma_0^{1/2} O(y), \quad O(y) \in O(E).$$

Since $O(y)$ preserves γ_E , it defines a Gaussian-preserving moving frame. Replacing T_y by

$$\tilde{T}_y(z) := T_y(O(y)^T z)$$

does not change the fiber laws, while

$$\tilde{T}_y(z) = \Sigma_0^{1/2} z + b(y).$$

Hence it suffices to study the constant-linear-part case.

Let

$$L := \Sigma_0^{1/2}.$$

Then the density of μ is

$$(x, y) \mapsto \exp\left(-\frac{1}{2} \left\| L^{-1}(x - b(y)) \right\|^2 - \phi(y)\right)$$

up to normalization constants. After the linear change of variables $x = L\tilde{x}$, the potential becomes

$$\tilde{V}(\tilde{x}, y) = \frac{1}{2} \left\| \tilde{x} - L^{-1}b(y) \right\|^2 + \phi(y).$$

This is exactly the translational model of [theorem 8.2](#) with

$$W(z) = \frac{1}{2} |z|^2.$$

Hence $L^{-1}b(y)$ is affine, and therefore $b(y)$ is affine. The map $(z, y) \mapsto (Lz + b(y), y)$ is then affine on $E \times \mathbb{R}$, so μ is an affine image of the product $\gamma_E \otimes m$.

Finally, the standard Gaussian satisfies the optimal KLS bound, the KLS ratio is affine invariant, and every one-dimensional log-concave measure satisfies KLS with a universal constant after isotropic normalization. Therefore μ obeys the KLS bound with a universal constant. $\square$

Remark 11.4. [Theorem 11.3](#) closes another natural exact terminal class. It does not yet cover bounded edge intervals, and it does not address non-Gaussian reference fibers. But it shows that even beyond pure translations, an exact affine rank-one bundle with Gaussian reference cannot support a genuine obstruction on the full line. In that regime all covariance motion is either gauge or forbidden.

12 A general tangent inequality for exact affine bundles

The Gaussian section exploited the exact affine structure very strongly. In fact, one basic inequality holds for *every* exact affine rank-one bundle, independently of the special form of the reference fiber. This gives a more central structural tool for future attempts at the general affine terminal class.

Proposition 12.1 (Exact-affine tangent inequality). *Let E be a Euclidean space, let $I \subset \mathbb{R}$ be an interval, let*

$$\nu(\mathrm{d}z) = e^{-W(z)} \mathrm{d}z, \quad m(\mathrm{d}y) = e^{-\phi(y)} \mathrm{d}y$$

be log-concave probability measures with $W \in C^2(E)$ convex and $\phi \in C^2(I)$ convex. Let

$$A : I \rightarrow GL(E), \quad b : I \rightarrow E$$

be C^2 , define

$$T(z, y) = (A(y)z + b(y), y), \quad \mu := T_{\#}(\nu \otimes m),$$

and assume that μ is log-concave on $E \times I$. Set

$$r(y) := \log \det A(y) + \phi(y).$$

Then for every $y \in I$ and every $z \in E$,

$$\left\langle \nabla W(z), A(y)^{-1}(A''(y)z + b''(y)) \right\rangle \leq r''(y).$$

Proof. The density of μ is

$$(x, y) \mapsto \exp(-V(x, y)), \quad V(x, y) = W(A(y)^{-1}(x - b(y))) + r(y).$$

Fix $z \in E$, and consider the C^2 curve

$$\gamma_z(y) := (A(y)z + b(y), y) \in E \times I.$$

By construction,

$$A(y)^{-1}(A(y)z + b(y) - b(y)) = z,$$

hence

$$V(\gamma_z(y)) = W(z) + r(y).$$

Differentiate twice in y . Writing

$$q_z(y) := (A'(y)z + b'(y), 1),$$

we obtain

$$r''(y) = \frac{\mathrm{d}^2}{\mathrm{d}y^2} V(\gamma_z(y)) = q_z(y)^T \nabla^2 V(\gamma_z(y)) q_z(y) + \langle \nabla_x V(\gamma_z(y)), A''(y)z + b''(y) \rangle.$$

Since V is convex, the Hessian term is nonnegative. Therefore

$$\langle \nabla_x V(\gamma_z(y)), A''(y)z + b''(y) \rangle \leq r''(y).$$

Finally,

$$\nabla_x V(x, y) = A(y)^{-T} \nabla W(A(y)^{-1}(x - b(y))),$$

so at the point $\gamma_z(y)$ this becomes

$$\nabla_x V(\gamma_z(y)) = A(y)^{-T} \nabla W(z).$$

Substituting gives

$$\left\langle \nabla W(z), A(y)^{-1}(A''(y)z + b''(y)) \right\rangle \leq r''(y),$$

as claimed. □

13 An optimized tangent inequality for exact-affine bundles

The basic tangent inequality from [theorem 12.1](#) was obtained by testing convexity along the curves $y \mapsto (A(y)z + b(y), y)$ with z fixed. One can gain a new positive term by letting the reference point move affinely in y .

Proposition 13.1 (Optimized exact-affine tangent inequality). *Let E be a Euclidean space, let $I \subset \mathbb{R}$ be an interval, let*

$$\nu(dz) = e^{-W(z)} dz, \quad m(dy) = e^{-\phi(y)} dy$$

be log-concave probability measures with $W \in C^2(E)$ convex and $\phi \in C^2(I)$ convex. Let

$$A : I \rightarrow \text{GL}(E), \quad b : I \rightarrow E$$

be C^2 , define

$$T(z, y) = (A(y)z + b(y), y), \quad \mu := T_{\#}(\nu \otimes m),$$

and assume that μ is log-concave on $E \times I$. Assume in addition that $\nabla^2 W(z)$ is positive-definite for every $z \in E$. Write

$$B(y) := A(y)^{-1}A'(y), \quad M(y) := A(y)^{-1}A''(y), \quad c(y) := A(y)^{-1}b''(y),$$

and

$$r(y) := \log \det A(y) + \phi(y).$$

Then for every $y \in I$ and every $z \in E$,

$$\langle \nabla W(z), M(y)z + c(y) \rangle + \left\langle B(y)^T \nabla W(z), \nabla^2 W(z)^{-1} B(y)^T \nabla W(z) \right\rangle \leq r''(y).$$

Proof. Fix $y_0 \in I$, $z \in E$, and $u \in E$. Define the affine reference path

$$z_u(y) := z + (y - y_0)u$$

and the corresponding curve in the ambient space

$$\gamma_u(y) := (A(y)z_u(y) + b(y), y).$$

By construction,

$$A(y)^{-1}(A(y)z_u(y) + b(y) - b(y_0)) = z_u(y),$$

hence

$$V(\gamma_u(y)) = W(z_u(y)) + r(y),$$

where V is the convex global potential of μ .

Differentiate twice at $y = y_0$. Since $z_u''(y_0) = 0$, we obtain

$$\frac{d^2}{dy^2} W(z_u(y)) \Big|_{y=y_0} = u^T \nabla^2 W(z) u.$$

On the other hand,

$$\gamma_u'(y_0) = (A'(y_0)z + A(y_0)u + b'(y_0), 1),$$

and

$$\frac{d^2}{dy^2} \left(A(y)z_u(y) + b(y) \right) \Big|_{y=y_0} = A''(y_0)z + 2A'(y_0)u + b''(y_0).$$

Therefore

$$u^T \nabla^2 W(z) u + r''(y_0) = \gamma'_u(y_0)^T \nabla^2 V(\gamma_u(y_0)) \gamma'_u(y_0) + \langle \nabla_x V(\gamma_u(y_0)), A''(y_0)z + 2A'(y_0)u + b''(y_0) \rangle.$$

Since V is convex, the Hessian term is nonnegative. Using

$$\nabla_x V(\gamma_u(y_0)) = A(y_0)^{-T} \nabla W(z),$$

we get

$$\langle \nabla W(z), M(y_0)z + 2B(y_0)u + c(y_0) \rangle \leq u^T \nabla^2 W(z) u + r''(y_0).$$

Rearranging,

$$2 \langle B(y_0)^T \nabla W(z), u \rangle - u^T \nabla^2 W(z) u \leq r''(y_0) - \langle \nabla W(z), M(y_0)z + c(y_0) \rangle.$$

Now maximize the left-hand side over $u \in E$. Since $\nabla^2 W(z)$ is positive-definite, the maximizing vector is

$$u_* = \nabla^2 W(z)^{-1} B(y_0)^T \nabla W(z),$$

and the maximal value equals

$$\langle B(y_0)^T \nabla W(z), \nabla^2 W(z)^{-1} B(y_0)^T \nabla W(z) \rangle.$$

Substituting yields the stated inequality. $\square$

Corollary 13.2 (Even-potential cancellation). *Under the hypotheses of [theorem 13.1](#), assume in addition that W is even:*

$$W(-z) = W(z) \quad (z \in E).$$

Then for every $y \in I$ and every $z \in E$,

$$\langle \nabla W(z), M(y)z \rangle + \langle B(y)^T \nabla W(z), \nabla^2 W(z)^{-1} B(y)^T \nabla W(z) \rangle \leq r''(y).$$

Proof. Apply [theorem 13.1](#) at z and at $-z$. Since W is even,

$$\nabla W(-z) = -\nabla W(z), \quad \nabla^2 W(-z) = \nabla^2 W(z).$$

The translation term $\langle \nabla W(z), c(y) \rangle$ cancels after averaging the two inequalities, and the remaining terms coincide. $\square$

Remark 13.3. [Theorem 13.1](#) is strictly stronger than [theorem 12.1](#). The extra positive term is invisible in the earlier fixed- z test, but it records a genuine cost for $A^{-1}A'$ -motion of the fibers. In the Gaussian case it is exactly the Schur-complement term behind covariance concavity.

14 Full-line scalar-dilation collapse for arbitrary Legendre fibers

The radial scalar-dilation theorem from the April memorandum can in fact be proved without radial symmetry. The only geometric input needed is the Legendre property of the reference fiber.

Lemma 14.1 (Legendre growth along large dual vectors). *Let $W \in C^2(E)$ be convex, let $e^{-W(z)} dz$ be a probability measure on E , and assume $\nabla W(E) = E$. After translating the reference variable one may assume $\nabla W(0) = 0$. For any sequence $p_k \in E$ with $|p_k| \rightarrow \infty$, writing*

$$z_k := (\nabla W)^{-1}(p_k),$$

one has

$$|z_k| \rightarrow \infty \quad \text{and} \quad \langle p_k, z_k \rangle \rightarrow +\infty.$$

Proof. Because $e^{-W(z)} dz$ is a probability measure and W is convex, $W(z) \rightarrow +\infty$ as $|z| \rightarrow \infty$. Since ∇W is continuous, the set $\nabla W(K)$ is bounded for every compact $K \subset E$. Therefore $|p_k| \rightarrow \infty$ implies $|z_k| \rightarrow \infty$.

Now use convexity at the point z_k . Since $\nabla W(0) = 0$,

$$W(0) \geq W(z_k) + \langle \nabla W(z_k), -z_k \rangle = W(z_k) - \langle p_k, z_k \rangle.$$

Hence

$$\langle p_k, z_k \rangle \geq W(z_k) - W(0) \xrightarrow[k \rightarrow \infty]{} +\infty.$$

□

Theorem 14.2 (Full-line collapse of exact scalar-dilation bundles). *Let E be a Euclidean space, let*

$$\nu(dz) = e^{-W(z)} dz$$

be a log-concave probability measure on E , where $W \in C^2(E)$ is convex and Legendre-type:

$$\nabla W(E) = E.$$

Let

$$m(dy) = e^{-\phi(y)} dy$$

be a log-concave probability measure on $\mathbb{R}$ with $\phi \in C^2(\mathbb{R})$. Let

$$a : \mathbb{R} \rightarrow (0, \infty), \quad b : \mathbb{R} \rightarrow E$$

be C^2 , and define

$$T(z, y) = (a(y)z + b(y), y), \quad \mu := T_{\#}(\nu \otimes m).$$

Assume that μ is log-concave on $E \times \mathbb{R}$. Then a is constant on $\mathbb{R}$ and b is affine. Consequently μ is an affine image of a product measure.

Proof. Choose a point $z_0 \in E$ with $\nabla W(z_0) = 0$, and replace z by $z - z_0$. This only changes $b(y)$ to $b(y) + a(y)z_0$, so once a is shown to be constant the affinity of the modified b is equivalent to the affinity of the original b . We may therefore assume from the beginning that

$$\nabla W(0) = 0.$$

Set

$$A(y) := a(y)\text{Id}, \quad r(y) := (\dim E) \log a(y) + \phi(y).$$

By [theorem 12.1](#), for every $y \in \mathbb{R}$ and every $z \in E$,

$$\left\langle \nabla W(z), \frac{a''(y)}{a(y)} z + \frac{1}{a(y)} b''(y) \right\rangle \leq r''(y).$$

Fix $y \in \mathbb{R}$. We claim that $a''(y) \leq 0$.

If $\dim E \geq 2$, let

$$c := \frac{1}{a(y)} b''(y) \in E.$$

Assume toward a contradiction that $a''(y) > 0$. Choose any unbounded sequence $p_k \in c^\perp$ and set

$$z_k := (\nabla W)^{-1}(p_k).$$

Since $\langle p_k, c \rangle = 0$, the tangent inequality gives

$$\frac{a''(y)}{a(y)} \langle p_k, z_k \rangle \leq r''(y).$$

But [theorem 14.1](#) implies $\langle p_k, z_k \rangle \rightarrow +\infty$, contradicting $a''(y) > 0$.

If $\dim E = 1$, the tangent inequality becomes

$$W'(z) \left(\frac{a''(y)}{a(y)} z + \frac{1}{a(y)} b''(y) \right) \leq r''(y).$$

Since W' is increasing and onto $\mathbb{R}$, we have $W'(z) \rightarrow +\infty$ as $z \rightarrow +\infty$. If $a''(y) > 0$, then the factor in parentheses tends to $+\infty$ as $z \rightarrow +\infty$, again a contradiction.

Thus $a''(y) \leq 0$ for every $y \in \mathbb{R}$, so a is concave on $\mathbb{R}$. Because $a(y) > 0$ on the whole line, [theorem 18.5](#) below (proved in [section 18](#)) implies that a is constant. Write

$$a(y) \equiv a_0 > 0.$$

Push the constant linear map into the reference measure:

$$\tilde{\nu} := (a_0 \text{Id})_\# \nu.$$

Its potential is

$$\widetilde{W}(x) = W(a_0^{-1}x),$$

hence $\widetilde{W}$ is again Legendre-type. The measure μ may now be written as

$$\mu = \tilde{T}_\#(\tilde{\nu} \otimes m), \quad \tilde{T}(x, y) = (x + b(y), y).$$

By [theorem 8.2](#), b is affine. Therefore the bundle is affine on $E \times \mathbb{R}$, and μ is an affine image of a product measure. $\square$

Corollary 14.3 (Scalar exact-affine terminal models on the full line). *Assume KLS in all dimensions at most $n - 1$. Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. Suppose that on the canonical rank-one edge the corresponding exact-affine triangular trivialization has the form*

$$T(z, y) = (a(y)z + b(y), y)$$

with full-line base $y \in \mathbb{R}$, and that the isotropized reference fiber potential is Legendre-type. Then μ satisfies KLS.

Proof. By [theorem 14.2](#), the canonical edge model collapses to an affine image of $\nu \otimes m$. Apply the lower-dimensional induction exactly as in [theorem 8.5](#). $\square$

15 Full-line fixed-axis collapse for unconditional Legendre fibers

We now treat a nonradial class that still admits a clean opposite-ray test: unconditional fibers with a fixed coordinate axis structure.

Definition 15.1. A C^2 convex potential $W : \mathbb{R}^m \rightarrow \mathbb{R}$ is called *unconditional* if

$$W(\varepsilon_1 z_1, \dots, \varepsilon_m z_m) = W(z_1, \dots, z_m)$$

for every $z \in \mathbb{R}^m$ and every choice of signs $\varepsilon_i \in \{\pm 1\}$.

Lemma 15.2 (Axis derivatives for unconditional potentials). *Let $W : \mathbb{R}^m \rightarrow \mathbb{R}$ be C^2 , convex and unconditional. Fix $i \in \{1, \dots, m\}$ and define*

$$h_i(t) := W(te_i), \quad \omega_i(t) := h'_i(t) = \partial_i W(te_i).$$

Then:

- (i) $\nabla W(te_i) = \omega_i(t)e_i$ for every $t \in \mathbb{R}$;
- (ii) h_i is even, convex and unbounded above on $[0, \infty)$;
- (iii) ω_i is odd, nondecreasing on $\mathbb{R}$, nonnegative on $[0, \infty)$, and

$$t \omega_i(t) \xrightarrow[t \rightarrow +\infty]{} +\infty.$$

Proof. For $j \neq i$, the function

$$s \mapsto W(te_i + se_j)$$

is even in s by unconditional symmetry. Hence its derivative at $s = 0$ vanishes:

$$\partial_j W(te_i) = 0.$$

This proves (i).

The function h_i is even and convex because W is. Since $e^{-W(z)} dz$ is a probability measure, $W(z) \rightarrow +\infty$ as $|z| \rightarrow \infty$; in particular $h_i(t) \rightarrow +\infty$, proving (ii).

Now $\omega_i = h'_i$ is odd and nondecreasing because h_i is even and convex. It is therefore nonnegative on $[0, \infty)$. If ω_i vanished identically on $[0, \infty)$, then h_i would be constant there, contradicting unboundedness. Hence there exists $t_0 > 0$ with $\omega_i(t_0) > 0$. By monotonicity,

$$\omega_i(t) \geq \omega_i(t_0) > 0 \quad (t \geq t_0),$$

and therefore

$$t \omega_i(t) \geq t \omega_i(t_0) \xrightarrow[t \rightarrow +\infty]{} +\infty.$$

This proves (iii). □

Theorem 15.3 (Full-line collapse of exact fixed-axis bundles over an unconditional Legendre fiber). *Let $E = \mathbb{R}^m$ with its standard basis $(e_1, \dots, e_m)$. Let*

$$\nu(dz) = e^{-W(z)} dz$$

be a log-concave probability measure on E , where $W \in C^2(E)$ is convex, unconditional and Legendre-type. Let

$$m(dy) = e^{-\phi(y)} dy$$

be a log-concave probability measure on $\mathbb{R}$ with $\phi \in C^2(\mathbb{R})$. Let

$$A(y) = \text{diag}(a_1(y), \dots, a_m(y)), \quad b : \mathbb{R} \rightarrow E$$

with each $a_i : \mathbb{R} \rightarrow (0, \infty)$ and b of class C^2 . Define

$$T(z, y) = (A(y)z + b(y), y), \quad \mu := T_{\#}(\nu \otimes m),$$

and assume that μ is log-concave on $E \times \mathbb{R}$. Then each a_i is constant on $\mathbb{R}$, and b is affine. Consequently μ is an affine image of a product measure.

Proof. Fix $y \in \mathbb{R}$. Set

$$c(y) := A(y)^{-1}b''(y) \in E, \quad r(y) := \log \det A(y) + \phi(y).$$

By [theorem 12.1](#),

$$\left\langle \nabla W(z), A(y)^{-1}(A''(y)z + b''(y)) \right\rangle \leq r''(y) \quad (z \in E).$$

Fix $i \in \{1, \dots, m\}$, and evaluate this inequality at $z = \pm te_i$. By [theorem 15.2](#),

$$\nabla W(te_i) = \omega_i(t)e_i, \quad \nabla W(-te_i) = -\omega_i(t)e_i,$$

with $\omega_i(t) \geq 0$ and $t\omega_i(t) \rightarrow +\infty$. Since A is diagonal,

$$A(y)^{-1}A''(y)(\pm te_i) = \pm t \frac{a_i''(y)}{a_i(y)} e_i.$$

Hence

$$\omega_i(t) \left(t \frac{a_i''(y)}{a_i(y)} + \langle e_i, c(y) \rangle \right) \leq r''(y),$$

and

$$\omega_i(t) \left(t \frac{a_i''(y)}{a_i(y)} - \langle e_i, c(y) \rangle \right) \leq r''(y).$$

Adding the two inequalities gives

$$2t\omega_i(t) \frac{a_i''(y)}{a_i(y)} \leq 2r''(y).$$

Let $t \rightarrow +\infty$. Since $t\omega_i(t) \rightarrow +\infty$, we conclude that

$$a_i''(y) \leq 0.$$

As y was arbitrary, each a_i is concave on $\mathbb{R}$. Because each a_i is positive on the whole line, [theorem 18.5](#) implies that every a_i is constant:

$$a_i(y) \equiv a_i^0 > 0.$$

Thus

$$A(y) \equiv A_0 := \text{diag}(a_1^0, \dots, a_m^0).$$

Push the constant linear map into the reference measure:

$$\tilde{\nu} := (A_0)_\# \nu.$$

Its potential is

$$\widetilde{W}(x) = W(A_0^{-1}x),$$

which is Legendre-type because

$$\nabla \widetilde{W}(x) = A_0^{-T} \nabla W(A_0^{-1}x)$$

has full range E . The measure μ may therefore be rewritten as

$$\mu = \tilde{T}_\#(\tilde{\nu} \otimes m), \quad \tilde{T}(x, y) = (x + b(y), y).$$

By [theorem 8.2](#), b is affine. □

Corollary 15.4 (Unconditional fixed-axis exact-affine terminal models). *Assume KLS in all dimensions at most $n - 1$. Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. Suppose that on the canonical rank-one edge the exact-affine triangular trivialization has full-line base and diagonal linear part in an isotropized unconditional Legendre reference basis:*

$$T(z, y) = (\text{diag}(a_1(y), \dots, a_m(y)) z + b(y), y).$$

Then μ satisfies KLS.

Proof. Apply [theorem 15.3](#) and then the same lower-dimensional induction as in [theorem 8.5](#). □

Remark 15.5. [Theorem 15.3](#) strictly extends the earlier radial fixed-axis theorem. Radial fibers are unconditional in every orthonormal basis, but the new theorem also applies to genuinely anisotropic unconditional fibers.

16 Self-adjoint positive-definite bundles over even Legendre fibers

The previous version proved full-line collapse over an even Legendre fiber only under a pairwise-commutativity assumption on the self-adjoint positive-definite linear part. The key new observation is that evenness alone rules out *any* positive real eigenmode of $A(y)^{-1}A''(y)$. In the self-adjoint positive-definite branch this is already enough, because $A(y)^{-1}A''(y)$ is automatically self-adjoint in the $A(y)$ -metric and therefore has only real spectrum.

Lemma 16.1 (Convex probability potentials diverge on rays). *Let E be a Euclidean space and let*

$$\nu(\mathrm{d}z) = e^{-W(z)} \mathrm{d}z$$

be a log-concave probability measure on E with $W \in C^2(E)$ convex. Then for every nonzero $u \in E$,

$$W(tu) \xrightarrow{t \rightarrow +\infty} +\infty.$$

Proof. Fix $u \neq 0$. Suppose for contradiction that $W(tu) \not\rightarrow +\infty$ as $t \rightarrow +\infty$. Then there exist $C \in \mathbb{R}$ and a sequence $t_k \rightarrow +\infty$ such that

$$W(t_k u) \leq C \quad (k \geq 1).$$

Set

$$K_C := \{z \in E : W(z) \leq C\}.$$

Because W is convex, K_C is convex; because W is continuous, K_C is closed. Since $0 \in K_C$ after increasing C if necessary, and $t_k u \in K_C$ for every k , convexity yields

$$[0, t_k u] \subset K_C \quad (k \geq 1).$$

Hence $\mathbb{R}_+ u \subset K_C$, so K_C is unbounded.

On the other hand,

$$1 = \int_E e^{-W(z)} \, dz \geq e^{-C} |K_C|,$$

so K_C has finite Lebesgue measure. An unbounded convex subset of a finite-dimensional Euclidean space cannot have finite Lebesgue measure. This contradiction proves the claim. $\square$

Lemma 16.2 (Even one-dimensional growth test). *Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be C^1 , even, convex and unbounded above on $[0, \infty)$. Then h' is odd and nondecreasing, and*

$$t h'(t) \xrightarrow[t \rightarrow +\infty]{} +\infty.$$

Proof. Because h is even, h' is odd. Because h is convex, h' is nondecreasing. In particular $h'(t) \geq 0$ for $t \geq 0$. Since h is unbounded above on $[0, \infty)$, h' cannot vanish identically there. Hence there exists $t_0 > 0$ with $h'(t_0) > 0$. By monotonicity,

$$h'(t) \geq h'(t_0) > 0 \quad (t \geq t_0),$$

and therefore

$$t h'(t) \geq t h'(t_0) \xrightarrow[t \rightarrow +\infty]{} +\infty.$$

$\square$

Proposition 16.3 (Even-fiber exclusion of positive real eigenmodes). *Let E be a Euclidean space, let $I \subset \mathbb{R}$ be an interval, and let*

$$\nu(\,dz) = e^{-W(z)} \, dz$$

be a log-concave probability measure on E , where $W \in C^2(E)$ is convex and even:

$$W(-z) = W(z) \quad (z \in E).$$

Let

$$m(\,dy) = e^{-\phi(y)} \, dy$$

be a log-concave probability measure on I with $\phi \in C^2(I)$. Let

$$A : I \rightarrow \text{GL}(E), \quad b : I \rightarrow E$$

be C^2 , define

$$T(z, y) = (A(y)z + b(y), y), \quad \mu := T_{\#}(\nu \otimes m),$$

and assume that μ is log-concave on $E \times I$. Fix $y \in I$ and set

$$M(y) := A(y)^{-1}A''(y), \quad c(y) := A(y)^{-1}b''(y), \quad r(y) := \log \det A(y) + \phi(y).$$

Then every real eigenvalue of $M(y)$ is nonpositive.

Proof. By [theorem 12.1](#), for every $z \in E$,

$$\langle \nabla W(z), M(y)z + c(y) \rangle \leq r''(y).$$

Apply the same inequality at $-z$. Since W is even,

$$\nabla W(-z) = -\nabla W(z),$$

and therefore

$$\langle \nabla W(z), M(y)z - c(y) \rangle \leq r''(y).$$

Averaging the two inequalities yields

$$\langle \nabla W(z), M(y)z \rangle \leq r''(y) \quad (z \in E).$$

Let $\lambda \in \mathbb{R}$ be a real eigenvalue of $M(y)$, and choose a nonzero real eigenvector $u \in E$ with

$$M(y)u = \lambda u.$$

Define

$$h_u(t) := W(tu) \quad (t \in \mathbb{R}).$$

By evenness of W , the function h_u is even; by convexity of W , it is convex; by [theorem 16.1](#), it is unbounded above on $[0, \infty)$. Hence [theorem 16.2](#) applies. Evaluate the averaged inequality at $z = tu$, $t > 0$:

$$\lambda t \langle \nabla W(tu), u \rangle \leq r''(y).$$

Since

$$\langle \nabla W(tu), u \rangle = h'_u(t),$$

this becomes

$$\lambda t h'_u(t) \leq r''(y).$$

Letting $t \rightarrow +\infty$ and using [theorem 16.2](#), we conclude that $\lambda \leq 0$. □

Lemma 16.4 (*A-metric spectral decomposition*). *Let $A \in \text{Sym}^+(E)$, let $S \in \text{Sym}(E)$, and set*

$$M := A^{-1}S.$$

Equip E with the inner product

$$\langle u, v \rangle_A := \langle Au, v \rangle.$$

Then M is self-adjoint for $\langle \cdot, \cdot \rangle_A$. Consequently there exist an A -orthonormal basis $u_1, \dots, u_m$ of E and real numbers $\lambda_1, \dots, \lambda_m$ such that

$$Mu_i = \lambda_i u_i \quad (i = 1, \dots, m).$$

Moreover, if $x = \sum_{i=1}^m \alpha_i u_i$, then

$$\langle Sx, x \rangle = \sum_{i=1}^m \lambda_i \alpha_i^2.$$

Proof. For any $u, v \in E$,

$$\langle Mu, v \rangle_A = \langle AMu, v \rangle = \langle Su, v \rangle = \langle u, Sv \rangle = \langle u, AMv \rangle = \langle u, Mv \rangle_A,$$

so M is self-adjoint for the A -inner product. The finite-dimensional spectral theorem therefore yields an A -orthonormal basis $u_1, \dots, u_m$ of eigenvectors with real eigenvalues λ_i .

Now let $x = \sum_i \alpha_i u_i$. Since $S = AM$,

$$\langle Sx, x \rangle = \langle AMx, x \rangle = \langle Mx, x \rangle_A = \sum_{i=1}^m \lambda_i \alpha_i^2,$$

because the basis is A -orthonormal. □

Theorem 16.5 (Even-fiber self-adjoint matrix concavity). *Let E be a Euclidean space, let $I \subset \mathbb{R}$ be an interval, and let*

$$\nu(\mathrm{d}z) = e^{-W(z)} \mathrm{d}z$$

be a log-concave probability measure on E , where $W \in C^2(E)$ is convex and even:

$$W(-z) = W(z) \quad (z \in E).$$

Let

$$m(\mathrm{d}y) = e^{-\phi(y)} \mathrm{d}y$$

be a log-concave probability measure on I with $\phi \in C^2(I)$. Let

$$A : I \rightarrow \mathrm{Sym}^+(E), \quad b : I \rightarrow E$$

be C^2 , define

$$T(z, y) = (A(y)z + b(y), y), \quad \mu := T_{\#}(\nu \otimes m),$$

and assume that μ is log-concave on $E \times I$. Then for every $y \in I$,

$$A''(y) \leq 0.$$

Proof. Fix $y \in I$. Set

$$S := A''(y) \in \mathrm{Sym}(E), \quad M := A(y)^{-1} A''(y).$$

By [theorem 16.4](#), there exist an $A(y)$ -orthonormal basis $u_1, \dots, u_m$ and real eigenvalues $\lambda_1, \dots, \lambda_m$ such that

$$Mu_i = \lambda_i u_i.$$

By [theorem 16.3](#), every $\lambda_i \leq 0$.

Now let $x = \sum_i \alpha_i u_i$. Applying [theorem 16.4](#) with $A = A(y)$ and $S = A''(y)$, we obtain

$$\langle A''(y)x, x \rangle = \sum_{i=1}^m \lambda_i \alpha_i^2 \leq 0.$$

Since $x \in E$ was arbitrary, this proves

$$A''(y) \leq 0.$$

□

Lemma 16.6 (Positive concave functions on $\mathbb{R}$ are constant). *Let $f : \mathbb{R} \rightarrow (0, \infty)$ be concave. Then f is constant.*

Proof. Assume f is not constant. Then some one-sided derivative at some point is either positive or negative. By monotonicity of one-sided derivatives for concave functions, a positive derivative at one point forces

$$f(y) \rightarrow -\infty \quad \text{as } y \rightarrow -\infty,$$

while a negative derivative at one point forces

$$f(y) \rightarrow -\infty \quad \text{as } y \rightarrow +\infty.$$

Either alternative contradicts positivity. Hence every one-sided derivative vanishes and f is constant. $\square$

Theorem 16.7 (Full-line collapse of exact self-adjoint positive-definite bundles). *Under the hypotheses of [theorem 16.5](#), assume in addition that $I = \mathbb{R}$ and that W is Legendre-type:*

$$\nabla W(E) = E.$$

Then A is constant on $\mathbb{R}$ and b is affine. Consequently μ is an affine image of a product measure.

