Heat Conduction in Solids with Fractal Cracks in the Path-Integral Formalism

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Abstract

We develop an approach to the description of steady-state heat conduction in solids containing a hierarchical (fractal) network of cracks. The temperature field and the crack geometry are treated as dynamical variables, and the full partition function is written as a double functional integral over the temperature field and the crack-surface configurations. Using the exact duality between the temperature field and an ensemble of Brownian streamlines, we show that averaging over fractal boundaries induces an effective two-dimensional quantum gravity of the Polyakov type on the worldsheets of the streamlines. A closed system of mean-field equations is derived: a nonlinear heat-conduction equation and a modified Liouville equation for the crack density. An exact analytical solution is found for the one-dimensional steady-state temperature and crack-density profiles, demonstrating the existence of a critical heat flux J_c that separates two qualitatively different regimes: for $J < J_c$ the cracks are suppressed by surface tension; for $J > J_c$ the heat flux concentrates the cracks into a localized cluster, which can lead to a percolation transition.

Keywords: heat conduction, fractal cracks, path integral, Liouville equation, two-dimensional quantum gravity, percolation transition.

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1 Introduction

The classical theory of heat conduction in heterogeneous media, based on the Wiener integral [16] or averaging methods (the Maxwell–Garnett and Bruggeman effective-medium theories [27]), encounters fundamental difficulties when describing systems with a fractal hierarchy of internal boundaries [21, 22]. Cracks in real materials—ceramics, rocks, concrete—often form complex branching structures with a fractional Hausdorff dimension D_f (for the basic concepts of fractal geometry, see [24, 25]). Direct numerical solution of the Laplace equation with boundary conditions on fractal surfaces is computationally expensive and does not yield insight into the critical behavior of the effective thermal conductivity κ_{eff}.

The approach developed in the present work is based on the formalism of **functional integration over surfaces**, proposed by Polyakov for string theory and two-dimensional quantum gravity [1, 2]. The path-integral method, originating from Feynman [10] and systematically presented in the monographs of Kleinert [8] and Zinn-Justin [9], is a universal tool of quantum and statistical physics. The key idea of the present work is to apply this formalism to the heat-transfer problem by treating *cracks* as "worldsurfaces" embedded in the bulk material ($\mathbb{R}^d$) and *heat-flux streamlines* as analogs of strings interacting with these boundaries.

We carry out the following program step by step. First, the partition function of the system Z is formulated as a functional integral over crack geometries Σ and the temperature field T (Sec. 2). Then the duality "field $T \leftrightarrow$ Brownian streamlines" in the presence of boundaries Σ is established and a loop-gas representation is obtained (Sec. 3). Next, it is demonstrated that averaging over fractal boundaries induces the Liouville action—an effective two-dimensional gravity—on the worldsheet of the streamlines (Sec. 4). From this, mean-field equations for macroscopic heat transfer are derived (Sec. 5). Finally, an exact solution in a one-dimensional geometry is found and the critical regimes are discussed (Sec. 6).

The geometric constructions used in this work—the metric, curvature, and conformal transformations on surfaces—are discussed in detail in [11]; the connection with quantum field theory is covered in [9, 2].

2 Partition function of the system

Consider a d-dimensional domain $\Omega \subset \mathbb{R}^d$ occupied by a solid with bare thermal conductivity κ_0. Inside it there is a macroscopically large fractal crack Σ; the generalization to an ensemble of cracks is straightforward.

In the steady state, the temperature field $T(x)$ obeys the Laplace equation $\Delta T = 0$ everywhere outside Σ (see, e.g., [26]). The cracks are assumed to be adiabatic, i.e., with zero normal heat flux:

$$n^\mu \partial_\mu T(x)\big|_{x\in\Sigma} = 0, \tag{1}$$

where n^μ is the unit normal to the surface.

The partition function of the system is written as a double functional integral (cf. the general construction in [8], Ch. 10):

$$Z = \int \mathcal{D}\Sigma\, e^{-S_{\text{geom}}[\Sigma]}\, Z_{\text{temp}}[\Sigma], \tag{2}$$

where the temperature part is the Gaussian integral over the field T for a fixed crack:

$$Z_{\text{temp}}[\Sigma] = \int \mathcal{D}T\, \exp\left(-\frac{1}{2}\int_{\Omega\setminus\Sigma} \kappa_0\, (\partial_\mu T)^2\, d^d x\right). \tag{3}$$

The measure $\mathcal{D}\Sigma$ implies summation over all surface configurations with the weight $e^{-S_{\text{geom}}}$. For the ensemble to generate fractal structures, the geometric action must ensure scale invariance and branching. In the spirit of Polyakov's approach [1, 2], the action for the crack is written in terms of the intrinsic metric g_{ab} on Σ:

$$S_{\text{geom}}[\Sigma] = \mu_0 \int d^2\xi \sqrt{g} + \beta \int d^2\xi \sqrt{g}\, R + \ldots \tag{4}$$

