Degree-One Boundary Decompositions for Hodge-Type Shimura Varieties at Hyperspecial Level

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Abstract

We study the degree-one étale cohomology of integral canonical models of Shimura varieties of Hodge type at hyperspecial level, with emphasis on the contribution of boundary strata in a Hecke-equivariant setting (For closely related constructions of regular integral canonical models in the Spin and orthogonal setting, see Madapusi Pera [13, Introduction, pp. 769–771; main theorem, p. 769]).

Using the Leray spectral sequence associated with the open immersion into a toroidal compactification, we analyze the low-degree structure of cohomology and obtain a natural exact sequence separating interior and boundary contributions under explicit geometric conditions.

We further relate the vanishing of the boundary contribution to structural properties of rational boundary tori, providing a criterion expressed in terms of their anisotropy.

The results isolate the mechanism governing boundary phenomena in degree one and offer a coherent framework for further arithmetic and geometric investigations.

Keywords Shimura varieties; integral canonical models; Hodge type; étale cohomology; Leray spectral sequence; Hecke correspondences; boundary cohomology; Hecke-equivariant splitting; anisotropy; toroidal compactification; degree-one Leray obstruction.

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1 Introduction and Main Results

Motivation

The cohomology of Shimura varieties carries rich arithmetic information, including automorphic representations and the associated Galois representations. A fundamental objective is to clarify the interaction between the geometry of integral canonical models and the structure of low-degree cohomology groups.

In degree one, cohomology typically receives contributions from two sources: cuspidal automorphic forms and Eisenstein phenomena arising from boundary strata. While such decompositions are well understood over the complex numbers or on generic fibers, a purely *integral* and Hecke-equivariant description of this separation for integral models has not been isolated in general.

The goal of this paper is to isolate a structural framework for the degree-one boundary contribution to

$$H^1_{\text{ét}}(S_K, \Lambda),$$

by introducing an interior kernel and boundary quotient, and establishing a Hecke-equivariant short exact sequence relating them.

While the results are formulated under explicit geometric and cohomological hypotheses, the purpose of the present work is to isolate the precise mechanism governing boundary contributions in degree one. Even in this form, the resulting structure provides a unified perspective on interior and boundary cohomology and serves as a basis for further refinements in more specialized settings.

The construction is motivated by the expectation that the boundary contribution in low-degree cohomology should be controlled by the geometry of rational boundary components rather than by the interior geometry alone. In the Hodge-type case, toroidal compactifications provide a natural setting in which this expectation can be made precise, because the boundary is described locally by toric data attached to rational parabolic subgroups (compare the PEL-type description in Lan [11, §1.2.3, pp. 38–40; §1.3.1–1.3.2, pp. 57–95]). The present paper isolates the degree-one part of this mechanism through the Leray spectral sequence for the open immersion

$$j : \mathscr{S}_K \hookrightarrow \mathscr{S}_K^{\text{tor}}.$$

This viewpoint is complementary to the study of automorphic cohomology, intersection cohomology, and nearby-cycle methods, since it focuses specifically on the formal obstruction produced by the transgression

$$d_2^{0,1} : H^0(\mathscr{S}_K^{\text{tor}}, R^1 j_* \Lambda) \longrightarrow H^2(\mathscr{S}_K^{\text{tor}}, \Lambda).$$

(see Lan–Stroh [12, Introduction, pp. 2073–2076; §5] for comparison results relating étale cohomology and nearby cycles on integral models of Shimura varieties, including the non-proper case, and compare Caraiani–Scholze [3, Thms. 1.10–1.15, pp. 654–657; §6.1, pp. 750–755] and the Hodge–Tate period-map framework of [16, Thm. 1.0.8, p. 950; Thm. 4.1.1, pp. 1018–1020]; Pilloni–Stroh [15], where deep p-adic cohomological structures and overconvergent techniques are used to analyze the interaction between boundary geometry and Galois representations in Hilbert and Siegel-type Shimura varieties, including the non-proper case).

Thus the paper should be read as a conditional structural analysis of the degree-one boundary term, rather than as an unconditional classification of automorphic or motivic cohomology.

Setting

Let (G, X) be a Shimura datum of Hodge type, let $E = E(G, X)$ be its reflex field, and let $v \mid p$ be a place of E above a rational prime p such that $G_{\mathbb{Q}_p}$ is unramified. Let $K = K^{(p)}K_p$ be a compact open subgroup with K_p hyperspecial. Denote by

$$\mathscr{S}_K$$

the integral canonical model over $\mathrm{Spec}(\mathcal{O}_{E,(v)})$ and by

$$j : \mathscr{S}_K \hookrightarrow \mathscr{S}_K^{\mathrm{tor}}$$

the open immersion into a toroidal compactification.

This viewpoint is also compatible with recent comparison-theoretic work on étale cohomology in rigid analytic geometry, especially Hansen's comparison theorems for analytifications and constructible sheaves [9, Thms. 1.8–1.10, pp. 303–304]. However, the present work operates entirely in the integral scheme-theoretic setting of Shimura varieties, and does not require any rigid-analytic input; the reference to Hansen is purely contextual.

Main results

Theorem 1.1 (Low-degree edge statement)**.** *Assume the boundary regularity, purity, and transgression-vanishing hypotheses of Assumption 4.1. Then for finite coefficients Λ with $\mathrm{char}(\Lambda) \neq p$, the Leray spectral sequence for*

$$j : \mathscr{S}_K \hookrightarrow \mathscr{S}_K^{\mathrm{tor}}$$

has vanishing differential $d_2^{0,1}$, and hence the edge morphism

$$H^1_{\acute{e}t}(\mathscr{S}_K, \Lambda) \longrightarrow H^0\big(\mathscr{S}_K^{\mathrm{tor}}, R^1 j_* \Lambda\big)$$

is surjective.

Theorem 1.2 (Hecke-equivariant splitting under localization)**.** *Assume the hypotheses of Theorem 1.1 together with a chosen Hecke-equivariant splitting hypothesis after inverting a finite set of primes Σ. Then the degree-one cohomology admits a Hecke-equivariant decomposition into an interior kernel and a chosen complementary summand.*

Theorem 1.3 (Boundary-torus criterion)**.** *Under the hypotheses of Theorems 1.1 and 1.2, vanishing of the degree-one boundary contribution is equivalent to vanishing of the $\mathbb{Q}$-split part of every rational boundary torus. In this case the boundary quotient vanishes; equivalently, after inverting Σ one has*

$$H^1_{\acute{e}t}(\mathscr{S}_K, \Lambda[1/\Sigma]) = H^1_{\mathrm{int}}(\mathscr{S}_K, \Lambda[1/\Sigma]),$$

and any chosen complementary summand is zero.

Structure of the paper

Section 2 recalls background material on Shimura data, integral models, and compactifications. Section 3 develops the cohomological tools needed for the analysis of the Leray spectral sequence. Section 4 establishes the geometric properties of the boundary needed for the main arguments. The proofs of the main statements are given in Section 5, followed by examples and a short discussion of arithmetic applications in Section 6.

2 Preliminaries

In this section we collect the background needed throughout the paper. All statements are classical and are included only once, with explicit references. Novel contributions begin only in later sections. Notation is fixed globally to ensure consistency.

2.1 Algebraic geometry background

Notation / Convention 2.1 (Schemes and morphisms)**.** All schemes are assumed to be separated and of finite type over the base scheme specified in the surrounding discussion. In the Shimura-variety sections, the relevant bases are typically $\mathrm{Spec}(\mathcal{O}_{E,(v)})$. For a scheme X, we denote its structure sheaf by $\mathcal{O}_X$, and write $\Gamma(X, \mathcal{O}_X)$ for its ring of global sections. If $f : X \to Y$ is a morphism, we write X_y for the fiber over a point $y \in Y$.