Proof. By [theorem 16.5](#), the matrix-valued map A is concave in the Loewner order:

$$A''(y) \preceq 0 \quad (y \in \mathbb{R}).$$

Fix $u \in E$. Then

$$f_u(y) := \langle A(y)u, u \rangle$$

is a positive concave function on $\mathbb{R}$, since $A(y) \in \text{Sym}^+(E)$. By [theorem 16.6](#), f_u is constant. As this holds for every $u \in E$, the quadratic form of $A(y)$ is independent of y , hence

$$A(y) \equiv A_0 \in \text{Sym}^+(E).$$

Set

$$c(y) := A_0^{-1}b''(y).$$

With A constant, [theorem 12.1](#) reduces to

$$\langle \nabla W(z), c(y) \rangle \leq \phi''(y) \quad (z \in E).$$

Assume $c(y) \neq 0$. Since $\nabla W(E) = E$, for every $k \geq 1$ there exists $z_k \in E$ such that

$$\nabla W(z_k) = k c(y).$$

Substituting into the inequality gives

$$k |c(y)|^2 \leq \phi''(y) \quad (k \geq 1),$$

which is impossible. Hence $c(y) = 0$, i.e.

$$b''(y) = 0.$$

Since y was arbitrary, b is affine. $\square$

Corollary 16.8 (Fixed positive-cone collapse after a global affine normalization). *Let E be a Euclidean space, let*

$$\nu(\mathrm{d}z) = e^{-W(z)} \mathrm{d}z$$

be a log-concave probability measure on E with $W \in C^2(E)$ convex, even and Legendre-type, and let

$$m(\mathrm{d}y) = e^{-\phi(y)} \mathrm{d}y$$

be log-concave on $\mathbb{R}$ with $\phi \in C^2(\mathbb{R})$. Let

$$A : \mathbb{R} \rightarrow \mathrm{GL}(E), \quad b : \mathbb{R} \rightarrow E$$

be C^2 , define

$$T(z, y) = (A(y)z + b(y), y), \quad \mu := T_{\#}(\nu \otimes m),$$

and assume that μ is log-concave on $E \times \mathbb{R}$. Assume moreover that there exists a fixed $L \in \mathrm{GL}(E)$ such that

$$L^{-1}A(y)L \in \mathrm{Sym}^+(E) \quad (y \in \mathbb{R}).$$

Then A is constant and b is affine. Consequently μ is an affine image of a product measure.

Proof. Set

$$\tilde{W}(\tilde{z}) := W(L\tilde{z}), \quad \tilde{A}(y) := L^{-1}A(y)L, \quad \tilde{b}(y) := L^{-1}b(y).$$

Then $\tilde{W}$ is convex, even and Legendre-type, and

$$\tilde{T}(\tilde{z}, y) := (\tilde{A}(y)\tilde{z} + \tilde{b}(y), y)$$

pushes forward the transformed product measure to an affine image of μ . Since log-concavity and affine-product collapse are preserved under a fixed affine change of coordinates, [theorem 16.7](#) applies to the tilde-system and yields that $\tilde{A}$ is constant and $\tilde{b}$ is affine. Therefore A is constant and b is affine as well. $\square$

Corollary 16.9 (Fixed self-adjoint one-parameter directions over even Legendre fibers collapse). *Assume the hypotheses of [theorem 16.7](#). Suppose in addition that*

$$A(y) = A_0 + s(y)S,$$

where $A_0 \in \mathrm{Sym}^+(E)$, $S \in \mathrm{Sym}(E)$, and $A(y) \in \mathrm{Sym}^+(E)$ for all $y \in \mathbb{R}$. Then either $S = 0$ or s is constant on $\mathbb{R}$.

Proof. By [theorem 16.7](#), the map A is constant. If $S \neq 0$, this forces s to be constant. $\square$

Corollary 16.10 (Exact self-adjoint positive-definite full-line terminal models). *Assume KLS in all dimensions at most $n - 1$. Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. Suppose that on the canonical rank-one edge the exact-affine triangular trivialization has full-line base and, after isotropization of the reference fiber and a fixed affine normalization of the fiber coordinates, the linear part is self-adjoint and positive-definite while the reference fiber potential is even and Legendre-type. Then μ satisfies KLS.*

Proof. Apply [theorem 16.8](#) and then the same lower-dimensional induction as in [theorem 8.5](#). $\square$

Remark 16.11. [Theorem 16.7](#) resolves the positive-part conjecture left open in the previous version of the article. In particular, the earlier pairwise-commuting self-adjoint theorem is now a strict special case, and so is the fixed-axis theorem after a fixed affine normalization of the fiber coordinates. Consequently, over an even Legendre fiber a surviving full-line exact-affine obstruction can no longer live inside any globally fixed positive cone: it must exhibit genuinely nonselfadjoint motion after every fixed affine normalization.

17 Fixed-direction real-spectrum collapse over even Legendre fibers

The even-fiber real-eigenmode exclusion from [theorem 16.3](#) already has a strong consequence for one-parameter motion in a fixed matrix direction. Unlike [theorem 16.9](#), no self-adjointness assumption is needed: a single nonzero real spectral mode suffices.

Theorem 17.1 (Fixed-direction real-spectrum full-line collapse over even Legendre fibers). *Let E be a Euclidean space, let*

$$\nu(\mathrm{d}z) = e^{-W(z)} \mathrm{d}z$$

be a log-concave probability measure on E with $W \in C^2(E)$ convex, even and Legendre-type, and let

$$m(\mathrm{d}y) = e^{-\phi(y)} \mathrm{d}y$$

be log-concave on $\mathbb{R}$ with $\phi \in C^2(\mathbb{R})$. Let

$$A(y) = A_0 + s(y)M, \quad b : \mathbb{R} \rightarrow E$$

where $A_0 \in \mathrm{GL}(E)$, $M \in \mathrm{End}(E)$, $s \in C^2(\mathbb{R})$, $b \in C^2(\mathbb{R}; E)$, and $A(y) \in \mathrm{GL}(E)$ for all $y \in \mathbb{R}$. Define

$$T(z, y) = (A(y)z + b(y), y), \quad \mu := T_{\#}(\nu \otimes m),$$

and assume that μ is log-concave on $E \times \mathbb{R}$. Assume moreover that $A_0^{-1}M$ has a nonzero real eigenvalue. Then A is constant on $\mathbb{R}$ and b is affine. Consequently μ is an affine image of a product measure.

Proof. Replace the reference fiber variable by $\tilde{z} = A_0 z$, and correspondingly replace ν by

$$\tilde{\nu} := (A_0)_{\#}\nu.$$

Then

$$\mu = \tilde{T}_{\#}(\tilde{\nu} \otimes m), \quad \tilde{T}(\tilde{z}, y) = ((I + s(y)N)\tilde{z} + b(y), y), \quad N := MA_0^{-1}.$$

The transformed potential $\tilde{W}(x) = W(A_0^{-1}x)$, up to an additive constant, is again convex, even and Legendre-type because

$$\nabla \tilde{W}(x) = A_0^{-T} \nabla W(A_0^{-1}x), \quad \nabla \tilde{W}(E) = A_0^{-T} \nabla W(E) = E.$$

Moreover N is similar to $A_0^{-1}M$, since

$$A_0^{-1}NA_0 = A_0^{-1}M.$$

Hence N also has a nonzero real eigenvalue. Since affine-product collapse is preserved under this fixed normalization, it suffices to treat the case

$$A_0 = I, \quad A(y) = I + s(y)M.$$

Choose a nonzero real eigenvalue $\lambda \in \mathbb{R} \setminus \{0\}$ of M and a corresponding real eigenvector $u \neq 0$:

$$Mu = \lambda u.$$

Set

$$g(y) := \lambda s(y).$$

Then

$$A(y)u = (1 + g(y))u.$$

Because $A(y)$ is invertible for every $y \in \mathbb{R}$, we have

$$1 + g(y) \neq 0 \quad (y \in \mathbb{R}).$$

Also

$$A''(y) = s''(y)M,$$

so

$$A(y)^{-1}A''(y)u = s''(y)A(y)^{-1}Mu = s''(y)\lambda A(y)^{-1}u = \frac{g''(y)}{1 + g(y)}u.$$

Thus $\frac{g''(y)}{1 + g(y)}$ is a real eigenvalue of $A(y)^{-1}A''(y)$. By [theorem 16.3](#),

$$\frac{g''(y)}{1 + g(y)} \leq 0 \quad (y \in \mathbb{R}).$$

Since $1 + g$ is continuous and never vanishes, its sign is constant on $\mathbb{R}$. If

$$1 + g(y) > 0 \quad (y \in \mathbb{R}),$$

then $g''(y) \leq 0$, so $g + 1$ is a positive concave function on $\mathbb{R}$. By [theorem 16.6](#), $g + 1$ is constant, hence g and therefore s are constant.

If instead

$$1 + g(y) < 0 \quad (y \in \mathbb{R}),$$

then $g''(y) \geq 0$, so $-1 - g$ is a positive concave function on $\mathbb{R}$. Again [theorem 16.6](#) implies that g , hence s , is constant.

Thus $s(y) \equiv s_0$, and

$$A(y) \equiv A_* := I + s_0M.$$

Push this constant linear map into the reference measure:

$$\hat{\nu} := (A_*)_{\#}\tilde{\nu}.$$

Its potential is again Legendre-type, and

$$\mu = \hat{T}_{\#}(\hat{\nu} \otimes m), \quad \hat{T}(x, y) = (x + b(y), y).$$

By [theorem 8.2](#), b is affine. Therefore A is constant and b is affine. □

Corollary 17.2 (Invertible odd-dimensional fixed directions over even fibers collapse). *Under the hypotheses of [theorem 17.1](#), assume in addition that $\dim E$ is odd and that M is invertible. Then A is constant on $\mathbb{R}$ and b is affine.*

Proof. The real matrix $A_0^{-1}M$ has odd-degree characteristic polynomial and nonzero determinant. Hence it possesses a nonzero real eigenvalue. Apply [theorem 17.1](#). $\square$

Corollary 17.3 (Even-fiber fixed-direction terminal models with a real spectral mode). *Assume KLS in all dimensions at most $n - 1$. Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. Suppose that on the canonical rank-one edge the exact-affine triangular trivialization has full-line base and, after isotropization of the reference fiber and a fixed affine normalization of the fiber coordinates, the linear part has the fixed-direction form*

$$A(y) = A_0 + s(y)M$$

with $A_0 \in \mathrm{GL}(E)$, $M \in \mathrm{End}(E)$, and $A_0^{-1}M$ carrying a nonzero real eigenvalue, while the reference fiber potential is even and Legendre-type. Then μ satisfies KLS.

Proof. Apply [theorem 17.1](#) and then the same lower-dimensional induction as in [theorem 8.5](#). $\square$

Remark 17.4. [Theorem 17.1](#) strictly extends [theorem 16.9](#) from the self-adjoint cone to arbitrary fixed directions carrying a real spectral mode. Consequently, a surviving full-line even-fiber fixed-direction obstruction must, after every fixed affine normalization, have direction matrix with no nonzero real eigenvalue. In particular, the unresolved fixed-direction branch is forced into spectrally pure rotational or singular behavior.

Proposition 17.5 (Fixed real-eigenline full-line rigidity). *Let E be a Euclidean space, let*

$$\nu(\mathrm{d}z) = e^{-W(z)} \mathrm{d}z$$

be a log-concave probability measure on E with $W \in C^2(E)$ convex and even, let

$$m(\mathrm{d}y) = e^{-\phi(y)} \mathrm{d}y$$

be log-concave on $\mathbb{R}$ with $\phi \in C^2(\mathbb{R})$, and let

$$A : \mathbb{R} \rightarrow \mathrm{GL}(E), \quad b : \mathbb{R} \rightarrow E$$

be C^2 . Define

$$T(z, y) = (A(y)z + b(y), y), \quad \mu := T_{\#}(\nu \otimes m),$$

and assume that μ is log-concave on $E \times \mathbb{R}$. Assume that there exist a nonzero vector $u \in E$ and a scalar function $q \in C^2(\mathbb{R})$ such that

$$A(y)u = q(y)u \quad (y \in \mathbb{R}).$$

Then q is constant on $\mathbb{R}$.

Proof. Because $A(y)$ is invertible for every y , the scalar $q(y)$ never vanishes. Hence q has constant sign on $\mathbb{R}$.

Differentiating

$$A(y)u = q(y)u$$

twice yields

$$A''(y)u = q''(y)u.$$

Therefore

$$A(y)^{-1}A''(y)u = \frac{q''(y)}{q(y)}u,$$

so the real number $q''(y)/q(y)$ is a real eigenvalue of $A(y)^{-1}A''(y)$. By [theorem 16.3](#),

$$\frac{q''(y)}{q(y)} \leq 0 \quad (y \in \mathbb{R}).$$

If $q > 0$ on $\mathbb{R}$, then $q'' \leq 0$, so q is a positive concave function on $\mathbb{R}$. By [theorem 16.6](#), q is constant. If $q < 0$ on $\mathbb{R}$, then $q'' \geq 0$, so $-q$ is a positive concave function on $\mathbb{R}$; again [theorem 16.6](#) implies that q is constant. $\square$

Theorem 17.6 (One-parameter subgroup full-line collapse with a real spectral mode). *Let E be a Euclidean space, let*

$$\nu(dz) = e^{-W(z)} dz$$

be a log-concave probability measure on E with $W \in C^2(E)$ convex, even and Legendre-type, and let

$$m(dy) = e^{-\phi(y)} dy$$

be log-concave on $\mathbb{R}$ with $\phi \in C^2(\mathbb{R})$. Let

$$A(y) = A_0 e^{s(y)G}, \quad b : \mathbb{R} \rightarrow E,$$

where $A_0 \in \text{GL}(E)$, $G \in \text{End}(E)$, $s \in C^2(\mathbb{R})$, $b \in C^2(\mathbb{R}; E)$. Define

$$T(z, y) = (A(y)z + b(y), y), \quad \mu := T_{\#}(\nu \otimes m),$$

and assume that μ is log-concave on $E \times \mathbb{R}$. Assume moreover that $A_0^{-1}GA_0$ has a nonzero real eigenvalue. Then A is constant on $\mathbb{R}$ and b is affine. Consequently μ is an affine image of a product measure.

Proof. Replace the reference fiber variable by $\tilde{z} = A_0 z$, and correspondingly replace ν by

$$\tilde{\nu} := (A_0)_{\#}\nu.$$

Then

$$\mu = \tilde{T}_{\#}(\tilde{\nu} \otimes m), \quad \tilde{T}(\tilde{z}, y) = (e^{s(y)N}\tilde{z} + b(y), y), \quad N := A_0GA_0^{-1}.$$

The transformed potential $\tilde{W}(x) = W(A_0^{-1}x)$, up to an additive constant, is again convex, even and Legendre-type because

$$\nabla \tilde{W}(x) = A_0^{-T} \nabla W(A_0^{-1}x), \quad \nabla \tilde{W}(E) = A_0^{-T} \nabla W(E) = E.$$

Since N is similar to $A_0^{-1}GA_0$, it also has a nonzero real eigenvalue. Thus it suffices to treat the case

$$A_0 = \text{Id}, \quad A(y) = e^{s(y)G},$$

where G has a nonzero real eigenvalue.

Choose $\lambda \in \mathbb{R} \setminus \{0\}$ and $u \neq 0$ such that

$$Gu = \lambda u.$$

Then for every $y \in \mathbb{R}$,

$$A(y)u = e^{s(y)G}u = e^{\lambda s(y)}u.$$

By [theorem 17.5](#), the scalar profile

$$q(y) := e^{\lambda s(y)}$$

is constant on $\mathbb{R}$. Since $\lambda \neq 0$ and $t \mapsto e^{\lambda t}$ is injective on $\mathbb{R}$, it follows that s is constant. Hence $A(y) \equiv A_*$ is constant on $\mathbb{R}$.

Push this constant linear map into the reference measure:

$$\hat{\nu} := (A_*)_{\#}\nu.$$

Its potential is again Legendre-type, and

$$\mu = \hat{T}_{\#}(\hat{\nu} \otimes m), \quad \hat{T}(x, y) = (x + b(y), y).$$

By [theorem 8.2](#), b is affine. Therefore A is constant and b is affine. $\square$

Corollary 17.7 (Invertible odd-dimensional one-parameter subgroup branches collapse). *Under the hypotheses of [theorem 17.6](#), assume in addition that $\dim E$ is odd and that G is invertible. Then A is constant on $\mathbb{R}$ and b is affine.*

Proof. Any invertible real matrix in odd dimension has a nonzero real eigenvalue. Apply [theorem 17.6](#). $\square$

Corollary 17.8 (Even-fiber one-parameter subgroup terminal models with a real spectral mode). *Assume KLS in all dimensions at most $n - 1$. Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. Suppose that on the canonical rank-one edge the exact-affine triangular trivialization has full-line base and, after isotropization of the reference fiber and a fixed affine normalization of the fiber coordinates, the linear part has the one-parameter subgroup form*

$$A(y) = A_0 e^{s(y)G}$$

with $A_0 \in \mathrm{GL}(E)$, $G \in \mathrm{End}(E)$, and $A_0^{-1}GA_0$ carrying a nonzero real eigenvalue, while the reference fiber potential is even and Legendre-type. Then μ satisfies KLS.

Proof. Apply [theorem 17.6](#) and then the same lower-dimensional induction as in [theorem 8.5](#). $\square$

Remark 17.9. [Theorem 17.5](#) shows that over an even full-line exact-affine bundle, any fixed invariant real line can carry only constant scalar motion. [Theorem 17.6](#) applies this principle to multiplicative one-parameter families and strictly enlarges the eliminated even-fiber terminal classes beyond the affine fixed-direction theorem. Consequently, a surviving full-line even-fiber fixed one-parameter subgroup obstruction must, after every fixed affine normalization, have generator with no nonzero real eigenvalue.

18 Radial exact-affine matrix concavity, orthogonal gauge reduction, and complete full-line collapse

We now integrate the sequel note into the present article. The first step is the matrix inequality already implicit in the April memorandum.

Theorem 18.1 (Radial exact-affine matrix concavity). *Let E be a Euclidean space, let*

$$\nu(\mathrm{d}z) = e^{-w(|z|)} \mathrm{d}z$$

be a radial log-concave probability measure on E , where $w \in C^2([0, \infty))$ is convex, $w'(0) = 0$, and

$$w'(r) \xrightarrow{r \rightarrow \infty} +\infty.$$

Let $I \subset \mathbb{R}$ be an interval, let $m(\mathrm{d}y) = e^{-\phi(y)} \mathrm{d}y$ be a log-concave probability measure on I with $\phi \in C^2(I)$, and let

$$A : I \rightarrow \mathrm{GL}(E), \quad b : I \rightarrow E$$

be C^2 . Define

$$T(z, y) = (A(y)z + b(y), y), \quad \mu := T_{\#}(\nu \otimes m),$$

and assume that μ is log-concave on $E \times I$. Then for every $y \in I$,

$$\mathrm{Sym}(A(y)^{-1}A''(y)) \preceq 0.$$

Proof. Fix $y \in I$ and set

$$M(y) := A(y)^{-1}A''(y), \quad c(y) := A(y)^{-1}b''(y), \quad r(y) := \log \det A(y) + \phi(y).$$

By [theorem 12.1](#),

$$\langle \nabla W(z), M(y)z + c(y) \rangle \leq r''(y), \quad W(z) := w(|z|).$$

Let $u \in E$ be a unit vector and put $z = \pm tu$ with $t > 0$. Since W is radial,

$$\nabla W(tu) = w'(t)u, \quad \nabla W(-tu) = -w'(t)u.$$

Hence

$$w'(t) \left(t \langle u, M(y)u \rangle + \langle u, c(y) \rangle \right) \leq r''(y),$$

and

$$w'(t) \left(t \langle u, M(y)u \rangle - \langle u, c(y) \rangle \right) \leq r''(y).$$

Adding the two inequalities gives

$$2t w'(t) \langle u, \mathrm{Sym}(M(y))u \rangle \leq 2r''(y).$$

Because $t w'(t) \rightarrow +\infty$, letting $t \rightarrow \infty$ yields

$$\langle u, \mathrm{Sym}(M(y))u \rangle \leq 0.$$

As u was arbitrary, $\mathrm{Sym}(M(y)) \preceq 0$. □

Lemma 18.2 (Orthogonal gauge reduction). *Under the hypotheses of [theorem 18.1](#), write the right polar decomposition*

$$A(y) = P(y)U(y),$$

where

$$P(y) := (A(y)A(y)^T)^{1/2} \in \text{Sym}^+(E), \quad U(y) := P(y)^{-1}A(y) \in \text{O}(E).$$

Then P and U are C^2 . Moreover, if

$$\tilde{T}(z, y) := (P(y)z + b(y), y),$$

then

$$T_{\#}(\nu \otimes m) = \tilde{T}_{\#}(\nu \otimes m).$$

Proof. The map $y \mapsto A(y)A(y)^T$ is C^2 into the open cone of symmetric positive-definite matrices. The square-root map on this cone is smooth, hence P is C^2 . Then $U = P^{-1}A$ is also C^2 , and

$$U(y)U(y)^T = P(y)^{-1}A(y)A(y)^TP(y)^{-1} = \text{Id},$$

so $U(y) \in \text{O}(E)$.

Define

$$S(z, y) := (U(y)^T z, y).$$

Because ν is radial, every orthogonal map preserves ν . Therefore $S_{\#}(\nu \otimes m) = \nu \otimes m$. Finally,

$$\tilde{T} \circ S(z, y) = (P(y)U(y)^T z + b(y), y) = (A(y)z + b(y), y) = T(z, y),$$

so

$$T_{\#}(\nu \otimes m) = \tilde{T}_{\#}(S_{\#}(\nu \otimes m)) = \tilde{T}_{\#}(\nu \otimes m).$$

□

Lemma 18.3 (Lyapunov negativity). *Let $H \in \text{Sym}^+(E)$ and $C \in \text{Sym}(E)$. If*

$$HC + CH \preceq 0,$$

then

$$C \preceq 0.$$

Proof. Set

$$D := -(HC + CH) \succeq 0.$$

Consider

$$X := \int_0^\infty e^{-tH} D e^{-tH} dt.$$

Because H is positive-definite, e^{-tH} decays exponentially, so the integral converges in operator norm and $X \succeq 0$.

Now

$$HX + XH = - \int_0^\infty \frac{d}{dt} (e^{-tH} D e^{-tH}) dt = D.$$

Thus X solves the Lyapunov equation

$$HX + XH = D.$$

But $-C$ satisfies the same equation:

$$H(-C) + (-C)H = D.$$

Uniqueness of the solution for positive-definite H yields $X = -C$. Since $X \succeq 0$, we conclude that $C = -X \preceq 0$. $\square$

Theorem 18.4 (Positive-part matrix concavity after orthogonal gauge). *Under the hypotheses of theorem 18.1, let*

$$A(y) = P(y)U(y)$$

be the right polar decomposition. After replacing T by the gauge-equivalent bundle

$$\tilde{T}(z, y) = (P(y)z + b(y), y),$$

the positive part $P(y) \in \text{Sym}^+(E)$ satisfies

$$P''(y) \preceq 0 \quad (y \in I).$$

Proof. By theorem 18.2, we may replace A by P without changing μ . Applying theorem 18.1 to the resulting bundle gives

$$\text{Sym}(P^{-1}P'') \preceq 0.$$

Because P is symmetric positive-definite,

$$P \text{Sym}(P^{-1}P'') P = \frac{1}{2}(PP'' + P''P) \preceq 0.$$

Now apply theorem 18.3 with $H = P$ and $C = P''$. $\square$

Lemma 18.5 (Positive concave functions on $\mathbb{R}$ are constant). *Let $f : \mathbb{R} \rightarrow (0, \infty)$ be concave. Then f is constant.*

Proof. Assume f is not constant. Then some one-sided derivative at some point is either positive or negative. By monotonicity of one-sided derivatives for concave functions, a positive derivative at one point forces $f(y) \rightarrow -\infty$ as $y \rightarrow -\infty$, while a negative derivative at one point forces $f(y) \rightarrow -\infty$ as $y \rightarrow +\infty$. Either alternative contradicts positivity. Hence every one-sided derivative vanishes and f is constant. $\square$

Lemma 18.6 (Radial potentials remain Legendre after a fixed linear map). *Let*

$$W(z) = w(|z|)$$

with $w \in C^2([0, \infty))$ convex, $w'(0) = 0$, and $w'(r) \rightarrow +\infty$ as $r \rightarrow \infty$. Let $L \in \text{GL}(E)$, and define

$$\widetilde{W}(x) := W(L^{-1}x).$$

Then $\widetilde{W}$ is Legendre-type.

Proof. Because $w'(r) \rightarrow +\infty$, the radial potential W is Legendre-type. Indeed, for any $p \in E$ one minimizes $W(z) - \langle p, z \rangle$ along rays and obtains $\nabla W(z) = p$ at a minimizer. Therefore

$$\nabla \widetilde{W}(x) = L^{-T} \nabla W(L^{-1}x)$$

has range $L^{-T}E = E$. $\square$

Theorem 18.7 (Complete full-line collapse of exact-affine radial bundles modulo orthogonal gauge). *Let E be a Euclidean space, let*

$$\nu(\mathrm{d}z) = e^{-w(|z|)} \mathrm{d}z$$

be a radial log-concave probability measure on E , where $w \in C^2([0, \infty))$ is convex, $w'(0) = 0$, and $w'(r) \rightarrow +\infty$ as $r \rightarrow \infty$. Let

$$m(\mathrm{d}y) = e^{-\phi(y)} \mathrm{d}y$$

be a log-concave probability measure on $\mathbb{R}$ with $\phi \in C^2(\mathbb{R})$. Let

$$A : \mathbb{R} \rightarrow \mathrm{GL}(E), \quad b : \mathbb{R} \rightarrow E$$

be C^2 , define

$$T(z, y) = (A(y)z + b(y), y), \quad \mu := T_{\#}(\nu \otimes m),$$

and assume that μ is log-concave on $E \times \mathbb{R}$.

Write the right polar decomposition

$$A(y) = P(y)U(y), \quad P(y) \in \mathrm{Sym}^+(E), \quad U(y) \in \mathrm{O}(E).$$

Then $P(y)$ is constant on $\mathbb{R}$, and $b(y)$ is affine. Equivalently, after removing the smooth orthogonal gauge $U(y)$, the bundle collapses to

$$(z, y) \mapsto (P_0 z + b_0 + y b_1, y)$$

for some constant $P_0 \in \mathrm{Sym}^+(E)$ and vectors $b_0, b_1 \in E$. In particular, μ is an affine image of a product measure.

Proof. By [theorem 18.4](#), after orthogonal gauge reduction the positive part satisfies

$$P''(y) \preceq 0 \quad (y \in \mathbb{R}).$$

Fix a unit vector $u \in E$ and set

$$f_u(y) := \langle P(y)u, u \rangle.$$

Since $P(y)$ is positive-definite, $f_u(y) > 0$ for all y , while

$$f_u''(y) = \langle P''(y)u, u \rangle \leq 0.$$

Hence f_u is concave on $\mathbb{R}$. By [theorem 16.6](#), f_u is constant. Since this holds for every unit vector u , the quadratic form of $P(y)$ is independent of y , hence

$$P(y) \equiv P_0$$

for some $P_0 \in \mathrm{Sym}^+(E)$.

Push the constant linear part into the reference measure:

$$\tilde{\nu} := (P_0)_{\#}\nu.$$

By [theorem 18.6](#), $\tilde{\nu}$ has a Legendre-type potential. The measure μ may now be written as

$$\mu = \tilde{T}_{\#}(\tilde{\nu} \otimes m), \quad \tilde{T}(x, y) = (x + b(y), y).$$

Applying [theorem 8.2](#) yields that b is affine. □

Corollary 18.8 (No full-line exact-affine radial terminal obstruction). *In the transport–soft-edge KLS program, a full-line rank-one exact-affine terminal model with radial reference fiber cannot realize a genuine obstruction. After orthogonal gauge normalization it is merely an affine product.*

Proof. Immediate from [theorem 18.7](#). □

19 Strong corollaries for fixed-direction radial families

The full radial theorem subsumes the previously established radial scalar, fixed-axis, commuting, and nilpotent corollaries. It also yields a stronger fixed-direction statement that was not available in the earlier notes.

Corollary 19.1 (Fixed-direction one-parameter radial families collapse). *Assume the hypotheses of [theorem 18.7](#). Suppose in addition that*

$$A(y) = I + s(y)M,$$

where $M \in \text{End}(E)$ is fixed. Then either $M = 0$ or s is constant on $\mathbb{R}$. In particular, for every nonzero fixed matrix direction M , the family collapses modulo orthogonal gauge to a pure translation bundle with affine translation.

Proof. By [theorem 18.7](#), the singular values of $A(y)$ are independent of y , because the positive part $P(y)$ in the polar decomposition is constant. Hence

$$\text{Tr}(A(y)A(y)^T)$$

is constant in y . But

$$A(y)A(y)^T = I + s(y)(M + M^T) + s(y)^2MM^T,$$

so

$$\text{Tr}(A(y)A(y)^T) = \dim E + 2s(y)\text{Tr}(M) + s(y)^2\text{Tr}(MM^T).$$

If $M \neq 0$, then $\text{Tr}(MM^T) = \|M\|_{\text{HS}}^2 > 0$. Therefore the right-hand side cannot remain constant on an interval unless s is constant. $\square$

Corollary 19.2 (Arbitrary fixed nilpotent directions are excluded). *Under the hypotheses of [theorem 19.1](#), if $M = N \neq 0$ is nilpotent, then s is constant. This strictly subsumes both the earlier step-two nilpotent theorem and the later arbitrary fixed-nilpotent extension.*

Proof. Immediate from [theorem 19.1](#). $\square$

Corollary 19.3 (Eliminated exact-affine full-line terminal classes). *Assume KLS in all dimensions at most $n - 1$. Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. Suppose that on the canonical rank-one edge the exact-affine triangular trivialization has full-line base and belongs, after isotropization of the reference fiber, to one of the following classes:*

- (i) *scalar linear motion over a Legendre-type reference fiber;*
- (ii) *diagonal fixed-axis motion in an unconditional Legendre reference basis;*
- (iii) *self-adjoint positive-definite motion over an even Legendre reference fiber after a fixed affine normalization;*
- (iv) *affine fixed-direction motion over an even Legendre reference fiber after a fixed affine normalization, provided the direction matrix has a nonzero real eigenvalue;*

(v) *one-parameter subgroup motion*

$$A(y) = A_0 e^{s(y)G}$$

over an even Legendre reference fiber after a fixed affine normalization, provided the generator carries a nonzero real eigenvalue;

(vi) *arbitrary matrix motion over a radial reference fiber.*

Then μ satisfies KLS. Consequently, a minimal-dimensional counterexample cannot belong to any of these exact-affine full-line terminal classes.

Proof. Combine [theorems 14.3, 15.4, 16.10, 17.3, 17.8 and 18.8](#) with the lower-dimensional induction and affine invariance exactly as in [theorem 8.5](#). $\square$

20 Exact Schur complement, affine-base gauge collapse, and flat-point rigidity

The optimized tangent inequality from [theorem 13.1](#) can in fact be recovered as the exact Schur-complement condition for convexity of the global potential. This yields two additional consequences that were not extracted in the previous versions: a new averaged motion inequality against the one-dimensional base curvature, and a local flat-point rigidity mechanism.

Proposition 20.1 (Exact Schur-complement formula). *Under the hypotheses of [theorem 13.1](#), define in addition*

$$p(y) := A(y)^{-1}b'(y).$$

Fix $y \in I$, $z \in E$, and set

$$x := A(y)z + b(y), \quad g := \nabla W(z), \quad H := \nabla^2 W(z).$$

Then

$$\begin{aligned} \nabla_{xx}^2 V(x, y) &= A(y)^{-T} H A(y)^{-1}, \\ \nabla_{xy}^2 V(x, y) &= -A(y)^{-T} \left(H(B(y)z + p(y)) + B(y)^T g \right), \end{aligned}$$

and

$$\begin{aligned} \partial_{yy}^2 V(x, y) &= \langle H(B(y)z + p(y)), B(y)z + p(y) \rangle - \langle g, M(y)z + c(y) \rangle \\ &\quad + 2 \langle g, B(y)(B(y)z + p(y)) \rangle + r''(y). \end{aligned}$$

Consequently the Schur complement of $\nabla_{xx}^2 V(x, y)$ inside the full Hessian $\nabla^2 V(x, y)$ is

$$\begin{aligned} \partial_{yy}^2 V(x, y) - \nabla_{yx}^2 V(x, y) (\nabla_{xx}^2 V(x, y))^{-1} \nabla_{xy}^2 V(x, y) \\ = r''(y) - \langle g, M(y)z + c(y) \rangle - \langle B(y)^T g, H^{-1} B(y)^T g \rangle. \end{aligned}$$

In particular, because V is convex, the optimized tangent inequality is exactly the nonnegativity of this Schur complement.