Here μ_0 is the bare crack tension; R is the scalar curvature of the intrinsic metric g_{ab} (for the definition and properties of surface curvature, see [11], Vol. 1, Ch. 3); β is a rigidity parameter. The second term is the topological Gauss–Bonnet invariant (see [11], Vol. 3, Ch. 3; also [12]). In the continuum limit, such an ensemble is described by a conformal field theory (CFT) with central charge c_Σ; for the foundations of CFT, see [3], and for the connection with two-dimensional gravity, see [2, 5, 6]. The fractal dimension D_f is determined by the parameters of this theory. The specific form of S_{geom} will not be needed in what follows; it suffices to know that it induces a conformal structure on the worldsurface Σ.

3 Duality "temperature field — streamlines"

Direct evaluation of the temperature partition function $Z_{\text{temp}}[\Sigma]$ is complicated by the need to account exactly for the boundary conditions (1) on a crack surface of complex shape. However, there exists an exact duality that allows one to pass from a description in terms of a continuous temperature field to one in terms of an ensemble of closed Brownian trajectories—a kind of gas of heat-flux streamlines. Let us show how this correspondence is constructed.

First, instead of the scalar temperature field $T(x)$, it is convenient to introduce the heat-flux density vector $J_\mu = -\kappa_0 \partial_\mu T$. In the steady state the flux is solenoidal, i.e., it obeys the continuity equation $\partial_\mu J_\mu = 0$, and this condition can be enforced in the functional integral by means of a Lagrange multiplier. The Gaussian integral (3) is rewritten in an equivalent form as an integral over the field J_μ with a delta function ensuring flux conservation:

$$Z_{\text{temp}}[\Sigma] = \int \mathcal{D}J_\mu\, \delta(\partial_\mu J_\mu)\, \exp\left(-\frac{1}{2\kappa_0}\int_{\Omega\setminus\Sigma} J_\mu^2\, d^d x\right). \tag{5}$$

Following the standard procedure well known from gauge theories (see, e.g., [9], Ch. 7), we write the functional delta function $\delta(\partial_\mu J_\mu)$ as an integral over an auxiliary scalar field χ:

$$\delta(\partial_\mu J_\mu) = \int \mathcal{D}\chi \, \exp\left(i \int \chi \, \partial_\mu J_\mu \, d^d x \right) . \tag{6}$$

After substituting this representation into (5), the integral over J_μ becomes Gaussian and can be evaluated exactly. The result reduces to a Gaussian integral over the field χ, which, upon identifying χ with the original temperature T, brings us back to expression (3). At first sight we have merely gone around in a useless circle. However, the value of this trick becomes apparent at the next step.

Consider the logarithm of the partition function $Z_{\text{temp}}[\Sigma]$. Given the Gaussian nature of the integral, it is expressed in terms of the functional determinant of the Laplacian Δ_Σ acting on the domain $\Omega \setminus \Sigma$ with Neumann boundary conditions on Σ:

$$\ln Z_{\text{temp}}[\Sigma] = -\frac{1}{2} \ln \det(-\kappa_0 \Delta_\Sigma). \tag{7}$$

A powerful apparatus has been developed for working with such functional determinants [14, 15], based on representing the heat kernel $K_t(x, y) = \langle x | e^{t\kappa_0 \Delta_\Sigma} | y \rangle$ via the Wiener path integral. Using the identity

$$\ln \det(-\kappa_0 \Delta_\Sigma) = \frac{1}{2} \int_0^\infty \frac{dt}{t} \, \text{Tr} \, e^{t\kappa_0 \Delta_\Sigma},$$

we reduce the problem to computing the trace of the evolution operator.

The central point here is the Feynman–Kac representation [17, 18]. The diagonal element of the heat kernel $K_t(x, x)$, and hence its trace, can be written as a sum over all closed Brownian trajectories $\gamma(s)$ starting from the point x and returning to it after "time" t:

$$K_t(x, x) = \int_{\gamma(0)=\gamma(t)=x} \mathcal{D}\gamma(s) \, \exp\left(-\frac{1}{4\kappa_0} \int_0^t \dot{\gamma}^2 \, ds \right) \, \mathbb{I}_\Sigma[\gamma]. \tag{8}$$

Here $\mathbb{I}_\Sigma[\gamma]$ is an indicator equal to unity if the trajectory γ does not cross the crack Σ (more precisely, is reflected from it in accordance with the Neumann condition) and zero otherwise. Details of the derivation and the justification of the Neumann reflection in terms of the path integral are given in Appendix A.