Definition 2.1 (Flatness and smoothness)**.** A morphism $f : X \to Y$ is *flat* if $\mathcal{O}_{X,x}$ is flat as an $\mathcal{O}_{Y,f(x)}$-module for every $x \in X$. It is *smooth* if it is locally of finite presentation, flat, and has geometrically regular fibres. Equivalently, for morphisms locally of finite presentation, smoothness is the usual infinitesimal lifting condition together with flatness [8, EGA IV$_4$, §17].

2.2 Number theoretic foundations

Notation / Convention 2.2 (Adeles and Galois groups)**.** We denote by $\mathbb{A}_f$ the finite adeles of $\mathbb{Q}$, and by $\widehat{\mathbb{Z}}$ the profinite completion of $\mathbb{Z}$. For a number field F, let $G_F = \mathrm{Gal}(\overline{F}/F)$ be its absolute Galois group (For background on discrete valuation rings, residue fields, and valuations, see Serre [17, Chap. I, §§1–3, pp. 5–12], For localization at primes, see [1, Ch. 3]).

2.3 Shimura varieties: notation and conventions

Definition 2.2 (Shimura datum)**.** A *Shimura datum* is a pair (G, X) where G is a connected reductive group over $\mathbb{Q}$ and X is a $G(\mathbb{R})$-conjugacy class of homomorphisms $h\colon \mathbb{S} \to G_{\mathbb{R}}$, with $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$, satisfying the conditions of Deligne defining Shimura data (see [4, §1.5, (1.5.1)–(1.5.3), pp. 127-128]).

Notation / Convention 2.3 (Reflex field and level structures)**.** Given a Shimura datum (G, X), we write $E(G, X)$ for its reflex field. For a compact open subgroup $K \subseteq G(\mathbb{A}_f)$, we denote the Shimura variety by

$$\mathrm{Sh}_K(G, X) = G(\mathbb{Q})\backslash\big(X \times G(\mathbb{A}_f)/K\big).$$

Proposition 2.1 (Existence of canonical models)**.** *Let (G, X) be a Shimura datum and $K \subseteq G(\mathbb{A}_f)$ compact open. Then $\mathrm{Sh}_K(G, X)$ admits a canonical model over the reflex field $E(G, X)$.*

Proof. The notion of canonical models is due to Deligne [4, §3]. Existence of canonical models for Shimura varieties is proved in the work of Deligne–Milne and later expositions, see [14, Chap. 2]. □

Construction 2.1 (Integral canonical models)**.** Suppose (G, X) is of Hodge type, let $E = E(G, X)$ be the reflex field, and let $v \mid p$ be a place of E above a prime p at which G is unramified (Here $v \mid p$ is understood in the usual valuation-theoretic sense; for the associated discrete valuation ring, uniformizer, residue field, and completion formalism, see Serre [17, Chap. I, §1, pp. 5–6; Chap. II, §1, p. 26]). For sufficiently small level with hyperspecial K_p, Kisin's construction provides an integral canonical model $\mathscr{S}_K(G, X)$ over $\mathrm{Spec}(\mathcal{O}_{E,(v)})$, characterized by the usual extension property for morphisms from regular schemes (see [10, Thm. 2.3.8 and surrounding construction]; In the Spin and orthogonal cases, an alternative construction in the GSpin/orthogonal setting of regular integral canonical models is given by Madapusi Pera [13, Introduction, pp. 769–771; §4, pp. 789–794]).

3 Cohomological Framework

This section develops the cohomological tools required for our analysis of integral models of Shimura varieties introduced in Section 2. The emphasis is on the interplay between sheaf-theoretic constructions, arithmetic cohomology, and comparison results that link étale, de Rham, and crystalline settings (see comparison results that link étale and de Rham settings. Throughout, notation is as in Section 2.

For a concise bibliographic orientation to the structure of SGA 4, SGA $4\frac{1}{2}$, and SGA 5, including the general placement of étale topology, proper base change, local acyclicity, and duality within these seminars, see Deligne's guide [5, pp. 1–2].

3.1 Sheaf-theoretic tools

Definition 3.1 (Constructible sheaves)**.** Let X be a scheme of finite type over $\mathrm{Spec}(\mathbb{Z})$. A sheaf $\mathcal{F}$ of abelian groups on $X_{\acute{e}t}$ is *constructible* if there exists a finite stratification of X into locally closed subschemes such that $\mathcal{F}$ restricts to a locally constant sheaf of finite type on each stratum [2, Exp. IX].

For the corresponding rigid-analytic notion of Zariski-constructible sheaves and its comparison with algebraic constructible sheaves in characteristic zero, see Hansen [9, Conj. 1.2 and Thm. 1.7, pp. 300–302]. We emphasize that the present work does not rely on rigid-analytic comparison results; the reference to Hansen serves only to situate the notion of constructibility in a broader cohomological context.

Notation / Convention 3.1 (Standing coefficients and invertibility)**.** Fix a rational prime p, and fix once and for all a prime $\ell \neq p$.

Throughout this section (and in any application of smooth/proper base change) we take Λ to be a finite ℓ-power ring so that $\mathrm{char}(\Lambda)$ is invertible on any base $S/\operatorname{Spec}(\mathcal{O}_{E,(v)})$ under consideration.

Lemma 3.1 (Well-known: proper base change in the étale setting)**.** *Let $f\colon X \to S$ be proper with S noetherian and Λ a finite ring with characteristic invertible on S. Then for any constructible $\mathcal{F}$ on $X_{\acute{e}t}$ and $s \in S$, the canonical specialization map*

$$(R^i f_{\acute{e}t*}\mathcal{F})_{\bar{s}} \longrightarrow H^i_{\acute{e}t}(X_{\bar{s}}, \mathcal{F}|_{X_{\bar{s}}})$$

is an isomorphism.

Outline via the proper base change formalism. **See [2, Exp. XII].**

Let $f : X \to S$ be a proper morphism with S noetherian, and let Λ be a finite ring whose characteristic is invertible on S. For a constructible sheaf $\mathcal{F}$ on $X_{\acute{e}t}$, the derived proper base change theorem [2, Exp. XII, Th. 5.1] furnishes a natural base–change morphism

$$\mathbf{L}s^*\,\mathbf{R}f_*\mathcal{F} \;\longrightarrow\; \mathbf{R}f_{s,*}\,\mathcal{F}|_{X_s}$$

for any morphism $s : \mathrm{Spec}(k(s)) \to S$, functorial in $\mathcal{F}$, compatible with long exact cohomology sequences, and compatible with change of the base S. When s is a geometric point $\bar{s} \to S$, the total derived morphism induces isomorphisms on cohomology sheaves, yielding

$$\big(R^i f_{\acute{e}t*}\mathcal{F}\big)_{\bar{s}} \xrightarrow{\;\sim\;} H^i_{\acute{e}t}\big(X_{\bar{s}},\, \mathcal{F}|_{X_{\bar{s}}}\big) \quad \text{for all } i \geq 0.$$

The key inputs are: (i) constructibility of $\mathbf{R}f_*\mathcal{F}$ under proper f; (ii) cohomological descent and the compatibility of the formation of higher direct images with proper base change; and (iii) the invertibility of $\mathrm{char}(\Lambda)$ on S ensuring the usual finiteness and continuity conditions.See [2, Exp. XII, Th. 5.1] for the proper base change theorem in the derived setting, from which the above stalkwise identification follows by evaluating at $\bar{s}$. (This is the proper base change theorem; see [2, Exp. XII].) □

3.2 Comparison lemmas and spectral sequences

Construction 3.1 (Leray spectral sequence)**.** Let $f\colon X \to Y$ be a morphism of schemes over $\mathrm{Spec}(\mathbb{Z})$. For a sheaf $\mathcal{F}$ on

$X_{ét}$ there is a spectral sequence

$$E_2^{p,q} = H^p_{ét}(Y, R^q f_{ét*}\mathcal{F}) \Rightarrow H^{p+q}_{ét}(X, \mathcal{F}),$$

natural in $\mathcal{F}$ [2, Exp. V].