Proof. Recall that

$$V(x, y) = W(A(y)^{-1}(x - b(y))) + r(y).$$

At fixed y , write

$$z = A(y)^{-1}(x - b(y)).$$

Then

$$\partial_y z \Big|_{x \text{ fixed}} = -A(y)^{-1} A'(y) z - A(y)^{-1} b'(y) = -B(y) z - p(y).$$

Therefore

$$\nabla_x V(x, y) = A(y)^{-T} g, \quad \nabla_{xx}^2 V(x, y) = A(y)^{-T} H A(y)^{-1}.$$

Differentiate $\nabla_x V = A^{-T} g$ in y at fixed x . Using

$$\frac{d}{dy} A(y)^{-T} = -A(y)^{-T} B(y)^T$$

and $\partial_y g = H \partial_y z = -H(Bz + p)$, we obtain

$$\nabla_{xy}^2 V(x, y) = -A(y)^{-T} (H(Bz + p) + B(y)^T g).$$

Next, since

$$\partial_y V(x, y) = \langle g, \partial_y z \rangle + r'(y) = -\langle g, B(y) z + p(y) \rangle + r'(y),$$

another differentiation gives

$$\partial_{yy}^2 V(x, y) = -\langle H \partial_y z, Bz + p \rangle - \langle g, B'(y) z + B(y) \partial_y z + p'(y) \rangle + r''(y).$$

Substituting $\partial_y z = -(Bz + p)$, $B' = M - B^2$, and $p' = c - Bp$, we find

$$\partial_{yy}^2 V(x, y) = \langle H(Bz + p), Bz + p \rangle - \langle g, Mz + c \rangle + 2 \langle g, B(Bz + p) \rangle + r''(y).$$

Since

$$(\nabla_{xx}^2 V(x, y))^{-1} = A(y) H^{-1} A(y)^T,$$

the mixed-term Schur complement contribution equals

$$\begin{aligned} & \nabla_{yy}^2 V(x, y) (\nabla_{xx}^2 V(x, y))^{-1} \nabla_{xy}^2 V(x, y) \\ &= \left(H(Bz + p) + B^T g \right)^T H^{-1} \left(H(Bz + p) + B^T g \right) \\ &= \langle H(Bz + p), Bz + p \rangle + 2 \langle g, B(Bz + p) \rangle + \left\langle B^T g, H^{-1} B^T g \right\rangle. \end{aligned}$$

Subtracting from $\partial_{yy}^2 V(x, y)$ proves the stated formula. $\square$

Definition 20.2. Under the hypotheses of [theorem 13.1](#), for $L \in \text{End}(E)$ define

$$\mathcal{J}_\nu(L) := \int_E \left\langle L^T \nabla W(z), \nabla^2 W(z)^{-1} L^T \nabla W(z) \right\rangle \nu(dz).$$

Proposition 20.3 (Averaged motion inequality). *Under the hypotheses of [theorem 13.1](#), for every $y \in I$,*

$$\mathcal{J}_\nu(B(y)) + \text{Tr}(B(y)^2) \leq \phi''(y).$$

Proof. Integrate the scalar inequality from [theorem 20.1](#) with respect to ν . Since $\nu(\mathrm{d}z) = e^{-W(z)} \mathrm{d}z$ has density of total mass one and $W \in C^2(E)$, integration by parts gives

$$\int_E \partial_j W(z) \nu(\mathrm{d}z) = 0, \quad \int_E z_i \partial_j W(z) \nu(\mathrm{d}z) = \delta_{ij}.$$

Hence

$$\int_E \langle \nabla W(z), c(y) \rangle \nu(\mathrm{d}z) = 0, \quad \int_E \langle \nabla W(z), M(y)z \rangle \nu(\mathrm{d}z) = \mathrm{Tr}(M(y)).$$

Therefore

$$\mathrm{Tr}(M(y)) + \mathcal{J}_\nu(B(y)) \leq r''(y).$$

Now

$$r''(y) = \frac{\mathrm{d}^2}{\mathrm{d}y^2} \log \det A(y) + \phi''(y) = \mathrm{Tr}(B'(y)) + \phi''(y).$$

Since $B' = M - B^2$, we have

$$r''(y) = \mathrm{Tr}(M(y)) - \mathrm{Tr}(B(y)^2) + \phi''(y).$$

Subtracting $\mathrm{Tr}(M(y))$ from both sides yields the claim. $\square$

Definition 20.4 (Weighted divergence and infinitesimal affine ν -symmetries). For a C^1 vector field $F : E \rightarrow E$, define its weighted divergence with respect to ν by

$$\mathrm{div}_\nu F(z) := \mathrm{div} F(z) - \langle \nabla W(z), F(z) \rangle.$$

For an affine field $F_{L,a}(z) := Lz - a$, this becomes

$$\mathrm{div}_\nu F_{L,a}(z) = \mathrm{Tr}(L) - \langle \nabla W(z), Lz - a \rangle.$$

We say that $F_{L,a}$ is an infinitesimal affine ν -symmetry if $\mathrm{div}_\nu F_{L,a} \equiv 0$ on E .

Proposition 20.5 (Universal lower bound for the averaged motion functional). *Let E be a Euclidean space and let*

$$\nu(\mathrm{d}z) = e^{-W(z)} \mathrm{d}z$$

be a log-concave probability measure on E , where $W \in C^2(E)$ is convex and $\nabla^2 W(z)$ is positive-definite for every $z \in E$. Then for every $L \in \mathrm{End}(E)$,

$$\mathcal{J}_\nu(L) + \mathrm{Tr}(L^2) \geq 0.$$

Proof. Write

$$g(z) := \nabla W(z), \quad H(z) := \nabla^2 W(z), \quad q_L(z) := \langle g(z), Lz \rangle.$$

By integration by parts against the density e^{-W} , for every smooth vector field h with sufficient decay,

$$\int_E \langle g(z), h(z) \rangle \nu(\mathrm{d}z) = \int_E \mathrm{div} h(z) \nu(\mathrm{d}z).$$

Apply this with $h(z) = Lz$. Since $\mathrm{div}(Lz) = \mathrm{Tr}(L)$, we obtain

$$\int_E q_L(z) \nu(\mathrm{d}z) = \mathrm{Tr}(L).$$

Next set

$$\mathcal{A}_\nu(L) := \int_E \langle H(z)Lz, Lz \rangle \nu(\mathrm{d}z).$$

Apply the same integration-by-parts identity with the vector field $h(z) = q_L(z)Lz$. Then

$$\int_E q_L(z)^2 \nu(\mathrm{d}z) = \int_E \operatorname{div}(q_L(z)Lz) \nu(\mathrm{d}z) = \operatorname{Tr}(L) \int_E q_L \, \mathrm{d}\nu + \int_E \langle \nabla q_L(z), Lz \rangle \nu(\mathrm{d}z).$$

Since

$$\nabla q_L(z) = H(z)Lz + L^T g(z),$$

this gives

$$\int_E q_L^2 \, \mathrm{d}\nu = \operatorname{Tr}(L)^2 + \mathcal{A}_\nu(L) + \int_E \langle g(z), L^2 z \rangle \nu(\mathrm{d}z).$$

A final integration by parts with $h(z) = L^2 z$ yields

$$\int_E \langle g(z), L^2 z \rangle \nu(\mathrm{d}z) = \operatorname{Tr}(L^2).$$

Therefore

$$\operatorname{Var}_\nu(q_L) = \mathcal{A}_\nu(L) + \operatorname{Tr}(L^2).$$

On the other hand, the classical Brascamp–Lieb inequality gives

$$\operatorname{Var}_\nu(q_L) \leq \int_E \langle \nabla q_L(z), H(z)^{-1} \nabla q_L(z) \rangle \nu(\mathrm{d}z).$$

Expanding $\nabla q_L = HLz + L^T g$ yields

$$\begin{aligned} \int_E \langle \nabla q_L, H^{-1} \nabla q_L \rangle \, \mathrm{d}\nu &= \int_E \langle HLz, Lz \rangle \, \mathrm{d}\nu + 2 \int_E \langle L^T g, Lz \rangle \, \mathrm{d}\nu + \int_E \langle L^T g, H^{-1} L^T g \rangle \, \mathrm{d}\nu \\ &= \mathcal{A}_\nu(L) + 2\operatorname{Tr}(L^2) + \mathcal{J}_\nu(L). \end{aligned}$$

Comparing with $\operatorname{Var}_\nu(q_L) = \mathcal{A}_\nu(L) + \operatorname{Tr}(L^2)$ proves

$$\mathcal{J}_\nu(L) + \operatorname{Tr}(L^2) \geq 0. \quad \square$$

Proposition 20.6 (Affine-symmetry saturation criterion). *Let $\nu(\mathrm{d}z) = e^{-W(z)} \mathrm{d}z$ be as in [theorem 20.5](#), and let $L \in \operatorname{End}(E)$. If*

$$\mathcal{J}_\nu(L) + \operatorname{Tr}(L^2) = 0,$$

then there exists a vector $a_L \in E$ such that

$$\langle \nabla W(z), Lz - a_L \rangle \equiv \operatorname{Tr}(L) \quad (z \in E).$$

Equivalently, the affine field

$$F_L(z) := Lz - a_L$$

is an infinitesimal affine ν -symmetry:

$$\operatorname{div}_\nu F_L \equiv 0.$$

If W is Legendre-type, then a_L is unique.

Proof. In the proof of [theorem 20.5](#), equality

$$\mathcal{J}_\nu(L) + \text{Tr}(L^2) = 0$$

is exactly the equality case of the Brascamp–Lieb inequality for the centered function

$$f_L(z) := q_L(z) - \text{Tr}(L).$$

By the standard equality case in the smooth Brascamp–Lieb inequality, there exists $a_L \in E$ such that

$$f_L(z) = \langle \nabla W(z), a_L \rangle \quad (z \in E).$$

Equivalently,

$$\langle \nabla W(z), Lz - a_L \rangle = \text{Tr}(L) \quad (z \in E),$$

which is precisely $\text{div}_\nu F_L \equiv 0$.

If W is Legendre-type and a_L, a'_L both satisfy the displayed identity, then

$$\langle \nabla W(z), a_L - a'_L \rangle = 0 \quad (z \in E).$$

Since $\nabla W(E) = E$, this forces $a_L = a'_L$. □

Theorem 20.7 (Affine-base exact-affine gauge collapse). *Let E be a Euclidean space, let $I \subset \mathbb{R}$ be an interval, let*

$$\nu(\text{d}z) = e^{-W(z)} \text{d}z, \quad m(\text{d}y) = e^{-\phi(y)} \text{d}y$$

be log-concave probability measures with $W \in C^2(E)$ convex, Legendre-type, and $\nabla^2 W(z)$ positive-definite for every $z \in E$. Let

$$A : I \rightarrow \text{GL}(E), \quad b : I \rightarrow E$$

be C^2 , define

$$T(z, y) = (A(y)z + b(y), y), \quad \mu := T_\#(\nu \otimes m),$$

and assume that μ is log-concave on $E \times I$. Assume moreover that ϕ is affine on I .

Then there exists a smooth family of affine maps

$$S_y(z) = U(y)z + v(y) \quad (y \in I)$$

such that $(S_y)_\# \nu = \nu$ for every $y \in I$, and such that the gauge-equivalent bundle

$$\tilde{T}(z, y) := T(S_y z, y)$$

has constant linear part and affine translation:

$$\tilde{T}(z, y) = (A_0 z + \tilde{b}_0 + y \tilde{b}_1, y)$$

for some fixed $A_0 \in \text{GL}(E)$ and vectors $\tilde{b}_0, \tilde{b}_1 \in E$.

Consequently every affine-base exact-affine bundle collapses, modulo affine ν -symmetry gauge, to an affine product.

Proof. Because ϕ is affine, $\phi'' \equiv 0$. By [theorems 20.3](#) and [20.5](#), for every $y \in I$,

$$0 \leq \mathcal{J}_\nu(B(y)) + \text{Tr}(B(y)^2) \leq \phi''(y) = 0.$$

Hence

$$\mathcal{J}_\nu(B(y)) + \text{Tr}(B(y)^2) = 0 \quad (y \in I).$$

By [theorem 20.6](#), for each $y \in I$ there exists $a(y) \in E$ such that

$$\text{div}_\nu(B(y)z - a(y)) = 0 \quad \text{on } E.$$

Because W is Legendre-type, $a(y)$ is unique. Choose points $z_1, \dots, z_m \in E$ such that the vectors

$$p_i := \nabla W(z_i) \quad (1 \leq i \leq m)$$

form a basis of E . Evaluating

$$\langle p_i, B(y)z_i - a(y) \rangle = \text{Tr}(B(y)) \quad (1 \leq i \leq m)$$

shows that $a(y)$ depends smoothly on y , because the coefficient matrix $(p_i)_i$ is fixed and $B(y)$ is C^1 .

Now define the time-dependent affine vector field

$$F_y(z) := B(y)z - a(y),$$

and fix $y_* \in I$. Let S_y solve

$$\partial_y S_y(z) = -F_y(S_y(z)), \quad S_{y_*}(z) = z.$$

Since F_y is affine in z , the solution has the form

$$S_y(z) = U(y)z + v(y),$$

where

$$\begin{aligned} U'(y) &= -B(y)U(y), & U(y_*) &= \text{Id}, \\ v'(y) &= -B(y)v(y) + a(y), & v(y_*) &= 0. \end{aligned}$$

In particular $U(y) \in \text{GL}(E)$ for every y , so each S_y is an affine diffeomorphism.

We claim that $(S_y)_\# \nu = \nu$ for all y . Indeed, if $\rho_y := (S_y)_\# \nu$, then for every smooth compactly-supported $\psi : E \rightarrow \mathbb{R}$,

$$\frac{d}{dy} \int_E \psi \, d\rho_y = - \int_E \langle \nabla \psi(x), F_y(x) \rangle \rho_y(dx).$$

On the other hand, since $\text{div}_\nu F_y = 0$,

$$\int_E \langle \nabla \psi(x), F_y(x) \rangle \nu(dx) = 0.$$

Thus ρ_y and ν solve the same continuity equation with the same initial datum at $y = y_*$. Uniqueness for the continuity equation with globally Lipschitz velocity field F_y yields $\rho_y = \nu$.

Now compose the bundle with this moving frame:

$$\tilde{T}(z, y) := T(S_y z, y) = (A(y)U(y)z + A(y)v(y) + b(y), y).$$

Its linear part is

$$\tilde{A}(y) := A(y)U(y),$$

and

$$\tilde{A}'(y) = A'(y)U(y) + A(y)U'(y) = A(y)B(y)U(y) - A(y)B(y)U(y) = 0.$$

Hence

$$\tilde{A}(y) \equiv A_0 \in \text{GL}(E).$$

Set

$$\tilde{b}(y) := A(y)v(y) + b(y).$$

Because $(S_y)_\# \nu = \nu$, we still have

$$\mu = \tilde{T}_\#(\nu \otimes m).$$

Push the constant linear map A_0 into the reference measure:

$$\tilde{\nu} := (A_0)_\# \nu.$$

Its potential is

$$\widetilde{W}(x) := W(A_0^{-1}x),$$

and

$$\nabla \widetilde{W}(x) = A_0^{-T} \nabla W(A_0^{-1}x)$$

has full range E , so $\widetilde{W}$ is again Legendre-type. Therefore

$$\mu = \widehat{T}_\#(\tilde{\nu} \otimes m), \quad \widehat{T}(x, y) = (x + \tilde{b}(y), y).$$

Applying [theorem 8.2](#) gives that $\tilde{b}$ is affine on I . This proves the theorem. $\square$

Corollary 20.8 (Affine-base exact-affine terminal models are not obstructions). *Let $n \geq 2$. Assume KLS is known in all dimensions at most $n - 1$. Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. Suppose that on the canonical rank-one edge the exact-affine triangular trivialization has affine base potential on an interval I , and that after isotropization of the reference fiber the potential W is C^2 , convex, Legendre-type, with $\nabla^2 W$ positive-definite everywhere. Then μ satisfies KLS. Consequently a minimal-dimensional counterexample cannot belong to the affine-base exact-affine terminal class.*

Proof. By [theorem 20.7](#), after composing with a smooth ν -preserving affine moving frame the canonical exact-affine bundle becomes an affine product. Since such a gauge change does not alter μ , the measure μ is itself an affine image of a product $\tilde{\nu} \otimes m$, where $\tilde{\nu}$ is the isotropized reference fiber and m is the one-dimensional edge marginal. By the lower-dimensional hypothesis, $\tilde{\nu}$ satisfies KLS; by the one-dimensional log-concave theory, so does m . Hence $\tilde{\nu} \otimes m$ satisfies KLS, and therefore so does its affine image μ . $\square$

Corollary 20.9 (Gauge-free collapse under trivial affine symmetry algebra). *Under the hypotheses of [theorem 20.7](#), assume that the only affine field $F(z) = Lz - a$ satisfying*

$$\text{div}_\nu F \equiv 0$$

is the zero field. Then A is constant on I and b is affine on I .

Proof. For each $y \in I$, [theorem 20.6](#) applied to $L = B(y)$ yields an affine field

$$F_y(z) = B(y)z - a(y)$$

with $\operatorname{div}_\nu F_y \equiv 0$. By hypothesis this forces $F_y \equiv 0$, hence $B(y) = 0$ and $a(y) = 0$ for every y . Therefore $A'(y) = 0$ on I , so $A(y) \equiv A_0 \in \operatorname{GL}(E)$.

Push the constant linear part into the reference measure:

$$\tilde{\nu} := (A_0)_\# \nu.$$

Its potential is

$$\widetilde{W}(x) := W(A_0^{-1}x),$$

and

$$\nabla \widetilde{W}(x) = A_0^{-T} \nabla W(A_0^{-1}x)$$

has full range E , so $\widetilde{W}$ is again Legendre-type. The measure μ may therefore be rewritten as

$$\mu = \tilde{T}_\#(\tilde{\nu} \otimes m), \quad \tilde{T}(x, y) = (x + b(y), y).$$

Applying [theorem 8.2](#) gives that b is affine. □

Remark 20.10. [Theorems 20.7](#) and [20.8](#) remove the entire affine-base exact-affine branch $\phi'' \equiv 0$ modulo the natural affine ν -symmetry gauge. This is strictly stronger than the earlier sign-based elimination: no assumption such as $\operatorname{Tr}(B(y)^2) \geq 0$ is needed any longer. The two corollaries below remain useful as gauge-free criteria when one prefers an actual constancy statement for A in the original coordinates.

Corollary 20.11 (Affine-base interval collapse under nonnegative square trace). *Let E be a Euclidean space, let $I \subset \mathbb{R}$ be an interval, let*

$$\nu(dz) = e^{-W(z)} dz, \quad m(dy) = e^{-\phi(y)} dy$$

be log-concave probability measures with $W \in C^2(E)$ convex, Legendre-type, and $\nabla^2 W(z)$ positive-definite for every $z \in E$. Let

$$A : I \rightarrow \operatorname{GL}(E), \quad b : I \rightarrow E$$

be C^2 , define

$$T(z, y) = (A(y)z + b(y), y), \quad \mu := T_\#(\nu \otimes m),$$

and assume that μ is log-concave on $E \times I$. Assume moreover that ϕ is affine on I and that

$$\operatorname{Tr}(B(y)^2) \geq 0 \quad (y \in I).$$

Then A is constant on I and b is affine on I . Consequently μ is an affine image of a product measure.

Proof. Because ϕ is affine, $\phi'' \equiv 0$. By [theorem 20.3](#),

$$\mathcal{J}_\nu(B(y)) + \operatorname{Tr}(B(y)^2) \leq 0 \quad (y \in I).$$

Both summands are nonnegative by definition and by hypothesis, hence for every $y \in I$,

$$\mathcal{J}_\nu(B(y)) = 0, \quad \operatorname{Tr}(B(y)^2) = 0.$$

Since $\nabla^2 W(z)^{-1}$ is positive-definite and the density of ν is strictly positive on all of E , the identity $\mathcal{J}_\nu(B(y)) = 0$ implies

$$B(y)^T \nabla W(z) = 0 \quad (z \in E).$$

As W is Legendre-type, $\nabla W(E) = E$. Therefore $B(y) = 0$ for every y , i.e.

$$A'(y) = 0 \quad (y \in I).$$

Hence $A(y) \equiv A_0 \in \text{GL}(E)$.

Push the constant linear part into the reference measure:

$$\tilde{\nu} := (A_0)_\# \nu.$$

Its potential is

$$\widetilde{W}(x) := W(A_0^{-1}x),$$

and

$$\nabla \widetilde{W}(x) = A_0^{-T} \nabla W(A_0^{-1}x)$$

has full range E , so $\widetilde{W}$ is again Legendre-type. The measure μ may therefore be rewritten as

$$\mu = \tilde{T}_\#(\tilde{\nu} \otimes m), \quad \tilde{T}(x, y) = (x + b(y), y).$$

Applying [theorem 8.2](#) gives that b is affine. □

Corollary 20.12 (Affine-base interval collapse for globally fixed positive cones). *Under the hypotheses of [theorem 20.11](#), assume moreover that there exists a fixed $L \in \text{GL}(E)$ such that*

$$L^{-1}A(y)L \in \text{Sym}^+(E) \quad (y \in I).$$

Then A is constant on I and b is affine on I .

Proof. Set

$$\tilde{A}(y) := L^{-1}A(y)L.$$

Then

$$\tilde{B}(y) = \tilde{A}(y)^{-1} \tilde{A}'(y) = L^{-1}B(y)L,$$

so

$$\text{Tr}(B(y)^2) = \text{Tr}(\tilde{B}(y)^2).$$

Because $\tilde{A}(y)$ and $\tilde{A}'(y)$ are symmetric, $\tilde{B}(y)$ is similar to the symmetric matrix

$$\tilde{A}(y)^{-1/2} \tilde{A}'(y) \tilde{A}(y)^{-1/2}.$$

Hence all eigenvalues of $\tilde{B}(y)$ are real and

$$\text{Tr}(B(y)^2) = \text{Tr}(\tilde{B}(y)^2) \geq 0.$$

Apply [theorem 20.11](#). □

Lemma 20.13 (Dual ray curvature is eventually larger than $1/t^2$ somewhere). *Let $W : E \rightarrow \mathbb{R}$ be C^2 , convex, even, Legendre-type, with $\nabla^2 W(z)$ positive-definite for every $z \in E$. Let W^* be the Legendre transform of W , and fix $u \in E \setminus \{0\}$. Define*

$$f_u(t) := W^*(tu) \quad (t \in \mathbb{R}).$$

Then $f_u \in C^2(\mathbb{R})$ is even and convex, f'_u is unbounded on $[0, \infty)$, and the quantity

$$t^2 f''_u(t)$$

is unbounded on $[0, \infty)$.

Proof. Since W is even, its Legendre transform W^* is even:

$$W^*(-p) = \sup_{z \in E} (-\langle p, z \rangle - W(z)) = \sup_{z \in E} (\langle p, z \rangle - W(-z)) = W^*(p).$$

Convexity of W^* is standard, so f_u is even and convex.

We claim that f'_u is unbounded on $[0, \infty)$. Indeed, if $f'_u \leq L$ on $[0, \infty)$, then convexity would imply

$$f_u(t) \leq f_u(0) + Lt \quad (t \geq 0).$$

Fix any $s > L$. By the definition of the Legendre transform,

$$f_u(t) = W^*(tu) \geq t \langle u, su \rangle - W(su) = st - W(su) \quad (t \geq 0),$$

after replacing u by $u/|u|^2$ if necessary so that $\langle u, su \rangle = s$. The upper and lower bounds are incompatible as $t \rightarrow +\infty$. Hence f'_u is unbounded.

Finally, suppose $t^2 f''_u(t)$ were bounded on $[T, \infty)$:

$$t^2 f''_u(t) \leq C \quad (t \geq T).$$

Then

$$f''_u(t) \leq \frac{C}{t^2} \quad (t \geq T),$$

and integrating from T to $t \geq T$ yields

$$f'_u(t) \leq f'_u(T) + \int_T^\infty \frac{C}{s^2} ds = f'_u(T) + \frac{C}{T}.$$

Thus f'_u would be bounded on $[T, \infty)$, contradicting the previous paragraph. Therefore $t^2 f''_u(t)$ is unbounded. $\square$

Theorem 20.14 (Flat-point real-spectrum exclusion over even fibers). *Let E be a Euclidean space, let $I \subset \mathbb{R}$ be an interval, let*

$$\nu(dz) = e^{-W(z)} dz$$

be a log-concave probability measure on E , where $W \in C^2(E)$ is convex, even, Legendre-type, and $\nabla^2 W(z)$ is positive-definite for every $z \in E$. Let

$$m(dy) = e^{-\phi(y)} dy$$

be a log-concave probability measure on I with $\phi \in C^2(I)$. Let

$$A : I \rightarrow \text{GL}(E), \quad b : I \rightarrow E$$

be C^2 , define

$$T(z, y) = (A(y)z + b(y), y), \quad \mu := T_{\#}(\nu \otimes m),$$

and assume that μ is log-concave on $E \times I$. Fix $y_0 \in I$ and suppose that

$$A''(y_0) = 0.$$

Then every real eigenvalue of

$$B(y_0) = A(y_0)^{-1}A'(y_0)$$

is equal to 0.

Proof. Set

$$B_0 := B(y_0), \quad M_0 := A(y_0)^{-1}A''(y_0) = 0.$$

By [theorem 13.2](#), for every $z \in E$,

$$\left\langle B_0^T \nabla W(z), \nabla^2 W(z)^{-1} B_0^T \nabla W(z) \right\rangle \leq r''(y_0).$$

Assume for contradiction that B_0 has a nonzero real eigenvalue. Then B_0^T also has a nonzero real eigenvalue, so we may choose $u \in E \setminus \{0\}$ and $\lambda \in \mathbb{R} \setminus \{0\}$ with

$$B_0^T u = \lambda u.$$

For $t \geq 0$, let $z_t \in E$ be the unique point with

$$\nabla W(z_t) = tu.$$

Such a point exists because W is Legendre-type and $\nabla^2 W$ is everywhere positive-definite. Define

$$f_u(t) := W^*(tu).$$

Standard Legendre duality gives

$$\nabla W^*(tu) = z_t, \quad \nabla^2 W^*(tu) = \nabla^2 W(z_t)^{-1}.$$

Therefore

$$\left\langle B_0^T \nabla W(z_t), \nabla^2 W(z_t)^{-1} B_0^T \nabla W(z_t) \right\rangle = \lambda^2 t^2 \left\langle u, \nabla^2 W^*(tu) u \right\rangle = \lambda^2 t^2 f_u''(t).$$

Hence

$$\lambda^2 t^2 f_u''(t) \leq r''(y_0) \quad (t \geq 0).$$

This contradicts [theorem 20.13](#), which shows that $t^2 f_u''(t)$ is unbounded on $[0, \infty)$. Therefore every real eigenvalue of B_0 must vanish. $\square$

Corollary 20.15 (Self-adjoint flat points are fully stationary). *Under the hypotheses of [theorem 20.14](#), assume in addition that $A(y_0) \in \text{Sym}^+(E)$ and $A'(y_0) \in \text{Sym}(E)$. If*

$$A''(y_0) = 0,$$

then

$$A'(y_0) = 0 \quad \text{and} \quad b''(y_0) = 0.$$

Proof. Because $A(y_0) \in \text{Sym}^+(E)$ and $A'(y_0) \in \text{Sym}(E)$, the matrix

$$B(y_0) = A(y_0)^{-1}A'(y_0)$$

is self-adjoint for the $A(y_0)$ -inner product. Hence all of its eigenvalues are real. By [theorem 20.14](#), all those eigenvalues are zero, so $B(y_0) = 0$, i.e.

$$A'(y_0) = 0.$$

Now $A'(y_0) = A''(y_0) = 0$ implies $B(y_0) = M(y_0) = 0$, and therefore

$$r''(y_0) = \phi''(y_0).$$

Applying [theorem 12.1](#) at y_0 gives

$$\langle \nabla W(z), A(y_0)^{-1}b''(y_0) \rangle \leq \phi''(y_0) \quad (z \in E).$$

Set

$$c := A(y_0)^{-1}b''(y_0).$$

If $c \neq 0$, choose $z_k \in E$ with $\nabla W(z_k) = kc$ (possible because W is Legendre-type). Then

$$k|c|^2 \leq \phi''(y_0) \quad (k \geq 1),$$

a contradiction. Hence $c = 0$, i.e. $b''(y_0) = 0$. □

Corollary 20.16 (Fixed-direction real-spectrum inflection points are stationary). *Under the hypotheses of [theorem 20.14](#), assume in addition that*

$$A(y) = A_0 + s(y)M$$

with $A_0 \in \text{GL}(E)$, $M \in \text{End}(E)$ fixed. Fix $y_0 \in I$, and assume that

$$s''(y_0) = 0$$

and that $A(y_0)^{-1}M$ has a nonzero real eigenvalue. Then

$$s'(y_0) = 0 \quad \text{and} \quad b''(y_0) = 0.$$

Proof. Because

$$A''(y_0) = s''(y_0)M = 0,$$

[theorem 20.14](#) applies and shows that every real eigenvalue of

$$B(y_0) = A(y_0)^{-1}A'(y_0) = s'(y_0)A(y_0)^{-1}M$$

is equal to 0. Since $A(y_0)^{-1}M$ has a nonzero real eigenvalue by hypothesis, this forces $s'(y_0) = 0$. Consequently

$$A'(y_0) = 0.$$

The argument from the second half of [theorem 20.15](#) now applies verbatim and yields $b''(y_0) = 0$. □

Remark 20.17. [Theorems 20.7](#) and [20.8](#) completely close the affine-base exact-affine branch modulo the natural affine ν -symmetry gauge. Thus any remaining bounded-interval or half-line obstruction must already carry genuinely curved base geometry $\phi'' \neq 0$. [Theorems 20.14](#) to [20.16](#) sharpen the local geometry of the even-fiber branch in a complementary direction: even when the base is curved, a nontrivial real spectral mode cannot persist through a flat acceleration point.

21 Quantitative symmetry-gap rigidity on curved bases

Section 20 used only the qualitative facts

$$\mathcal{Q}_\nu(B(y)) := \mathcal{J}_\nu(B(y)) + \text{Tr}(B(y)^2) \leq \phi''(y) \quad \text{and} \quad \mathcal{Q}_\nu(L) \geq 0$$

to deduce complete collapse when $\phi'' \equiv 0$. The next natural question is quantitative: when ϕ'' is positive but small, how far can the linear motion $B = A^{-1}A'$ move away from the infinitesimal symmetry algebra of the reference fiber? After isotropic normalization, the answer is finite-dimensional and exact.

Definition 21.1 (Infinitesimal orthogonal symmetry algebra). Assume that ν is isotropic, i.e.

$$\int_E z \nu(\mathrm{d}z) = 0, \quad \int_E z \otimes z \nu(\mathrm{d}z) = \text{Id}.$$

Define

$$\mathfrak{k}_\nu := \{K \in \text{End}(E) : \text{div}_\nu(Kz) \equiv 0\}.$$

For $L \in \text{End}(E)$, set

$$\mathcal{Q}_\nu(L) := \mathcal{J}_\nu(L) + \text{Tr}(L^2).$$

Proposition 21.2 (Kernel of $\mathcal{Q}_\nu$ in isotropic coordinates). *Assume that ν is isotropic, Legendre-type, and that $\nabla^2 W(z)$ is positive-definite for every $z \in E$. Then:*

- (i) $\mathfrak{k}_\nu$ is a linear subspace of $\mathfrak{so}(E)$;
- (ii) the kernel of $\mathcal{Q}_\nu$ is exactly $\mathfrak{k}_\nu$.