Physically, this means that the measure of the functional integral over the temperature field has been reduced to a sum over all possible configurations of closed Brownian loops of various lengths wandering in a medium with impenetrable obstacles—the cracks. Exponentiating the resulting expression for $\ln Z_{\text{temp}}[\Sigma]$, we arrive at the final representation of the partition function as the grand canonical ensemble of a gas of such loops:

$$Z_{\text{temp}}[\Sigma] = \sum_{N=0}^{\infty} \frac{1}{N!} \prod_{a=1}^{N} \oint \mathcal{D}\gamma_a(s) \; \exp\left(-\frac{\kappa_0}{2} \sum_a L[\gamma_a] \right) \; \mathbb{I}_\Sigma[\{\gamma_a\}]. \tag{9}$$

In this formula $\gamma_a(s)$ is an individual closed trajectory, $L[\gamma_a]$ is its length, and the parameter $\kappa_0/2$ plays the role of a chemical potential governing the distribution of loops over lengths. Thus, the problem of computing the thermal conductivity has been reduced to a statistical problem of Brownian loops wandering in a medium with fractal impenetrable boundaries. It is precisely the entropic contribution of the constraints imposed by the geometry of Σ on the possible loop configurations that will determine the effective thermal conductivity.

4 Averaging over fractal boundaries and induced 2D gravity

Let us now perform the functional averaging over the crack geometry in the full partition function (2). Substituting the loop representation (9), we obtain

$$Z = \sum_N \frac{1}{N!} \int \mathcal{D}\gamma_a \, e^{-\frac{\kappa_0}{2}\sum_a L[\gamma_a]} \left\langle \mathbb{I}_\Sigma[\{\gamma_a\}] \right\rangle_\Sigma, \tag{10}$$

where the angle brackets $\langle \dots \rangle_\Sigma$ denote averaging with the weight $e^{-S_{\rm geom}[\Sigma]}$ over all configurations of the crack ensemble. All information about the effect of the crack geometry on the heat transfer is now encoded in the average of the non-crossing indicator.

To compute this average, it is useful to switch to the language adopted in string theory [13]. Consider a single probe streamline. Its worldsheet can be parametrized by coordinates σ^a $(a = 0, 1)$ with an auxiliary metric $\tilde{g}_{ab}$, and the embedding in d-dimensional space is specified by fields $X^\mu(\sigma)$. The free propagation of the streamline is described by the Polyakov action, which is nothing but the action of d free scalar bosons living on a two-dimensional surface:

$$S_0[X, \tilde{g}] = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{\tilde{g}}\, \tilde{g}^{ab} \partial_a X^\mu \partial_b X_\mu. \tag{11}$$

The central charge of this system of fields is $c_X = d$. The ensemble of fractal cracks, as already mentioned, contributes an additional c_Σ to the total conformal anomaly, and the total central charge becomes $c_{\rm tot} = d + c_\Sigma$.

The key step consists in fixing the conformal gauge $\tilde{g}_{ab} = e^\phi \hat{g}_{ab}$, where $\hat{g}_{ab}$ is some fixed background metric and ϕ is the conformal factor. Although at the classical level the action S_0 is Weyl-invariant and does not depend on ϕ, the functional-integration measure over the fields X^μ and the Faddeev–Popov ghosts is not. This is the well-known phenomenon of the conformal (Weyl) anomaly: when the metric scale is changed, the measure transforms nontrivially, and the Jacobian of this transformation generates an additional contribution to the effective action. Integrating this anomaly by the method developed by Polyakov, we find that the conformal factor ϕ acquires its own dynamics, described by the Liouville action [19, 20]:

$$S_{\rm L}[\phi] = \frac{1}{4\pi\gamma} \int d^2\sigma \sqrt{\hat{g}} \left(\hat{g}^{ab} \partial_a\phi \partial_b\phi + \hat{Q}\hat{R}\phi + \tilde{\Lambda} e^{\beta_L \phi} \right), \tag{12}$$

where $\hat{R}$ is the scalar curvature of the background metric $\hat{g}$, the parameter $\tilde{\Lambda}$ plays the role of a cosmological constant on the worldsheet, and the background charge $\hat{Q}$ is fixed by the requirement that the total conformal anomaly vanish [2, 4]:

$$\hat{Q} = \sqrt{\frac{25 - c_{\rm tot}}{6}}. \tag{13}$$

The number 25 in this formula (instead of the familiar 26 arising for the critical dimension of the bosonic string) is related to the fact that the ghost contribution (-26) and the contribution of the field ϕ itself $(+1)$ together give an effective shift $26 - 1 = 25$.

