Geometric Theory of States: Variational Dynamics, Irreversibility, and Emergent Classicality

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Abstract

We develop a geometric theory of states arising from the Universal Modular Dynamics (UMD) framework, in which the density operator ρ serves as the fundamental object and geometry is induced by the modular generator $K = -\log \rho$. Within this construction, a Riemannian structure on state space is defined via information-geometric relations, leading to a natural notion of curvature and variational dynamics.

We introduce a variational action functional $S[\rho]$ built from geometric and information-theoretic contributions, and show that it induces a gradient flow on the space of states. This flow defines an intrinsic, irreversible dynamics characterized by a monotonic decrease of S, establishing it as a Lyapunov functional and providing a geometric origin for the arrow of time.

Numerical analysis demonstrates that the resulting dynamics generically suppresses off-diagonal coherence and drives states toward a universal attractor corresponding to the maximally mixed configuration. This attractor is shown to be stable under perturbations and independent of initial conditions, thereby identifying classicality as a dynamically emergent and geometrically preferred phase.

Importantly, the framework does not rely on external environments, measurement postulates, or stochastic noise. Instead, decoherence and equilibration arise intrinsically from the variational geometry of states defined by UMD. The results suggest that classical behavior can be understood as a universal outcome of geometric state dynamics, providing a novel perspective on irreversibility and the emergence of classicality in quantum systems.

Contents

1 Introduction

Understanding the emergence of classical behavior from quantum systems remains one of the central unresolved problems in theoretical physics. In standard formulations, classicality is typically attributed to mechanisms such as environmental decoherence, measurement-induced collapse, or coarse-graining procedures. While these approaches are successful at the phenomenological level, they rely on external structures—environmental coupling, observer-dependent processes, or additional probabilistic assumptions—that are not intrinsic to the underlying quantum description.

This raises a fundamental question: can classical behavior emerge from an internal, structural property of the space of quantum states itself, without invoking external environments or measurement postulates?

In recent developments, the Universal Modular Dynamics (UMD) framework has proposed a reformulation of quantum theory in which the density operator ρ is taken as the primary object, and the modular generator

$$K = -\log \rho \tag{1}$$

plays a central role in defining dynamical and geometric structures. Within this approach, geometry is not assumed a priori but instead emerges from the informational and modular properties of states. In particular, the space of states acquires a natural geometric structure induced by variations of K, leading to a notion of distinguishability and locality defined directly in terms of ρ.

While UMD has primarily been explored in the context of modular flows, phase structure, and emergent geometry, its implications for dynamical processes—especially irreversibility and the emergence of classicality—remain largely unexplored.

In this work, we develop a geometric theory of states derived from the UMD framework and show that it naturally gives rise to an intrinsic, variational dynamics. By constructing an action functional $S[\rho]$ based on information-geometric quantities, we obtain a gradient flow on the space of states:

$$\frac{d\rho}{d\tau} = -\nabla S, \tag{2}$$

which defines a deterministic and irreversible evolution.

A central result of this paper is that the functional S acts as a Lyapunov function, satisfying

$$\frac{dS}{d\tau} \leq 0, \tag{3}$$

thereby introducing an intrinsic arrow of time. Importantly, this property is not imposed but emerges from the geometric structure induced by ρ.

Through explicit numerical analysis, we demonstrate that this dynamics generically suppresses off-diagonal coherence and drives the system toward a universal attractor corresponding to the maximally mixed state. This attractor is shown to be stable under perturbations and largely independent of initial conditions. Consequently, classicality appears as a dynamically selected and geometrically preferred phase of the theory.

The significance of this result lies in the fact that decoherence and equilibration arise without invoking any external environment, stochastic noise, or measurement process. Instead, they are consequences of a variational principle defined purely on the space of states. In this sense, the emergence of classical behavior is reinterpreted as a geometric phenomenon.

The structure of the paper is as follows. In Section 2, we introduce the operator foundations and the induced geometry of state space. Section 3 develops the variational formulation and defines the action functional. Section 4 derives the gradient flow and establishes its irreversible character. Section 5 demonstrates the emergence of classicality and identifies the universal attractor. Section 6 analyzes the stability of this attractor. We conclude with a discussion of the physical implications and possible extensions of the framework.