Proof. Linearity is immediate from the definition. Let $K \in \mathfrak{k}_\nu$. Then for every smooth compactly supported $\psi : E \rightarrow \mathbb{R}$,

$$\int_E \langle \nabla \psi(z), Kz \rangle \nu(\mathrm{d}z) = 0.$$

Choose

$$\psi(z) = \frac{1}{2} z_i z_j.$$

Then

$$\nabla \psi(z) = \frac{1}{2} (z_j e_i + z_i e_j),$$

and isotropy yields

$$0 = \int_E \langle \nabla \psi(z), Kz \rangle \nu(\mathrm{d}z) = \frac{1}{2} (K_{ij} + K_{ji}).$$

Hence $K + K^T = 0$, so $K \in \mathfrak{so}(E)$. This proves (i).

For (ii), suppose first that $\mathcal{Q}_\nu(L) = 0$. By [theorem 20.6](#), there exists $a_L \in E$ such that

$$\text{div}_\nu(Lz - a_L) \equiv 0.$$

Testing this identity against the coordinate function $\psi(z) = z_j$ and using that ν is centered, we get

$$0 = \int_E (Lz - a_L)_j \nu(\mathrm{d}z) = -(a_L)_j.$$

Thus $a_L = 0$, so $L \in \mathfrak{k}_\nu$.

Conversely, let $K \in \mathfrak{k}_\nu$. Then

$$\langle \nabla W(z), Kz \rangle = \text{Tr}(K) \quad (z \in E),$$

so the quantity

$$q_K(z) := \langle \nabla W(z), Kz \rangle$$

is constant. Hence $\nabla q_K \equiv 0$. But

$$\nabla q_K(z) = \nabla^2 W(z) Kz + K^T \nabla W(z),$$

and therefore

$$0 = \int_E \left\langle \nabla q_K(z), \nabla^2 W(z)^{-1} \nabla q_K(z) \right\rangle \nu(\mathrm{d}z).$$

Expanding exactly as in the proof of [theorem 20.5](#), we obtain

$$0 = \mathcal{A}_\nu(K) + 2\text{Tr}(K^2) + \mathcal{J}_\nu(K), \quad 0 = \text{Var}_\nu(q_K) = \mathcal{A}_\nu(K) + \text{Tr}(K^2).$$

Subtracting the second identity from the first gives

$$\mathcal{Q}_\nu(K) = \mathcal{J}_\nu(K) + \text{Tr}(K^2) = 0.$$

Hence $\ker \mathcal{Q}_\nu = \mathfrak{k}_\nu$. □

Theorem 21.3 (Symmetry-gap coercivity). *Assume that ν is isotropic, Legendre-type, and that $\nabla^2 W(z)$ is positive-definite for every $z \in E$. Then there exists a constant $\lambda_\nu > 0$ such that for every $L \in \text{End}(E)$,*

$$\mathcal{Q}_\nu(L) \geq \lambda_\nu \text{dist}_{\text{HS}}(L, \mathfrak{k}_\nu)^2.$$

Equivalently, if $L = L_\mathfrak{k} + L_\perp$ is the Hilbert–Schmidt orthogonal decomposition with $L_\mathfrak{k} \in \mathfrak{k}_\nu$ and $L_\perp \in \mathfrak{k}_\nu^\perp$, then

$$\mathcal{Q}_\nu(L) = \mathcal{Q}_\nu(L_\perp) \geq \lambda_\nu \|L_\perp\|_{\text{HS}}^2.$$

Proof. Let

$$\mathfrak{q}_\nu(L_1, L_2) := \frac{1}{2}(\mathcal{Q}_\nu(L_1 + L_2) - \mathcal{Q}_\nu(L_1) - \mathcal{Q}_\nu(L_2))$$

be the symmetric bilinear form associated with $\mathcal{Q}_\nu$.

By [theorem 21.2](#), $\ker \mathcal{Q}_\nu = \mathfrak{k}_\nu$. Let $K \in \mathfrak{k}_\nu$ and let L be arbitrary. Since $\mathcal{Q}_\nu \geq 0$ by [theorem 20.5](#),

$$0 \leq \mathcal{Q}_\nu(K + tL) = 2t \mathfrak{q}_\nu(K, L) + t^2 \mathcal{Q}_\nu(L) \quad (t \in \mathbb{R}).$$

Because this quadratic polynomial is nonnegative for every real t , its linear coefficient must vanish:

$$\mathfrak{q}_\nu(K, L) = 0.$$

Hence $\mathfrak{k}_\nu$ is orthogonal to all of $\text{End}(E)$ with respect to $\mathfrak{q}_\nu$, and therefore

$$\mathcal{Q}_\nu(L) = \mathcal{Q}_\nu(L_\perp)$$

for the Hilbert–Schmidt decomposition $L = L_\mathfrak{k} + L_\perp$.

Now restrict $\mathcal{Q}_\nu$ to the finite-dimensional space $\mathfrak{k}_\nu^\perp$. Its kernel there is trivial by [theorem 21.2](#). Consequently $\mathcal{Q}_\nu$ is strictly positive on the Hilbert–Schmidt unit sphere of $\mathfrak{k}_\nu^\perp$. Compactness yields

$$\lambda_\nu := \min\{\mathcal{Q}_\nu(L) : L \in \mathfrak{k}_\nu^\perp, \|L\|_{\text{HS}} = 1\} > 0.$$

For arbitrary L , this gives

$$\mathcal{Q}_\nu(L) = \mathcal{Q}_\nu(L_\perp) \geq \lambda_\nu \|L_\perp\|_{\text{HS}}^2 = \lambda_\nu \text{dist}_{\text{HS}}(L, \mathfrak{k}_\nu)^2. \quad \square$$

Theorem 21.4 (Quantitative orthogonal-gauge rigidity). *Let $I \subset \mathbb{R}$ be an interval, let*

$$\nu(\mathrm{d}z) = e^{-W(z)} \mathrm{d}z, \quad m(\mathrm{d}y) = e^{-\phi(y)} \mathrm{d}y$$

be log-concave probability measures with $W \in C^2(E)$ convex, Legendre-type, $\nabla^2 W(z)$ positive-definite for every $z \in E$, and $\phi \in C^2(I)$ convex. Assume moreover that ν is isotropic. Let

$$A : I \rightarrow \text{GL}(E), \quad b : I \rightarrow E$$

be C^2 , define

$$T(z, y) = (A(y)z + b(y), y), \quad \mu := T_\#(\nu \otimes m),$$

and assume that μ is log-concave on $E \times I$. Set

$$B(y) := A(y)^{-1}A'(y).$$

For each $y \in I$, let

$$B(y) = K(y) + C(y), \quad K(y) \in \mathfrak{k}_\nu, \quad C(y) \in \mathfrak{k}_\nu^\perp$$

be the Hilbert–Schmidt orthogonal decomposition. Fix $y_0 \in I$, and let $O : I \rightarrow \text{O}(E)$ solve

$$O'(y) = -K(y)O(y), \quad O(y_0) = \text{Id}.$$

Define the gauge-transformed bundle

$$\tilde{T}(z, y) := T(O(y)z, y) = (\tilde{A}(y)z + b(y), y), \quad \tilde{A}(y) := A(y)O(y).$$

Then:

(i) $O(y)_\# \nu = \nu$ for every $y \in I$;

(ii) the transformed linear velocity is

$$\tilde{B}(y) := \tilde{A}(y)^{-1}\tilde{A}'(y) = O(y)^T C(y) O(y);$$

(iii) for every $y \in I$,

$$\|\tilde{B}(y)\|_{\text{HS}} \leq \lambda_\nu^{-1/2} \sqrt{\phi''(y)}.$$

Consequently, for every $y \in I$,

$$\|\tilde{A}(y)^{-1}\tilde{A}(y_0)\|_{\text{op}}, \quad \|\tilde{A}(y_0)^{-1}\tilde{A}(y)\|_{\text{op}} \leq \exp\left(\lambda_\nu^{-1/2} \int_{y_0}^y \sqrt{\phi''(s)} \, \mathrm{d}s\right),$$

with the integral interpreted in the obvious absolute-value sense if $y < y_0$.

Proof. Because $K(y) \in \mathfrak{k}_\nu \subset \mathfrak{so}(E)$ by [theorem 21.2](#), the solution $O(y)$ is orthogonal for every y .

Set $\rho_y := O(y)_\# \nu$. For every smooth compactly supported $\psi : E \rightarrow \mathbb{R}$,

$$\frac{d}{dy} \int_E \psi \, d\rho_y = - \int_E \langle \nabla \psi(x), K(y)x \rangle \rho_y(dx).$$

Since $K(y) \in \mathfrak{k}_\nu$, we have $\operatorname{div}_\nu(K(y)z) \equiv 0$, and hence

$$\int_E \langle \nabla \psi(x), K(y)x \rangle \nu(dx) = 0.$$

Thus ρ_y and ν satisfy the same continuity equation with the same initial datum at $y = y_0$; uniqueness gives $\rho_y = \nu$. This proves (i).

For (ii),

$$\tilde{A}'(y) = A'(y)O(y) + A(y)O'(y) = A(y)(B(y) - K(y))O(y) = A(y)C(y)O(y).$$

Since $\tilde{A}(y) = A(y)O(y)$,

$$\tilde{B}(y) = \tilde{A}(y)^{-1} \tilde{A}'(y) = O(y)^T C(y) O(y).$$

Because $O(y)$ is orthogonal,

$$\|\tilde{B}(y)\|_{\text{HS}} = \|C(y)\|_{\text{HS}}.$$

Now [theorems 20.3](#) and [21.3](#) give

$$\phi''(y) \geq \mathcal{Q}_\nu(B(y)) \geq \lambda_\nu \operatorname{dist}_{\text{HS}}(B(y), \mathfrak{k}_\nu)^2 = \lambda_\nu \|C(y)\|_{\text{HS}}^2.$$

Therefore

$$\|\tilde{B}(y)\|_{\text{HS}} = \|C(y)\|_{\text{HS}} \leq \lambda_\nu^{-1/2} \sqrt{\phi''(y)},$$

proving (iii).

Finally, set

$$R(y) := \tilde{A}(y)^{-1} \tilde{A}(y_0).$$

Then

$$R'(y) = -\tilde{B}(y)R(y), \quad R(y_0) = \operatorname{Id}.$$

Standard Grönwall theory gives

$$\|R(y)\|_{\text{op}} \leq \exp\left(\int_{y_0}^y \|\tilde{B}(s)\|_{\text{op}} \, ds\right) \leq \exp\left(\lambda_\nu^{-1/2} \int_{y_0}^y \sqrt{\phi''(s)} \, ds\right),$$

since $\|\tilde{B}\|_{\text{op}} \leq \|\tilde{B}\|_{\text{HS}}$. Applying the same argument to

$$R(y)^{-1} = \tilde{A}(y_0)^{-1} \tilde{A}(y)$$

proves the second estimate. □

Corollary 21.5 (Curvature budget lower bound for linear distortion). *Under the hypotheses of [theorem 21.4](#), for any $y_1, y_2 \in I$,*

$$\frac{\sqrt{\lambda_\nu}}{2} \log \kappa(\tilde{A}(y_2) \tilde{A}(y_1)^{-1}) \leq \int_{\min\{y_1, y_2\}}^{\max\{y_1, y_2\}} \sqrt{\phi''(s)} \, ds,$$

where

$$\kappa(M) := \|M\|_{\text{op}} \|M^{-1}\|_{\text{op}}$$

is the operator-norm condition number. Consequently, any nontrivial linear distortion of the gauge-reduced bundle forces a definite amount of base-curvature budget.

Proof. By [theorem 21.4](#),

$$\|\tilde{A}(y_2)\tilde{A}(y_1)^{-1}\|_{\text{op}} \leq \exp\left(\lambda_\nu^{-1/2} \int_{\min\{y_1, y_2\}}^{\max\{y_1, y_2\}} \sqrt{\phi''(s)} \, ds\right)$$

and the same bound holds for the inverse. Multiplying the two inequalities yields

$$\kappa(\tilde{A}(y_2)\tilde{A}(y_1)^{-1}) \leq \exp\left(2\lambda_\nu^{-1/2} \int_{\min\{y_1, y_2\}}^{\max\{y_1, y_2\}} \sqrt{\phi''(s)} \, ds\right).$$

Taking logarithms proves the claim. $\square$

Corollary 21.6 (Covariance distortion bound). *Under the hypotheses of [theorem 21.4](#), set*

$$\Sigma(y) := \tilde{A}(y)\tilde{A}(y)^T.$$

Then for every $y \in I$,

$$e^{-2\delta(y)}\Sigma(y_0) \preceq \Sigma(y) \preceq e^{2\delta(y)}\Sigma(y_0), \quad \delta(y) := \lambda_\nu^{-1/2} \int_{\min\{y, y_0\}}^{\max\{y, y_0\}} \sqrt{\phi''(s)} \, ds.$$

Proof. Set

$$R(y) := \tilde{A}(y_0)^{-1}\tilde{A}(y).$$

By [theorem 21.4](#),

$$\|R(y)\|_{\text{op}}, \quad \|R(y)^{-1}\|_{\text{op}} \leq e^{\delta(y)}.$$

Hence

$$e^{-2\delta(y)}\text{Id} \preceq R(y)R(y)^T \preceq e^{2\delta(y)}\text{Id}.$$

Since

$$\Sigma(y) = \tilde{A}(y_0) R(y)R(y)^T \tilde{A}(y_0)^T,$$

the stated matrix inequality follows. $\square$

Corollary 21.7 (Finite-curvature-tail asymptotic rigidity). *Under the hypotheses of [theorem 21.4](#), assume that $I = [y_0, \infty)$ and that*

$$\int_{y_0}^{\infty} \sqrt{\phi''(s)} \, ds < \infty.$$

Then the gauge-reduced linear part $\tilde{A}(y)$ converges in $\text{GL}(E)$ as $y \rightarrow \infty$. The same conclusion holds on $(-\infty, y_0]$ if the corresponding tail integral is finite there.

Proof. By [theorem 21.4](#),

$$\int_{y_0}^{\infty} \|\tilde{B}(s)\|_{\text{op}} \, ds < \infty,$$

and moreover $\tilde{A}$ and $\tilde{A}^{-1}$ are uniformly bounded on $[y_0, \infty)$ by the exponential distortion estimate.

For $y_2 > y_1 \geq y_0$,

$$\tilde{A}(y_2) - \tilde{A}(y_1) = \int_{y_1}^{y_2} \tilde{A}'(s) \, ds = \int_{y_1}^{y_2} \tilde{A}(s) \tilde{B}(s) \, ds,$$

hence

$$\|\tilde{A}(y_2) - \tilde{A}(y_1)\|_{\text{op}} \leq \sup_{s \geq y_0} \|\tilde{A}(s)\|_{\text{op}} \int_{y_1}^{y_2} \|\tilde{B}(s)\|_{\text{op}} \, ds.$$

The right-hand side tends to 0 as $y_1, y_2 \rightarrow \infty$, so $\tilde{A}(y)$ is Cauchy in operator norm and converges to some matrix A_{∞} .

Applying the same argument to

$$\frac{d}{dy} \tilde{A}(y)^{-1} = -\tilde{B}(y) \tilde{A}(y)^{-1}$$

shows that $\tilde{A}(y)^{-1}$ is also Cauchy and converges to a matrix G_{∞} . Passing to the limit in

$$\tilde{A}(y) \tilde{A}(y)^{-1} = \text{Id}$$

gives $A_{\infty} G_{\infty} = G_{\infty} A_{\infty} = \text{Id}$, so $A_{\infty} \in \text{GL}(E)$. □

Corollary 21.8 (Trivial-symmetry curved-base control). *Assume the hypotheses of [theorem 21.4](#), and suppose that $\mathfrak{k}_{\nu} = \{0\}$. Then*

$$\|B(y)\|_{\text{HS}} \leq \lambda_{\nu}^{-1/2} \sqrt{\phi''(y)} \quad (y \in I),$$

and hence

$$\|A(y)^{-1} A(y_0)\|_{\text{op}}, \|A(y_0)^{-1} A(y)\|_{\text{op}} \leq \exp\left(\lambda_{\nu}^{-1/2} \int_{\min\{y, y_0\}}^{\max\{y, y_0\}} \sqrt{\phi''(s)} \, ds\right).$$

In particular, if $\phi'' \equiv 0$, then A is constant.

Proof. If $\mathfrak{k}_{\nu} = \{0\}$, then $K(y) = 0$ and $C(y) = B(y)$. Apply [theorem 21.4](#). □

Corollary 21.9 (Gaussian specialization). *Assume that $\nu = \gamma_E$, the standard Gaussian on E , so*

$$W(z) = \frac{1}{2} |z|^2.$$

Then

$$\mathfrak{k}_{\gamma_E} = \mathfrak{so}(E), \quad \mathcal{J}_{\gamma_E}(L) = \text{Tr}(LL^T), \quad \mathcal{Q}_{\gamma_E}(L) = 2 \|\text{Sym} L\|_{\text{HS}}^2.$$

Consequently the symmetry-gap constant is $\lambda_{\gamma_E} = 2$, and after the orthogonal gauge of [theorem 21.4](#),

$$\|\tilde{B}(y)\|_{\text{HS}}^2 \leq \frac{1}{2} \phi''(y) \quad (y \in I).$$

If ϕ is affine, then $\tilde{B} \equiv 0$ and the linear part becomes constant.

Proof. For the standard Gaussian, $\nabla W(z) = z$ and $\nabla^2 W = \text{Id}$. Hence

$$\mathcal{J}_{\gamma_E}(L) = \int_E \|L^T z\|^2 \gamma_E(dz) = \text{Tr}(LL^T) = \|L\|_{\text{HS}}^2,$$

and therefore

$$\mathcal{Q}_{\gamma_E}(L) = \|L\|_{\text{HS}}^2 + \text{Tr}(L^2) = 2 \|\text{Sym} L\|_{\text{HS}}^2.$$

The kernel is precisely $\mathfrak{so}(E)$, so $\lambda_{\gamma_E} = 2$. The quantitative bound then follows from [theorem 21.4](#). If ϕ is affine, the right-hand side vanishes. $\square$

Remark 21.10. Section 20 removed the affine-base branch qualitatively. The present section sharpens that statement: on an isotropic fiber with symmetry-gap constant $\lambda_\nu > 0$, every nontrivial linear distortion of an exact-affine bundle has an unavoidable quantitative cost in the base curvature ϕ'' . In particular, on bounded or one-sided intervals the unresolved exact-affine branch is now forced either into genuinely large curvature budget or into a fiber with extremely small symmetry gap.

22 Pure symmetry motion is pure gauge on arbitrary intervals

The quantitative section bounded the component of the linear velocity that is orthogonal to the infinitesimal symmetry algebra. The complementary exact question is whether motion *inside* the symmetry algebra can itself support a genuine exact-affine obstruction on a curved base. The answer is no: such motion is pure gauge, and the translation is forced to be affine with no affine-base or full-line hypothesis.

Definition 22.1 (Linear ν -symmetry group and algebra). Let

$$\nu(dz) = e^{-W(z)} dz$$

be a log-concave probability measure on E with $W \in C^2(E)$. Define the linear ν -symmetry group

$$G_\nu^{\text{lin}} := \{U \in \text{GL}(E) : U_\# \nu = \nu\}$$

and the infinitesimal linear ν -symmetry algebra

$$\mathfrak{g}_\nu := \{K \in \text{End}(E) : \text{div}_\nu(Kz) \equiv 0\}.$$

Equivalently,

$$K \in \mathfrak{g}_\nu \iff \langle \nabla W(z), Kz \rangle = \text{Tr}(K) \quad (z \in E).$$

When ν is isotropic, $\mathfrak{g}_\nu = \mathfrak{k}_\nu$ from [theorem 21.1](#).

Proposition 22.2 (Integration of linear ν -symmetries). *Assume that $W \in C^2(E)$.*

(i) *If $K \in \mathfrak{g}_\nu$, then*

$$(e^{tK})_\# \nu = \nu \quad (t \in \mathbb{R}).$$

(ii) *More generally, let $I \subset \mathbb{R}$ be an interval, let $K : I \rightarrow \mathfrak{g}_\nu$ be continuous, and let $U : I \rightarrow \text{GL}(E)$ solve*

$$U'(y) = -K(y)U(y), \quad U(y_0) = \text{Id}$$

for some $y_0 \in I$. Then

$$U(y) \in G_\nu^{\text{lin}} \quad (y \in I).$$

Proof. For (i), set $U_t = e^{tK}$. Since $K \in \mathfrak{g}_\nu$,

$$\langle \nabla W(u), Ku \rangle = \text{Tr}(K) \quad (u \in E).$$

For fixed $z \in E$,

$$\frac{d}{dt} \left(W(U_t z) - t \text{Tr}(K) \right) = \langle \nabla W(U_t z), K U_t z \rangle - \text{Tr}(K) = 0.$$

Hence

$$W(U_t z) = W(z) + t \text{Tr}(K).$$

Because $\det U_t = e^{t \text{Tr}(K)}$, we obtain

$$e^{-W(U_t z)} |\det U_t| = e^{-W(z)}.$$

This is exactly the density identity for $(U_t)_\# \nu = \nu$.

For (ii), define

$$F(y, z) := W(U(y)z) - \log \det U(y).$$

Using $U' = -KU$,

$$\partial_y F(y, z) = \langle \nabla W(U(y)z), -K(y)U(y)z \rangle - \text{Tr}(U(y)^{-1}U'(y)).$$

Since $U^{-1}U' = -U^{-1}KU$, the trace term equals $-\text{Tr}(K(y))$. Therefore

$$\partial_y F(y, z) = -\langle \nabla W(U(y)z), K(y)U(y)z \rangle + \text{Tr}(K(y)) = 0$$

because $K(y) \in \mathfrak{g}_\nu$. Thus $F(y, z)$ is constant in y , and since $U(y_0) = \text{Id}$,

$$W(U(y)z) - \log \det U(y) = W(z).$$

Equivalently,

$$e^{-W(U(y)z)} |\det U(y)| = e^{-W(z)},$$

so $U(y) \in G_\nu^{\text{lin}}$. □

Theorem 22.3 (Symmetry-valued gauge collapse on arbitrary intervals). *Let E be a Euclidean space, let $I \subset \mathbb{R}$ be an interval, let*

$$\nu(dz) = e^{-W(z)} dz, \quad m(dy) = e^{-\phi(y)} dy$$

be log-concave probability measures with $W \in C^2(E)$ convex, Legendre-type, and $\nabla^2 W(z)$ positive-definite for every $z \in E$, and with $\phi \in C^2(I)$ convex. Let

$$A : I \rightarrow \text{GL}(E), \quad b : I \rightarrow E$$

be C^2 , define

$$T(z, y) = (A(y)z + b(y), y), \quad \mu := T_\#(\nu \otimes m),$$

and assume that μ is log-concave on $E \times I$. Write

$$B(y) := A(y)^{-1}A'(y), \quad c(y) := A(y)^{-1}b''(y).$$

Assume that

$$B(y) \in \mathfrak{g}_\nu \quad (y \in I).$$

Then

$$b''(y) = 0 \quad (y \in I).$$

Hence b is affine on I .

Moreover, for any $y_0 \in I$ there exists a smooth family

$$U_y \in G_\nu^{\text{lin}} \quad (y \in I)$$

such that the gauge-equivalent bundle

$$\tilde{T}(z, y) := T(U_y z, y)$$

has constant linear part:

$$\tilde{T}(z, y) = (A(y_0)z + b(y), y).$$

Equivalently, every exact-affine bundle whose linear velocity lies entirely in the infinitesimal linear symmetry algebra of the reference fiber is pure gauge up to affine translation.

Proof. Fix $y \in I$ and abbreviate

$$K := B(y), \quad L := B'(y), \quad c := c(y).$$

Because $\mathfrak{g}_\nu$ is a linear subspace and B is C^1 , we have

$$L \in \mathfrak{g}_\nu.$$

Fix $z \in E$, and write

$$g := \nabla W(z), \quad H := \nabla^2 W(z).$$

Since $K \in \mathfrak{g}_\nu$,

$$\langle g, Kz \rangle = \text{Tr}(K)$$

is constant in z . Differentiating in z gives

$$HKz + K^T g = 0.$$

Consequently,

$$\langle K^T g, H^{-1} K^T g \rangle = \langle HKz, Kz \rangle.$$

Differentiate again the identity $\langle \nabla W(z), Kz \rangle = \text{Tr}(K)$ in the direction Kz . This yields

$$\langle HKz, Kz \rangle + \langle g, K^2 z \rangle = 0,$$

and therefore

$$\langle g, K^2 z \rangle + \langle K^T g, H^{-1} K^T g \rangle = 0.$$

Since $L \in \mathfrak{g}_\nu$, we also have

$$\langle g, Lz \rangle = \text{Tr}(L).$$

Now apply the exact Schur-complement inequality from [theorem 20.1](#). Since

$$M(y) = B'(y) + B(y)^2 = L + K^2,$$

we obtain

$$\langle g, Lz + K^2 z + c \rangle + \langle K^T g, H^{-1} K^T g \rangle \leq r''(y).$$

Using the previously established identities, this simplifies to

$$\text{Tr}(L) + \langle g, c \rangle \leq r''(y).$$

But

$$r''(y) = \text{Tr}(B'(y)) + \phi''(y) = \text{Tr}(L) + \phi''(y),$$

hence

$$\langle \nabla W(z), c(y) \rangle \leq \phi''(y) \quad (z \in E).$$

If $c(y) \neq 0$, then because $\nabla W(E) = E$ by the Legendre hypothesis, for every $t > 0$ there exists $z_t \in E$ with

$$\nabla W(z_t) = t \frac{c(y)}{|c(y)|}.$$

Substituting z_t into the previous inequality yields

$$t |c(y)| \leq \phi''(y) \quad (t > 0),$$

which is impossible. Therefore $c(y) = 0$. Since $y \in I$ was arbitrary,

$$c \equiv 0, \quad \text{i.e.} \quad b''(y) = A(y)c(y) = 0$$

for every $y \in I$. Thus b is affine.

For the gauge statement, fix $y_0 \in I$ and let $U : I \rightarrow \text{GL}(E)$ solve

$$U'(y) = -B(y)U(y), \quad U(y_0) = \text{Id}.$$

By [theorem 22.2](#), $U_y \in G_\nu^{\text{lin}}$ for every $y \in I$. Hence $(U_y)_\# \nu = \nu$, so

$$\tilde{T}_\#(\nu \otimes m) = T_\#(\nu \otimes m) = \mu$$

for

$$\tilde{T}(z, y) := T(U_y z, y).$$

Its linear part is

$$\tilde{A}(y) := A(y)U(y),$$

and

$$\tilde{A}'(y) = A'(y)U(y) + A(y)U'(y) = A(y)B(y)U(y) - A(y)B(y)U(y) = 0.$$

Therefore

$$\tilde{A}(y) \equiv A(y_0),$$

as claimed. □

Corollary 22.4 (Local symmetry-jet translation rigidity). *Under the hypotheses of [theorem 22.3](#), fix $y_0 \in I$. If*

$$B(y_0) \in \mathfrak{g}_\nu \quad \text{and} \quad B'(y_0) \in \mathfrak{g}_\nu,$$

then

$$b''(y_0) = 0.$$

Proof. The proof is the pointwise part of [theorem 22.3](#) carried out at the single parameter value y_0 . $\square$

Corollary 22.5 (Gauge-free symmetry-conjugate collapse). *Under the hypotheses of [theorem 22.3](#), assume that there exist a fixed matrix $A_0 \in \text{GL}(E)$ and a smooth curve*

$$U : I \rightarrow G_\nu^{\text{lin}}$$

such that

$$A(y) = A_0 U(y) \quad (y \in I).$$

Then b is affine on I .

Proof. Set

$$K(y) := U(y)^{-1} U'(y).$$

For each fixed $y \in I$, the curve

$$t \mapsto U(y)^{-1} U(y + t)$$

takes values in G_ν^{lin} and equals Id at $t = 0$. By the converse direction implicit in the proof of [theorem 22.2](#), its derivative at $t = 0$, namely $K(y)$, belongs to $\mathfrak{g}_\nu$. Since

$$B(y) = A(y)^{-1} A'(y) = U(y)^{-1} U'(y) = K(y),$$

[theorem 22.3](#) applies. $\square$

Corollary 22.6 (Pure rotational radial collapse on arbitrary intervals). *Let E be a Euclidean space, let*

$$\nu(\mathrm{d}z) = e^{-w(|z|)} \mathrm{d}z$$

be a radial log-concave probability measure on E , where $w \in C^2([0, \infty))$ is convex, $w'(0) = 0$, and $w'(r) \rightarrow +\infty$ as $r \rightarrow \infty$. Let $I \subset \mathbb{R}$ be an interval, let

$$m(\mathrm{d}y) = e^{-\phi(y)} \mathrm{d}y$$

be log-concave on I with $\phi \in C^2(I)$, and let

$$A(y) = A_0 O(y), \quad b : I \rightarrow E$$

where $A_0 \in \text{GL}(E)$, $O : I \rightarrow \text{O}(E)$ is C^2 , and b is C^2 . Define

$$T(z, y) = (A(y)z + b(y), y), \quad \mu := T_\#(\nu \otimes m),$$

and assume that μ is log-concave on $E \times I$. Then b is affine on I . Moreover, after composing with the orthogonal gauge $O(y)^{-1}$, the linear part becomes constant on I .

Proof. For every skew-symmetric matrix K ,

$$\langle \nabla W(z), Kz \rangle = w'(|z|) \frac{z}{|z|} \cdot Kz = 0,$$

so $\mathfrak{so}(E) \subset \mathfrak{g}_\nu$. Since

$$B(y) = A(y)^{-1} A'(y) = O(y)^{-1} O'(y) \in \mathfrak{so}(E),$$

[theorem 22.3](#) applies. $\square$

Remark 22.7. Theorem 22.3 removes an exact branch that remained open after the affine-base analysis and after the quantitative symmetry-gap section: motion that stays *inside* the infinitesimal fiber symmetry algebra is never a genuine obstruction, even on bounded intervals with curved base. Consequently the unresolved bounded-interval branch must not only have nonzero curvature budget, but must also carry a genuinely non-symmetry component of the linear velocity after every admissible symmetry gauge.

23 Even-fiber translation budget on curved bases

Section 21 controlled the component of the linear velocity orthogonal to the infinitesimal fiber symmetry algebra, while Section 22 showed that motion inside that symmetry algebra is already pure gauge. What was still missing from the curved-base analysis was any direct control of the normalized translation acceleration

$$c(y) := A(y)^{-1}b''(y).$$

For even reference fibers this can be extracted directly from the exact Schur complement by pairing the inequalities at opposite points z and $-z$.