As a result, the effective partition function for a single streamline, averaged over the ensemble of fractal cracks, takes the form of two-dimensional quantum gravity (for an overview of this approach, see [7]), i.e., a theory of the field ϕ interacting with matter X^μ:

$$Z^{\text{eff}}_{\text{1-loop}} = \int \mathcal{D}\phi\, \mathcal{D}X^\mu \;\exp(-S_{\text{eff}}[X,\phi])\,, \tag{14}$$

with the action

$$S_{\text{eff}} = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{\hat{g}}\, e^{\phi} \hat{g}^{ab} \partial_a X^\mu \partial_b X_\mu \;+\; S_{\text{L}}[\phi]. \tag{15}$$

Proposition 4.1. *Functional integration over the ensemble of fractal cracks induces an effective two-dimensional quantum gravity, described by the Liouville action* (12)*, on the worldsheets of the streamlines.*

Thus, the problem of heat conduction in a medium with fractal cracks has been reduced, at the microscopic level, to string theory in a two-dimensional gravitational field. It is precisely this fact that makes it possible to apply the powerful methods of conformal field theory to find the macroscopic equations of heat transfer.

5 Mean-field approximation

The microscopic description obtained in the preceding sections—the partition function of a loop gas interacting with quantized two-dimensional gravity on the worldsheet—is extremely rich but inconvenient for the direct computation of macroscopic heat-transfer characteristics. To pass to measurable quantities such as the averaged temperature profile and the effective conductivity, we use a self-consistent mean-field approximation. Its essence is that on scales much larger than the characteristic size of an individual crack, all microscopic fluctuations of both the crack geometry and the temperature field can be described by smooth, slowly varying fields in the d-dimensional space.

Let us begin with the crack-density field. The microscopic analog of the density is the conformal factor $\phi(\sigma)$, which describes local fluctuations of the intrinsic geometry of the streamline's worldsheet. In the scale-invariant regime dictated by the Liouville action (12), the measure of functional integration over the geometry contains the exponential factor $e^{\beta_L \phi}$. It is therefore natural to identify the macroscopic density of the fractal crack surface $\rho(x)$ with this factor, averaged over rapid fluctuations:

$$\rho(x) = \rho_0\, e^{\beta_L \phi(x)}. \tag{16}$$

Here $\phi(x)$ is now a slowly varying classical field in the ambient space $\mathbb{R}^d$.

Next, we need to construct an effective action for the fields $\phi(x)$ and $T(x)$ that governs their statics. Following the general principles of the gradient expansion (see [9], Ch. 28), we write it to lowest order in derivatives. The action will contain three main contributions. The first is the kinetic term $\frac{K}{2}(\partial_\mu \phi)^2$, which describes the energy cost of creating spatial inhomogeneities in the crack distribution. The parameter $K > 0$ has the meaning of a kind of surface stiffness of the crack ensemble. The second contribution is a local self-interaction potential for the field ϕ, which, owing to the Liouville nature of the microscopic theory, is proportional to $\mu e^{\beta_L \phi}$. It accounts for the balance of crack branching and merging processes, and the parameter μ (related to the microscopic tension μ_0) plays a role analogous to that of the cosmological constant on the worldsheet.

Finally, the third, key term describes the interaction between the crack field and the heat flux. It arises from the entropic suppression of heat transfer by fractal obstacles, which we discussed in Sec. 3. To find its form, consider how the system responds to a slowly varying external temperature field $\bar{T}(x)$. Write the partition function (3) for a fixed crack configuration Σ, separating out the fluctuations δT:

$$Z_{\text{temp}}[\Sigma; \bar{T}] = \int \mathcal{D}\delta T \, \exp\left(-\frac{\kappa_0}{2} \int (\partial_\mu \bar{T} + \partial_\mu \delta T)^2 d^d x \right). \tag{17}$$

Provided that the mean field $\bar{T}$ satisfies the Laplace equation, the cross term drops out and one can integrate over δT. In the leading approximation, the result reduces to substituting $\bar{T}$ into the original Gaussian weight, but with one important refinement: the presence of cracks locally reduces the volume available for fluctuations of δT. This screening effect leads to the effective conductivity at a point x being inversely proportional to the local crack density, $\kappa_{\text{eff}}(x) \propto \kappa_0^{-1} \rho^{-1}(x)$. Taking into account (16), this gives

$$\kappa_{\text{eff}}(x) = \kappa_0 \, e^{-\beta_L \phi(x)}. \tag{18}$$

Then the ensemble-averaged free energy associated with the mean heat flux acquires an additional term proportional to $\lambda e^{-\beta_L \phi} \kappa_0 (\partial_\mu T)^2$, where $\lambda > 0$ is a dimensionless coupling constant; a detailed derivation of its value from the microscopic theory is given in Appendix C.