Proposition 3.1 (Leray spectral sequence for the structure morphism)**.** *Let (G, X) be a Shimura datum of Hodge type with integral canonical model $\mathscr{S}_K$ over $\mathrm{Spec}(\mathcal{O}_{E,(v)})$, and let*

$$f : \mathscr{S}_K \to \mathrm{Spec}(\mathcal{O}_{E,(v)})$$

be the structure morphism. For every torsion étale sheaf Λ on $(\mathscr{S}_K)_{ét}$ whose order is prime to p, there is a natural Leray spectral sequence

$$E_2^{a,b} = H^a_{ét}\big(\mathrm{Spec}(\mathcal{O}_{E,(v)}), R^b f_{ét*}\Lambda\big) \Longrightarrow H^{a+b}_{ét}(\mathscr{S}_K, \Lambda).$$

Proof. This is the standard Leray spectral sequence for the morphism f; see Construction 3.1. □

4 Integral Models over the Localized Reflex Ring

We keep the global conventions of Section 2, in particular the notation of Notations 2.1 and 2.3 and definition 2.2 and the existence statement for integral canonical models in Construction 2.1. Cohomological tools from Section 3 (e.g. Lemma 3.1, construction 3.1, and proposition 3.1) will be used repeatedly.

4.1 Construction of integral models

Notation / Convention 4.1 (Neat level, hyperspecial factor, and reflex-ring base)**.** Let (G, X) be a Shimura datum of Hodge type. Fix a rational prime p, let $E = E(G, X)$ be the reflex field, and let $v \mid p$ be a place of E. Factor $K = K^{(p)}K_p$ with $K^{(p)} \subset G(\mathbb{A}_f^{(p)})$ neat and $K_p \subset G(\mathbb{Q}_p)$ hyperspecial. Write $\mathscr{S}_K = \mathscr{S}_K(G, X)$ for the integral canonical model over $\mathrm{Spec}(\mathcal{O}_{E,(v)})$.

Lemma 4.1 (Well-known: extension property)**.** *Assume (G, X) is of Hodge type and K_p hyperspecial. Then $\mathscr{S}_K$ is normal and satisfies the (regular) extension property: for any regular, locally noetherian $\mathbb{Z}_{(p)}$-scheme T with function field $K(T)$, every morphism $T_\eta = \mathrm{Spec}\, K(T) \to \mathscr{S}_K$ extends uniquely to $T \to \mathscr{S}_K$.*

Proof. This is a fundamental characterization of integral canonical models. For Hodge type, it follows from Kisin's construction; see [10, Thm. 2.3.8]. The formulation of canonical models over the reflex field goes back to Deligne [4, §3]. □

Proposition 4.1 (Smoothness at hyperspecial level)**.** *If K_p is hyperspecial and (G, X) is unramified at p, then the integral canonical model $\mathscr{S}_K$ is smooth over $\mathrm{Spec}(\mathcal{O}_{E,(v)})$.*

Proof. For Hodge type, smoothness follows from Kisin's construction of integral canonical models, which are smooth over $\mathrm{Spec}(\mathcal{O}_{E,(v)})$ under the hyperspecial and unramified hypotheses; see [10, Thm. 2.3.8].

In the Siegel case, this reduces to the smoothness of the moduli stack of abelian schemes, which in turn follows from the smoothness of abelian schemes; see [6, Ch. I, §1]. □

4.2 Properties under base change

Proposition 4.2 (Proper base change on the toroidal compactification)**.** *Let*

$$\bar{f} : \mathscr{S}_K^{\mathrm{tor}} \to \mathrm{Spec}(\mathcal{O}_{E,(v)})$$

be the structural morphism of a toroidal compactification, and let $\mathcal{F}$ be a constructible Λ-sheaf on $(\mathscr{S}_K^{\mathrm{tor}})_{ét}$, where Λ is finite of characteristic prime to p. Then for each $i \geq 0$ and each geometric point $\bar{s} \to \mathrm{Spec}(\mathcal{O}_{E,(v)})$, the canonical specialization map

$$\big(R^i \bar{f}_{ét*}\mathcal{F}\big)_{\bar{s}} \xrightarrow{\sim} H^i_{ét}\big((\mathscr{S}_K^{\mathrm{tor}})_{\bar{s}}, \mathcal{F}|_{(\mathscr{S}_K^{\mathrm{tor}})_{\bar{s}}}\big)$$

is an isomorphism.

Proof. This is the proper base change theorem applied to the proper morphism $\bar{f}$; see Lemma 3.1, together with standard flat/base-change exactness. □

4.3 Boundary components and compactifications

Definition 4.1 (Toroidal compactification)**.** Let $\mathscr{S}_K$ be as above. We choose an admissible rational polyhedral cone decomposition, refined if necessary, such that the associated toroidal compactification $\mathscr{S}_K^{\mathrm{tor}}$ is regular and proper over $\mathcal{O}_{E,(v)}$, contains $\mathscr{S}_K$ as an open dense subscheme, and has boundary $D := \mathscr{S}_K^{\mathrm{tor}} \setminus \mathscr{S}_K$ a relative strict normal crossings Cartier divisor over $\mathrm{Spec}(\mathcal{O}_{E,(v)})$. For the PEL construction and refinement formalism see Lan [11, Defs. 1.2.2.13–1.2.2.16 and Prop. 1.2.2.17, pp. 37–38]; for the Hodge-type case see Lan–Stroh [12, Assumption 2.1(Hdg), pp. 2078–2079; Proposition 2.2, pp. 2079–2081]; see also Pilloni–Stroh [15] for related p-adic and cohomological structures on Hilbert-type Shimura varieties.

Notation / Convention 4.2 (Standing boundary regularity)**.** We henceforth work with a choice of S_K^{tor} as in Definition 4.1; in particular S_K^{tor} is regular and D is a relative SNC Cartier divisor.

Proposition 4.3 (Standard: extension of Hecke correspondences)**.** *For neat level, Hecke correspondences at primes $\ell \neq p$ extend to finite correspondences on $\mathscr{S}_K^{\mathrm{tor}}$ compatible with the open immersion $\mathscr{S}_K \hookrightarrow \mathscr{S}_K^{\mathrm{tor}}$.*

Proof. This is standard for neat level using the moduli interpretation and normality of $\mathscr{S}_K^{\mathrm{tor}}$; see Lan [11, §1.4.3, p. 123; Prop. 1.3.2.91, p. 95] for the PEL case.

For compatibility of Hecke correspondences with toroidal and minimal compactifications in the nearby-cycle setting, see also Lan–Stroh [12, Remark 5.41, pp. 2097–2098]. □

Lemma 4.2 (Boundary purity in codimension one; well known)**.** *Let $X = \mathscr{S}_K^{\mathrm{tor}}$ be regular, and let $i : D \hookrightarrow X$ be the boundary divisor, assumed to be a relative strict normal crossings Cartier divisor. Let Λ be a finite coefficient ring of order prime to p. Then, by the codimension-one case of Gabber's absolute cohomological purity theorem [7, Thm. 2.1.1, p. 159], one has*

$$i^!\Lambda \simeq \Lambda_D(-1)[-2]$$

in the derived category on D. Equivalently, for the local-cohomology sheaves on X,

$$\mathcal{H}^q_D(X,\Lambda)=0 \quad \textit{for } q\neq 2, \qquad \mathcal{H}^2_D(X,\Lambda)\cong i_*\Lambda_D(-1).$$

Consequently, after taking global sections one obtains the corresponding global local-cohomology groups through the usual local-to-global spectral sequence.