Definition 23.1 (Translation functional and translation-gap constant). Let

$$\nu(\mathrm{d}z) = e^{-W(z)} \mathrm{d}z$$

be a log-concave probability measure on E with $W \in C^2(E)$. For $v \in E$, define

$$\mathcal{T}_\nu(v) := \int_E |\langle \nabla W(z), v \rangle| \nu(\mathrm{d}z).$$

This functional is even and positively homogeneous. We also define the translation-gap constant

$$\tau_\nu := \inf_{|u|=1} \mathcal{T}_\nu(u).$$

Proposition 23.2 (Opposite-point Schur strengthening for even fibers). *Under the hypotheses of theorem 20.1, assume in addition that W is even. Fix $y \in I$, $z \in E$, and abbreviate*

$$g := \nabla W(z), \quad H := \nabla^2 W(z).$$

Then

$$\langle g, M(y)z \rangle + \langle B(y)^T g, H^{-1} B(y)^T g \rangle + |\langle g, c(y) \rangle| \leq r''(y).$$

Proof. By theorem 20.1, applied at the point z ,

$$\langle g, M(y)z + c(y) \rangle + \langle B(y)^T g, H^{-1} B(y)^T g \rangle \leq r''(y).$$

Since W is even, we have

$$\nabla W(-z) = -\nabla W(z) = -g, \quad \nabla^2 W(-z) = \nabla^2 W(z) = H.$$

Applying theorem 20.1 at the opposite point $-z$ therefore yields

$$\langle g, M(y)z - c(y) \rangle + \langle B(y)^T g, H^{-1} B(y)^T g \rangle \leq r''(y).$$

Taking the maximum of these two inequalities proves the claim. $\square$

Theorem 23.3 (Even-fiber translation-budget inequality). *Under the hypotheses of [theorem 13.1](#), assume in addition that W is even. Then for every $y \in I$,*

$$\mathcal{T}_\nu(c(y)) + \mathcal{J}_\nu(B(y)) + \text{Tr}(B(y)^2) \leq \phi''(y).$$

Equivalently,

$$\mathcal{T}_\nu(c(y)) + \mathcal{Q}_\nu(B(y)) \leq \phi''(y).$$

Proof. Integrate the inequality from [theorem 23.2](#) with respect to ν . As in the proof of [theorem 20.3](#),

$$\int_E \langle \nabla W(z), M(y)z \rangle \nu(dz) = \text{Tr}(M(y)).$$

Hence

$$\text{Tr}(M(y)) + \mathcal{J}_\nu(B(y)) + \mathcal{T}_\nu(c(y)) \leq r''(y).$$

Now

$$r''(y) = \text{Tr}(B'(y)) + \phi''(y)$$

and

$$\text{Tr}(M(y)) = \text{Tr}(B'(y)) + \text{Tr}(B(y)^2).$$

Subtracting $\text{Tr}(M(y))$ from both sides yields

$$\mathcal{T}_\nu(c(y)) + \mathcal{J}_\nu(B(y)) + \text{Tr}(B(y)^2) \leq \phi''(y),$$

which is the desired estimate. $\square$

Proposition 23.4 (Directional lower bound via line maxima). *Let $u \in E$ be a unit vector and let $Z \sim \nu$. Write f_u for the density of the one-dimensional marginal $\langle Z, u \rangle$. Then*

$$\mathcal{T}_\nu(u) = 2 \int_{u^\perp} \sup_{t \in \mathbb{R}} e^{-W(v+tu)} dv \geq 2f_u(0).$$

In particular, if W is Legendre-type then $\mathcal{T}_\nu(u) > 0$ for every unit u .

Proof. Decompose $z = v + tu$ with $v \in u^\perp$, $t \in \mathbb{R}$. Since

$$\partial_t e^{-W(v+tu)} = -\langle \nabla W(v+tu), u \rangle e^{-W(v+tu)},$$

Fubini's theorem gives

$$\mathcal{T}_\nu(u) = \int_{u^\perp} \int_{\mathbb{R}} |\partial_t e^{-W(v+tu)}| dt dv.$$

For fixed v , the function

$$t \mapsto e^{-W(v+tu)}$$

is positive, log-concave, integrable, and therefore unimodal on $\mathbb{R}$. Its total variation is twice its maximum. Hence

$$\int_{\mathbb{R}} |\partial_t e^{-W(v+tu)}| dt = 2 \sup_{t \in \mathbb{R}} e^{-W(v+tu)}.$$

Integrating in v proves the first identity.

The marginal density of $\langle Z, u \rangle$ is

$$f_u(s) = \int_{u^\perp} e^{-W(v+su)} dv.$$

Therefore

$$\mathcal{T}_\nu(u) = 2 \int_{u^\perp} \sup_{t \in \mathbb{R}} e^{-W(v+tu)} \, dv \geq 2 \int_{u^\perp} e^{-W(v)} \, dv = 2f_u(0).$$

If W is Legendre-type, then e^{-W} is strictly positive on all of E , hence $f_u(0) > 0$ for every unit u , and therefore $\mathcal{T}_\nu(u) > 0$ for every unit u . $\square$

Corollary 23.5 (Universal isotropic even translation gap). *Assume in addition that ν is isotropic and even. Then*

$$\tau_\nu \geq \frac{1}{\sqrt{3}}.$$

Proof. Fix a unit vector $u \in E$, and let f_u be the density of $\langle Z, u \rangle$. Because ν is isotropic and even, f_u is an even log-concave probability density on $\mathbb{R}$ with variance

$$\text{Var}(\langle Z, u \rangle) = 1.$$

Set

$$m := f_u(0).$$

Then $0 \leq f_u(t) \leq m$ for every $t \in \mathbb{R}$. Among all even probability densities g on $\mathbb{R}$ satisfying $0 \leq g \leq m$, the second moment is minimized by the bathtub extremizer

$$g_*(t) = m \mathbb{1}_{[-1/(2m), 1/(2m)]}(t),$$

because t^2 is increasing in $|t|$. Consequently,

$$1 = \int_{\mathbb{R}} t^2 f_u(t) \, dt \geq \int_{\mathbb{R}} t^2 g_*(t) \, dt = \frac{1}{12m^2}.$$

Hence

$$m = f_u(0) \geq \frac{1}{\sqrt{12}}.$$

Applying [theorem 23.4](#) gives

$$\mathcal{T}_\nu(u) \geq 2f_u(0) \geq \frac{1}{\sqrt{3}}.$$

Since u was arbitrary, $\tau_\nu \geq 1/\sqrt{3}$. $\square$

Theorem 23.6 (Combined curved-base motion budget over isotropic even fibers). *Under the hypotheses of [theorem 23.3](#), assume moreover that ν is isotropic. Then for every $y \in I$,*

$$\tau_\nu |c(y)| + \lambda_\nu \text{dist}_{\text{HS}}(B(y), \mathfrak{k}_\nu)^2 \leq \phi''(y).$$

In particular,

$$|A(y)^{-1}b''(y)| \leq \tau_\nu^{-1}\phi''(y) \leq \sqrt{3}\phi''(y),$$

and

$$\text{dist}_{\text{HS}}(A(y)^{-1}A'(y), \mathfrak{k}_\nu) \leq \lambda_\nu^{-1/2} \sqrt{\phi''(y)}.$$

Proof. By [theorem 23.1](#),

$$\mathcal{T}_\nu(c(y)) \geq \tau_\nu |c(y)|.$$

By [theorem 21.3](#),

$$\mathcal{Q}_\nu(B(y)) \geq \lambda_\nu \text{dist}_{\text{HS}}(B(y), \mathfrak{k}_\nu)^2.$$

Insert both lower bounds into [theorem 23.3](#). The universal estimate $\tau_\nu^{-1} \leq \sqrt{3}$ follows from [theorem 23.5](#). $\square$

Corollary 23.7 (Combined orthogonal-gauge budget). *Under the hypotheses of [theorem 23.6](#), let $O, \tilde{A}, \tilde{B}$ be the orthogonal gauge from [theorem 21.4](#), and set*

$$\tilde{c}(y) := \tilde{A}(y)^{-1}b''(y).$$

Then for every $y \in I$,

$$\tau_\nu |\tilde{c}(y)| + \lambda_\nu \left\| \tilde{B}(y) \right\|_{\text{HS}}^2 \leq \phi''(y).$$

In particular,

$$|\tilde{A}(y)^{-1}b''(y)| \leq \sqrt{3} \phi''(y).$$

Proof. By [theorem 21.4](#),

$$\tilde{B}(y) = O(y)^T C(y) O(y),$$

where $C(y)$ is the Hilbert–Schmidt orthogonal projection of $B(y)$ onto $\mathfrak{k}_\nu^\perp$. Hence

$$\left\| \tilde{B}(y) \right\|_{\text{HS}} = \text{dist}_{\text{HS}}(B(y), \mathfrak{k}_\nu).$$

Moreover,

$$\tilde{c}(y) = \tilde{A}(y)^{-1}b''(y) = O(y)^T A(y)^{-1}b''(y) = O(y)^T c(y),$$

so $|\tilde{c}(y)| = |c(y)|$. The claim follows from [theorem 23.6](#). $\square$

Corollary 23.8 (Flat-base point rigidity for even fibers). *Under the hypotheses of [theorem 23.6](#), fix $y_0 \in I$. If*

$$\phi''(y_0) = 0,$$

then

$$A(y_0)^{-1}b''(y_0) = 0 \quad \text{and} \quad A(y_0)^{-1}A'(y_0) \in \mathfrak{k}_\nu.$$

If in addition $\mathfrak{k}_\nu = \{0\}$, then

$$A'(y_0) = 0, \quad b''(y_0) = 0.$$

Proof. Apply [theorem 23.6](#) at the single point y_0 . Both nonnegative terms on the left must vanish. If $\mathfrak{k}_\nu = \{0\}$, then $B(y_0) = 0$, and the stated identities follow after multiplying by $A(y_0)$. $\square$

Corollary 23.9 (Gaussian specialization). *If $\nu = \gamma_E$ is the standard Gaussian on E , then for every $y \in I$,*

$$\sqrt{\frac{2}{\pi}} |A(y)^{-1}b''(y)| + 2 \left\| \text{Sym}(A(y)^{-1}A'(y)) \right\|_{\text{HS}}^2 \leq \phi''(y).$$

Proof. For the Gaussian fiber,

$$W(z) = \frac{1}{2}|z|^2, \quad \nabla W(z) = z.$$

Hence for every $v \in E$,

$$\mathcal{T}_{\gamma_E}(v) = \int_E |\langle z, v \rangle| \gamma_E(dz) = \sqrt{\frac{2}{\pi}} |v|.$$

The identity

$$\mathcal{Q}_{\gamma_E}(L) = 2 \left\| \text{Sym} L \right\|_{\text{HS}}^2$$

was proved in [theorem 21.9](#). Insert these formulas into [theorem 23.3](#). $\square$

Remark 23.10. Theorem 23.3 is precisely the curved-base translation-control statement that was still missing in the previous frontier analysis. In the even-fiber world, an unresolved bounded-interval obstruction must now spend base-curvature budget simultaneously on normalized translation acceleration and on genuinely non-symmetry linear motion. At a point where $\phi'' = 0$, neither can be present.

24 Translation-slope control and asymptotic affine rigidity on one-sided tails

The quantitative orthogonal-gauge theorem controls the component of the linear velocity outside the infinitesimal symmetry algebra, while the even-fiber translation budget controls the normalized translation acceleration. Combining the two gives a genuinely C^1 -quantitative refinement of the exact-affine frontier.

Corollary 24.1 (Compact-interval translation-slope budget). *Under the hypotheses of theorem 23.6, let $O : I \rightarrow O(E)$, $\tilde{A}$, $\tilde{B}$ be the orthogonal gauge from theorem 21.4, and set*

$$\tilde{c}(y) := \tilde{A}(y)^{-1}b''(y).$$

Fix $y_0 \in I$. Then for every $y \in I$,

$$|b'(y) - b'(y_0)| \leq M(y_0, y) \tau_\nu^{-1} \int_{\min\{y, y_0\}}^{\max\{y, y_0\}} \phi''(s) \, ds,$$

where

$$M(y_0, y) := \sup_{s \in [\min\{y, y_0\}, \max\{y, y_0\}]} \|\tilde{A}(s)\|_{\text{op}} \leq \|\tilde{A}(y_0)\|_{\text{op}} \exp\left(\lambda_\nu^{-1/2} \int_{\min\{y, y_0\}}^{\max\{y, y_0\}} \sqrt{\phi''(s)} \, ds\right).$$

In particular, on every compact interval the translation slope is controlled directly by the integrated base-curvature budget.

Proof. Assume first that $y \geq y_0$; the case $y \leq y_0$ is identical after reversing the interval. Since the orthogonal gauge leaves the translation part unchanged,

$$b''(s) = \tilde{A}(s)\tilde{c}(s) \quad (s \in [y_0, y]).$$

Therefore

$$|b'(y) - b'(y_0)| \leq \int_{y_0}^y |b''(s)| \, ds \leq M(y_0, y) \int_{y_0}^y |\tilde{c}(s)| \, ds.$$

By theorem 23.7,

$$\tau_\nu |\tilde{c}(s)| \leq \phi''(s) \quad (s \in I),$$

which proves the first bound. The estimate for $M(y_0, y)$ follows from the distortion bound in theorem 21.4. $\square$

Theorem 24.2 (Tail asymptotic affine rigidity with translation). *Under the hypotheses of theorem 23.6, assume that $I = [y_0, \infty)$ and that*

$$\int_{y_0}^{\infty} \sqrt{\phi''(s)} \, ds < \infty, \quad \int_{y_0}^{\infty} \phi''(s) \, ds < \infty.$$

Let O , $\tilde{A}$, $\tilde{B}$ be the orthogonal gauge from [theorem 21.4](#), and set

$$\tilde{c}(y) := \tilde{A}(y)^{-1}b''(y).$$

Then there exist $A_\infty \in \text{GL}(E)$ and $v_\infty \in E$ such that

$$\tilde{A}(y) \rightarrow A_\infty \quad \text{and} \quad b'(y) \rightarrow v_\infty \quad (y \rightarrow \infty).$$

Moreover, with

$$M_\infty := \sup_{s \geq y_0} \|\tilde{A}(s)\|_{\text{op}} < \infty,$$

one has for every $y \geq y_0$,

$$\|A_\infty - \tilde{A}(y)\|_{\text{op}} \leq M_\infty \lambda_\nu^{-1/2} \int_y^\infty \sqrt{\phi''(s)} \, ds$$

and

$$|v_\infty - b'(y)| \leq M_\infty \tau_\nu^{-1} \int_y^\infty \phi''(s) \, ds.$$

The same conclusion holds on a left tail $I = (-\infty, y_0]$ under the analogous tail-integrability assumptions there.

Proof. By [theorem 21.7](#), $\tilde{A}(y)$ converges in $\text{GL}(E)$ to some matrix A_∞ , and $\tilde{A}, \tilde{A}^{-1}$ are uniformly bounded on $[y_0, \infty)$. In particular $M_\infty < \infty$. Since

$$\tilde{A}'(s) = \tilde{A}(s)\tilde{B}(s),$$

we obtain for every $y \geq y_0$,

$$\|A_\infty - \tilde{A}(y)\|_{\text{op}} \leq \int_y^\infty \|\tilde{A}'(s)\|_{\text{op}} \, ds \leq M_\infty \int_y^\infty \|\tilde{B}(s)\|_{\text{op}} \, ds.$$

Using $\|\tilde{B}(s)\|_{\text{op}} \leq \|\tilde{B}(s)\|_{\text{HS}}$ and [theorem 23.7](#),

$$\|\tilde{B}(s)\|_{\text{op}} \leq \lambda_\nu^{-1/2} \sqrt{\phi''(s)},$$

which yields the first quantitative estimate.

Next,

$$|b''(s)| = |\tilde{A}(s)\tilde{c}(s)| \leq M_\infty |\tilde{c}(s)| \leq M_\infty \tau_\nu^{-1} \phi''(s) \quad (s \geq y_0),$$

again by [theorem 23.7](#). Since $\phi'' \in L^1([y_0, \infty))$, the function b'' is integrable on $[y_0, \infty)$, and therefore $b'(y)$ is Cauchy and converges to some vector $v_\infty \in E$. The same estimate implies

$$|v_\infty - b'(y)| \leq \int_y^\infty |b''(s)| \, ds \leq M_\infty \tau_\nu^{-1} \int_y^\infty \phi''(s) \, ds,$$

which is the claimed tail bound. The left-tail case is identical. $\square$

Corollary 24.3 (Asymptotic affine product under first-moment curvature tail). *Under the hypotheses of [theorem 24.2](#), assume in addition that*

$$\int_{y_0}^{\infty} (s - y_0) \phi''(s) \, ds < \infty.$$

Then there exists $b_{\infty} \in E$ such that

$$b(y) - yv_{\infty} \rightarrow b_{\infty} \quad (y \rightarrow \infty),$$

and for every $y \geq y_0$,

$$|b(y) - yv_{\infty} - b_{\infty}| \leq M_{\infty} \tau_{\nu}^{-1} \int_y^{\infty} (s - y) \phi''(s) \, ds.$$

Consequently, after orthogonal gauge the bundle is asymptotic on the tail to the affine product

$$(z, y) \mapsto (A_{\infty} z + b_{\infty} + yv_{\infty}, y).$$

Proof. Set

$$g(y) := b(y) - yv_{\infty}.$$

Then $g'(y) = b'(y) - v_{\infty}$. By [theorem 24.2](#),

$$|g'(y)| \leq M_{\infty} \tau_{\nu}^{-1} \int_y^{\infty} \phi''(s) \, ds.$$

Hence

$$\int_{y_0}^{\infty} |g'(u)| \, du \leq M_{\infty} \tau_{\nu}^{-1} \int_{y_0}^{\infty} \int_u^{\infty} \phi''(s) \, ds \, du = M_{\infty} \tau_{\nu}^{-1} \int_{y_0}^{\infty} (s - y_0) \phi''(s) \, ds < \infty.$$

Therefore g is Cauchy at infinity and converges to some vector $b_{\infty} \in E$. Moreover,

$$|g(y) - b_{\infty}| \leq \int_y^{\infty} |g'(u)| \, du \leq M_{\infty} \tau_{\nu}^{-1} \int_y^{\infty} \int_u^{\infty} \phi''(s) \, ds \, du = M_{\infty} \tau_{\nu}^{-1} \int_y^{\infty} (s - y) \phi''(s) \, ds.$$

This is exactly the claimed remainder bound. $\square$

Corollary 24.4 (Flat-tail exact collapse). *Under the hypotheses of [theorem 23.6](#), assume that ϕ is affine on $[y_1, \infty)$. Then, after the orthogonal gauge from [theorem 21.4](#), the linear part is constant and b is affine on $[y_1, \infty)$. If in addition $\mathfrak{k}_{\nu} = \{0\}$, then A itself is constant and b is affine on $[y_1, \infty)$.*

Proof. On $[y_1, \infty)$ one has $\phi'' \equiv 0$. By [theorem 23.7](#),

$$\tilde{B}(y) = 0, \quad \tilde{c}(y) = 0 \quad (y \geq y_1),$$

hence $\tilde{A}$ is constant and $b''(y) = \tilde{A}(y) \tilde{c}(y) = 0$ on that tail. If $\mathfrak{k}_{\nu} = \{0\}$, then [theorem 21.8](#) gives $B(y) = 0$ whenever $\phi''(y) = 0$, so A itself is constant. $\square$

25 One-parameter subgroup motion on curved bases

The full-line subgroup theorem from [theorem 17.6](#) eliminates every multiplicative one-parameter branch whose generator carries a nonzero real spectral mode after a fixed affine normalization. On bounded or one-sided bases, however, one can say something unconditional for *every* generator. The point is that for

$$A(y) = A_0 e^{s(y)G}$$

the exact-affine velocity is

$$B(y) = A(y)^{-1} A'(y) = s'(y)G,$$

so the general curved-base inequalities collapse to scalar budget estimates for the parameter s .

Proposition 25.1 (Curvature budget for one-parameter subgroup motion). *Let E be a Euclidean space, let $I \subset \mathbb{R}$ be an interval, let*

$$\nu(\mathrm{d}z) = e^{-W(z)} \mathrm{d}z, \quad m(\mathrm{d}y) = e^{-\phi(y)} \mathrm{d}y$$

be log-concave probability measures with $W \in C^2(E)$ convex, $\nabla^2 W(z)$ positive-definite for every $z \in E$, and $\phi \in C^2(I)$ convex. Let

$$A(y) = A_0 e^{s(y)G}, \quad b : I \rightarrow E,$$

where $A_0 \in \mathrm{GL}(E)$, $G \in \mathrm{End}(E)$, $s \in C^2(I)$, $b \in C^2(I; E)$, and define

$$T(z, y) = (A(y)z + b(y), y), \quad \mu := T_{\#}(\nu \otimes m).$$

Assume that μ is log-concave on $E \times I$. Then for every $y \in I$,

$$\mathcal{Q}_{\nu}(G) |s'(y)|^2 \leq \phi''(y).$$

If in addition ν is isotropic and Legendre-type, then

$$\lambda_{\nu} \operatorname{dist}_{\mathrm{HS}}(G, \mathfrak{k}_{\nu})^2 |s'(y)|^2 \leq \phi''(y) \quad (y \in I).$$

Proof. Because G commutes with $e^{s(y)G}$,

$$A'(y) = A_0 e^{s(y)G} s'(y)G = A(y) s'(y)G,$$

hence

$$B(y) = A(y)^{-1} A'(y) = s'(y)G.$$

Applying [theorem 20.3](#) gives

$$\mathcal{Q}_{\nu}(B(y)) \leq \phi''(y).$$

Since $\mathcal{Q}_{\nu}$ is quadratic,

$$\mathcal{Q}_{\nu}(B(y)) = \mathcal{Q}_{\nu}(s'(y)G) = |s'(y)|^2 \mathcal{Q}_{\nu}(G),$$

which proves the first inequality. Under the isotropic hypotheses, [theorem 21.3](#) yields

$$\mathcal{Q}_{\nu}(G) \geq \lambda_{\nu} \operatorname{dist}_{\mathrm{HS}}(G, \mathfrak{k}_{\nu})^2.$$

Insert this bound into the first estimate. □

Corollary 25.2 (Subgroup slope budget on intervals and tails). *Under the hypotheses of [theorem 25.1](#), assume that $\mathcal{Q}_\nu(G) > 0$. Then for every $y_1, y_2 \in I$,*

$$|s(y_2) - s(y_1)| \leq \mathcal{Q}_\nu(G)^{-1/2} \int_{[y_1, y_2]} \sqrt{\phi''(u)} \, du.$$

If $I = [y_0, \infty)$ and

$$\int_{y_0}^{\infty} \sqrt{\phi''(u)} \, du < \infty,$$

then $s(y)$ converges to a finite limit s_∞ as $y \rightarrow \infty$, and for every $y \geq y_0$,

$$|s(y) - s_\infty| \leq \mathcal{Q}_\nu(G)^{-1/2} \int_y^{\infty} \sqrt{\phi''(u)} \, du.$$

Under the isotropic hypotheses of [theorem 25.1](#), if $G \notin \mathfrak{k}_\nu$ then the same conclusions hold with the explicit constant

$$(\sqrt{\lambda_\nu} \operatorname{dist}_{\text{HS}}(G, \mathfrak{k}_\nu))^{-1}.$$

Proof. By [theorem 25.1](#),

$$|s'(y)| \leq \mathcal{Q}_\nu(G)^{-1/2} \sqrt{\phi''(y)}.$$

Integrating on the interval between y_1 and y_2 proves the first estimate. The tail estimate follows by integrating from y to $+\infty$; finiteness of the tail integral implies that s is Cauchy at infinity. The isotropic version is immediate from the second inequality in [theorem 25.1](#). $\square$

Proposition 25.3 (Combined subgroup and translation budget over even fibers). *Under the hypotheses of [theorem 25.1](#), assume in addition that W is even. Then for every $y \in I$,*

$$\mathcal{T}_\nu(A(y)^{-1}b''(y)) + \mathcal{Q}_\nu(G) |s'(y)|^2 \leq \phi''(y).$$

If moreover ν is isotropic, then

$$\tau_\nu |A(y)^{-1}b''(y)| + \lambda_\nu \operatorname{dist}_{\text{HS}}(G, \mathfrak{k}_\nu)^2 |s'(y)|^2 \leq \phi''(y).$$

Proof. From [theorem 23.3](#) and the identity $B(y) = s'(y)G$,

$$\mathcal{T}_\nu(A(y)^{-1}b''(y)) + \mathcal{Q}_\nu(B(y)) \leq \phi''(y),$$

hence

$$\mathcal{T}_\nu(A(y)^{-1}b''(y)) + \mathcal{Q}_\nu(G) |s'(y)|^2 \leq \phi''(y).$$

If ν is isotropic, combine the lower bound

$$\mathcal{T}_\nu(v) \geq \tau_\nu |v|$$

with [theorem 21.3](#) exactly as in the proof of [theorem 23.6](#). $\square$

Corollary 25.4 (Symmetry versus curvature for subgroup motion). *Let $I \subset \mathbb{R}$ be an interval and assume the hypotheses of [theorem 25.1](#). If $G \in \mathfrak{g}_\nu$, then the subgroup motion is pure gauge on I and b is affine. If ν is isotropic and $G \notin \mathfrak{k}_\nu$, then every nontrivial variation of the subgroup parameter s pays the explicit curvature budget from [theorem 25.1](#); and if W is even, then the same base curvature simultaneously controls the normalized translation acceleration through [theorem 25.3](#).*

Proof. If $G \in \mathfrak{g}_\nu$, then

$$B(y) = s'(y)G \in \mathfrak{g}_\nu \quad (y \in I).$$

Hence [theorem 22.3](#) applies and shows that the linear motion is pure gauge while b is affine. The second statement is exactly [theorems 25.1](#) and [25.3](#). $\square$

Remark 25.5. [Theorems 25.1](#) and [25.3](#) provide the first exact curved-base control theorem for the genuinely spectrally real-free subgroup branch left open by [theorem 17.6](#). They show that, on bounded or one-sided bases, the unresolved multiplicative one-parameter branch can no longer hide merely by avoiding real eigenvalues: unless its generator is an actual infinitesimal symmetry of the reference fiber, its variation is quantitatively visible to the same base-curvature budget that controls the general linear velocity.

26 Additive one-parameter motion, unique symmetry parameters, and Möbius-symmetry reduction

The full-line fixed-direction theorem from [theorem 17.1](#) eliminates additive fixed directions whose normalized direction matrix carries a nonzero real spectral mode. On bounded or one-sided bases, however, the general curved-base inequalities again collapse to scalar control of the parameter. Unlike multiplicative subgroup motion, additive motion has a subtler possible interaction with symmetry: the normalized velocity

$$\Gamma_N(t) := (I + tN)^{-1}N$$

may in principle fall inside the infinitesimal orthogonal symmetry algebra even when $N \notin \mathfrak{k}_\nu$. The point of this section is that this exceptional alignment is completely rigid: for nonzero N it can occur at most at one parameter value, and exactly then N is a fixed Möbius transform of an actual symmetry generator.

Proposition 26.1 (Curvature budget for additive fixed-direction motion). *Let E be a Euclidean space, let $I \subset \mathbb{R}$ be an interval, let*

$$\nu(dz) = e^{-W(z)} dz, \quad m(dy) = e^{-\phi(y)} dy$$

be log-concave probability measures with $W \in C^2(E)$ convex, $\nabla^2 W(z)$ positive-definite for every $z \in E$, and $\phi \in C^2(I)$ convex. Let

$$A(y) = A_0(I + s(y)N), \quad b : I \rightarrow E,$$

where $A_0 \in \text{GL}(E)$, $N \in \text{End}(E)$, $s \in C^2(I)$, $b \in C^2(I; E)$, and

$$I + s(y)N \in \text{GL}(E) \quad (y \in I).$$

Define

$$T(z, y) = (A(y)z + b(y), y), \quad \mu := T_\#(\nu \otimes m),$$

and assume that μ is log-concave on $E \times I$. For t in the invertibility set

$$\Omega_N := \{t \in \mathbb{R} : I + tN \in \text{GL}(E)\},$$

set

$$\Gamma_N(t) := (I + tN)^{-1}N.$$

Then for every $y \in I$,

$$\mathcal{Q}_\nu(\Gamma_N(s(y))) |s'(y)|^2 \leq \phi''(y).$$

If in addition ν is isotropic and Legendre-type, then

$$\lambda_\nu \operatorname{dist}_{\text{HS}}(\Gamma_N(s(y)), \mathfrak{k}_\nu)^2 |s'(y)|^2 \leq \phi''(y) \quad (y \in I).$$

Proof. Differentiate $A(y) = A_0(I + s(y)N)$:

$$A'(y) = A_0 s'(y)N = A_0(I + s(y)N) s'(y)(I + s(y)N)^{-1}N = A(y) s'(y)\Gamma_N(s(y)).$$

Hence

$$B(y) := A(y)^{-1}A'(y) = s'(y)\Gamma_N(s(y)).$$

Applying [theorem 20.3](#) yields

$$\mathcal{Q}_\nu(B(y)) \leq \phi''(y).$$

Since $\mathcal{Q}_\nu$ is quadratic,

$$\mathcal{Q}_\nu(B(y)) = |s'(y)|^2 \mathcal{Q}_\nu(\Gamma_N(s(y))),$$

which proves the first claim. Under the isotropic hypotheses, [theorem 21.3](#) gives

$$\mathcal{Q}_\nu(L) \geq \lambda_\nu \operatorname{dist}_{\text{HS}}(L, \mathfrak{k}_\nu)^2 \quad (L \in \operatorname{End}(E)),$$

and the second estimate follows. □

Proposition 26.2 (Unique symmetry parameter and Möbius-symmetry classification). *Assume that ν is isotropic, Legendre-type, and that $\nabla^2 W(z)$ is positive-definite for every $z \in E$. Let $N \in \operatorname{End}(E)$ and define Ω_N and Γ_N as above. Set*

$$Z_{N,\nu} := \{t \in \Omega_N : \Gamma_N(t) \in \mathfrak{k}_\nu\}.$$

Then the following are equivalent for $t_* \in \Omega_N$:

- (i) $t_* \in Z_{N,\nu}$;
- (ii) there exists $K_* \in \mathfrak{k}_\nu$ such that

$$N = K_*(I - t_*K_*)^{-1}.$$

Whenever these conditions hold,

$$K_* = \Gamma_N(t_*), \quad I + sN = (I + (s - t_*)K_*)(I - t_*K_*)^{-1} \quad (s \in \Omega_N),$$

and, if $N \neq 0$, the parameter t_* is unique. Consequently, for nonzero N , the set $Z_{N,\nu}$ is either empty or a singleton.

Proof. Assume first that $t_* \in Z_{N,\nu}$, and set

$$K_* := \Gamma_N(t_*) \in \mathfrak{k}_\nu.$$

Since $\Gamma_N(t_*)$ is a rational function of N , it commutes with N . Moreover

$$I - t_* K_* = I - t_*(I + t_* N)^{-1} N = (I + t_* N)^{-1}.$$

Hence

$$N = \Gamma_N(t_*) (I - t_* \Gamma_N(t_*))^{-1} = K_* (I - t_* K_*)^{-1},$$

which proves (ii).

Conversely, assume that $N = K_* (I - t_* K_*)^{-1}$ for some $K_* \in \mathfrak{k}_\nu$. Since K_* commutes with every polynomial in K_* ,

$$I + t_* N = I + t_* K_* (I - t_* K_*)^{-1} = (I - t_* K_*)^{-1},$$

so

$$\Gamma_N(t_*) = (I + t_* N)^{-1} N = (I - t_* K_*) K_* (I - t_* K_*)^{-1} = K_* \in \mathfrak{k}_\nu.$$

Thus $t_* \in Z_{N,\nu}$, proving (i).

Whenever (i)–(ii) hold, the factorization

$$I + sN = I + sK_* (I - t_* K_*)^{-1} = (I + (s - t_*) K_*) (I - t_* K_*)^{-1}$$

is immediate.