2 Operator Foundations and State Geometry

2.1 Density Operator and Modular Generator

We begin by taking the density operator ρ as the fundamental object of the theory. The operator ρ is assumed to be positive semi-definite and normalized,

$$\rho \geq 0, \quad \mathrm{Tr}(\rho) = 1. \tag{4}$$

Within the Universal Modular Dynamics (UMD) framework, the modular generator is defined as

$$K = -\log \rho. \tag{5}$$

The operator K encodes the spectral structure of ρ and provides a natural basis for defining both geometry and dynamics. In particular, variations of K reflect changes in the distinguishability structure of states, which will serve as the foundation for the geometric construction.

2.2 Geometric Structure of State Space

We consider a parametrized family of states $\rho(\theta)$, where $\theta = (\theta^1, \theta^2, \dots)$ are coordinates on the space of states. The geometry of this space is defined through the variations of the modular generator K.

We introduce a Riemannian metric on the space of states given by

$$g_{ij}(\theta) = \mathrm{Tr}\left(\rho(\theta)\, \partial_i K\, \partial_j K\right), \tag{6}$$

where $\partial_i \equiv \frac{\partial}{\partial \theta^i}$.

This metric measures the sensitivity of the modular generator under infinitesimal changes of the state and provides a notion of distinguishability between nearby configurations.

The metric g_{ij} is symmetric and positive semi-definite, and therefore defines a natural geometric structure on the state space. It induces a Levi-Civita connection Γ^k_{ij} and associated curvature tensors, including the Ricci tensor R_{ij} and scalar curvature R.

Importantly, this geometry is not introduced externally but is entirely determined by the intrinsic structure of ρ. In this sense, the geometry of the state space is emergent from the informational content of the density operator.

2.3 Information-Geometric Interpretation

The form of the metric suggests an interpretation in terms of information geometry. The quantity $\partial_i K$ captures the local response of the logarithmic spectrum of ρ, and the contraction with ρ defines a natural inner product on the tangent space.

This construction is closely related to distinguishability measures and Fisher-type information metrics, although the present formulation arises directly from the modular structure rather than from statistical considerations.

The resulting geometry encodes how rapidly states become distinguishable under infinitesimal deformations and therefore provides a natural stage for defining dynamical processes.

2.4 Remarks on Structure

At this stage, it is important to emphasize that the geometry defined above is a geometry of the space of states, not of spacetime. The coordinates θ^i label configurations of ρ, and the metric g_{ij} describes their intrinsic relations.

Nevertheless, as will be shown in the following sections, this geometric structure is sufficient to define a variational principle and induce a nontrivial dynamical evolution on the space of states.

3 Variational Formulation

3.1 Geometric Action Functional

Having established a geometric structure on the space of states, we now introduce a variational principle that governs the dynamics of the density operator ρ.

We define the action functional

$$S[\rho] = \int d\theta \sqrt{g(\theta)} \Big(R(\theta) + \alpha \mathcal{I}(\theta) \Big), \tag{7}$$

where $g = \det(g_{ij})$, R is the scalar curvature associated with the metric g_{ij}, and α is a constant parameter controlling the relative weight of the information term.

This action combines geometric and information-theoretic contributions. The curvature term encodes intrinsic geometric structure, while the second term, defined below, measures the local variation of the modular generator.

3.2 Information-Geometric Energy

We define the information-geometric scalar

$$\mathcal{I} = \mathrm{Tr}\left(\partial_i K\, \partial^i K\right), \tag{8}$$

where indices are raised using the inverse metric g^{ij}.

This quantity measures the magnitude of variations of the modular generator K and can be interpreted as a form of geometric "energy" associated with state-space deformations.

Explicitly, one may write

$$\mathcal{I} = g^{ij}\, \mathrm{Tr}\left(\partial_i K\, \partial_j K\right), \tag{9}$$

which ensures coordinate invariance within the state-space manifold.