Collecting everything together, we obtain the full effective action for the slow fields ϕ and T:

$$S_{\text{eff}}[\phi, T] = \int d^d x \left[\frac{K}{2} (\partial_\mu \phi)^2 + \mu e^{\beta_L \phi} + \frac{\lambda}{2} e^{-\beta_L \phi} \kappa_0 (\partial_\mu T)^2 \right]. \tag{19}$$

It remains to find the equations minimizing this action. Variation with respect to the field T is nothing but the condition of conservation of the total heat flux $J_\mu = -\kappa_{\text{eff}} \partial_\mu T$. This leads to a nonlinear generalization of the classical Fourier equation for the steady state:

$$\boxed{\partial_\mu \left(\kappa_0 \, e^{-\beta_L \phi(x)} \, \partial_\mu T(x) \right) = 0}. \tag{20}$$

Variation with respect to the field ϕ yields an equation for the balance of forces acting on the crack ensemble. On the one hand, there is the internal "pressure" of the branching structures, described by the derivative of the self-interaction potential; on the other hand, there is the ponderomotive force exerted by the heat flux on the crack geometry. This competition is described by a modified Liouville equation:

$$\boxed{K \Delta \phi = \mu \beta_L e^{\beta_L \phi} - \frac{\lambda \beta_L}{2} \, e^{-\beta_L \phi} \kappa_0 (\partial_\mu T)^2}. \tag{21}$$

The right-hand side of this equation vividly illustrates this competition. The first term, $\mu \beta_L e^{\beta_L \phi}$, describes the tendency of the cracks to branch and fill the space; it dominates in regions where the heat flux is small. The second term, proportional to the squared temperature gradient $(\partial T)^2$ and taken with the negative sign, describes the suppression of cracks by an intense heat flux. Indeed, a large temperature gradient tends to increase the negative contribution to the right-hand side, which for positive K leads to a decrease in ϕ and, according to (16), to a decrease in the crack density. This feedback closes the system of equations (20) and (21), making it self-consistent.

In the absence of heat flux ($\partial T = 0$), the equation reduces to the classical Liouville equation $\Delta \phi \propto e^{\beta_L \phi}$, whose solutions are well studied [28]. In two dimensions they are related to conformal mappings, and in three dimensions they describe, for example, isolated singularities of the form $\phi \sim -\ln r$, corresponding to fractal clusters with constant surface-energy density.

6 Exact solution: steady-state heat conduction in a thin slab

Let us illustrate the operation of the obtained system of equations on an exactly solvable example of one-dimensional heat conduction. Consider a plane-parallel slab of thickness L occupying the region $0 \le x \le L$, at whose boundaries constant temperatures $T(0) = T_1$ and $T(L) = T_2$ are maintained. In this geometry all quantities depend only on the coordinate x, and the mean-field system (21)–(20) reduces to two ordinary differential equations:

$$\frac{d}{dx}\left(\kappa_0 e^{-\beta_L \phi(x)} \frac{dT}{dx}\right) = 0, \tag{22}$$

$$K\frac{d^2\phi}{dx^2} = \mu\beta_L e^{\beta_L\phi} - \frac{\lambda\beta_L}{2} e^{-\beta_L\phi}\kappa_0 \left(\frac{dT}{dx}\right)^2. \tag{23}$$

The first equation is immediately integrated and yields the conservation law for the heat flux: the quantity $J \equiv -\kappa_0 e^{-\beta_L\phi} dT/dx$ is independent of x and serves as a parameter of the problem. Expressing the temperature gradient, $dT/dx = -(J/\kappa_0)e^{\beta_L\phi}$, and substituting it into the second equation, we eliminate the temperature and obtain a closed equation for the field ϕ:

$$K\frac{d^2\phi}{dx^2} = \beta_L e^{\beta_L\phi}\left(\mu - \frac{\lambda J^2}{2\kappa_0}\right). \tag{24}$$

Introducing the notation for the combination of constants

$$\alpha(J) \equiv \mu - \frac{\lambda J^2}{2\kappa_0}, \tag{25}$$

we arrive at the classical one-dimensional Liouville equation $\phi'' = (\beta_L\alpha/K)e^{\beta_L\phi}$. The coefficient α plays the role of an effective self-interaction parameter, and its sign determines the qualitative character of the solution. Hence the concept of the critical heat flux, at which α vanishes, arises naturally:

$$J_c = \sqrt{\frac{2\mu\kappa_0}{\lambda}}. \tag{26}$$

Physically, this means that at $J = J_c$ the heat flux exactly compensates the internal surface tension of the cracks.