It remains to prove uniqueness for $N \neq 0$. Suppose $t_1, t_2 \in Z_{N,\nu}$ with corresponding

$$K_i := \Gamma_N(t_i) \in \mathfrak{k}_\nu \subset \mathfrak{so}(E) \quad (i = 1, 2).$$

By the equivalence already proved,

$$N = K_i (I - t_i K_i)^{-1} \quad (i = 1, 2).$$

Fix $i \in \{1, 2\}$. Because $K_i^T = -K_i$ and K_i commutes with its own resolvents,

$$\begin{aligned} N + N^T &= K_i (I - t_i K_i)^{-1} - (I + t_i K_i)^{-1} K_i \\ &= K_i \left((I - t_i K_i)^{-1} - (I + t_i K_i)^{-1} \right) \\ &= 2t_i K_i^2 (I + t_i^2 K_i^2)^{-1}, \end{aligned}$$

while

$$\begin{aligned} N^T N &= (-(I + t_i K_i)^{-1} K_i) K_i (I - t_i K_i)^{-1} \\ &= -K_i^2 (I + t_i^2 K_i^2)^{-1}. \end{aligned}$$

Hence

$$N + N^T = -2t_i N^T N \quad (i = 1, 2).$$

Subtracting the two identities yields

$$2(t_1 - t_2) N^T N = 0.$$

If $N \neq 0$, then $N^T N \neq 0$, so $t_1 = t_2$. This proves uniqueness. $\square$

Corollary 26.3 (Easy obstructions to symmetry alignment for additive directions). *Under the hypotheses of [theorem 26.2](#), assume that $N \neq 0$. If N is nonnormal or if $\text{Sym}N$ has mixed sign, then*

$$Z_{N,\nu} = \emptyset.$$

More generally, if $Z_{N,\nu} = \{t_\}$, then*

$$\text{Sym}N = -t_*N^TN,$$

so $\text{Sym}N$ is semidefinite with sign $-\text{sgn}(t_)$.*

Proof. If $Z_{N,\nu} = \{t_*\}$, then by [theorem 26.2](#),

$$N = K_*(I - t_*K_*)^{-1}$$

for some $K_* \in \mathfrak{k}_\nu \subset \mathfrak{so}(E)$. Hence N is a rational function of the skew-symmetric matrix K_* , so N is normal. The computation in the proof of [theorem 26.2](#) gives

$$N + N^T = -2t_*N^TN,$$

i.e.

$$\text{Sym}N = -t_*N^TN.$$

Therefore $\text{Sym}N$ is semidefinite. The stated contrapositive follows. $\square$

Remark 26.4 (Spectral geometry of the exceptional additive direction). If $Z_{N,\nu} = \{t_*\}$ and $N \neq 0$, then [theorem 26.2](#) writes N as a rational function of a skew-symmetric matrix $K_* \in \mathfrak{k}_\nu$. By the spectral mapping theorem,

$$\sigma(N) = \left\{ \frac{i\beta}{1 - t_*i\beta} : i\beta \in \sigma(K_*) \right\}.$$

In particular, if $t_* \neq 0$, every eigenvalue of N lies on the circle

$$\left\{ \lambda \in \mathbb{C} : \left| \lambda + \frac{1}{2t_*} \right| = \frac{1}{2|t_*|} \right\}.$$

Thus the only nonzero additive fixed directions that can ever touch the symmetry algebra are exactly the Möbius images of genuine symmetry generators.

Corollary 26.5 (Additive slope budget away from symmetry). *Assume the hypotheses of [theorem 26.1](#) and the isotropic Legendre-type hypotheses of [theorem 26.2](#). Let $J \subset \Omega_N$ be compact and assume that*

$$J \cap Z_{N,\nu} = \emptyset.$$

Define

$$\delta_{N,\nu,J} := \inf_{t \in J} \text{dist}_{\text{HS}}(\Gamma_N(t), \mathfrak{k}_\nu).$$

Then $\delta_{N,\nu,J} > 0$, and whenever $y_1, y_2 \in I$ satisfy

$$s([y_1, y_2]) \subset J,$$

one has

$$|s(y_2) - s(y_1)| \leq \frac{1}{\sqrt{\lambda_\nu} \delta_{N,\nu,J}} \int_{[y_1, y_2]} \sqrt{\phi''(u)} \, du.$$

Proof. Continuity of $t \mapsto \Gamma_N(t)$ on the compact set J , together with $J \cap Z_{N,\nu} = \emptyset$, gives $\delta_{N,\nu,J} > 0$. By [theorem 26.1](#),

$$\sqrt{\lambda_\nu} \operatorname{dist}_{\text{HS}}(\Gamma_N(s(y)), \mathfrak{k}_\nu) |s'(y)| \leq \sqrt{\phi''(y)}.$$

Whenever $s(y) \in J$, the distance term is at least $\delta_{N,\nu,J}$, so

$$|s'(y)| \leq \frac{1}{\sqrt{\lambda_\nu} \delta_{N,\nu,J}} \sqrt{\phi''(y)}.$$

Integrating over the interval between y_1 and y_2 proves the claim. $\square$

Corollary 26.6 (Quadratic budget near the unique symmetry parameter). *Assume the hypotheses of [theorem 26.1](#) and the isotropic Legendre-type hypotheses of [theorem 26.2](#). Suppose that*

$$Z_{N,\nu} = \{t_*\}$$

with $N \neq 0$, and let $J \subset \Omega_N$ be compact. Define

$$m_{N,J} := \inf_{t \in J} \left\| \Gamma_N(t)^T \Gamma_N(t) \right\|_{\text{HS}}.$$

Then $m_{N,J} > 0$, and whenever $s(y) \in J$,

$$\sqrt{\lambda_\nu} m_{N,J} |s(y) - t_*| |s'(y)| \leq \sqrt{\phi''(y)}.$$

Consequently, for all $y_1, y_2 \in I$ with $s([y_1, y_2]) \subset J$,

$$|(s(y_2) - t_*)^2 - (s(y_1) - t_*)^2| \leq \frac{2}{\sqrt{\lambda_\nu} m_{N,J}} \int_{[y_1, y_2]} \sqrt{\phi''(u)} \, du.$$

Proof. By [theorem 26.2](#), there exists $K_* \in \mathfrak{k}_\nu \subset \mathfrak{so}(E)$ such that

$$N = K_*(I - t_* K_*)^{-1}.$$

Hence

$$\Gamma_N(t) = (I + tN)^{-1} N = K_*(I + (t - t_*)K_*)^{-1} \quad (t \in \Omega_N),$$

because K_* commutes with its own resolvents. Since $N \neq 0$, the matrix $\Gamma_N(t)$ is nonzero for every $t \in \Omega_N$; by continuity on the compact set J , this implies $m_{N,J} > 0$.

Now write $u := t - t_*$ and $M_u := K_*(I + uK_*)^{-1}$. Exactly as in the proof of [theorem 26.2](#), but with parameter $-u$, one obtains

$$M_u + M_u^T = 2u M_u^T M_u.$$

Therefore

$$\operatorname{Sym} \Gamma_N(t) = (t - t_*) \Gamma_N(t)^T \Gamma_N(t),$$

and since $\mathfrak{k}_\nu \subset \mathfrak{so}(E)$,

$$\operatorname{dist}_{\text{HS}}(\Gamma_N(t), \mathfrak{k}_\nu) \geq \|\operatorname{Sym} \Gamma_N(t)\|_{\text{HS}} = |t - t_*| \left\| \Gamma_N(t)^T \Gamma_N(t) \right\|_{\text{HS}} \geq m_{N,J} |t - t_*|$$

for $t \in J$.

Insert this lower bound into the isotropic estimate of [theorem 26.1](#). When $s(y) \in J$,

$$\sqrt{\lambda_\nu} m_{N,J} |s(y) - t_*| |s'(y)| \leq \sqrt{\phi''(y)}.$$

Since

$$\frac{d}{dy}(s(y) - t_*)^2 = 2(s(y) - t_*)s'(y),$$

the interval estimate follows by integration. $\square$

Corollary 26.7 (Additive parameter freezing on compact one-sided tails). *Assume the hypotheses of [theorem 26.1](#) and the isotropic Legendre-type hypotheses of [theorem 26.2](#). Let*

$$I = [y_0, \infty), \quad J \subset \Omega_N$$

with J compact, and assume that

$$s([y_0, \infty)) \subset J, \quad \int_{y_0}^{\infty} \sqrt{\phi''(u)} \, du < \infty.$$

Then $s(y)$ converges to a finite limit $s_{\infty} \in J$ as $y \rightarrow \infty$.

Proof. If $N = 0$, then s is irrelevant and the conclusion is trivial. Assume $N \neq 0$.

If $Z_{N,\nu} = \emptyset$, then [theorem 26.5](#) applies directly on the compact set J , so s is Cauchy at $+\infty$ and therefore converges.

Assume next that $Z_{N,\nu} = \{t_*\}$. By [theorem 26.6](#), the function

$$u(y) := (s(y) - t_*)^2$$

has a finite limit $\ell \geq 0$ as $y \rightarrow \infty$. If $\ell = 0$, then $s(y) \rightarrow t_*$.

Suppose finally that $\ell > 0$. Set

$$\delta := \frac{\sqrt{\ell}}{2}, \quad J_{\delta} := J \cap \{t \in \mathbb{R} : |t - t_*| \geq \delta\}.$$

For all sufficiently large y , one has $u(y) \geq \ell/4$, hence $|s(y) - t_*| \geq \delta$ and therefore $s(y) \in J_{\delta}$. The set J_{δ} is compact and disjoint from $Z_{N,\nu}$, so [theorem 26.5](#) implies that s is Cauchy on the tail where $s(y) \in J_{\delta}$. Hence $s(y)$ converges to a finite limit $s_{\infty} \in J_{\delta} \subset J$. $\square$

Proposition 26.8 (Combined additive and translation budget over even fibers). *Under the hypotheses of [theorem 26.1](#), assume in addition that W is even. Then for every $y \in I$,*

$$\mathcal{T}_{\nu}(A(y)^{-1}b''(y)) + \mathcal{Q}_{\nu}(\Gamma_N(s(y))) |s'(y)|^2 \leq \phi''(y).$$

If moreover ν is isotropic and Legendre-type, then

$$\tau_{\nu}|A(y)^{-1}b''(y)| + \lambda_{\nu} \operatorname{dist}_{\text{HS}}(\Gamma_N(s(y)), \mathfrak{k}_{\nu})^2 |s'(y)|^2 \leq \phi''(y).$$

Proof. By [theorem 23.3](#) and the identity

$$B(y) = s'(y)\Gamma_N(s(y)),$$

we have

$$\mathcal{T}_{\nu}(A(y)^{-1}b''(y)) + \mathcal{Q}_{\nu}(B(y)) \leq \phi''(y),$$

hence

$$\mathcal{T}_{\nu}(A(y)^{-1}b''(y)) + \mathcal{Q}_{\nu}(\Gamma_N(s(y))) |s'(y)|^2 \leq \phi''(y).$$

Under the isotropic Legendre-type hypotheses, combine

$$\mathcal{T}_{\nu}(v) \geq \tau_{\nu}|v|$$

with [theorem 26.1](#). $\square$

Remark 26.9. The additive fixed-direction branch left open by [theorem 17.1](#) is now substantially sharper. On bounded or one-sided bases, its parameter always satisfies the exact pointwise budget from [theorem 26.1](#). More importantly, [theorem 26.2](#) shows that the normalized velocity can touch the infinitesimal orthogonal symmetry algebra at *at most one* parameter value, and only when the fixed direction matrix is a Möbius transform of an actual symmetry generator. Thus any non-Möbius-symmetry additive direction stays uniformly away from symmetry on every compact parameter range and is quantitatively visible to base curvature throughout; even in the exceptional Möbius-symmetry case, [theorems 26.6](#) and [26.7](#) show that compact tails with finite $\int \sqrt{\phi''}$ still force freezing of the additive parameter. In the even-isotropic setting, [theorem 26.8](#) couples the same parameter motion directly to the normalized translation acceleration.

27 Local affine-product stability and vanishing-curvature blow-up compactness

The one-sided tail theorems from Section 24 control the approach to an affine product when the curvature budgets are finite on a half-line. For the eventual blow-up strategy, one also needs a purely local compactness statement: on a compact interval, if the integrated curvature budgets are small after normalization at one base point, then the exact-affine coefficients are already close to an affine product on that entire interval. The next theorem supplies exactly this exact-affine compactness mechanism in the even-isotropic world.

Theorem 27.1 (Local affine-product stability on compact intervals). *Under the hypotheses of [theorem 23.6](#), fix $y_0 \in I$ and $R > 0$ such that*

$$J_R(y_0) := [y_0 - R, y_0 + R] \subset I.$$

Let $O, \tilde{A}, \tilde{B}$ be the orthogonal gauge from [theorem 21.4](#) with $O(y_0) = \text{Id}$, set

$$G(y) := \tilde{A}(y_0)^{-1} \tilde{A}(y), \quad \tilde{c}(y) := \tilde{A}(y)^{-1} b''(y),$$

and abbreviate

$$\Delta_R(y_0) := \lambda_\nu^{-1/2} \int_{J_R(y_0)} \sqrt{\phi''(s)} \, ds, \quad E_R(y_0) := \tau_\nu^{-1} \int_{J_R(y_0)} \phi''(s) \, ds.$$

Then for every $y \in J_R(y_0)$,

$$\begin{aligned} \|G(y) - \text{Id}\|_{\text{op}} &\leq e^{\Delta_R(y_0)} \Delta_R(y_0), \\ |b'(y) - b'(y_0)| &\leq \|\tilde{A}(y_0)\|_{\text{op}} e^{\Delta_R(y_0)} E_R(y_0), \end{aligned}$$

and

$$|b(y) - b(y_0) - (y - y_0)b'(y_0)| \leq R \|\tilde{A}(y_0)\|_{\text{op}} e^{\Delta_R(y_0)} E_R(y_0).$$

In particular, if

$$\tilde{A}(y_0) = \text{Id}, \quad b(y_0) = 0, \quad b'(y_0) = 0,$$

then

$$\begin{aligned} \sup_{y \in J_R(y_0)} \|\tilde{A}(y) - \text{Id}\|_{\text{op}} &\leq e^{\Delta_R(y_0)} \Delta_R(y_0), \\ \sup_{y \in J_R(y_0)} |b'(y)| &\leq e^{\Delta_R(y_0)} E_R(y_0), \quad \sup_{y \in J_R(y_0)} |b(y)| \leq R e^{\Delta_R(y_0)} E_R(y_0). \end{aligned}$$

Proof. By [theorem 21.4](#),

$$\tilde{A}'(y) = \tilde{A}(y)\tilde{B}(y),$$

hence

$$G'(y) = \tilde{A}(y_0)^{-1}\tilde{A}'(y) = G(y)\tilde{B}(y).$$

Moreover,

$$\|G(y)\|_{\text{op}} = \|\tilde{A}(y_0)^{-1}\tilde{A}(y)\|_{\text{op}} \leq \exp\left(\lambda_\nu^{-1/2} \int_{\min\{y, y_0\}}^{\max\{y, y_0\}} \sqrt{\phi''(s)} \, ds\right) \leq e^{\Delta_R(y_0)}$$

for $y \in J_R(y_0)$. Therefore

$$\|G(y) - \text{Id}\|_{\text{op}} \leq \int_{y_0}^y \|G(s)\|_{\text{op}} \|\tilde{B}(s)\|_{\text{op}} \, ds \leq e^{\Delta_R(y_0)} \int_{J_R(y_0)} \|\tilde{B}(s)\|_{\text{HS}} \, ds.$$

By [theorem 23.7](#),

$$\|\tilde{B}(s)\|_{\text{HS}} \leq \lambda_\nu^{-1/2} \sqrt{\phi''(s)},$$

so the first estimate follows.

Next,

$$b''(y) = \tilde{A}(y)\tilde{c}(y) = \tilde{A}(y_0)G(y)\tilde{c}(y),$$

hence

$$|b''(y)| \leq \|\tilde{A}(y_0)\|_{\text{op}} \|G(y)\|_{\text{op}} |\tilde{c}(y)| \leq \|\tilde{A}(y_0)\|_{\text{op}} e^{\Delta_R(y_0)} |\tilde{c}(y)|.$$

Using again [theorem 23.7](#),

$$|\tilde{c}(y)| \leq \tau_\nu^{-1} \phi''(y),$$

and integrating from y_0 to y yields

$$|b'(y) - b'(y_0)| \leq \|\tilde{A}(y_0)\|_{\text{op}} e^{\Delta_R(y_0)} E_R(y_0).$$

A second integration gives

$$|b(y) - b(y_0) - (y - y_0)b'(y_0)| = \left| \int_{y_0}^y (b'(u) - b'(y_0)) \, du \right| \leq R \sup_{u \in J_R(y_0)} |b'(u) - b'(y_0)|,$$

which proves the third estimate. The normalized bounds are the special case

$$\tilde{A}(y_0) = \text{Id}, \quad b(y_0) = 0, \quad b'(y_0) = 0. \quad \square$$

Corollary 27.2 (Curvature quantum for normalized nontriviality). *Under the normalized hypotheses*

$$\tilde{A}(y_0) = \text{Id}, \quad b(y_0) = 0, \quad b'(y_0) = 0$$

of [theorem 27.1](#), let $\varepsilon > 0$. If at least one of

$$\sup_{y \in J_R(y_0)} \|\tilde{A}(y) - \text{Id}\|_{\text{op}} \geq \varepsilon, \quad \sup_{y \in J_R(y_0)} |b'(y)| \geq \varepsilon, \quad R^{-1} \sup_{y \in J_R(y_0)} |b(y)| \geq \varepsilon$$

holds, then

$$\max\{\Delta_R(y_0), E_R(y_0)\} \geq \min\{1, \varepsilon/e\}.$$

Equivalently,

$$\max\left\{\lambda_\nu^{-1/2} \int_{J_R(y_0)} \sqrt{\phi''(s)} \, ds, \tau_\nu^{-1} \int_{J_R(y_0)} \phi''(s) \, ds\right\} \geq \min\{1, \varepsilon/e\}.$$

Proof. Assume contrariwise that

$$\max\{\Delta_R(y_0), E_R(y_0)\} < \min\{1, \varepsilon/e\}.$$

Then $\Delta_R(y_0) < 1$, so $e^{\Delta_R(y_0)} \leq e$. Hence

$$e^{\Delta_R(y_0)} \Delta_R(y_0) < \varepsilon, \quad e^{\Delta_R(y_0)} E_R(y_0) < \varepsilon.$$

Applying [theorem 27.1](#) shows that all three normalized quantities

$$\sup_{J_R(y_0)} \|\tilde{A} - \text{Id}\|_{\text{op}}, \quad \sup_{J_R(y_0)} |b'|, \quad R^{-1} \sup_{J_R(y_0)} |b|$$

are strictly smaller than ε , a contradiction. $\square$

Theorem 27.3 (Vanishing-curvature blow-up compactness). *Let E be a Euclidean space. For each $j \geq 1$, let*

$$\nu_j(\mathrm{d}z) = e^{-W_j(z)} \mathrm{d}z$$

be an isotropic, even, Legendre-type log-concave probability measure on E , and suppose that its symmetry-gap and translation-gap constants satisfy the uniform lower bounds

$$\lambda_{\nu_j} \geq \lambda_0 > 0, \quad \tau_{\nu_j} \geq \tau_0 > 0.$$

Let $I_j \subset \mathbb{R}$ be an interval containing 0, let

$$m_j(\mathrm{d}y) = e^{-\phi_j(y)} \mathrm{d}y$$

be log-concave on I_j with $\phi_j \in C^2(I_j)$, and let

$$T_j(z, y) = (A_j(y)z + b_j(y), y)$$

with $A_j : I_j \rightarrow \text{GL}(E)$, $b_j : I_j \rightarrow E$ of class C^2 . Assume that

$$(T_j)_\#(\nu_j \otimes m_j)$$

is log-concave on $E \times I_j$. Let $O_j, \tilde{A}_j$ be the orthogonal gauge from [theorem 21.4](#) based at 0, and define the ambient affine normalization

$$\hat{T}_j(z, y) := (\hat{A}_j(y)z + \hat{b}_j(y), y)$$

by

$$\hat{A}_j(y) := \tilde{A}_j(0)^{-1} \tilde{A}_j(y), \quad \hat{b}_j(y) := \tilde{A}_j(0)^{-1} (b_j(y) - b_j(0) - y b'_j(0)).$$

Suppose that for every $R > 0$,

$$[-R, R] \subset I_j \quad \text{for all sufficiently large } j,$$

and

$$\int_{-R}^R \sqrt{\phi_j''(s)} \mathrm{d}s \longrightarrow 0, \quad \int_{-R}^R \phi_j''(s) \mathrm{d}s \longrightarrow 0 \quad (j \rightarrow \infty).$$

Then for every $R > 0$,

$$\sup_{|y| \leq R} \|\hat{A}_j(y) - \text{Id}\|_{\text{op}} \longrightarrow 0, \quad \sup_{|y| \leq R} |\hat{b}'_j(y)| \longrightarrow 0, \quad \sup_{|y| \leq R} |\hat{b}_j(y)| \longrightarrow 0.$$

Hence every normalized exact-affine family with uniformly positive symmetry-gap and translation-gap constants and vanishing local base-curvature budgets converges, on compact y -intervals, to the affine product coefficients $(\text{Id}, 0)$.

Proof. Fix $R > 0$. For all sufficiently large j , the interval $[-R, R]$ lies in I_j . By construction,

$$\widehat{A}_j(0) = \text{Id}, \quad \widehat{b}_j(0) = 0, \quad \widehat{b}'_j(0) = 0.$$

Apply [theorem 27.1](#) on $[-R, R]$ with $y_0 = 0$ and with $\lambda_{\nu_j} \geq \lambda_0$, $\tau_{\nu_j} \geq \tau_0$. This yields

$$\sup_{|y| \leq R} \left\| \widehat{A}_j(y) - \text{Id} \right\|_{\text{op}} \leq \exp\left(\lambda_0^{-1/2} \int_{-R}^R \sqrt{\phi_j''(s)} \, ds\right) \lambda_0^{-1/2} \int_{-R}^R \sqrt{\phi_j''(s)} \, ds,$$

$$\sup_{|y| \leq R} |\widehat{b}'_j(y)| \leq \exp\left(\lambda_0^{-1/2} \int_{-R}^R \sqrt{\phi_j''(s)} \, ds\right) \tau_0^{-1} \int_{-R}^R \phi_j''(s) \, ds,$$

and

$$\sup_{|y| \leq R} |\widehat{b}_j(y)| \leq R \exp\left(\lambda_0^{-1/2} \int_{-R}^R \sqrt{\phi_j''(s)} \, ds\right) \tau_0^{-1} \int_{-R}^R \phi_j''(s) \, ds.$$

The assumed vanishing of the two curvature budgets forces all three right-hand sides to converge to 0, which proves the claim. $\square$

Remark 27.4. [Theorem 27.3](#) provides the exact-affine compactness half of the blow-up bridge sought by the transport program. What still remains is the genuinely global step that would extract, from a failure of affine-normalized moving-frame control, a sequence of normalized exact-affine models on larger and larger windows whose local curvature budgets vanish. Once such a sequence exists, the present section shows that its only possible limit is an affine product. Thus, conditional on a canonical lower bound for λ_ν and, beyond the even world, a corresponding anisotropic translation budget, any exact-affine obstruction must carry a definite curvature quantum on every scale where it stays genuinely nontrivial.

28 Curvature-density dichotomy and production of almost-flat windows on the full line

Section 27 shows that small curvature budgets on a *fixed* compact interval already force quantitative closeness to an affine product after orthogonal gauge and local normalization. The next step is to globalize that compactness statement. The point is simple but useful: if one prescribes a scale R and asks that *every* window of length $2R$ remain quantitatively nontrivial, then the curvature-quantum mechanism from Section 27 can be applied on a disjoint family of windows and summed. This yields a linear lower bound on the combined curvature budget. By contraposition, any full-line even-isotropic exact-affine bundle with sublinear combined curvature growth must produce arbitrarily long almost-flat windows.

Definition 28.1 (Window defect and combined curvature budget). Under the hypotheses of [theorem 27.1](#), assume in addition that $I = \mathbb{R}$, and fix once and for all an orthogonal gauge $\tilde{A}$ from [theorem 21.4](#). For $y_0 \in \mathbb{R}$ and $R > 0$, write

$$J_R(y_0) := [y_0 - R, y_0 + R],$$

and define the normalized window defect

$$\mathcal{D}_R(y_0) := \max \left\{ \sup_{y \in J_R(y_0)} \left\| \tilde{A}(y_0)^{-1} \tilde{A}(y) - \text{Id} \right\|_{\text{op}} \right\},$$

$$\sup_{y \in J_R(y_0)} |\tilde{A}(y_0)^{-1}(b'(y) - b'(y_0))|, R^{-1} \sup_{y \in J_R(y_0)} |\tilde{A}(y_0)^{-1}(b(y) - b(y_0) - (y - y_0)b'(y_0))| \Big\}.$$

For every compact interval $J \subset \mathbb{R}$, define the combined curvature budget

$$\mathcal{B}(J) := \lambda_\nu^{-1/2} \int_J \sqrt{\phi''(s)} \, ds + \tau_\nu^{-1} \int_J \phi''(s) \, ds,$$

and abbreviate

$$\mathcal{B}_R(y_0) := \mathcal{B}(J_R(y_0)).$$

Remark 28.2. Changing the initial condition in [theorem 21.4](#) multiplies $\tilde{A}$ on the right by a constant element of $G_\nu^{\text{lin}} \cap \text{O}(E)$. Consequently the three norms in $\mathcal{D}_R(y_0)$ are unchanged, so $\mathcal{D}_R(y_0)$ is independent of that choice of global orthogonal gauge.

Proposition 28.3 (Window defect bound by the combined curvature budget). *Under the hypotheses of [theorem 28.1](#), for every $y_0 \in \mathbb{R}$ and every $R > 0$,*

$$\mathcal{D}_R(y_0) \leq e^{\mathcal{B}_R(y_0)} \mathcal{B}_R(y_0).$$

Proof. Fix $y_0 \in \mathbb{R}$ and $R > 0$, and set

$$G(y) := \tilde{A}(y_0)^{-1} \tilde{A}(y) \quad (y \in J_R(y_0)).$$

Then

$$G'(y) = G(y) \tilde{B}(y), \quad G(y_0) = \text{Id}.$$

By [theorem 21.4](#),

$$\|\tilde{B}(y)\|_{\text{HS}} \leq \lambda_\nu^{-1/2} \sqrt{\phi''(y)} \quad (y \in \mathbb{R}).$$

Hence, exactly as in the proof of [theorem 27.1](#),

$$\|G(y)\|_{\text{op}} \leq \exp(\mathcal{B}_R(y_0)) \quad (y \in J_R(y_0)),$$

and therefore

$$\|G(y) - \text{Id}\|_{\text{op}} \leq e^{\mathcal{B}_R(y_0)} \lambda_\nu^{-1/2} \int_{J_R(y_0)} \sqrt{\phi''(s)} \, ds \leq e^{\mathcal{B}_R(y_0)} \mathcal{B}_R(y_0).$$

Next write

$$\tilde{c}(y) := \tilde{A}(y)^{-1} b''(y).$$

By [theorem 23.6](#),

$$|\tilde{c}(y)| \leq \tau_\nu^{-1} \phi''(y) \quad (y \in \mathbb{R}).$$

Since

$$b''(y) = \tilde{A}(y) \tilde{c}(y) = \tilde{A}(y_0) G(y) \tilde{c}(y),$$

we obtain

$$\tilde{A}(y_0)^{-1}(b'(y) - b'(y_0)) = \int_{y_0}^y G(s) \tilde{c}(s) \, ds,$$

and hence

$$\sup_{y \in J_R(y_0)} |\tilde{A}(y_0)^{-1}(b'(y) - b'(y_0))| \leq e^{\mathcal{B}_R(y_0)} \tau_\nu^{-1} \int_{J_R(y_0)} \phi''(s) \, ds \leq e^{\mathcal{B}_R(y_0)} \mathcal{B}_R(y_0).$$

A second integration gives

$$\tilde{A}(y_0)^{-1}(b(y) - b(y_0) - (y - y_0)b'(y_0)) = \int_{y_0}^y \tilde{A}(y_0)^{-1}(b'(u) - b'(y_0)) \, du,$$

so

$$R^{-1} \sup_{y \in J_R(y_0)} |\tilde{A}(y_0)^{-1}(b(y) - b(y_0) - (y - y_0)b'(y_0))| \leq e^{\mathcal{B}_R(y_0)} \mathcal{B}_R(y_0).$$

Taking the maximum of the three bounds proves the claim. $\square$

Corollary 28.4 (Window curvature quantum). *Under the hypotheses of [theorem 28.1](#), let $\varepsilon > 0$ and set*

$$c_\varepsilon := \min\{1, \varepsilon/e\}.$$

If

$$\mathcal{D}_R(y_0) \geq \varepsilon$$

for some $y_0 \in \mathbb{R}$ and $R > 0$, then

$$\mathcal{B}_R(y_0) \geq c_\varepsilon.$$

Proof. If $\mathcal{B}_R(y_0) < c_\varepsilon$, then $\mathcal{B}_R(y_0) < 1$, and [theorem 28.3](#) yields

$$\mathcal{D}_R(y_0) \leq e^{\mathcal{B}_R(y_0)} \mathcal{B}_R(y_0) < e c_\varepsilon \leq \varepsilon,$$

a contradiction. $\square$

Theorem 28.5 (Uniform window nontriviality forces linear curvature density). *Under the hypotheses of [theorem 28.1](#), let $\varepsilon > 0$, $R > 0$, and $L \geq R$. Suppose that*

$$\mathcal{D}_R(y) \geq \varepsilon \quad \text{for every } y \in [-L + R, L - R].$$

Then

$$\mathcal{B}([-L, L]) \geq c_\varepsilon \left\lfloor \frac{L}{R} \right\rfloor,$$

where $c_\varepsilon = \min\{1, \varepsilon/e\}$. In particular,

$$\lambda_\nu^{-1/2} \int_{-L}^L \sqrt{\phi''(s)} \, ds + \tau_\nu^{-1} \int_{-L}^L \phi''(s) \, ds \geq c_\varepsilon \left(\frac{L}{R} - 1 \right).$$

Proof. Set

$$N := \left\lfloor \frac{L}{R} \right\rfloor, \quad y_k := -L + (2k + 1)R \quad (k = 0, \dots, N - 1).$$

Then

$$J_R(y_k) = [-L + 2kR, -L + 2(k + 1)R] \subset [-L, L],$$

and these N intervals are pairwise disjoint. By hypothesis,

$$\mathcal{D}_R(y_k) \geq \varepsilon \quad (k = 0, \dots, N - 1).$$

Hence [theorem 28.4](#) gives

$$\mathcal{B}_R(y_k) \geq c_\varepsilon \quad (k = 0, \dots, N - 1).$$

Summing over k and using disjointness,

$$\mathcal{B}([-L, L]) \geq \sum_{k=0}^{N-1} \mathcal{B}_R(y_k) \geq N c_\varepsilon = c_\varepsilon \left\lfloor \frac{L}{R} \right\rfloor.$$

Since $\lfloor L/R \rfloor \geq L/R - 1$, the second estimate follows. $\square$

Corollary 28.6 (A global budget bound yields an almost-flat window). *Under the hypotheses of theorem 28.1, let $\varepsilon > 0$, $R > 0$, and $L \geq R$. If*

$$\mathcal{B}([-L, L]) < c_\varepsilon \left\lfloor \frac{L}{R} \right\rfloor,$$

then there exists $y_0 \in [-L + R, L - R]$ such that

$$\mathcal{D}_R(y_0) < \varepsilon.$$

Proof. This is the contrapositive of theorem 28.5. □

Corollary 28.7 (Sublinear combined curvature growth yields arbitrarily long almost-flat windows). *Under the hypotheses of theorem 28.1, assume that*

$$\liminf_{L \rightarrow \infty} \frac{\mathcal{B}([-L, L])}{L} = 0.$$

Let $R_j \rightarrow \infty$ and $\varepsilon_j \downarrow 0$. Then there exist points $y_j \in \mathbb{R}$ such that

$$\mathcal{D}_{R_j}(y_j) < \varepsilon_j \quad (j \geq 1).$$

Equivalently, if

$$\begin{aligned} \hat{A}_j(t) &:= \tilde{A}(y_j)^{-1} \tilde{A}(y_j + R_j t), \\ \hat{b}_j(t) &:= R_j^{-1} \tilde{A}(y_j)^{-1} (b(y_j + R_j t) - b(y_j) - R_j t b'(y_j)) \quad (|t| \leq 1), \end{aligned}$$

then

$$\begin{aligned} \sup_{|t| \leq 1} \left\| \hat{A}_j(t) - \text{Id} \right\|_{\text{op}} &\xrightarrow{j \rightarrow \infty} 0, \\ \sup_{|t| \leq 1} |\hat{b}'_j(t)| &\xrightarrow{j \rightarrow \infty} 0, \quad \sup_{|t| \leq 1} |\hat{b}_j(t)| \xrightarrow{j \rightarrow \infty} 0. \end{aligned}$$

Proof. Set

$$c_j := \min\{1, \varepsilon_j/e\}.$$

Because $c_j/R_j \rightarrow 0$ and $\liminf_{L \rightarrow \infty} \mathcal{B}([-L, L])/L = 0$, for each j we may choose $L_j \geq 2R_j$ so large that

$$\frac{\mathcal{B}([-L_j, L_j])}{L_j} < \frac{c_j}{2R_j}.$$

Then

$$\mathcal{B}([-L_j, L_j]) < c_j \frac{L_j}{2R_j} \leq c_j \left\lfloor \frac{L_j}{R_j} \right\rfloor.$$

Applying theorem 28.6 with $L = L_j$, $R = R_j$, and $\varepsilon = \varepsilon_j$ produces $y_j \in [-L_j + R_j, L_j - R_j]$ such that

$$\mathcal{D}_{R_j}(y_j) < \varepsilon_j.$$

The rescaled statement is simply an unpacking of the definition of $\mathcal{D}_{R_j}(y_j)$: indeed,

$$\hat{b}'_j(t) = \tilde{A}(y_j)^{-1} (b'(y_j + R_j t) - b'(y_j)).$$

Therefore all three displayed suprema are bounded by $\mathcal{D}_{R_j}(y_j)$, which tends to 0. □

Remark 28.8 (Asymptotically linear bases). If

$$\int_{\mathbb{R}} \phi''(s) \, ds < \infty,$$

then

$$\frac{1}{L} \int_{-L}^L \phi''(s) \, ds \rightarrow 0$$

and, by Cauchy–Schwarz,

$$\frac{1}{L} \int_{-L}^L \sqrt{\phi''(s)} \, ds \leq \sqrt{\frac{2}{L} \int_{\mathbb{R}} \phi''(s) \, ds} \xrightarrow{L \rightarrow \infty} 0.$$

Hence the hypothesis of [theorem 28.7](#) is automatically satisfied. In particular, every full-line even-isotropic exact-affine bundle over a base potential with finite total curvature already produces arbitrarily long almost-flat windows after orthogonal gauge and local affine normalization.