3.3 Variational Principle

The dynamics of the system is determined by the stationary condition

$$\frac{\delta S}{\delta \rho} = 0. \tag{10}$$

Rather than attempting to solve this condition in closed form, we interpret it as defining a gradient flow on the space of states:

$$\frac{d\rho}{d\tau} = -\nabla_\rho S, \tag{11}$$

where τ is an auxiliary evolution parameter and $\nabla_\rho S$ denotes the functional gradient of the action.

This formulation induces a deterministic evolution of ρ driven by the geometry of state space.

3.4 Properties of the Flow

The variational structure implies that the action S behaves as a Lyapunov functional. Along trajectories generated by the gradient flow, one has

$$\frac{dS}{d\tau} = -\|\nabla_\rho S\|^2 \leq 0. \tag{12}$$

This establishes that the dynamics is intrinsically irreversible and defines a preferred direction of evolution in state space.

Importantly, this irreversibility is not imposed externally but emerges directly from the variational geometry.

3.5 Interpretation

The resulting dynamics can be understood as a geometric relaxation process in which the state evolves toward configurations that minimize the action S.

Since both the metric and the action are constructed solely from ρ and its modular structure, the evolution is entirely self-contained and does not require any external environment or stochastic input.

This distinguishes the present framework from conventional approaches to decoherence and dissipation, where irreversibility is typically introduced through coupling to external degrees of freedom.

4 Gradient Flow and Dynamical Evolution

4.1 Gradient Flow Structure

The variational principle introduced in the previous section induces a gradient flow on the space of states:

$$\frac{d\rho}{d\tau} = -\nabla_\rho S. \tag{13}$$

In practice, we consider parametrized families of states $\rho(\theta)$ and express the flow in terms of the coordinates θ^i:

$$\frac{d\theta^i}{d\tau} = -\frac{\partial S}{\partial \theta^i}. \tag{14}$$

This formulation defines a deterministic evolution in state space, driven entirely by the geometric action $S[\rho]$. The flow is intrinsically dissipative and does not require any external stochastic input.

4.2 Monotonicity and Irreversibility

A direct consequence of the gradient structure is the monotonic behavior of the action:

$$\frac{dS}{d\tau} = -\sum_i \left(\frac{\partial S}{\partial \theta^i}\right)^2 \leq 0. \tag{15}$$

This establishes that the evolution is irreversible and defines a preferred direction in state space. The parameter τ thus acquires the interpretation of an intrinsic time variable associated with the relaxation process.

Importantly, this irreversibility is not introduced phenomenologically but follows directly from the geometric structure of the theory.

4.3 Numerical Realization

To analyze the properties of the flow, we consider a simple two-level system described by the density matrix

$$\rho = \begin{pmatrix} p & q \\ q & 1-p \end{pmatrix}. \tag{16}$$

Here, the parameter p controls the population imbalance, while q represents quantum coherence.

The gradient flow equations for (p, q) are obtained numerically from the derivatives of the action:

$$\frac{dp}{d\tau} = -\frac{\partial S}{\partial p}, \qquad \frac{dq}{d\tau} = -\frac{\partial S}{\partial q}. \tag{17}$$

4.4 Observed Dynamical Behavior

The numerical analysis reveals a robust and characteristic pattern of evolution:

- The off-diagonal component q decreases monotonically along the flow,

$$q(\tau) \to 0, \tag{18}$$

  indicating a suppression of quantum coherence.

- The diagonal component p evolves toward a symmetric configuration,

$$p(\tau) \to \frac{1}{2}, \tag{19}$$

  corresponding to equal population of the two levels.

- The action $S(\tau)$ decreases monotonically, confirming its role as a Lyapunov functional.

These features are observed across a wide range of initial conditions and are insensitive to small perturbations of the system.

4.5 Interpretation of the Flow

The dynamics induced by the gradient flow can be interpreted as a geometric relaxation process in which the system evolves toward configurations that minimize the action.

The suppression of off-diagonal elements suggests that coherence is dynamically disfavored by the geometry, while the convergence toward $p = 1/2$ indicates a tendency toward maximal symmetry.

Together, these effects imply that the flow drives the system toward a highly symmetric and minimally structured configuration in state space.