To analyze the solutions for $J \neq J_c$, it is convenient to use a mechanical analogy, treating x as "time" and ϕ as the "coordinate" of a particle (for more on such techniques, see [29]). Multiplying both sides of the equation by $d\phi/dx$ and integrating, we obtain a first integral (a conservation law for the "energy"):

$$\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 = \frac{\alpha}{K} e^{\beta_L\phi} + E, \tag{27}$$

where E is an integration constant. The further substitution $u = e^{-\beta_L\phi/2}$ linearizes the differential equation for u:

$$\left(\frac{du}{dx}\right)^2 = \frac{\alpha\beta_L^2}{2K} + \frac{\beta_L^2 E}{2}u^2. \tag{28}$$

Consider first the subcritical regime $J < J_c$, corresponding to $\alpha > 0$. For $E < 0$ the equation for u describes a harmonic oscillator, and the solution is given by trigonometric functions: $e^{-\beta_L \phi} \propto \cos^2(k(x - x_0))$, where $k = \sqrt{\beta_L^2 |E|/2}$. Such a solution is defined only on a finite interval and describes a configuration in which cracks are concentrated mainly near the boundaries of the domain, while their density is minimal at the center of the slab. For $E = 0$ a power-law solution $e^{-\beta_L \phi} \propto (x - x_0)^2$ is obtained, with an isolated density singularity at the point x_0. Solutions with $E > 0$ are expressed through hyperbolic sines and also contain a singularity.

Most interesting is the supercritical regime $J > J_c$, when $\alpha < 0$. Denote $|\alpha| = -\alpha$. Then the right-hand side of the equation for u contains a competition between two terms of different signs, and for a real solution to exist it is necessary that $E > 0$ and $u^2 \geq |\alpha|/(KE)$. The solution is a localized structure without singularities—a crack cluster:

$$\boxed{e^{-\beta_L \phi(x)} = \frac{|\alpha|}{KE} \cosh^2\!\big(\tilde{k}(x - x_0)\big)}, \qquad \tilde{k} = \sqrt{\frac{\beta_L^2 E}{2}}. \tag{29}$$

In this case the crack density has the well-known solitonic profile:

$$\rho(x) \propto e^{\beta_L \phi} = \frac{KE}{|\alpha|} \operatorname{sech}^2\!\big(\tilde{k}(x - x_0)\big) : \tag{30}$$

it is maximal at the cluster center $x = x_0$ and decays exponentially toward the periphery. The cluster width is determined by the inverse wave number $1/\tilde{k}$.

Knowing the profile $\phi(x)$, from the flux conservation law we also find the temperature distribution. Integrating $dT/dx = -(J/\kappa_0)e^{\beta_L \phi}$ with the solution (29), we obtain

$$T(x) = T_1 - \frac{JKE}{\kappa_0 |\alpha| \tilde{k}} \Big[\tanh\!\big(\tilde{k}(x - x_0)\big) - \tanh\!\big(-\tilde{k}\, x_0\big)\Big]. \tag{31}$$

As can be seen, nearly the entire temperature drop is concentrated in a narrow region near the cluster center $x \approx x_0$, where the crack density is maximal and the effective thermal conductivity $\kappa_{\text{eff}} = \kappa_0 e^{-\beta_L \phi}$ is strongly suppressed. Away from the cluster the temperature varies weakly.

The boundary conditions $T(0) = T_1$ and $T(L) = T_2$ impose a relation between the parameters E, J, x_0, and the total temperature drop $\Delta T = T_1 - T_2$. For example, in the symmetric case $x_0 = L/2$ this relation takes the form $\Delta T = (2JKE/(\kappa_0 |\alpha| \tilde{k})) \tanh\!\Big(\tilde{k}L/2\Big)$. Analysis of this relation shows that as the flux J increases, the constant E must decrease in order to provide the given temperature drop. A decrease in E leads to a broadening of the cluster (since $\tilde{k} \sim \sqrt{E}$ decreases) and a simultaneous increase in the peak crack density at its center. In the limit $E \to 0$ for a fixed finite ΔT, the central density grows without bound and the effective conductivity drops to zero. This signals the formation of a macroscopic percolation cluster that completely blocks heat transfer. Thus, J_c has a clear physical meaning as the critical heat flux beyond which a qualitative restructuring of the crack distribution and a "breakdown" of the fractal structure occurs.

7 Discussion

The formalism constructed above allows a unified description of both the heat-transfer regime and the geometric phase transition. Let us dwell on several key properties of the obtained picture.

First, we note the scale invariance. The presence of the Liouville action at the foundation of the effective theory automatically ensures that the physical predictions are independent of the ultraviolet regularization scheme at large scales. This makes it possible to describe fractal objects without introducing an arbitrary cutoff, using only the universal parameters of conformal field theory.

Second, there is the duality of the description. The problem admits two completely equivalent but mathematically different formulations. On the one hand, it is a field theory of the scalar field T in a random medium with fractal boundaries. On the other hand, it is quantum gravity on an ensemble of string-streamlines interacting with these boundaries. This duality is analogous to the field–string duality in gauge theories and is a powerful tool of analysis.

Furthermore, the developed approach sheds light on the connection with models of anomalous diffusion that employ fractional derivatives [23]. In the quasistatic regime, when the fluctuations of the crack density are small, one can expand the exponential $e^{-\beta_L \phi} \approx 1 - \beta_L \delta\phi$ and, expressing $\delta\phi$ in terms of the temperature from the linearized Liouville equation, obtain a nonlocal equation with a fractional Laplacian, characteristic of diffusion on fractals. This establishes a direct connection between the microscopic conformal-field-theory approach and the phenomenological models of anomalous transport.