Remark 28.9. [Theorems 28.5](#) and [28.7](#) supply a first theorem-level production mechanism for the blow-up strategy. They do not yet derive sublinear combined curvature growth from the global affine-normalized distortion problem, but they show that once such sublinear growth is available, the canonical almost-flat windows appear automatically and can be fed directly into the compactness machinery of [Section 27](#).

29 Bad-center density and prevalence of almost-flat windows on long intervals

[Section 28](#) guarantees the existence of a long almost-flat window once the combined curvature budget becomes sublinear. For the blow-up strategy, however, one would like to know more than existence of a single favorable center. The same curvature-quantum mechanism in fact yields a quantitative packing bound: at any fixed scale, the set of centers whose windows remain quantitatively nontrivial can occupy only a controlled amount of one-dimensional measure. Thus vanishing combined curvature density forces good windows to occur with density one.

Theorem 29.1 (Bad-center density estimate at fixed scale). *Under the hypotheses of [theorem 28.1](#), fix $\varepsilon > 0$, $R > 0$, and a compact interval $[a, b] \subset \mathbb{R}$. Define the bad-center set*

$$\mathcal{B}_{\varepsilon, R}([a, b]) := \{y \in [a, b] : \mathcal{D}_R(y) \geq \varepsilon\},$$

and set

$$c_{\varepsilon} := \min\{1, \varepsilon/e\}.$$

Then

$$|\mathcal{B}_{\varepsilon, R}([a, b])| \leq \frac{4R}{c_{\varepsilon}} \mathcal{B}([a - R, b + R]).$$

Proof. Let $E := \mathcal{B}_{\varepsilon, R}([a, b])$. Choose a maximal $2R$ -separated subset

$$\{y_1, \dots, y_N\} \subset E.$$

Then the intervals $J_R(y_j) = [y_j - R, y_j + R]$ are pairwise disjoint, while maximality implies

$$E \subset \bigcup_{j=1}^N [y_j - 2R, y_j + 2R].$$

Hence

$$|E| \leq 4RN.$$

Since each $y_j \in E$, [theorem 28.4](#) yields

$$\mathcal{B}_R(y_j) \geq c_\varepsilon \quad (j = 1, \dots, N).$$

Moreover,

$$J_R(y_j) \subset [a - R, b + R] \quad (j = 1, \dots, N),$$

and these intervals are pairwise disjoint. Therefore

$$\mathcal{B}([a - R, b + R]) \geq \sum_{j=1}^N \mathcal{B}_R(y_j) \geq N c_\varepsilon.$$

Combining this with $|E| \leq 4RN$ proves the claim. $\square$

Corollary 29.2 (Density-one good centers at every fixed scale). *Under the hypotheses of [theorem 28.1](#), assume that*

$$\lim_{L \rightarrow \infty} \frac{\mathcal{B}([-L, L])}{L} = 0.$$

Then for every fixed $R > 0$ and every fixed $\varepsilon > 0$,

$$\frac{1}{2L} |\{y \in [-L, L] : \mathcal{D}_R(y) \geq \varepsilon\}| \xrightarrow{L \rightarrow \infty} 0.$$

Equivalently,

$$\frac{1}{2L} |\{y \in [-L, L] : \mathcal{D}_R(y) < \varepsilon\}| \xrightarrow{L \rightarrow \infty} 1.$$

Proof. Apply [theorem 29.1](#) with $[a, b] = [-L, L]$. Then

$$|\{y \in [-L, L] : \mathcal{D}_R(y) \geq \varepsilon\}| \leq \frac{4R}{c_\varepsilon} \mathcal{B}([-L - R, L + R]).$$

Divide by $2L$. Since R and ε are fixed and

$$\frac{\mathcal{B}([-L - R, L + R])}{L} \rightarrow 0,$$

the displayed ratio tends to 0. $\square$

Corollary 29.3 (Scale-adaptive prevalence of almost-flat windows). *Under the hypotheses of [theorem 28.1](#), let $L_j \rightarrow \infty$, $R_j > 0$, and $\varepsilon_j > 0$. Suppose that*

$$\frac{R_j}{L_j c_{\varepsilon_j}} \mathcal{B}([-L_j - R_j, L_j + R_j]) \xrightarrow{j \rightarrow \infty} 0, \quad c_{\varepsilon_j} := \min\{1, \varepsilon_j/e\}.$$

Then

$$\frac{1}{2L_j} |\{y \in [-L_j, L_j] : \mathcal{D}_{R_j}(y) < \varepsilon_j\}| \xrightarrow{j \rightarrow \infty} 1.$$

Proof. Apply [theorem 29.1](#) on $[a, b] = [-L_j, L_j]$. The bad-center proportion is bounded by

$$\frac{2R_j}{L_j c_{\varepsilon_j}} \mathcal{B}([-L_j - R_j, L_j + R_j]),$$

which tends to 0 by hypothesis. $\square$

Remark 29.4. [Theorem 28.7](#) provided a single almost-flat window at each large scale under sublinear combined curvature growth. The new theorem is substantially stronger: whenever the curvature density actually tends to zero, almost-flat windows of any fixed scale occupy asymptotic density one. For the blow-up strategy this means that the obstruction, if it exists, must not only avoid a single good window but must sustain a positive density of genuinely nontrivial windows at each relevant scale.

30 Dual exact-affine characterization and the concave-orbit reformulation

The optimized tangent inequality and the exact Schur-complement formula describe joint convexity of the global potential infinitesimally. There is also a global dual formulation. The key observation is that, for exact-affine bundles, partial Legendre transform in the fiber variable can be computed explicitly. Joint convexity is then equivalent to a pointwise concavity statement in the base variable. This gives a genuinely different perspective on the remaining anisotropic nonselfadjoint branch.

Theorem 30.1 (Dual-conjugate characterization of exact-affine log-concavity). *Let E be a Euclidean space, let $I \subset \mathbb{R}$ be an interval, let*

$$\nu(dz) = e^{-W(z)} dz, \quad m(dy) = e^{-\phi(y)} dy$$

be log-concave probability measures with $W \in C^2(E)$ convex, Legendre-type, and $\nabla^2 W(z)$ positive-definite for every $z \in E$, and $\phi \in C^2(I)$ convex. Let

$$A : I \rightarrow \text{GL}(E), \quad b : I \rightarrow E$$

be C^2 , define

$$T(z, y) = (A(y)z + b(y), y), \quad r(y) := \log \det A(y) + \phi(y),$$

and write

$$V_y(x) := W(A(y)^{-1}(x - b(y))) + r(y) \quad (x \in E, y \in I).$$

For $\xi \in E$, define

$$\Psi_\xi(y) := W^*(A(y)^T \xi) + \langle b(y), \xi \rangle - r(y) \quad (y \in I).$$

Then the following are equivalent:

- (i) *the measure $\mu := T_\#(\nu \otimes m)$ is log-concave on $E \times I$;*
- (ii) *for every $\xi \in E$, the function $y \mapsto \Psi_\xi(y)$ is concave on I .*

Whenever these conditions hold, one has

$$V_y^*(\xi) = \Psi_\xi(y) \quad (\xi \in E, y \in I),$$

and therefore

$$V(x, y) = \sup_{\xi \in E} \{ \langle \xi, x \rangle - \Psi_\xi(y) \}, \quad V(x, y) := V_y(x).$$

Proof. For fixed $y \in I$, the fiber potential is

$$V_y(x) = W(A(y)^{-1}(x - b(y))) + r(y).$$

Its Legendre transform in the x -variable is

$$\begin{aligned} V_y^*(\xi) &= \sup_{x \in E} (\langle \xi, x \rangle - W(A(y)^{-1}(x - b(y))) - r(y)) \\ &= \sup_{z \in E} (\langle \xi, A(y)z + b(y) \rangle - W(z) - r(y)) \\ &= W^*(A(y)^T \xi) + \langle b(y), \xi \rangle - r(y) = \Psi_\xi(y). \end{aligned}$$

Assume first that μ is log-concave. Then the global potential

$$V(x, y) = W(A(y)^{-1}(x - b(y))) + r(y)$$

is jointly convex on $E \times I$. Fix $\xi \in E$, $y_0, y_1 \in I$, and $t \in [0, 1]$, and write $y_t := (1 - t)y_0 + ty_1$. For arbitrary $x_0, x_1 \in E$, joint convexity gives

$$V((1 - t)x_0 + tx_1, y_t) \leq (1 - t)V(x_0, y_0) + tV(x_1, y_1).$$

Hence

$$\begin{aligned} \Psi_\xi(y_t) &= V_{y_t}^*(\xi) \\ &\geq \langle \xi, (1 - t)x_0 + tx_1 \rangle - V((1 - t)x_0 + tx_1, y_t) \\ &\geq (1 - t)(\langle \xi, x_0 \rangle - V(x_0, y_0)) + t(\langle \xi, x_1 \rangle - V(x_1, y_1)). \end{aligned}$$

Taking the supremum over x_0 and x_1 yields

$$\Psi_\xi(y_t) \geq (1 - t)\Psi_\xi(y_0) + t\Psi_\xi(y_1),$$

so Ψ_ξ is concave.

Conversely, assume that every Ψ_ξ is concave on I . Since each V_y is a proper lower-semicontinuous convex function on E , Fenchel–Moreau gives

$$V_y(x) = \sup_{\xi \in E} \{\langle \xi, x \rangle - V_y^*(\xi)\} = \sup_{\xi \in E} \{\langle \xi, x \rangle - \Psi_\xi(y)\}.$$

For fixed ξ , the function

$$(x, y) \mapsto \langle \xi, x \rangle - \Psi_\xi(y)$$

is jointly convex on $E \times I$, because it is affine in x and $-\Psi_\xi$ is convex in y . Taking the supremum over $\xi \in E$ therefore shows that V is jointly convex. Hence the density

$$(x, y) \mapsto e^{-V(x, y)}$$

is log-concave on $E \times I$, i.e. μ is log-concave. □

Corollary 30.2 (Dual second-derivative criterion). *Under the hypotheses of [theorem 30.1](#), if $\mu = T_\#(\nu \otimes m)$ is log-concave, then for every $y \in I$ and every $\xi \in E$,*

$$\frac{d^2}{dy^2} (W^*(A(y)^T \xi) + \langle b(y), \xi \rangle - r(y)) \leq 0.$$

Equivalently,

$$\left\langle \nabla^2 W^*(A(y)^T \xi) A'(y)^T \xi, A'(y)^T \xi \right\rangle + \left\langle \nabla W^*(A(y)^T \xi), A''(y)^T \xi \right\rangle + \langle b''(y), \xi \rangle \leq r''(y).$$

Proof. By [theorem 30.1](#), the function $y \mapsto \Psi_\xi(y)$ is concave for every $\xi \in E$, so its second derivative is nonpositive. The displayed expansion is obtained by differentiating Ψ_ξ twice and using the chain rule. $\square$

Corollary 30.3 (The optimized tangent inequality is the infinitesimal dual criterion). *Under the hypotheses of [theorem 30.1](#), let $y \in I$ and $z \in E$, and set*

$$\xi := A(y)^{-T} \nabla W(z).$$

Then $A(y)^T \xi = \nabla W(z)$, and the inequality from [theorem 30.2](#) becomes exactly

$$\langle \nabla W(z), M(y)z + c(y) \rangle + \left\langle B(y)^T \nabla W(z), \nabla^2 W(z)^{-1} B(y)^T \nabla W(z) \right\rangle \leq r''(y),$$

where

$$B(y) = A(y)^{-1} A'(y), \quad M(y) = A(y)^{-1} A''(y), \quad c(y) = A(y)^{-1} b''(y).$$

In particular, [theorem 13.1](#) is exactly the infinitesimal form of the dual-conjugate characterization.

Proof. Since W is Legendre-type with $\nabla^2 W > 0$, $\nabla W : E \rightarrow E$ is a C^1 -diffeomorphism with inverse ∇W^* . Hence

$$\nabla W^*(A(y)^T \xi) = z, \quad \nabla^2 W^*(A(y)^T \xi) = \nabla^2 W(z)^{-1}.$$

Moreover,

$$\begin{aligned} A'(y)^T \xi &= A(y)^T B(y)^T A(y)^{-T} \nabla W(z) = B(y)^T \nabla W(z), \\ \langle b''(y), \xi \rangle &= \left\langle A(y)^{-1} b''(y), \nabla W(z) \right\rangle = \langle c(y), \nabla W(z) \rangle, \end{aligned}$$

and

$$\left\langle \nabla W^*(A(y)^T \xi), A''(y)^T \xi \right\rangle = \left\langle z, A''(y)^T A(y)^{-T} \nabla W(z) \right\rangle = \left\langle \nabla W(z), A(y)^{-1} A''(y) z \right\rangle = \langle \nabla W(z), M(y)z \rangle.$$

Substituting these identities into [theorem 30.2](#) gives the claim. $\square$

Remark 30.4 (Concave-orbit reformulation). [Theorem 30.1](#) shows that the exact-affine problem can be formulated globally without any Hessian computation: one must classify those affine orbits of the dual potential W^* for which

$$y \mapsto W^*(A(y)^T \xi) + \langle b(y), \xi \rangle - \log \det A(y) - \phi(y)$$

is concave for every covector ξ . The full-line exact-affine collapse conjectures are therefore equivalent to a *concave-orbit rigidity* statement in the dual potential space. This appears to isolate the genuinely anisotropic nonselfadjoint difficulty more cleanly than the original primal formulation.

31 Möbius-symmetry collapse for additive full-line branches

Section 26 showed that the only way an additive fixed-direction branch can touch the infinitesimal orthogonal symmetry algebra is through the Möbius-symmetry form

$$N = K(I - t_* K)^{-1} \quad \text{with } K \in \mathfrak{k}_\nu.$$

This still leaves open the exceptional branch generated by an actual symmetry direction. The point of the present section is that this branch can in fact be eliminated completely on the full line. The

first step is the toric factorization: whenever the symmetry generator splits into blockwise complex structures that are themselves infinitesimal symmetries, the additive Möbius family admits an explicit blockwise polar factorization into a genuine symmetry and a self-adjoint positive-definite part. The latter falls directly into the even-fiber full-line positive-cone collapse theorem from Section 16. The second step is a dual large-ray argument showing that, for an arbitrary compact symmetry generator, the singular radii on its nonzero skew-spectral blocks are concave along the base variable. This closes the remaining resonant case as well.

Definition 31.1 (Torically split symmetry generator). Let E be a Euclidean space and let ν be an isotropic log-concave probability measure on E . We say that $K \in \mathfrak{k}_\nu$ is *torically split* if there exist an orthogonal decomposition

$$E = E_0 \oplus E_1 \oplus \cdots \oplus E_m,$$

positive numbers $\omega_1, \dots, \omega_m$, and linear maps $J_j \in \mathfrak{k}_\nu \cap \text{End}(E_j)$ such that

$$J_j^2 = -\text{Id}_{E_j} \quad (j = 1, \dots, m),$$

and

$$K|_{E_0} = 0, \quad K|_{E_j} = \omega_j J_j \quad (j = 1, \dots, m).$$

Equivalently, K acts as zero on E_0 and as a single-frequency complex structure on each nontrivial block E_j , with each blockwise complex structure already belonging to the infinitesimal linear symmetry algebra of ν .

Proposition 31.2 (Blockwise polar factorization for torically split generators). *Let ν be isotropic and let $K \in \mathfrak{k}_\nu$ be torically split as in [theorem 31.1](#). For $u \in \mathbb{R}$, define*

$$\alpha_j(u) := \sqrt{1 + \omega_j^2 u^2}, \quad \theta_j(u) := \arctan(\omega_j u) \quad (j = 1, \dots, m),$$

and set

$$\begin{aligned} P(u) &:= \text{Id}_{E_0} \oplus \alpha_1(u) \text{Id}_{E_1} \oplus \cdots \oplus \alpha_m(u) \text{Id}_{E_m}, \\ R(u) &:= \text{Id}_{E_0} \oplus e^{\theta_1(u) J_1} \oplus \cdots \oplus e^{\theta_m(u) J_m}. \end{aligned}$$

Then

$$P(u) \in \text{Sym}^+(E), \quad R(u) \in G_\nu^{\text{lin}},$$

and

$$I + uK = R(u)P(u) = P(u)R(u) \quad (u \in \mathbb{R}).$$

Proof. Fix $j \in \{1, \dots, m\}$. Since $J_j^2 = -\text{Id}_{E_j}$, one has

$$e^{\theta J_j} = \cos \theta \text{Id}_{E_j} + \sin \theta J_j \quad (\theta \in \mathbb{R}).$$

With $\theta = \theta_j(u) = \arctan(\omega_j u)$, the trigonometric identities

$$\cos \theta_j(u) = \frac{1}{\sqrt{1 + \omega_j^2 u^2}}, \quad \sin \theta_j(u) = \frac{\omega_j u}{\sqrt{1 + \omega_j^2 u^2}}$$

give

$$\alpha_j(u) e^{\theta_j(u) J_j} = \text{Id}_{E_j} + u \omega_j J_j = \text{Id}_{E_j} + u K|_{E_j}.$$

On E_0 , both $P(u)$ and $R(u)$ are the identity and $K|_{E_0} = 0$. Hence

$$I + uK = R(u)P(u) = P(u)R(u)$$

on each summand, and therefore on all of E . Since each $J_j \in \mathfrak{k}_\nu$, the one-parameter subgroup e^{tJ_j} preserves ν . Their direct-sum product $R(u)$ therefore lies in G_ν^{lin} . Finally, $P(u)$ is blockwise scalar with strictly positive eigenvalues $1, \alpha_1(u), \dots, \alpha_m(u)$, so $P(u) \in \text{Sym}^+(E)$. $\square$

Theorem 31.3 (Toric Möbius-symmetry collapse on the full line). *Let E be a Euclidean space and let*

$$\nu(\mathrm{d}z) = e^{-W(z)} \mathrm{d}z$$

be an isotropic log-concave probability measure on E , where $W \in C^2(E)$ is convex, even, and Legendre-type. Let

$$m(\mathrm{d}y) = e^{-\phi(y)} \mathrm{d}y$$

be a log-concave probability measure on $\mathbb{R}$ with $\phi \in C^2(\mathbb{R})$. Let

$$A(y) = A_0(I + s(y)N), \quad b : \mathbb{R} \rightarrow E$$

with $A_0 \in \text{GL}(E)$, $N \in \text{End}(E)$, and $s, b \in C^2(\mathbb{R})$. Define

$$T(z, y) = (A(y)z + b(y), y), \quad \mu := T_\#(\nu \otimes m),$$

and assume that μ is log-concave on $E \times \mathbb{R}$. Suppose moreover that there exist $t_ \in \mathbb{R}$ and a torically split $K \in \mathfrak{k}_\nu$ such that*

$$N = K(I - t_*K)^{-1}.$$

Then A is constant on $\mathbb{R}$ and b is affine. Consequently μ is an affine image of a product measure.

Proof. If $K = 0$, then $N = 0$, hence $A(y) \equiv A_0$ is already constant and the claim follows from [theorem 22.3](#) applied to the zero symmetry-valued velocity. We may therefore assume $K \neq 0$.

Set

$$u(y) := s(y) - t_*, \quad C := A_0(I - t_*K)^{-1}.$$

Then

$$A(y) = A_0(I + s(y)K(I - t_*K)^{-1}) = C(I + u(y)K).$$

By [theorem 31.2](#), for each $y \in \mathbb{R}$,

$$I + u(y)K = R(u(y))P(u(y)),$$

where

$$R(u(y)) \in G_\nu^{\text{lin}}, \quad P(u(y)) \in \text{Sym}^+(E).$$

Now compose the bundle on the source side with the ν -preserving moving frame

$$S_y := R(u(y))^{-1} \in G_\nu^{\text{lin}}.$$

By [theorem 4.4](#), this does not change the underlying family of fibers except by gauge. The resulting exact-affine bundle still pushes $\nu \otimes m$ to μ , and its linear part is

$$A(y)S_y = C P(u(y)).$$

Next apply the fixed ambient affine normalization $x \mapsto C^{-1}x$. The transformed measure remains log-concave, and the transformed exact-affine bundle has the form

$$\tilde{T}(z, y) = (P(u(y))z + \tilde{b}(y), y), \quad \tilde{b}(y) := C^{-1}b(y).$$

Since $P(u(y)) \in \text{Sym}^+(E)$ for all y , [theorem 16.7](#) yields that $P(u(y))$ is constant on $\mathbb{R}$ and $\tilde{b}$ is affine.

Thus each blockwise scalar factor

$$\alpha_j(u(y)) = \sqrt{1 + \omega_j^2 u(y)^2} \quad (j = 1, \dots, m)$$

is constant on $\mathbb{R}$. Since $K \neq 0$, at least one ω_j is positive, and hence $|u(y)|$ is constant on $\mathbb{R}$. Because $u : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $\mathbb{R}$ is connected, u itself is constant. Therefore $s = u + t_*$ is constant, so

$$A(y) = A_0(I + s(y)N)$$

is constant on $\mathbb{R}$. Since $\tilde{b}$ is affine and $b = C\tilde{b}$, the original translation b is affine as well. $\square$

Corollary 31.4 (Single-frequency complex-structure Möbius branches collapse). *Under the hypotheses of [theorem 31.3](#), assume in addition that*

$$K^2 = -\omega^2 \text{Id}_E \quad \text{for some } \omega > 0,$$

and that

$$N = K(I - t_*K)^{-1}.$$

Then every additive full-line exact-affine branch

$$A(y) = A_0(I + s(y)N)$$

collapses: A is constant and b is affine.

Proof. Take $E_0 = \{0\}$, $E_1 = E$, and $J_1 := \omega^{-1}K$. Then $J_1 \in \mathfrak{k}_\nu$, $J_1^2 = -\text{Id}_E$, and K is torically split. Apply [theorem 31.3](#). $\square$

Remark 31.5 (What survives after toric collapse). [Theorem 31.3](#) removes a substantial part of the exceptional additive frontier left by [theorem 26.2](#). Indeed, once an additive branch can touch symmetry only through

$$N = K(I - t_*K)^{-1},$$

the new theorem eliminates the entire subclass for which the symmetry generator K decomposes into blockwise complex structures that are themselves infinitesimal symmetries. Hence any remaining additive symmetry-touching obstruction must already come from a *spectrally entangled* symmetry generator: a generator whose distinct rotation blocks cannot all be gauged independently inside the fiber symmetry algebra. The new torus-rank reduction above sharpens this further: such an obstruction must in fact be *arithmetically resonant*, in the precise sense that the nonzero block frequencies of the underlying compact symmetry generator satisfy a nontrivial integer relation.

Proposition 31.6 (Weight-lattice decomposition of compact symmetry generators). *Let E be a Euclidean space, let ν be an isotropic log-concave probability measure on E , and let*

$$K \in \mathfrak{k}_\nu.$$

Write the orthogonal skew-spectral decomposition of K as

$$E = E_0 \oplus E_1 \oplus \cdots \oplus E_m,$$

with

$$K|_{E_0} = 0, \quad K|_{E_j} = \omega_j J_j \quad (j = 1, \dots, m),$$

where $\omega_j > 0$, $J_j \in \text{End}(E_j)$, and

$$J_j^2 = -\text{Id}_{E_j}.$$

Let

$$\Gamma_K := \sum_{j=1}^m \mathbb{Z} \omega_j \subset \mathbb{R},$$

and choose a $\mathbb{Z}$ -basis $\Omega_1, \dots, \Omega_r$ of the free abelian group Γ_K . Then there exist integers $n_{j\alpha} \in \mathbb{Z}$ such that

$$\omega_j = \sum_{\alpha=1}^r n_{j\alpha} \Omega_\alpha \quad (j = 1, \dots, m).$$

Define $L_\alpha \in \text{End}(E)$ by

$$L_\alpha|_{E_0} = 0, \quad L_\alpha|_{E_j} = n_{j\alpha} J_j \quad (j = 1, \dots, m).$$

Then:

- (i) $L_\alpha \in \mathfrak{k}_\nu$ for every $\alpha = 1, \dots, r$;
- (ii) the endomorphisms $L_1, \dots, L_r$ commute pairwise;
- (iii)

$$K = \sum_{\alpha=1}^r \Omega_\alpha L_\alpha;$$

- (iv) the compact torus

$$\mathbb{T}_K := \{\exp(\theta_1 L_1 + \cdots + \theta_r L_r) : \theta \in \mathbb{R}^r\}$$

is contained in G_ν^{lin} .

Proof. Because ν is isotropic, every linear symmetry of ν is orthogonal, hence

$$\mathfrak{k}_\nu \subset \mathfrak{so}(E).$$

The stated orthogonal decomposition and the blockwise complex structures J_j are therefore the standard spectral decomposition of the skew-symmetric map K .

Since $\Gamma_K \subset \mathbb{R}$ is a finitely generated torsion-free abelian group, it is free abelian of some rank r , and we may choose a $\mathbb{Z}$ -basis $\Omega_1, \dots, \Omega_r$. The integer coefficients $n_{j\alpha}$ are then uniquely determined by

$$\omega_j = \sum_{\alpha=1}^r n_{j\alpha} \Omega_\alpha.$$

By construction,

$$\sum_{\alpha=1}^r \Omega_\alpha L_\alpha$$

acts on E_j as

$$\sum_{\alpha=1}^r \Omega_{\alpha} n_{j\alpha} J_j = \omega_j J_j = K|_{E_j},$$

and vanishes on E_0 . Hence

$$K = \sum_{\alpha=1}^r \Omega_{\alpha} L_{\alpha},$$

which proves (iii). Since the L_{α} are block diagonal and scalar multiples of the same J_j on each block E_j , they commute pairwise, proving (ii).

For $t \in \mathbb{R}$,

$$e^{tK} = \exp\left(t \sum_{\alpha=1}^r \Omega_{\alpha} L_{\alpha}\right) = \prod_{\alpha=1}^r e^{t\Omega_{\alpha} L_{\alpha}},$$

because the L_{α} commute. Since the Ω_{α} form a $\mathbb{Z}$ -basis of Γ_K , they are linearly independent over $\mathbb{Q}$. By the Kronecker density theorem, the set

$$\left\{ \left(e^{t\Omega_1 L_1}, \dots, e^{t\Omega_r L_r} \right) : t \in \mathbb{R} \right\}$$

is dense in the product torus

$$\prod_{\alpha=1}^r \{e^{\theta L_{\alpha}} : \theta \in \mathbb{R}\}.$$

After applying the multiplication map

$$(R_1, \dots, R_r) \longmapsto R_1 \cdots R_r,$$

whose image is exactly $\mathbb{T}_K$, we find that $\{e^{tK} : t \in \mathbb{R}\}$ is dense in $\mathbb{T}_K$.

Now $e^{tK} \in G_{\nu}^{\text{lin}}$ for every t , because $K \in \mathfrak{k}_{\nu}$, and G_{ν}^{lin} is closed in $O(E)$. Therefore

$$\mathbb{T}_K \subset G_{\nu}^{\text{lin}},$$

which proves (iv). Finally, for each α , the path

$$\theta \longmapsto e^{\theta L_{\alpha}}$$

lies in G_{ν}^{lin} ; differentiating at $\theta = 0$ gives

$$L_{\alpha} \in \mathfrak{k}_{\nu},$$

proving (i). □

Corollary 31.7 (Full $\mathbb{Q}$ -rank forces toric splitting). *Under the hypotheses of [theorem 31.6](#), assume that*

$$\text{rank}_{\mathbb{Q}}\{\omega_1, \dots, \omega_m\} = m.$$

Then K is torically split in the sense of [theorem 31.1](#).

Proof. Let $N = (n_{j\alpha})_{1 \leq j, \alpha \leq m}$. The full $\mathbb{Q}$ -rank hypothesis means exactly that the matrix N has rank m , hence N is invertible over $\mathbb{Q}$. Let $C = (c_{\alpha j}) = N^{-1}$. For each $j \in \{1, \dots, m\}$, set

$$\tilde{J}_j := \sum_{\alpha=1}^m c_{\alpha j} L_{\alpha}.$$

Since each $L_\alpha \in \mathfrak{k}_\nu$ and $\mathfrak{k}_\nu$ is a real vector space, we have

$$\tilde{J}_j \in \mathfrak{k}_\nu.$$

On the block E_i ,

$$\tilde{J}_j|_{E_i} = \left(\sum_{\alpha=1}^m c_{\alpha j} n_{i\alpha} \right) J_i = \delta_{ij} J_i.$$

Thus $\tilde{J}_j$ vanishes on E_i for $i \neq j$ and equals J_j on E_j . In particular,

$$J_j \in \mathfrak{k}_\nu \quad (j = 1, \dots, m).$$

Therefore the original skew-spectral decomposition

$$K|_{E_0} = 0, \quad K|_{E_j} = \omega_j J_j$$

already satisfies the definition of a torically split generator. $\square$

Theorem 31.8 (Nonresonant Möbius-symmetry collapse on the full line). *Under the hypotheses of [theorem 31.3](#), replace the toric-splitting assumption by the weaker requirement that, for the nonzero skew-spectral blocks of K ,*

$$\text{rank}_{\mathbb{Q}}\{\omega_1, \dots, \omega_m\} = m.$$

Then A is constant on $\mathbb{R}$ and b is affine. Consequently μ is an affine image of a product measure.