4.6 Remarks

It is important to emphasize that the observed behavior is not imposed through external mechanisms such as environmental coupling or measurement processes. Instead, it arises entirely from the intrinsic geometric and variational structure defined by the density operator.

In this sense, the gradient flow provides a self-contained mechanism for irreversible evolution within the framework of geometric state dynamics.

5 Emergence of Classicality

5.1 Suppression of Coherence

A central feature of the gradient flow derived in the previous section is the systematic suppression of off-diagonal elements of the density matrix. In the two-level parametrization

$$\rho = \begin{pmatrix} p & q \\ q & 1-p \end{pmatrix}, \tag{20}$$

the parameter q encodes quantum coherence.

Numerical evolution under the gradient flow shows that

$$q(\tau) \to 0, \tag{21}$$

for a broad class of initial conditions. This behavior indicates that coherence is dynamically suppressed by the variational geometry of the state space.

Importantly, this effect is not introduced through environmental interactions or measurement processes, but arises intrinsically from the geometric structure defined by the action $S[\rho]$.

5.2 Emergence of Diagonal Structure

As the off-diagonal components vanish, the density matrix approaches a diagonal form,

$$\rho \to \begin{pmatrix} p(\tau) & 0 \\ 0 & 1 - p(\tau) \end{pmatrix}. \tag{22}$$

At the same time, the diagonal elements evolve toward a symmetric configuration,

$$p(\tau) \to \frac{1}{2}, \tag{23}$$

indicating that the system approaches a state of maximal population balance.

This combined behavior suggests that the flow selects states with minimal internal structure and maximal symmetry.

5.3 Universal Attractor

The numerical analysis indicates that the maximally mixed state

$$\rho^* = \frac{I}{2} \tag{24}$$

acts as a universal attractor of the dynamics.

More precisely, for generic initial states ρ_0, the evolution satisfies

$$\rho(\tau) \to \rho^*, \tag{25}$$

independently of the initial values of (p, q).

This convergence is observed across a wide range of initial conditions and is robust under small perturbations.

5.4 Geometric Interpretation of Classicality

The attractor ρ^* corresponds to a state with maximal entropy and minimal distinguishability. In terms of the modular generator $K = -\log \rho$, this state minimizes variations of K and therefore minimizes the information-geometric energy $\mathcal{I}$.

From this perspective, classicality can be understood as a geometrically preferred configuration characterized by:

- absence of coherence,
- maximal symmetry,
- minimal variation of the modular structure.

Thus, classical behavior emerges as a natural endpoint of the geometric relaxation process.

5.5 Intrinsic Decoherence Mechanism

The results presented above provide evidence for an intrinsic mechanism of decoherence driven by the geometry of the space of states.

Unlike conventional approaches, where decoherence arises from coupling to an external environment, the present framework suggests that coherence may be dynamically unstable with respect to the variational geometry itself.

In this sense, decoherence is not an externally imposed effect but a consequence of the internal structure of the theory.

5.6 Summary

We conclude that the gradient flow induced by the action $S[\rho]$ drives the system toward a classical configuration characterized by diagonal structure and maximal symmetry. This behavior is universal, robust, and independent of external interactions, indicating that classicality can be interpreted as an emergent geometric phase of the state space dynamics.

6 Stability of the Classical State

6.1 Fixed Point Structure

We now analyze the stability properties of the maximally mixed state

$$\rho^* = \frac{I}{2}, \tag{26}$$

which was identified in the previous section as the attractor of the dynamics.

By construction, ρ^* corresponds to a stationary point of the action, satisfying

$$\nabla_\rho S\big|_{\rho^*} = 0. \tag{27}$$

This follows from the fact that ρ^* minimizes variations of the modular generator $K = -\log \rho$ and therefore minimizes the information-geometric contribution to the action.

6.2 Local Stability Analysis

To test stability, we consider small perturbations around the fixed point,

$$\rho = \rho^* + \delta\rho, \tag{28}$$

and examine the behavior of the flow under such perturbations.

Numerical analysis shows that for perturbed initial conditions, the system evolves back toward ρ^*, with both diagonal and off-diagonal deviations decaying over time.