Proof. This is the codimension-one case of Gabber's absolute cohomological purity theorem; see Fujiwara [7, Thm. 2.1.1, p. 159]. Since D is a Cartier divisor on the regular scheme X, the immersion $i : D \hookrightarrow X$ is a regular immersion of pure codimension one (see [8, EGA IV_4, §16]).

Hence Gabber's absolute purity theorem applies and gives

$$i^!\Lambda \simeq \Lambda_D(-1)[-2].$$

Applying the local-cohomology functor sheaf-theoretically gives

$$\mathcal{H}^q_D(X,\Lambda)=0 \quad \text{for } q\neq 2, \qquad \mathcal{H}^2_D(X,\Lambda)\cong i_*\Lambda_D(-1).$$

The corresponding statements for global local cohomology follow by taking global sections, equivalently through the local-to-global spectral sequence for cohomology with support. ☐

Assumption 4.1 (Purity–Kummer fence for the open immersion j)**.** *Let $j : U = \mathscr{S}_K \hookrightarrow X = \mathscr{S}_K^{\mathrm{tor}}$ be the open immersion with boundary $i : D = X\setminus U \hookrightarrow X$. Assume:*

1. *X is regular and D is a* relative Cartier divisor with (strict) normal crossings *over* $\mathrm{Spec}(\mathcal{O}_{E,(v)})$ *(as in Definition 4.1). (The regular/SNC boundary condition is standard in the smooth toroidal PEL setting (and expected analogously in Hodge type); compare Lan [11, §1.3.1, pp. 57–66; Overview, pp. 3–4] and Lan–Stroh [12, Proposition 2.2(4)–(9), pp. 2079–2081; Corollary 2.4, pp. 2081–2082]).*
2. *Coefficients Λ are finite of order prime to p (equivalently, every prime dividing $|\Lambda|$ is $\neq p$).*
3. ***Absolute cohomological purity in codimension** 1: for $i : D \hookrightarrow X$ one has $H^q_D(X,\Lambda)=0$ for $q\neq 2$ and $H^2_D(X,\Lambda)\cong i_*\Lambda_D(-1)$ (see [7, Thm. 2.1.1, p. 159]).*
4. ***Low-degree boundary control.** For the chosen coefficients Λ, the degree-one boundary sheaf $R^1j_*\Lambda$ is constructible, and the transgression*

$$d_2^{0,1} : H^0_{\acute{e}t}\big(X, R^1j_*\Lambda\big) \longrightarrow H^2_{\acute{e}t}(X,\Lambda)$$

vanishes.

(The constructibility requirement should be viewed as an explicit hypothesis in the present integral setting; compare Hansen's treatment of Zariski-constructibility and the preservation of constructibility under open immersions in characteristic-zero rigid analytic geometry [9, Thm. 1.11(i), p. 304].)

The above hypotheses are natural in the context of toroidal compactifications and are compatible with the expected behavior of low-degree cohomology in situations where boundary contributions are controlled by codimension-one phenomena. The formulation isolates the minimal conditions under which the Leray spectral sequence yields an exact description in degree one.

Remark 4.1. The low-degree results below rely on an explicit vanishing hypothesis for the relevant boundary transgression. We do not claim that such a vanishing is automatic in general Hodge-type compactifications; rather, this formulation isolates the precise obstruction that must be controlled in any future refinement.

Theorem 4.1 (Low-degree edge surjectivity under vanishing of the transgression)**.** *Assume Assumption 4.1. Let (G,X) be of Hodge type with $\mathscr{S}_K$ smooth over* $\mathrm{Spec}(\mathcal{O}_{E,(v)})$*, and let*

$$j : \mathscr{S}_K \hookrightarrow \mathscr{S}_K^{\mathrm{tor}}$$

be a toroidal compactification as above. Then the edge morphism

$$H^1_{\acute{e}t}(\mathscr{S}_K,\Lambda) \longrightarrow H^0_{\acute{e}t}\big(\mathscr{S}_K^{\mathrm{tor}}, R^1j_{\acute{e}t*}\Lambda\big)$$

is surjective.

Proof. This is the low-degree exactness coming from the Leray spectral sequence of j once the transgression $d_2^{0,1}$ is assumed to vanish. ☐

Remark 4.2. We do not assert any general vanishing of the higher direct images $R^bj_*\Lambda$ for $b>1$. The only input used later is the vanishing of the specific transgression $d_2^{0,1}$ in the Leray spectral sequence.

5 Main Results

We retain the global setup and notation from Sections 2 to 4. Fix a Shimura datum (G,X) of Hodge type, a neat compact open $K=K^{(p)}K_p$ with K_p hyperspecial, and write $f : \mathscr{S}_K \to \mathrm{Spec}(\mathcal{O}_{E,(v)})$ for the integral canonical model (Construction 2.1 and notation 4.1) and $j : \mathscr{S}_K \hookrightarrow \mathscr{S}_K^{\mathrm{tor}}$ for a toroidal compactification (Definition 4.1). Throughout Λ denotes a finite coefficient ring with $\mathrm{char}(\Lambda)\neq p$. Hecke correspondences at primes $\ell\neq p$ extend to $\mathscr{S}_K^{\mathrm{tor}}$ (Proposition 4.3).

5.1 First main theorem: structural statement

Definition 5.1 (Boundary quotient and interior kernel in degree one)**.** Let

$$\mathrm{edge} :\ H^1_{\acute{e}t}(\mathscr{S}_K,\Lambda) \longrightarrow H^0_{\acute{e}t}\big(\mathscr{S}_K^{\mathrm{tor}}, R^1j_{\acute{e}t*}\Lambda\big)$$

be the edge morphism of Theorem 4.1. Define the degree-one interior kernel by

$$H^1_{\mathrm{int}}(\mathscr{S}_K,\Lambda) := \ker(\mathrm{edge}),$$

and the degree-one boundary quotient by

$$H^1_{\mathrm{bdry}}(\mathscr{S}_K,\Lambda) := H^0_{\acute{e}t}\big(\mathscr{S}_K^{\mathrm{tor}}, R^1j_{\acute{e}t*}\Lambda\big).$$

If a Hecke-equivariant splitting of the edge sequence is chosen, we write $H^1_{\mathrm{comp}}(\mathscr{S}_K,\Lambda)$ for the corresponding chosen complement to H^1_{int} inside $H^1_{\acute{e}t}(\mathscr{S}_K,\Lambda)$.

Theorem 5.1 (Hecke-equivariant splitting after a chosen localization hypothesis)**.** *Assume (G,X) is of Hodge type, K is neat with K_p hyperspecial, and Λ as above. Then:*

(i) As a direct consequence of the edge surjectivity of Theorem 4.1, there is a Hecke-equivariant short exact sequence

$$\begin{aligned} 0 &\longrightarrow H^1_{\mathrm{int}}(\mathscr{S}_K,\Lambda)\\ &\longrightarrow H^1_{\acute{e}t}(\mathscr{S}_K,\Lambda)\\ &\xrightarrow{\text{edge}} H^1_{\mathrm{bdry}}(\mathscr{S}_K,\Lambda)\\ &\longrightarrow 0\end{aligned}$$

functorial under level change away from p.

Here $H^1_{\mathrm{int}}(\mathscr{S}_K,\Lambda) := \ker(\mathrm{edge})$ and

$$H^1_{\mathrm{bdry}}(\mathscr{S}_K,\Lambda) := H^0_{\acute{e}t}\big(\mathscr{S}_K^{\mathrm{tor}}, R^1 j_{\acute{e}t*}\Lambda\big)$$

are the interior kernel and boundary quotient introduced in Definition 5.1. We do not identify H^1_{int} a priori with automorphic interior cohomology or with an Eisenstein/cuspidal decomposition in the classical sense.

This exact sequence is thus a reformulation of the low-degree edge exactness under the vanishing hypothesis of Notation 4.1.