Proof. By [theorem 31.7](#), the compact symmetry generator K is automatically torically split. The conclusion now follows immediately from [theorem 31.3](#). $\square$

Remark 31.9 (Resonance before the final compact-symmetry step). Combine [theorem 26.2](#) with [theorem 31.8](#). This shows that, after the toric and weight-lattice reductions alone, only arithmetically resonant compact symmetry generators can still remain in the additive symmetry-touching full-line branch. The dual large-ray argument below removes this residual resonant case as well.

Proposition 31.10 (Block singular-radius concavity for compact-symmetry additive branches). *Let E be a Euclidean space, let $I \subset \mathbb{R}$ be an interval, and let*

$$\nu(dz) = e^{-W(z)} dz, \quad m(dy) = e^{-\phi(y)} dy$$

be log-concave probability measures, where ν is isotropic, $W \in C^2(E)$ is convex, even, Legendre-type, and $\nabla^2 W(z)$ is positive-definite for every $z \in E$, while $\phi \in C^2(I)$ is convex. Let $K \in \mathfrak{k}_\nu$, let

$$u : I \rightarrow \mathbb{R}, \quad b : I \rightarrow E$$

be C^2 , and define

$$T(z, y) = ((I + u(y)K)z + b(y), y), \quad \mu := T_\#(\nu \otimes m).$$

Assume that μ is log-concave on $E \times I$. Let

$$E = E_0 \oplus E_1 \oplus \dots \oplus E_m$$

be the orthogonal decomposition into the eigenspaces of $-K^2$, so that

$$K|_{E_0} = 0, \quad K|_{E_j} = \omega_j J_j \quad (j = 1, \dots, m),$$

with $\omega_j > 0$ and $J_j^2 = -\text{Id}_{E_j}$. For $j = 1, \dots, m$, define

$$a_j(y) := \sqrt{1 + \omega_j^2 u(y)^2}.$$

Then each a_j is concave on I .

Proof. If $m = 0$, there is nothing to prove. Fix $j \in \{1, \dots, m\}$ and choose a unit vector $e_j \in E_j$. Define

$$h_j(r) := W^*(re_j) \quad (r \in \mathbb{R}).$$

Because W is even, W^* is even; because W^* is convex and C^2 , the function h_j is even, convex, and C^2 on $\mathbb{R}$. Moreover, for $r \geq 0$,

$$h_j(r) \geq \langle re_j, e_j \rangle - W(e_j) = r - W(e_j),$$

so h_j is unbounded above on $[0, \infty)$. Hence [theorem 16.2](#) gives

$$r h_j'(r) \xrightarrow{r \rightarrow +\infty} +\infty.$$

Set

$$\theta_j(y) := \arctan(\omega_j u(y)).$$

Since $K \in \mathfrak{k}_\nu$, [theorem 21.2](#) shows that $K \in \mathfrak{so}(E)$. Therefore

$$U_j(y) := \exp\left(-\frac{\theta_j(y)}{\omega_j} K\right) \in \text{O}(E).$$

By [theorem 22.2](#), $U_j(y) \in G_\nu^{\text{lin}}$ for every $y \in I$. Because $U_j(y)$ is orthogonal and preserves ν , one has

$$W(U_j(y)z) = W(z) \quad (z \in E),$$

and consequently

$$W^*(U_j(y)\xi) = W^*(\xi) \quad (\xi \in E).$$

On the block E_j , one has $K = \omega_j J_j$, so

$$(I + u(y)K)^T e_j = (I - u(y)K)e_j = (I - \omega_j u(y)J_j)e_j = a_j(y)e^{-\theta_j(y)J_j} e_j = U_j(y)(a_j(y)e_j).$$

Hence, for every $t > 0$,

$$W^*(t(I + u(y)K)^T e_j) = W^*(ta_j(y)e_j) = h_j(ta_j(y)).$$

By [theorem 30.1](#), for every $t > 0$ the functions

$$\Psi_{\pm te_j}(y) = W^*(\pm t(I + u(y)K)^T e_j) \pm t \langle b(y), e_j \rangle - \log \det(I + u(y)K) - \phi(y)$$

are concave on I . Averaging them and using the evenness of W^* gives another concave function,

$$F_{j,t}(y) := \frac{1}{2}(\Psi_{te_j}(y) + \Psi_{-te_j}(y)) = h_j(ta_j(y)) - \log \det(I + u(y)K) - \phi(y).$$

Fix $y_0 \in I$. Differentiating twice at y_0 yields

$$F_{j,t}''(y_0) = t^2 h_j''(ta_j(y_0)) a_j'(y_0)^2 + t h_j'(ta_j(y_0)) a_j''(y_0) - \frac{d^2}{dy^2} \Big|_{y=y_0} (\log \det(I + u(y)K) + \phi(y)).$$

The first term is nonnegative. If $a_j''(y_0) > 0$, then $a_j(y_0) > 0$ and

$$t h_j'(ta_j(y_0)) = \frac{1}{a_j(y_0)} \left(ta_j(y_0) h_j'(ta_j(y_0)) \right) \xrightarrow{t \rightarrow +\infty} +\infty.$$

Since the last term is independent of t , this would imply $F_{j,t}''(y_0) > 0$ for all sufficiently large t , contradicting concavity of $F_{j,t}$. Therefore $a_j''(y_0) \leq 0$. As y_0 was arbitrary, a_j is concave on I . $\square$

Theorem 31.11 (Full compact-symmetry Möbius collapse on the full line). *Let E be a Euclidean space and let*

$$\nu(dz) = e^{-W(z)} dz$$

be an isotropic log-concave probability measure on E , where $W \in C^2(E)$ is convex, even, Legendre-type, and $\nabla^2 W(z)$ is positive-definite for every $z \in E$. Let

$$m(dy) = e^{-\phi(y)} dy$$

be a log-concave probability measure on $\mathbb{R}$ with $\phi \in C^2(\mathbb{R})$. Let

$$A(y) = A_0(I + s(y)N), \quad b : \mathbb{R} \rightarrow E$$

with $A_0 \in \text{GL}(E)$, $N \in \text{End}(E)$, and $s, b \in C^2(\mathbb{R})$. Define

$$T(z, y) = (A(y)z + b(y), y), \quad \mu := T_{\#}(\nu \otimes m),$$

and assume that μ is log-concave on $E \times \mathbb{R}$. Suppose moreover that there exist $t_ \in \mathbb{R}$ and $K \in \mathfrak{k}_{\nu}$ such that*

$$N = K(I - t_*K)^{-1}.$$

Then A is constant on $\mathbb{R}$ and b is affine. Consequently μ is an affine image of a product measure.

Proof. If $K = 0$, then $N = 0$, so $A(y) \equiv A_0$ is already constant. The conclusion then follows from [theorem 8.2](#) after pushing the fixed linear map A_0 into the reference fiber. We may therefore assume that $K \neq 0$.

Set

$$u(y) := s(y) - t_*, \quad C := A_0(I - t_*K)^{-1}.$$

Then

$$A(y) = A_0(I + s(y)K(I - t_*K)^{-1}) = C(I + u(y)K).$$

Apply the fixed ambient affine normalization $x \mapsto C^{-1}x$. The transformed measure remains log-concave, and the transformed exact-affine bundle has the form

$$\tilde{T}(z, y) = ((I + u(y)K)z + \tilde{b}(y), y), \quad \tilde{b}(y) := C^{-1}b(y).$$

Let

$$E = E_0 \oplus E_1 \oplus \cdots \oplus E_m$$

be the orthogonal decomposition into the eigenspaces of $-K^2$, so that

$$K|_{E_0} = 0, \quad K|_{E_j} = \omega_j J_j \quad (j = 1, \dots, m),$$

with $\omega_j > 0$ and $J_j^2 = -\text{Id}_{E_j}$. By [theorem 31.10](#), each function

$$a_j(y) := \sqrt{1 + \omega_j^2 u(y)^2} \quad (j = 1, \dots, m)$$

is concave on $\mathbb{R}$. Since $a_j(y) > 0$, [theorem 16.6](#) implies that every a_j is constant on $\mathbb{R}$. Because $K \neq 0$, at least one ω_j is positive. For such a block,

$$1 + \omega_j^2 u(y)^2$$

is constant, so $|u(y)|$ is constant on $\mathbb{R}$. As $u : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $\mathbb{R}$ is connected, u itself is constant. Therefore A is constant on $\mathbb{R}$.

Let u_0 denote this constant value and set

$$L := I + u_0 K \in \text{GL}(E).$$

Push the fixed linear map L into the reference fiber:

$$\nu_*(dx) := L_{\#}\nu.$$

Its potential is

$$W_*(x) = W(L^{-1}x) + \log \det L,$$

so

$$\nabla W_*(x) = L^{-T} \nabla W(L^{-1}x)$$

has range E ; hence W_* is again Legendre-type. The normalized bundle now becomes

$$\tilde{T}(z, y) = (Lz + \tilde{b}(y), y),$$

which is exactly a translational branch over ν_* after pushing L into the reference measure. By [theorem 8.2](#), $\tilde{b}$ is affine on $\mathbb{R}$. Since $b = C\tilde{b}$, the original translation b is affine as well. $\square$

Corollary 31.12 (Complete elimination of the additive symmetry-touching full-line branch). *Let E be a Euclidean space and let*

$$\nu(dz) = e^{-W(z)} dz$$

be an isotropic log-concave probability measure on E , where $W \in C^2(E)$ is convex, even, Legendre-type, and $\nabla^2 W(z)$ is positive-definite for every $z \in E$. Let

$$m(dy) = e^{-\phi(y)} dy$$

be a log-concave probability measure on $\mathbb{R}$ with $\phi \in C^2(\mathbb{R})$. Let

$$A(y) = A_0(I + s(y)N), \quad b : \mathbb{R} \rightarrow E$$

with $A_0 \in \text{GL}(E)$, $N \in \text{End}(E)$, and $s, b \in C^2(\mathbb{R})$. Define

$$T(z, y) = (A(y)z + b(y), y), \quad \mu := T_{\#}(\nu \otimes m),$$

and assume that μ is log-concave on $E \times \mathbb{R}$. If

$$Z_{N,\nu} \neq \emptyset,$$

equivalently if $\Gamma_N(t_) \in \mathfrak{k}_\nu$ for some $t_* \in \Omega_N$, then A is constant on $\mathbb{R}$ and b is affine.*

Proof. Choose $t_* \in Z_{N,\nu}$. By [theorem 26.2](#), there exists $K \in \mathfrak{k}_\nu$ such that

$$N = K(I - t_*K)^{-1}.$$

Now apply [theorem 31.11](#). □

Remark 31.13 (What survives after the full Möbius collapse). Combine [theorem 17.1](#) with [theorem 31.12](#). Then, over an even isotropic Legendre fiber, every additive full-line exact-affine branch

$$A(y) = A_0(I + s(y)N)$$

collapses whenever either the fixed direction carries a nonzero real spectral mode after a fixed affine normalization or the normalized velocity can touch the symmetry algebra at some parameter. Consequently a surviving additive full-line obstruction would have to be simultaneously spectrally real-free after every fixed affine normalization and uniformly separated from the symmetry algebra on every compact parameter range. In particular, the old exceptional symmetry-touching Möbius branch no longer survives the full-line exact-affine elimination theory.

32 What still remains after even-fiber positive-part collapse, and the next exact-affine conjectures

The article still does not close KLS, but it now sharpens the exact-affine frontier more substantially than the previous revision. The tail-rigidity results of Section 24 show that, in the even-isotropic exact-affine world, finite one-sided curvature budget forces not only asymptotic freezing of the linear part but also asymptotic freezing of the translation slope, and under a first-moment curvature-tail bound the entire bundle approaches an affine product on the tail. Section 25 shows, in addition, that multiplicative one-parameter motion is already quantitatively pinned to the same curvature budget unless its generator is a genuine infinitesimal symmetry. Section 26 proves the additive companion: fixed-direction motion on curved bases obeys an exact scalar budget through $\Gamma_N(s) = (I + sN)^{-1}N$, and the normalized velocity can enter the orthogonal symmetry algebra at most at a unique Möbius-symmetry parameter t_* , where the direction matrix is necessarily $K(I - t_*K)^{-1}$ for an actual symmetry generator $K \in \mathfrak{k}_\nu$. Thus every non-Möbius-symmetry additive direction remains uniformly away from symmetry on compact parameter ranges, while even the exceptional additive branch freezes on compact tails with finite $\int \sqrt{\phi''}$. Section 27 supplies the exact-affine blow-up half of the program: vanishing local curvature budgets force convergence to an affine product after orthogonal gauge and normalization, and therefore any normalized nontrivial exact-affine motion must carry a definite curvature quantum on every compact scale. Section 28 globalizes that principle on the full line: if prescribed-scale almost-flat windows fail to appear, then the combined curvature budget

$$\lambda_\nu^{-1/2} \int \sqrt{\phi''} + \tau_\nu^{-1} \int \phi''$$

must already accumulate linearly in the number of windows; conversely, sublinear combined curvature growth automatically produces arbitrarily long almost-flat windows. Section 29 strengthens this further from existence to prevalence: at every fixed scale the bad centers occupy measure at most $O(R\mathcal{B}/c_\epsilon)$, so vanishing curvature density yields almost-flat windows with density one. Section 30 then recasts exact-affine log-concavity itself as a pointwise concavity problem for partial Legendre conjugates, i.e. as a concave-orbit rigidity problem in the dual potential space. Finally, Section 31 removes the exceptional additive symmetry-touching full-line frontier altogether: the toric and

nonresonant subbranches collapse by explicit factorization and weight-lattice reduction, while a dual large-ray argument forces concavity of the block singular radii for every compact symmetry generator and hence collapses the remaining resonant case as well.

After the present theorem package, a surviving obstruction would have to use at least one of the following mechanisms.

- (1) **Bounded or one-sided base geometry.** The full-line hypothesis is still crucial in the scalar, fixed-direction real-spectrum, self-adjoint positive-definite, and radial collapse theorems, because positivity plus concavity forces constancy only on the whole line. However, [theorems 20.7](#) and [20.8](#) remove the entire affine-base subbranch $\phi'' \equiv 0$ modulo affine ν -symmetry gauge, with no sign restriction on $\text{Tr}(B^2)$, and [theorem 22.3](#) removes the whole branch $A^{-1}A' \in \mathfrak{g}_\nu$ on arbitrary intervals. Moreover, in the even-isotropic setting [theorems 23.6](#) and [23.7](#) show that any residual normalized translation acceleration $A^{-1}b''$ and any genuinely non-symmetry linear motion must each pay pointwise base-curvature budget. Thus any remaining bounded-interval or half-line obstruction must already carry genuinely curved base geometry $\phi'' \not\equiv 0$ together with genuinely non-symmetry linear motion; and over even fibers it must also carry genuinely nontrivial translation budget. By [theorems 27.1](#) to [27.3](#), a normalized nontrivial exact-affine motion cannot persist on a compact interval with arbitrarily small values of

$$\int \sqrt{\phi''} \quad \text{and} \quad \int \phi'' :$$

such intervals already collapse to affine products after orthogonal gauge. By [theorems 28.5](#) and [28.7](#), this local statement globalizes on the full line: if prescribed-scale almost-flat windows fail to occur, then the combined curvature budget

$$\lambda_\nu^{-1/2} \int \sqrt{\phi''} + \tau_\nu^{-1} \int \phi''$$

must grow at least linearly with the number of disjoint windows, whereas any sublinear combined curvature growth automatically produces arbitrarily long almost-flat windows. The new density theorem [theorems 29.1](#) and [29.2](#) goes further: once the combined curvature density tends to zero, almost-flat windows of any fixed scale occur with density one, so a genuine obstruction must sustain a positive-density population of bad centers rather than merely dodge a sparse exceptional subsequence. By [theorems 24.2](#) and [24.3](#), if the one-sided curvature budgets $\int \sqrt{\phi''}$ and $\int \phi''$ are finite then the tail is already asymptotically affine after orthogonal gauge, and under the first-moment tail condition it is asymptotic to an actual affine product. Moreover, by [theorems 25.1](#) and [25.2](#), any bounded-interval or one-sided multiplicative one-parameter branch $A(y) = A_0 e^{s(y)G}$ with generator outside the symmetry algebra must pay explicit curvature budget proportional to $\text{dist}_{\text{HS}}(G, \mathfrak{k}_\nu)^2$, and on one-sided tails with finite $\int \sqrt{\phi''}$ the subgroup parameter already freezes asymptotically. In particular, for fibers with trivial affine symmetry algebra, both the affine-base branch and the exact symmetry-valued branch collapse outright in the original coordinates, while flat base points are instantaneously rigid.

- (2) **Large curvature budget or very small symmetry gap.** By [theorems 21.4](#) and [21.5](#), once the reference fiber is isotropic, any nontrivial linear distortion of the exact-affine bundle on an interval $[y_1, y_2]$ forces

$$\int_{y_1}^{y_2} \sqrt{\phi''(s)} \, ds$$

to be at least of order $\sqrt{\lambda_\nu}$ times the logarithm of the resulting condition number, after optimal orthogonal gauge. By [theorem 23.6](#), in the even-isotropic case the same base curvature also controls the normalized translation acceleration through the universal translation gap $\tau_\nu \geq 1/\sqrt{3}$. The new compact-interval stability theorem and curvature-quantum corollary strengthen this further: after normalization at a base point, a bounded-interval obstruction cannot remain genuinely nontrivial while both

$$\int \sqrt{\phi''} \quad \text{and} \quad \int \phi''$$

are arbitrarily small. On the full line, [theorems 28.5, 28.7, 29.1 and 29.3](#) sharpen the same message into a quantitative density statement: persistent nontriviality on a prescribed window scale already forces linear growth of the combined budget $\lambda_\nu^{-1/2} \int \sqrt{\phi''} + \tau_\nu^{-1} \int \phi''$, and if the scale-dependent ratio $R\mathcal{B}/L$ tends to zero then the corresponding good windows occupy asymptotic density one on intervals of length $2L$. Combined with [theorem 22.3](#), this shows that an unresolved bounded-interval or full-line obstruction cannot hide entirely in the symmetry algebra: it must either accumulate genuinely large curvature budget or arise from a canonical fiber whose symmetry-gap constant λ_ν is anomalously small while simultaneously carrying a genuine non-symmetry component of the linear velocity. There is no analogous even-fiber translation degeneracy left, because τ_ν is already universally bounded below.

- (3) **Genuinely nonselfadjoint, spectrally real-free, and symmetry-avoiding linear motion.** Over an even Legendre fiber, [theorem 16.7](#) closes the entire full-line branch whose linear part is self-adjoint positive-definite after a fixed affine normalization, [theorem 17.1](#) eliminates the affine fixed-direction branch carrying a nonzero real spectral mode, [theorem 17.6](#) eliminates the multiplicative one-parameter subgroup branch

$$A(y) = A_0 e^{s(y)G}$$

whenever the generator has a nonzero real spectral mode after fixed affine normalization, and [theorem 31.12](#) eliminates the additive fixed-direction branch whenever the normalized velocity can touch the symmetry algebra at some parameter. More generally, [theorem 16.3](#) shows that even without self-adjointness the matrix $A(y)^{-1}A''(y)$ cannot carry a positive real eigenmode, and [theorem 20.14](#) shows that at a flat point $A''(y_0) = 0$ the velocity matrix $A(y_0)^{-1}A'(y_0)$ cannot carry a nonzero real eigenmode either. Hence the remaining exact-affine full-line branch is forced into genuinely nonselfadjoint or rotational behavior not reducible either to a globally fixed positive cone, to an affine fixed-direction real mode, to a multiplicative one-parameter real mode, to an additive symmetry-touching Möbius branch, or to a real-spectrum flat-point crossing. In the additive fixed-direction subworld, any surviving full-line obstruction must therefore be both spectrally real-free after every fixed affine normalization and uniformly separated from the symmetry algebra on every compact parameter range. On bounded or one-sided bases, [theorems 25.1 and 25.3](#) further show that even these real-free subgroup branches are quantitatively visible unless the generator is an actual symmetry.

- (4) **Non-even or nonradial fiber geometry.** The even-fiber and radial arguments rely on opposite-ray or orthogonal-orbit tests that are unavailable for a general anisotropic Legendre potential. The new translation-budget theorem closes the missing curved-base translation-control problem in the even world; the remaining local exact-affine difficulty is therefore genuinely anisotropic and non-even.

- (5) **Non-exact-affine terminal deformation.** A minimal obstruction might still involve higher-order internal shape change beyond the affine ansatz $A(y)z + b(y)$.
- (6) **The global rank-one problems.** Even if exact-affine terminal models are eliminated, one still needs a universal Poincaré bound for the isotropized canonical reference fiber and a universal affine-normalized moving-frame distortion bound.

The conjecture called “even-fiber positive-part collapse” in the previous version is now a theorem, and the previously missing curved-base translation-control theorem is now also proved in the even setting. The new subgroup-budget theorem closes the last missing bounded-interval control statement for fixed multiplicative one-parameter motion: outside the symmetry algebra, the subgroup parameter is itself curvature-budgeted. The next natural exact-affine target is therefore the genuinely nonselfadjoint anisotropic branch which is spectrally real-free after every fixed affine normalization, both in the additive fixed-direction sense and in the multiplicative one-parameter subgroup sense; in the additive case it must also stay uniformly away from the symmetry algebra on compact parameter ranges. The new quantitative sections suggest two finite-dimensional obstruction parameters: the symmetry-gap constant λ_ν and, beyond evenness, a possible anisotropic translation-gap functional.

At the exact-affine level, the compactness half of the blow-up strategy is now settled by [theorem 27.3](#): once one can produce normalized scales with vanishing local curvature budgets, the only possible limit is an affine product. The remaining bridge to a genuine partial solution is therefore no longer an exact-affine compactness theorem but the global mechanism that extracts such almost-flat scales from a true failure of affine-normalized moving-frame control.

The dual characterization from [theorems 30.1](#) and [30.4](#) refines that bridge conceptually. It shows that the unresolved exact-affine branch is not just a matrix-ODE problem for $A^{-1}A'$; it is equivalently a problem of classifying full-line pointwise-concave families

$$\Psi_\xi(y) = W^*(A(y)^T \xi) + \langle b(y), \xi \rangle - \log \det A(y) - \phi(y) \quad (\xi \in E),$$

modulo the natural ν -preserving gauge action. Any decisive future advance must therefore either force density-one almost-flat scales from the transport side or exclude nontrivial concave dual orbits by exploiting anisotropic structure of W^* .

Conjecture 32.1 (Canonical symmetry-gap lower bound). For canonical rank-one fibers arising from a genuine minimal KLS obstruction, after isotropic normalization the symmetry-gap constant λ_ν is bounded below by a universal constant.

In the even-isotropic setting, [theorem 23.6](#) now proves the missing translation-control theorem on curved bases:

$$\tau_\nu |A^{-1}b''| + \lambda_\nu \text{dist}_{\text{HS}}(A^{-1}A', \mathfrak{k}_\nu)^2 \leq \phi''.$$

Consequently every unresolved even exact-affine obstruction on a bounded interval must spend a definite amount of base-curvature budget simultaneously on normalized translation acceleration and on *genuinely non-symmetry* linear distortion. Extending this local budget theorem beyond even fibers would, in our view, be the next decisive bounded-interval step.

Conjecture 32.2 (Anisotropic curved-base translation budget). Let E be a Euclidean space, let

$$\nu(dz) = e^{-W(z)} dz$$

be an isotropic log-concave probability measure on E with $W \in C^2(E)$ convex and Legendre-type. Then there exists a positively homogeneous, coercive functional

$$\mathcal{T}_\nu : E \rightarrow [0, \infty)$$

depending only on the reference fiber, such that for every exact-affine bundle

$$T(z, y) = (A(y)z + b(y), y)$$

with $T_\#(\nu \otimes m)$ log-concave on $E \times I$,

$$\mathcal{T}_\nu(A(y)^{-1}b''(y)) + \mathcal{Q}_\nu(A(y)^{-1}A'(y)) \leq \phi''(y) \quad (y \in I).$$

For even fibers, one may take

$$\mathcal{T}_\nu(v) = \int_E |\langle \nabla W(z), v \rangle| \nu(\mathrm{d}z)$$

by [theorem 23.3](#).

Conjecture 32.3 (Even-fiber full-line exact-affine collapse). Let E be a Euclidean space, let

$$\nu(\mathrm{d}z) = e^{-W(z)} \mathrm{d}z$$

be a log-concave probability measure on E with $W \in C^2(E)$ convex, even and Legendre-type, let

$$m(\mathrm{d}y) = e^{-\phi(y)} \mathrm{d}y$$

be log-concave on $\mathbb{R}$, and let

$$T(z, y) = (A(y)z + b(y), y)$$

with $A : \mathbb{R} \rightarrow \mathrm{GL}(E)$, $b : \mathbb{R} \rightarrow E$ of class C^2 . If $T_\#(\nu \otimes m)$ is log-concave on $E \times \mathbb{R}$, then after composing with a smooth family of ν -preserving diffeomorphisms of the reference fiber, the linear part becomes constant and the translation becomes affine.

By [theorems 16.7](#) and [16.8](#), [theorem 32.3](#) is already verified whenever, after a fixed affine normalization of the fiber coordinates, the linear part lies in the positive self-adjoint cone. By [theorem 17.1](#), it is also verified for every affine fixed-direction branch whose normalized direction matrix carries a nonzero real eigenvalue, by [theorem 17.6](#) it is further verified for every multiplicative one-parameter subgroup branch

$$A(y) = A_0 e^{s(y)G}$$

whose normalized generator carries a nonzero real eigenvalue, and by [theorem 31.12](#) it is also verified for every additive fixed-direction branch whose normalized velocity can touch the symmetry algebra at some parameter. By [theorem 20.14](#), even a local flat-point crossing cannot carry such a real mode. By [theorem 20.7](#), the affine-base branch $\phi'' \equiv 0$ is also gone modulo natural gauge. Hence any counterexample would have to produce a genuinely nonselfadjoint acceleration mechanism rather than a positive real eigenmode; in the additive fixed-direction subworld it would moreover have to stay uniformly away from the symmetry algebra on every compact parameter range, while in the bounded/one-sided world it would have to carry genuinely curved base geometry.

The most natural global exact-affine conjecture left by the present article is the following.

Conjecture 32.4 (Full-line exact-affine Legendre collapse). Let E be a Euclidean space, let

$$\nu(\mathrm{d}z) = e^{-W(z)} \mathrm{d}z$$

be a log-concave probability measure on E with $W \in C^2(E)$ convex and Legendre-type, let

$$m(\mathrm{d}y) = e^{-\phi(y)} \mathrm{d}y$$

be log-concave on $\mathbb{R}$, and let

$$T(z, y) = (A(y)z + b(y), y)$$

with $A : \mathbb{R} \rightarrow \mathrm{GL}(E)$, $b : \mathbb{R} \rightarrow E$ of class C^2 . If $T_{\#}(\nu \otimes m)$ is log-concave on $E \times \mathbb{R}$, then after composing with a smooth family of ν -preserving diffeomorphisms of the reference fiber, the linear part becomes constant and the translation becomes affine.

The results proved here verify [theorem 32.4](#) in the Gaussian case, in the scalar case, in the unconditional fixed-axis case, in the self-adjoint positive-definite even case, in the entire radial case, and in the even-isotropic additive fixed-direction branch whenever the normalized velocity can touch the symmetry algebra. They also eliminate an affine-base interval subbranch and sharpen the local inflection geometry of the even-fiber branch. Closing the remaining anisotropic nonselfadjoint branch would, in our view, be the next decisive exact-affine step toward a real partial solution of KLS.

Status of the KLS conjecture

The results proved here constitute substantial unconditional structural advances inside the transport–soft-edge program toward the Kannan–Lovász–Simonovits conjecture, but they do not constitute a proof of the full conjecture. The KLS conjecture remains open in full generality; the strongest known general bound is the $\sqrt{\log n}$ -isoperimetric (equivalently $\log n$ -Poincaré) estimate of Klartag [8].

What the present article achieves is a systematic and materially stronger elimination of exact-affine terminal geometries. The eliminated classes now include: scalar dilations, fixed-axis unconditional, self-adjoint positive-definite (over even fibers), fixed-direction with real spectral mode, complete radial, affine-base (modulo gauge), toric and nonresonant Möbius subbranches, and in the even-isotropic fixed-direction setting the entire additive symmetry-touching Möbius branch on the full line. In addition, the article establishes quantitative structural controls: the symmetry-gap coercivity theorem, the even-fiber translation-budget theorem, the local affine-product stability theorem, and the curvature-density dichotomy on the full line.

The remaining exact-affine frontier is precisely characterized in Section 32: any surviving obstruction must be genuinely nonselfadjoint and spectrally real-free after every fixed affine normalization, must carry genuinely curved base geometry, and in the additive fixed-direction full-line world must remain uniformly away from the symmetry algebra on every compact parameter range; in the even bounded-interval world it must also spend base-curvature budget simultaneously on translation acceleration and genuinely non-symmetry linear motion. Resolving this remaining branch, together with the global rank-one problems (universal fiber Poincaré bound and affine-normalized moving-frame distortion bound), would complete the transport–soft-edge approach to the KLS conjecture.

A Retained theorem statements from the earlier memoranda

For completeness, we record the principal statements from the previous notes that survive unchanged and that are not reproved here.

Theorem A.1 (Retained first-eigenfunction bridge). *Let $\mu = e^{-\psi} dx$ be regular, isotropic and strictly log-concave on $\mathbb{R}^n$, let*

$$L = \Delta - \nabla \psi \cdot \nabla,$$

and let f be a normalized first eigenfunction of $-L$ with eigenvalue

$$\lambda = \frac{1}{C_P(\mu)}.$$

Then the linear part

$$a_f := \int f(x)x \, d\mu(x)$$

satisfies

$$a_f \otimes a_f \preceq \lambda \mathcal{H}_\mu.$$

Moreover, if $a_f = 0$, then

$$\nabla^2 \psi(x) \nabla f(x) = 0 \quad \text{for } \mu\text{-almost every } x.$$

Theorem A.2 (Retained Poisson-gauge obstruction). *Let μ be isotropic, let $\mathbb{R}^n = E \oplus F$ be an orthogonal splitting, and let $\mathcal{K}(y)$ be the interaction matrix attached to the Poisson gauge. If*

$$P_F \mathcal{H}_\mu P_F \succeq K \text{Id}_F$$

and

$$\sup_y \|\mathcal{K}(y)\|_{\text{op}} \leq S,$$

then

$$S \geq \frac{K}{C_P(m)} - 1.$$

In particular, on the canonical transport edge F_K , one has $S_K \gtrsim K$.

Remark A.3. The present article does not contradict [theorem A.2](#). It reinterprets it. The theorem remains correct, but it governs only the Poisson gauge, i.e. the minimal kinetic lift of the rank-one transport motion.

B Bibliographic note

The external mathematical inputs used or quoted in the present note are:

- the original KLS paper and localization lemma;
- the classical Brascamp–Lieb inequality and its equality case for smooth log-concave measures;
- the stochastic-localization program of Eldan, Lee–Vempala, Chen, Klartag–Lehec and others;
- the improved log-concave Lichnerowicz inequality and the $\sqrt{\log n}$ -KLS bound of Klartag;

- the affirmative resolution of Bourgain’s slicing problem by Klartag and Lehec;
- the thin-shell theorem of Klartag and Lehec;
- basic one-dimensional extremal facts for even log-concave densities, used only to lower-bound the translation-gap constant in isotropic coordinates.

The proofs in the present article only use these results through the retained transport-spectrum theorem and the previously verified identities recorded in the earlier memoranda.

Computer-verified proofs. Several foundational lemmas from this article have been formally verified in the Lean 4 proof assistant with the Mathlib mathematical library, including the positive-concave-implies-constant lemma ([theorems 16.6](#) and [18.5](#)) and its nonnegative variant. The Lean source files are available in the supplementary materials.

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