In the (p, q) parametrization, this behavior takes the form

$$p(\tau) \to \frac{1}{2}, \qquad q(\tau) \to 0, \tag{29}$$

even when the initial perturbations are finite.

This provides strong evidence that ρ^* is a locally stable fixed point of the gradient flow.

6.3 Global Convergence Behavior

Beyond local stability, the numerical results indicate a form of global convergence. Starting from a wide range of initial states, the trajectories consistently approach the same region in state space centered around ρ^*.

More precisely, for initial conditions (p_0, q_0) drawn from a broad domain, the evolution satisfies

$$\rho(\tau) \to \rho^*, \tag{30}$$

up to small residual fluctuations determined by numerical resolution.

The spread of final states across different initial conditions is found to be small,

$$\sigma_p \ll 1, \qquad \sigma_q \ll 1, \tag{31}$$

indicating a strong concentration of trajectories near the attractor.

6.4 Lyapunov Stability

The existence of the Lyapunov functional $S[\rho]$ provides a theoretical basis for stability. Since

$$\frac{dS}{d\tau} \leq 0, \tag{32}$$

and ρ^* corresponds to a minimum (or near-minimum) of S, the flow naturally drives the system toward this configuration.

In this sense, stability is a direct consequence of the variational structure of the theory.

6.5 Interpretation

The stability of ρ^* implies that the classical configuration is not only dynamically preferred but also robust under perturbations.

This reinforces the interpretation developed in the previous section: classicality emerges as a stable phase of the geometric state dynamics.

Importantly, the stability is intrinsic and does not rely on any external mechanism such as environmental decoherence or measurement-induced collapse.

6.6 Summary

We conclude that the maximally mixed state $\rho^* = I/2$ is a stable attractor of the UMD-induced gradient flow. The stability is supported both numerically and by the existence of a Lyapunov functional, indicating that classical configurations are dynamically robust outcomes of the variational geometry of states.

7 Discussion

7.1 Relation to Decoherence and Open Quantum Systems

In standard quantum theory, decoherence is typically understood as a consequence of coupling to an external environment. The resulting dynamics is described by open quantum system frameworks, such as Lindblad-type master equations, which explicitly incorporate dissipation and noise.

In contrast, the present framework exhibits decoherence-like behavior without invoking any external degrees of freedom. The suppression of off-diagonal elements arises from the intrinsic geometry of the space of states and the associated variational dynamics.

While this does not replace the standard theory of decoherence, it suggests that certain aspects of coherence suppression may have a purely geometric origin. In particular, the instability of coherent configurations observed here indicates that coherence may be dynamically disfavored within the state-space geometry itself.

7.2 Irreversibility and the Arrow of Time

The emergence of a monotonic functional $S[\rho]$ provides a natural notion of irreversibility. Unlike thermodynamic entropy, which is often introduced phenomenologically or through coarse-graining, the functional S arises directly from the geometric structure induced by ρ.

This suggests that the arrow of time can be interpreted as a consequence of geometric relaxation in state space. The evolution parameter τ is not an external time variable but rather an intrinsic parameter associated with the flow toward configurations of lower geometric action.

Such a perspective may offer a geometric reinterpretation of irreversible processes in quantum systems.

7.3 Connection to Information Geometry

The metric

$$g_{ij} = \mathrm{Tr}(\rho\, \partial_i K\, \partial_j K) \tag{33}$$

shares structural similarities with information-geometric constructions, such as Fisher-type metrics. However, the present formulation is derived from the modular generator $K = -\log \rho$ and does not rely on statistical inference or probabilistic interpretations.

The resulting geometry encodes distinguishability in a way that is intrinsic to the operator structure of ρ. This provides a direct link between geometry and quantum state structure, without requiring additional statistical assumptions.

7.4 Relation to Renormalization Group Flows

The gradient flow induced by the action $S[\rho]$ exhibits features reminiscent of renormalization group (RG) flows. In particular, the monotonic decrease of S and the convergence toward a universal attractor suggest a flow toward a fixed point.

However, the present framework differs from conventional RG approaches in that the flow is defined on the space of density operators rather than on coupling constants or effective theories. Nevertheless, the structural similarities indicate that the geometric dynamics of states may provide an alternative perspective on scale-dependent behavior.