(ii) Assume in addition that, after inverting a finite set Σ of rational primes disjoint from $\{p\}$, the short exact sequence in (i)

$$\begin{aligned} 0 &\longrightarrow H^1_{\mathrm{int}}(\mathscr{S}_K,\Lambda[1/\Sigma])\\ &\longrightarrow H^1_{\acute{e}t}(\mathscr{S}_K,\Lambda[1/\Sigma])\\ &\xrightarrow{\text{edge}} H^1_{\mathrm{bdry}}(\mathscr{S}_K,\Lambda[1/\Sigma])\\ &\longrightarrow 0\end{aligned}$$

splits in the category of $\mathbb{T}^{(\Sigma)}$-modules. Then there exists a $\mathbb{T}^{(\Sigma)}$-equivariant idempotent endomorphism of $H^1_{\acute{e}t}(\mathscr{S}_K,\Lambda[1/\Sigma])$ whose image is a chosen Hecke-stable complement to $H^1_{\mathrm{int}}(\mathscr{S}_K,\Lambda[1/\Sigma])$.

Proof. **Step 0: Reduction to the edge surjectivity.** The existence of the short exact sequence in (i) is a formal consequence of Theorem 4.1, which provides the surjectivity of the edge morphism

$$H^1_{\acute{e}t}(\mathscr{S}_K,\Lambda) \longrightarrow H^0_{\acute{e}t}\big(\mathscr{S}_K^{\mathrm{tor}}, R^1 j_{\acute{e}t*}\Lambda\big)\,.$$

By definition (Definition 5.1), the kernel of this morphism is $H^1_{\mathrm{int}}(\mathscr{S}_K,\Lambda)$, and its target is $H^1_{\mathrm{bdry}}(\mathscr{S}_K,\Lambda)$, yielding the exact sequence.

Step 1: Hecke action on the Leray package. Let $j : \mathscr{S}_K \hookrightarrow \mathscr{S}_K^{\mathrm{tor}}$ be the open immersion and $i : D \hookrightarrow \mathscr{S}_K^{\mathrm{tor}}$ the boundary. By Proposition 4.3, for each prime $\ell \neq p$ and each double coset $[K_\ell g K_\ell]$, there is a finite correspondence

$$\mathscr{S}_K^{\mathrm{tor}} \xleftarrow{\;p_1\;} \mathcal{H}_g^{\mathrm{tor}} \xrightarrow{\;p_2\;} \mathscr{S}_K^{\mathrm{tor}}$$

extending the Hecke correspondence on $\mathscr{S}_K$, with p_1, p_2 finite and étale over the open. The induced functors $(p_i)_{\acute{e}t*}$ act on $R^b j_{\acute{e}t*}\Lambda$ and commute with the differentials of the Leray spectral sequence. Hence both $H^1_{\acute{e}t}(\mathscr{S}_K,\Lambda)$ and $H^0_{\acute{e}t}(\mathscr{S}_K^{\mathrm{tor}}, R^1 j_{\acute{e}t*}\Lambda)$ are $\Lambda[\mathrm{Hecke}^{(p)}]$-modules and the edge map is Hecke-equivariant.

Step 2: Proof of Item (i). Consider the Leray spectral sequence

$$E_2^{a,b} = H^a_{\acute{e}t}(\mathscr{S}_K^{\mathrm{tor}}, R^b j_{\acute{e}t*}\Lambda) \Rightarrow H^{a+b}_{\acute{e}t}(\mathscr{S}_K,\Lambda).$$

By Theorem 4.1, the specific transgression

$$d_2^{0,1} : E_2^{0,1} \longrightarrow E_2^{2,0}$$

vanishes. Therefore the standard low-degree exact sequence attached to Leray yields

$$0 \to E_2^{1,0} \longrightarrow H^1_{\acute{e}t}(\mathscr{S}_K,\Lambda) \xrightarrow{\text{edge}} E_2^{0,1} \to 0.$$

Since $E_2^{1,0} = E_\infty^{1,0}$ and $E_2^{0,1} = E_\infty^{0,1}$ in total degree 1, this gives

$$0 \longrightarrow E_\infty^{1,0} \longrightarrow H^1_{\mathrm{et}}(S_K,\Lambda) \longrightarrow E_2^{0,1} = H^0_{\mathrm{et}}(S_K^{\mathrm{tor}}, R^1 j_*\Lambda) \longrightarrow 0. \qquad (1)$$

Step 3: Splitting hypothesis after localization. After inverting Σ, assume that the short exact sequence of Theorem 5.1 – Item (i) splits in the category of $\mathbb{T}^{(\Sigma)}$-modules. No general semisimplicity claim for $\mathbb{T}^{(\Sigma)}$-modules is used here.

Step 4: Construction of a Hecke-equivariant splitting. Write

$$M := H^1_{\acute{e}t}(\mathscr{S}_K,\Lambda[1/\Sigma]), \qquad B := H^0_{\acute{e}t}\big(\mathscr{S}_K^{\mathrm{tor}}, R^1 j_*\Lambda\big)[1/\Sigma].$$

By Step 3, the sequence

$$0 \to H^1_{\mathrm{int}}(\mathscr{S}_K,\Lambda)[1/\Sigma] \longrightarrow M \xrightarrow{\text{edge}} B \to 0$$

is exact in the abelian category of $\mathbb{T}^{(\Sigma)}$-modules. By the splitting hypothesis assumed in Theorem 5.1 – Item (ii), this exact sequence admits a $\mathbb{T}^{(\Sigma)}$-equivariant section. Choose such a splitting

$$s : B \longrightarrow M$$

of the edge map. Then

$$e := s \circ \mathrm{edge} \in \mathrm{End}_{\mathbb{T}^{(\Sigma)}}(M)$$

is an idempotent endomorphism of M, because

$$e^2 = s \circ \mathrm{edge} \circ s \circ \mathrm{edge} = s \circ \mathrm{id}_B \circ \mathrm{edge} = e.$$

Moreover,

$$\ker(e) = \ker(\mathrm{edge}) = H^1_{\mathrm{int}}(\mathscr{S}_K,\Lambda)[1/\Sigma],$$

and

$$\mathrm{im}(e) = s(B) \cong B.$$

Therefore

$$M = H^1_{\mathrm{int}}(\mathscr{S}_K, \Lambda)[1/\Sigma] \oplus s(B)$$

as $\mathbb{T}^{(\Sigma)}$-modules. The idempotent e is thus the projector onto the chosen complementary summand $s(B)$ along

$$H^1_{\mathrm{int}}(\mathscr{S}_K, \Lambda)[1/\Sigma].$$

Accordingly, define

$$\mathbf{P}_{\mathrm{comp}} := e, \qquad \mathbf{P}_{\mathrm{int}} := 1 - e.$$

Then both $\mathbf{P}_{\mathrm{comp}}$ and $\mathbf{P}_{\mathrm{int}}$ are $\mathbb{T}^{(\Sigma)}$-equivariant idempotents, with

$$\mathrm{im}(\mathbf{P}_{\mathrm{comp}}) = s(B), \qquad \mathrm{im}(\mathbf{P}_{\mathrm{int}}) = H^1_{\mathrm{int}}(\mathscr{S}_K, \Lambda)[1/\Sigma].$$

Equivalently,

$$\ker(\mathbf{P}_{\mathrm{comp}}) = H^1_{\mathrm{int}}(\mathscr{S}_K, \Lambda)[1/\Sigma], \qquad \ker(\mathbf{P}_{\mathrm{int}}) = s(B).$$

Thus the short exact sequence splits after inverting Σ, and one obtains a Hecke-stable complement to

$$H^1_{\mathrm{int}}(\mathscr{S}_K, \Lambda)[1/\Sigma].$$

We emphasize, however, that this complementary summand depends on the choice of $\mathbb{T}^{(\Sigma)}$-equivariant splitting s unless additional hypotheses are imposed to make the splitting canonical. □

Remark 5.1 (Non-boundary localization under vanishing hypothesis)**.** Suppose that, in a given arithmetic situation, one knows by independent input that the localized boundary module

$$H^1_{\mathrm{bdry}}(\mathscr{S}_K, \Lambda[1/\Sigma])_{\mathfrak{m}}$$

vanishes for a maximal ideal $\mathfrak{m} \subset \mathbb{T}^{(\Sigma)}$. Then the localized edge map is zero, and the $\mathfrak{m}$-localization of $H^1_{\acute{e}t}(\mathscr{S}_K, \Lambda[1/\Sigma])$ identifies with the localized interior kernel.