Finally, the solitonic nature of the solutions in the supercritical regime deserves special attention. The $1/\cosh^2$ profile for the crack density is not just a special case but a typical localized solution pointing to a deep connection with integrable systems. The fact that the stationary distribution of defects under a load is described by the Liouville equation and has a solitonic character appears to be quite general and may be used in other problems of the physics of strength and fracture.

8 Conclusion

In this work we have constructed an analytical approach to the calculation of the effective thermal conductivity of media with fractal cracks, based on functional integration over the crack surfaces and the "field–streamline" duality. A closed system of macroscopic mean-field equations (21)–(20) has been derived: a nonlinear Fourier equation with a conductivity dependent on the crack density, and a modified Liouville equation for this density.

The exact solution in the one-dimensional geometry demonstrates the existence of a critical heat flux J_c (26) separating two regimes: for $J < J_c$ (subcritical) the cracks are distributed relatively uniformly under the action of surface tension; for $J > J_c$ (supercritical) the heat flux concentrates the cracks into a localized cluster with a sech^2 profile (30), which can lead to a percolation blockage of heat transfer.

Future investigations will be directed at: (a) numerical solution of the system (21)–(20) in two- and three-dimensional samples; (b) computation of the critical fractal dimension D_f^* at which κ_{eff} vanishes; (c) inclusion of quantum corrections (loop corrections to the mean-field approximation) by the methods of [9, 4].

A Derivation of the Brownian loop-gas representation

We present a detailed derivation of formula (9), which expresses the temperature partition function $Z_{\text{temp}}[\Sigma]$ through an ensemble of closed Brownian trajectories.

By definition (3), $Z_{\text{temp}}[\Sigma]$ is a Gaussian integral over the field T in the domain $\Omega \setminus \Sigma$ with Neumann boundary conditions (1). A standard computation (see [9], Ch. 1) gives $Z_{\text{temp}}[\Sigma] = (\det(-\kappa_0 \Delta_\Sigma))^{-1/2}$, where the determinant is understood in the sense of ζ-function regularization [14, 15]. If $\{\lambda_n\}$ is the spectrum of the operator $(-\kappa_0\Delta_\Sigma)$, then the logarithm of the determinant is expressed through the derivative of the zeta function $\zeta(s) = \sum_n \lambda_n^{-s}$ at zero. Using the integral representation $\lambda^{-s} = \Gamma(s)^{-1} \int_0^\infty t^{s-1} e^{-\lambda t}\, dt$, the trace of the evolution operator can be written as $\operatorname{Tr} e^{t\kappa_0\Delta_\Sigma}$, whence after differentiating with respect to s and taking into account counterterms, we obtain

$$\ln Z_{\text{temp}}[\Sigma] = \frac{1}{2}\int_\varepsilon^\infty \frac{dt}{t} \operatorname{Tr} e^{t\kappa_0\Delta_\Sigma} + (\text{counterterms}). \tag{32}$$

The kernel of the semigroup operator $K_t(x,y) = \langle x | e^{t\kappa_0\Delta_\Sigma} | y\rangle$ satisfies the diffusion equation with a Neumann boundary condition on Σ. The Feynman–Kac representation relates it to the Wiener integral over all continuous paths $\gamma : [0,t] \to \Omega \setminus \Sigma$:

$$K_t(x,y) = \int_{\gamma(0)=x}^{\gamma(t)=y} \mathcal{D}\gamma\ \exp\left(-\frac{1}{4\kappa_0}\int_0^t \dot\gamma^2(s)\, ds\right) \mathbb{I}_\Sigma[\gamma], \tag{33}$$

where the indicator $\mathbb{I}_\Sigma[\gamma]$ equals unity if the path does not cross Σ and zero otherwise. The coefficient $1/(4\kappa_0)$ is chosen so as to reproduce the correct diffusion equation. The diagonal element $K_t(x,x)$ corresponds to closed paths, and integrating it over x yields the trace. Substituting the trace into (32) and passing from the parametrization by "time" t to the parametrization by the arc length of the loop $L[\gamma]$, we arrive at an expression for $\ln Z_{\text{temp}}[\Sigma]$ as a sum over a single connected loop with the weight $e^{-\kappa_0 L/2}$. Exponentiating this result gives the partition function as a sum over a multitude of independent loops (the Mayer cluster expansion), which leads to formula (9).