Example 5.1 (Basic rank computation in a single-cusp case)**.** Consider a situation in which the Shimura variety admits exactly one rational boundary component c, and suppose that the associated boundary torus T_c has character group satisfying

$$\mathrm{rank}_{\mathbb{Z}} X^*(T_c)^{G_{\mathbb{Q}}} = 1.$$

Under the boundary-module description used in Theorem 5.6, this yields

$$H^1_{\mathrm{bdry}}(S_K, \Lambda[1/\Sigma]) \cong \Lambda[1/\Sigma](-1),$$

and hence

$$\delta_{\Lambda[1/\Sigma]}(S_K) = 1.$$

In particular, the edge morphism

$$H^1_{\acute{e}t}(S_K, \Lambda[1/\Sigma]) \longrightarrow H^1_{\mathrm{bdry}}(S_K, \Lambda[1/\Sigma])$$

is nontrivial, and the boundary contribution does not vanish. This illustrates concretely how the presence of a nontrivial $\mathbb{Q}$-split character in the boundary torus produces a nonzero degree-one boundary term.

5.2 Second main theorem: cohomological invariants

Definition 5.2 (Degree-one boundary rank)**.** Define the degree-one boundary rank by

$$\delta_\Lambda(\mathscr{S}_K) := \mathrm{rank}_\Lambda\ H^0_{\acute{e}t}\big(\mathscr{S}_K^{\mathrm{tor}}, R^1 j_{\acute{e}t*}\Lambda\big).$$

Lemma 5.1 (Diagrammatic edge package for (S^{tor}, D))**.** *Let $j : U := \mathscr{S}_K \hookrightarrow X := \mathscr{S}_K^{\mathrm{tor}}$ be the open immersion and $i : D := X \setminus U \hookrightarrow X$ the boundary divisor. For any finite Λ with $\mathrm{char}(\Lambda) \neq p$, the localization triangle*

$$j_!\Lambda \longrightarrow \Lambda \longrightarrow i_* i^*\Lambda \xrightarrow{+1}$$

induces the long exact sequence in cohomology

$$\cdots \to H^1(X, \Lambda) \to H^1(U, \Lambda) \xrightarrow{\partial} H^2_D(X, \Lambda) \to H^2(X, \Lambda) \to \cdots,$$

where ∂ is the boundary map.

By the codimension-one case of Gabber's absolute purity theorem [7, Thm. 2.1.1, p. 159], applied to the regular immersion $i : D \hookrightarrow X$, one has

$$H^2_D(X, \Lambda) \cong H^0(X, i_*\Lambda(-1)).$$

Moreover, étale locally along a strict normal crossings boundary $D = \bigcup_k D_k$, the Kummer description gives

$$R^1 j_*\Lambda \simeq \bigoplus_k i_{k*}\Lambda(-1).$$

The connecting map

$$\partial :\ H^1(U, \Lambda) \longrightarrow H^2_D(X, \Lambda)$$

from the localization long exact sequence is related to the transgression in the Leray spectral sequence of j, but we do not identify the Leray edge map itself with a map into $H^2(X, \Lambda)$.

Theorem 5.2 (Boundary-torus criterion)**.** *Assume the setup of Theorem 5.1, and assume in addition that there is an identification*

$$H^1_{\mathrm{bdry}}(\mathscr{S}_K, \Lambda[1/\Sigma]) \ \cong\ \bigoplus_c \mathrm{Hom}_{\mathbb{Z}}\big(X^*(T_c)^{G_{\mathbb{Q}}}, \Lambda[1/\Sigma](-1)\big),$$

where c runs over the relevant rational boundary components. Then the following are equivalent:

(a) $\delta_{\Lambda[1/\Sigma]}(\mathscr{S}_K) = 0$;

(b) every rational boundary torus T_c associated with a maximal parabolic is anisotropic over $\mathbb{Q}$;

(c)

$$H^1_{\mathrm{bdry}}(\mathscr{S}_K, \Lambda[1/\Sigma]) = 0.$$

The relevant cusp and rational-boundary-component language is modeled on the PEL description in Lan [11, §1.2.3, pp. 38–40].

Proof. We recall the *boundary defect* δ_Λ from Definition 5.2:

$$\delta_\Lambda(\mathscr{S}_K) := \operatorname{rank}_\Lambda H^0_{\acute{e}t}(\mathscr{S}_K^{\mathrm{tor}}, R^1 j_{\acute{e}t*}\Lambda), \quad j: \mathscr{S}_K \hookrightarrow \mathscr{S}_K^{\mathrm{tor}}.$$

By Theorem 4.1 there is a Hecke-equivariant surjection (edge map)

$$\mathrm{edge}: H^1_{\acute{e}t}(\mathscr{S}_K, \Lambda) \twoheadrightarrow H^0_{\acute{e}t}(\mathscr{S}_K^{\mathrm{tor}}, R^1 j_{\acute{e}t*}\Lambda).$$

Step 1: Use of the boundary-module hypothesis. The substantive geometric input in the present theorem is the additional hypothesis that

$$H^1_{\mathrm{bdry}}(\mathscr{S}_K, \Lambda[1/\Sigma]) \cong \bigoplus_c \operatorname{Hom}_{\mathbb{Z}}\big(X^*(T_c)^{G_{\mathbb{Q}}}, \Lambda[1/\Sigma](-1)\big).$$

The identification of the boundary module appearing below is consistent with the expected description arising from local Kummer theory and the geometry of toroidal compactifications. For the purposes of the present work, we treat this description as a structural input and focus on the formal consequences that follow from it.

In particular, under this hypothesis one has

$$\delta_{\Lambda[1/\Sigma]}(\mathscr{S}_K) = \sum_c \operatorname{rank}_{\mathbb{Z}}\big(X^*(T_c)^{G_{\mathbb{Q}}}\big),$$

namely the total $\mathbb{Q}$-split rank of the relevant rational boundary tori.

Step 2: Item (a) $\Rightarrow$ Item (b). If $\delta_{\Lambda[1/\Sigma]}(\mathscr{S}_K) = 0$, then the equality established in Step 1,

$$\delta_{\Lambda[1/\Sigma]}(\mathscr{S}_K) = \sum_c \operatorname{rank}_{\mathbb{Z}}\big(X^*(T_c)^{G_{\mathbb{Q}}}\big),$$

forces

$$\operatorname{rank}_{\mathbb{Z}}\big(X^*(T_c)^{G_{\mathbb{Q}}}\big) = 0$$

for every cusp c. Hence

$$X^*(T_c)^{G_{\mathbb{Q}}} = 0$$

for every c. Equivalently, the $\mathbb{Q}$-split subtorus of T_c is trivial, so T_c is anisotropic over $\mathbb{Q}$.

Step 3: Item (b) $\Rightarrow$ Item (c). If each T_c is anisotropic, then $X^*(T_c)^{G_{\mathbb{Q}}} = 0$ for every cusp c. By the assumed boundary-torus description, this implies

$$H^1_{\mathrm{bdry}}(\mathscr{S}_K, \Lambda[1/\Sigma]) = 0,$$

which is exactly Item (c).

Step 4: Item (c) $\Rightarrow$ Item (a). If

$$H^1_{\mathrm{bdry}}(\mathscr{S}_K, \Lambda[1/\Sigma]) = 0,$$

then by definition its rank is zero, so

$$\delta_{\Lambda[1/\Sigma]}(\mathscr{S}_K) = 0.$$

Step 5: Final remark. Any precise comparison with intersection cohomology on the minimal compactification would require additional input from intermediate extension, perverse truncation, and decomposition results that are not developed in the present manuscript. Accordingly, no such identification is claimed here. ◻

Example 5.2 (Illustrative Hilbert modular surface case under additional boundary-module hypotheses)**.** Let $G = \operatorname{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$ for a real quadratic field F and take $K = K^{(p)}K_p$ with K_p hyperspecial and K neat. Then $\mathscr{S}_K = \mathscr{S}_K(G, X)$ is a Hilbert modular surface over $S = \operatorname{Spec}(\mathcal{O}_{E,(v)})$, whose minimal and toroidal compactifications fit into the setup of Theorem 5.2 (compare the geometric compactification framework in Lan [11, §1.3.1–1.3.2, pp. 57–95]). The following calculation is intended only as an illustrative model under the boundary-module hypothesis of Theorem 5.2; it is not used as an unconditional computation of boundary cohomology.

(1) Boundary tori and their $\mathbb{Q}$-split rank. Each cusp c of $\mathscr{S}_K$ corresponds to a rational parabolic of G with Levi quotient $M_c \simeq \operatorname{Res}_{F/\mathbb{Q}} \mathrm{GL}_1 \times \mathrm{GL}_1 / \mathbb{G}_m$. The associated boundary torus is

$$T_c \simeq \operatorname{Res}_{F/\mathbb{Q}} \mathbb{G}_m / \mathbb{G}_m,$$

whose character lattice is $X^*(T_c) \cong \mathbb{Z}^{[F:\mathbb{Q}]}/\mathbb{Z}$ endowed with a nontrivial $G_{\mathbb{Q}}$–fixed line. Hence T_c has $\mathbb{Q}$–split rank 1 and is *not anisotropic*. In particular,

$$\delta_{\Lambda[1/\Sigma]}(\mathscr{S}_K) = \sum_c \operatorname{rank}_{\mathbb{Z}} X^*(T_c)^{G_{\mathbb{Q}}} = \#\{\text{cusps}\} \neq 0,$$

so Item (a) of Theorem 5.2 fails.

(2) Boundary computation under the hypothesis of Theorem 5.2. Assuming the boundary-module identification appearing in Theorem 5.2, one obtains

$$H^1_{\mathrm{bdry}}(\mathscr{S}_K, \Lambda[1/\Sigma]) \cong \bigoplus_c \Lambda[1/\Sigma](-1),$$

since in the Hilbert modular surface case each relevant boundary torus contributes a one-dimensional $\mathbb{Q}$-split character space. This boundary term records the failure of anisotropy: each cusp contributes a nonzero boundary summand, and therefore the edge quotient does not vanish.

(3) Interpretation under Theorem 5.2.

- Item (a) fails since $\delta_{\Lambda[1/\Sigma]} \neq 0$.
- Item (b) fails because T_c is $\mathbb{Q}$-split of rank 1.
- Consequently, Item (c) fails: the boundary quotient

  $$H^1_{\mathrm{bdry}}(\mathscr{S}_K, \Lambda[1/\Sigma])$$

  is nonzero, so the edge quotient does not vanish.

Thus, under the stated boundary-module hypothesis, the Hilbert modular surface illustrates why non-anisotropic boundary tori are expected to produce nonzero degree-one boundary contributions.

(4) Contrast (compact anisotropic case). If G is replaced by a quaternionic inner form of $\operatorname{Res}_{F/\mathbb{Q}} \mathrm{GL}_2$ that is anisotropic modulo its center, then $\mathscr{S}_K$ is proper, so there is no boundary contribution in degree one. In that case

$$H^1_{\mathrm{bdry}}(\mathscr{S}_K, \Lambda[1/\Sigma]) = 0$$

and the edge sequence identifies

$$H^1_{\acute{e}t}(\mathscr{S}_K, \Lambda[1/\Sigma]) = H^1_{\mathrm{int}}(\mathscr{S}_K, \Lambda[1/\Sigma]).$$

This gives the compact model for the vanishing statement predicted by Theorem 5.2. Any further comparison with automorphic cuspidal cohomology or intersection cohomology lies beyond the scope of the present manuscript.

Remark 5.2 (Motivic interpretation and Siegel-type examples)**.** Although the present paper is formulated in étale cohomology, the boundary quotient appearing in Theorems 5.1 and 5.2 is compatible with the general philosophy that boundary strata encode degenerations of motives attached to Shimura varieties. In Siegel and Hodge-type settings, toroidal boundary components are governed by semi-abelian degenerations, and their toric parts contribute precisely the character-lattice terms that appear in the boundary-module hypothesis of Theorem 5.2. Thus the term

$$\mathrm{Hom}_{\mathbb{Z}}\big(X^*(T_c)^{G_{\mathbb{Q}}}, \Lambda[1/\Sigma](-1)\big)$$

may be viewed as the degree-one étale shadow of the toric part of the degeneration. This remark is only interpretive: no motivic category or comparison theorem is used in the proofs.

5.3 Equivalences and classification results

Diagrammatic equivalence. The long exact sequence and edge–purity package used in this subsection are summarized in Lemma 5.1.

Remark 5.3 (Localization heuristic)**.** Let $\mathfrak{m} \subset \mathbb{T}^{(\Sigma)}$ be a maximal ideal. If one knows by independent input that

$$H^1_{\mathrm{bdry}}(\mathscr{S}_K, \Lambda[1/\Sigma])_{\mathfrak{m}} = 0,$$

then Remark 5.1 shows that the localized edge map vanishes and

$$H^1_{ét}(\mathscr{S}_K, \Lambda[1/\Sigma])_{\mathfrak{m}} = H^1_{\mathrm{int}}(\mathscr{S}_K, \Lambda[1/\Sigma])_{\mathfrak{m}}.$$

Thus any genuine Eisenstein/non-Eisenstein classification at $\mathfrak{m}$ requires additional arithmetic input on the localized boundary module (For comparison with localized Hecke eigenspaces in compact unitary Shimura varieties, see [3, Thm. 1.1 and Rem. 1.3, pp. 651–652; Def. 1.9, pp. 652–653]) and is not proved in the present paper.

Corollary 5.1 (Stability under level change away from p)**.** *Let $K' \subset K$ be neat with the same hyperspecial K_p. The formation of H^1_{int}, H^1_{bdry}, and, after choosing a splitting as in Item (ii), the projector $\mathbf{P}_{\mathrm{comp}}$, is compatible with pullback along finite étale level maps and with Hecke operators away from $\{p\} \cup \Sigma$.*

Remark 5.4 (Relation with Hodge-type decompositions)**.** The direct-sum decomposition obtained after choosing a Hecke-equivariant splitting should not be confused with a Hodge decomposition. It is a decomposition of the localized étale cohomology module into an interior kernel and a chosen Hecke-stable complement:

$$H^1_{ét}(\mathscr{S}_K, \Lambda[1/\Sigma]) = H^1_{\mathrm{int}}(\mathscr{S}_K, \Lambda[1/\Sigma]) \oplus H^1_{\mathrm{comp}}(\mathscr{S}_K, \Lambda[1/\Sigma]).$$

Its origin is the Leray filtration for the open immersion into the toroidal compactification, not the classical Hodge filtration. Nevertheless, in comparison settings, one may expect the boundary summand to reflect the toric part of the degeneration along the boundary.

In particular, the results show that the entire contribution of boundary cohomology in degree one is governed by a single obstruction arising from the interaction between the Leray filtration and the geometry of rational boundary tori.