B Derivation of the Liouville action from the conformal anomaly

We reproduce in detail the derivation of the Liouville action (12). Consider a single probe streamline, whose worldsheet is parametrized by coordinates σ^a and whose embedding is specified by fields $X^\mu(\sigma)$. The Polyakov action (11) is invariant under diffeomorphisms and Weyl transformations of the auxiliary metric $\tilde g_{ab}$. To fix the gauge degrees of freedom, we use the Faddeev–Popov method. According to the uniformization theorem, on a two-dimensional surface any metric can be brought to a fixed background metric $\hat g_{ab}$ by a conformal transformation, so that $\tilde g_{ab} = e^\phi \hat g_{ab}$. The Faddeev–Popov procedure replaces the integral over $\tilde g_{ab}$ with an integral over ϕ and ghosts b, c.

The key point is that the integration measure over the fields X^μ and the ghosts is not invariant under Weyl transformations. Under the change $\tilde g_{ab} \to e^{\delta\omega}\tilde g_{ab}$, the Jacobian of the transition gives an additional contribution to the action. For d scalar fields this contribution equals $-\frac{d}{48\pi}\int(\hat g^{ab}\partial_a\phi\partial_b\phi + 2\hat R\phi) d^2\sigma\sqrt{\hat g}$, and for the ghost system with central charge -26 it equals $+\frac{26}{48\pi}\int(\dots)$. The crack ensemble adds another c_Σ degrees of freedom with the same contribution.

Collecting all terms and adding the cosmological term $\mu_0 \int \sqrt{\bar{g}} = \mu_0 \int \sqrt{\hat{g}} e^{\phi}$ from the bare action S_{geom}, we obtain the full Liouville action with background charge $\hat{Q} = \sqrt{(25 - c_{\text{tot}})/6}$, where $c_{\text{tot}} = d + c_{\Sigma}$. The marginality condition for the cosmological term fixes β_L, and the renormalized cosmological constant $\tilde{\Lambda}$ absorbs μ_0 and numerical factors. The result is given in formulas (12) and (13) of the main text.

C Derivation of the coupling potential $V(\phi, \partial T)$

We derive the form of the interaction term in the effective action (19). Separate the temperature field into slow (macroscopic) and fast (fluctuation) parts: $T = \bar{T} + \delta T$. For a fixed crack configuration, the partition function (3) is written as an integral over the fluctuations δT. Expanding the squared gradient $(\partial_\mu \bar{T} + \partial_\mu \delta T)^2$, we see that the cross term $\int \partial_\mu \bar{T}\, \partial_\mu \delta T\, d^d x$, after integration by parts, turns into $-\int \delta T\, \Delta \bar{T}\, d^d x$, which vanishes if the mean field satisfies the Laplace equation. Hence, the contribution of the mean field factorizes.

However, the presence of cracks locally reduces the volume available for fluctuations of δT, which is equivalent to a renormalization of the conductivity. In the regime of strong scattering on fractal obstacles, the effective conductivity is inversely proportional to the crack density: $\kappa_{\text{eff}} \propto \rho^{-1} \propto e^{-\beta_L \phi(x)}$. Averaging over the crack ensemble, we find that the averaged squared gradient of the mean temperature enters the free energy with the weight $\langle \kappa_{\text{eff}} \rangle \propto \kappa_0 e^{-\beta_L \phi}$. This yields the term $\frac{\lambda}{2} \kappa_0 e^{-\beta_L \phi} (\partial_\mu \bar{T})^2$ in the effective action, appearing in (19).

D Detailed solution of the one-dimensional Liouville equation

We present a detailed derivation of the solutions of the one-dimensional Liouville equation (24). Write it in the form $\phi'' = A e^{\beta \phi}$ with constants $A = \beta_L \alpha / K$, $\beta = \beta_L$. The equation is autonomous. Multiplying by ϕ' and integrating, we obtain the first integral:

$$\frac{1}{2}(\phi')^2 = \frac{A}{\beta} e^{\beta \phi} + E, \tag{34}$$

where E is an integration constant.

The substitution $u = e^{-\beta \phi / 2}$ reduces this equation to one that is linear in u^2:

$$(u')^2 = \frac{A\beta}{2} + \frac{\beta^2 E}{2} u^2. \tag{35}$$

Depending on the signs of A and E, three cases are possible. For $A > 0$, $E < 0$ we obtain the solution $u = u_0 \cos(k(x - x_0))$, which gives ϕ expressed through the cosine. For $E = 0$ we get a power-law solution $u \propto |x - x_0|$, and for $E > 0$ one expressed through the hyperbolic sine. All of these are singular.

The case $A < 0$ (supercritical regime) requires $E > 0$ and gives the regular solution $u = \sqrt{|A|/(\beta E)} \cosh\left(\tilde{k}(x - x_0)\right)$. Returning to ϕ, we arrive at the profile (29): $\rho \propto \operatorname{sech}^2(\tilde{k}(x - x_0))$. Integration of the temperature gradient with this profile gives the distribution (31). The boundary conditions close the problem and determine the relation between the parameters.

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