6 Arithmetic and Geometric Applications

The preceding results have three immediate formal applications.

First, they give a precise criterion for when degree-one cohomology is entirely interior after localization. Namely, if the boundary quotient

$$H^1_{\mathrm{bdry}}(\mathscr{S}_K, \Lambda[1/\Sigma])$$

vanishes, then the edge sequence identifies

$$H^1_{ét}(\mathscr{S}_K, \Lambda[1/\Sigma]) = H^1_{\mathrm{int}}(\mathscr{S}_K, \Lambda[1/\Sigma]).$$

This provides a useful test for excluding Eisenstein-type boundary contributions in degree one.

Second, after localization at a maximal ideal of the Hecke algebra (cf. the compact unitary setting of [3, Thm. 1.1, p. 651; Rem. 1.3, p. 651]), the same mechanism separates interior and boundary phenomena whenever the localized boundary module is known independently to vanish. This provides a formal tool for comparing low-degree cohomology with automorphic packets detected after Hecke localization. For related results on the structure of localized cohomology and its concentration under genericity assumptions, see [3, Thm. 1.1; §6.3, pp. 750–756].)

Third, the boundary-torus criterion translates a cohomological vanishing problem into a geometric condition on rational boundary tori. In particular, anisotropy of the relevant boundary tori implies the disappearance of the degree-one boundary quotient under the boundary-module hypothesis of Theorem 5.2. This gives a geometric explanation for why compact or anisotropic situations behave differently from Hilbert modular or Siegel-type examples with nontrivial cuspidal boundary.

Example 6.1 (Application to compact Shimura varieties)**.** If G is anisotropic modulo center over $\mathbb{Q}$, then $\mathscr{S}_K$ is proper and admits no boundary. Hence

$$H^1_{\mathrm{bdry}}(\mathscr{S}_K, \Lambda) = 0,$$

and the edge sequence reduces to

$$H^1_{ét}(\mathscr{S}_K, \Lambda) = H^1_{\mathrm{int}}(\mathscr{S}_K, \Lambda).$$

This shows that in compact cases the entire degree-one cohomology is interior.

Conclusion of the present note.

The paper establishes a conditional structural description of degree-one étale cohomology for integral canonical models of Hodge-type Shimura varieties at hyperspecial level. The main contribution is the isolation of the precise low-degree Leray obstruction governing the passage from total cohomology to the boundary quotient. Under the stated purity, regularity, and transgression-vanishing hypotheses, this yields a Hecke-equivariant short exact sequence separating the interior kernel from the degree-one boundary term.

The results also clarify the geometric meaning of boundary vanishing. Under the boundary-module hypothesis, the disappearance of the boundary quotient is equivalent to the anisotropy over $\mathbb{Q}$ of the relevant rational boundary tori. Hence the paper converts a cohomological question into a concrete condition on the toric geometry of the boundary.

Several directions remain open. One should seek geometric criteria ensuring the vanishing of the transgression $d_2^{0,1}$ without imposing it as a hypothesis. A second direction is to compare the interior kernel defined here with automorphic interior cohomology, intersection cohomology on the minimal compactification, completed cohomology and Hecke-theoretic structures (in the sense of [16, Thm. 4.2.1, pp. 1020–1024; Thm. 4.3.1, pp. 1024–1031]); see also [3, Thm. 1.1, p. 651; §6.3, pp. 756–760]. A third direction is to develop explicit computations for Siegel, Hilbert, orthogonal, and unitary Shimura varieties, where the boundary tori can often be described directly (For the integral-model background specific to orthogonal and GSpin Shimura varieties, see Madapusi Pera [13, pp. 769–771; §5–§7, pp. 796–822]). These refinements would turn the conditional framework of the present paper into a more effective arithmetic tool.

Declarations

Usage of AI

The authors declare that they have used AI for grammatical mistakes correction and sentence structuring, have not used AI for any other purpose.

Availability of data and material

No datasets were generated or analyzed during the current study. All arguments and constructions are purely theoretical and contained within the manuscript.

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Authors' contributions

Kundnani Rahul Thakurdas conceived the main results, developed the proofs, and wrote the manuscript.

Vinoth Marimuthu contributed to the verification of arguments, provided mathematical feedback on the structure of the results, and assisted in proofreading the manuscript.

Dr. Khursheed Alam supervised the research work, provided conceptual guidance, and reviewed the manuscript for mathematical clarity and presentation.

Dr. Shri Kant Ojha co-supervised the research work, and contributed to the overall review of the manuscript, suggested improvements in exposition, and assisted in refining the presentation of the results.

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REFERENCES

[1] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley Publishing Company, Reading, MA, 1969.

[2] M. Artin, A. Grothendieck, and J.-L. Verdier (eds.), *Théorie des topos et cohomologie étale des schémas (SGA 4) [Theory of Topoi and Étale Cohomology of Schemes]*, Lecture Notes in Mathematics, vols. 269, 270, 305, Springer, Berlin, 1972–73.

[3] A. Caraiani and P. Scholze, "On the generic part of the cohomology of compact unitary Shimura varieties," *Annals of Mathematics*, vol. 186, no. 3, pp. 649–766, 2017.

[4] P. Deligne, "Travaux de Shimura [Shimura's Work]," *Séminaire Bourbaki, 23ème année (1970/71), Exp. 389*, Lecture Notes in Mathematics, vol. 244, Springer, Berlin, 1971, pp. 123–165.

[5] P. Deligne (ed.), *Cohomologie Étale (SGA $4\frac{1}{2}$) [Étale Cohomology]*, Lecture Notes in Mathematics, vol. 569, Springer, Berlin, 1977.

[6] G. Faltings and C.-L. Chai, *Degeneration of Abelian Varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 22, Springer, Berlin, 1990.

[7] K. Fujiwara, "A Proof of the Absolute Purity Conjecture (after Gabber)," in *Algebraic Geometry 2000, Azumino*, Advanced Studies in Pure Mathematics, vol. 36, Mathematical Society of Japan, Tokyo, 2002, pp. 153–183.

[8] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique IV: Étude locale des schémas et des morphismes de schémas, Quatrième partie*, Publications Mathématiques de l'IHÉS, no. 32, 1967, pp. 5–361.

[9] D. Hansen, "Vanishing and comparison theorems in rigid analytic geometry," *Compositio Mathematica*, vol. 156, 2020, pp. 299–324.

[10] M. Kisin, "Integral models for Shimura varieties of abelian type," *Journal of the American Mathematical Society*, vol. 23, no. 4, pp. 967–1012, 2010.

[11] K.-W. Lan, *Compactifications of PEL-type Shimura varieties and Kuga families with ordinary loci*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2018.

[12] K.-W. Lan and B. Stroh, "Nearby cycles of automorphic étale sheaves," *Compositio Mathematica*, vol. 154, no. 10, pp. 2073–2103, 2018.

[13] K. Madapusi Pera, "Integral canonical models for Spin Shimura varieties," *Compositio Mathematica*, vol. 152, no. 4, pp. 769–824, 2016.

[14] J. S. Milne, "Introduction to Shimura Varieties," in *Harmonic Analysis, the Trace Formula, and Shimura Varieties*, Clay Mathematics Proceedings, vol. 4, American Mathematical Society, Providence, RI, 2005, pp. 265–378; revised version (2017), available at https://www.jmilne.org/math/xnotes/svi.pdf.

[15] V. Pilloni and B. Stroh, "Surconvergence, ramification and cohomology of Siegel modular varieties," *Astérisque [Asterisque]*, no. 416, 2020.

[16] P. Scholze, "On torsion in the cohomology of locally symmetric varieties," *Annals of Mathematics*, vol. 182, no. 3, pp. 945–1066, 2015.

[17] J.-P. Serre, *Local Fields*, Graduate Texts in Mathematics, vol. 67, Springer, New York, 1979.