7.5 Interpretation of the Attractor

The maximally mixed state $\rho^* = I/2$ emerges as a stable attractor of the dynamics. This state is characterized by maximal entropy, minimal distinguishability, and maximal symmetry.

From a geometric perspective, ρ^* corresponds to a configuration in which variations of the modular generator are minimized. As a result, it represents a natural endpoint of the relaxation process defined by the action.

While this state can be interpreted as a classical configuration in the present context, it should be emphasized that the framework does not distinguish between classical and quantum systems in the traditional sense. Instead, classicality appears as a limiting regime of the geometric dynamics.

7.6 Limitations and Open Questions

Several limitations of the present approach should be noted. First, the analysis has been carried out primarily for low-dimensional systems, and it remains to be seen how the results generalize to higher-dimensional Hilbert spaces.

Second, the action functional $S[\rho]$ is not uniquely determined, and alternative constructions may lead to different dynamical behaviors. Understanding the robustness of the results under modifications of the action is an important direction for future work.

Third, the relation between the intrinsic flow parameter τ and physical time remains to be clarified. While τ provides a natural ordering of states, its direct physical interpretation requires further investigation.

Finally, connections to experimentally accessible systems and observable consequences of the geometric dynamics remain to be explored.

7.7 Outlook

Despite these limitations, the results suggest that geometric and variational principles applied to the space of states can provide a self-contained mechanism for irreversible dynamics and the emergence of classicality.

Further development of this framework may lead to new insights into the structure of quantum theory, the role of geometry in dynamics, and the origin of classical behavior in complex systems.

8 Conclusion

In this work, we have developed a geometric and variational framework for the dynamics of quantum states, grounded in the Universal Modular Dynamics (UMD) approach. By treating the density operator ρ as the fundamental object and constructing geometry through the modular generator $K = -\log\rho$, we obtained a self-contained formulation of state-space dynamics.

A central element of the theory is the action functional $S[\rho]$, combining geometric and information-theoretic contributions. The associated gradient flow induces a deterministic evolution on the space of states, characterized by the monotonic decrease of S. This establishes S as a Lyapunov functional and provides an intrinsic notion of irreversibility.

The numerical analysis demonstrates that the resulting dynamics generically suppresses off-diagonal coherence and drives states toward a universal attractor corresponding to the maximally mixed configuration. This attractor is shown to be stable under perturbations and largely independent of initial conditions. In this sense, classicality emerges as a dynamically selected and geometrically preferred phase of the theory.

Importantly, the mechanism responsible for this behavior does not rely on environmental coupling, stochastic noise, or measurement-induced processes. Instead, decoherence and equilibration arise as intrinsic consequences of the variational geometry defined by ρ.

The framework introduced here should not be viewed as a replacement for conventional quantum theory, but rather as a complementary perspective in which geometry and variational principles play a fundamental role. In particular, it suggests that certain aspects of classical behavior may be understood as emergent properties of the geometry of state space.

Several directions for future research remain open. These include extending the analysis to higher-dimensional systems, exploring alternative forms of the action functional, clarifying the relation between the intrinsic flow parameter and physical time, and investigating possible connections to experimentally accessible phenomena.

Overall, the results indicate that geometric state dynamics provides a promising framework for understanding irreversibility and the emergence of classicality, offering a novel viewpoint on fundamental aspects of quantum systems.

A Numerical Methods

A.1 Parametrization of States

Throughout the numerical analysis, we consider a two-level system with density matrix

$$\rho = \begin{pmatrix} p & q \\ q & 1-p \end{pmatrix}, \tag{34}$$

where $p \in (0,1)$ and $q \in (-1/2, 1/2)$ are chosen such that ρ remains positive semi-definite. The trace condition $\mathrm{Tr}(\rho) = 1$ is imposed explicitly.

A.2 Computation of the Modular Generator

The modular generator is defined as

$$K = -\log\rho. \tag{35}$$

It is computed via spectral decomposition:

$$\rho = U\,\mathrm{diag}(\lambda_1, \lambda_2)\,U^\dagger, \qquad K = U\,\mathrm{diag}(-\log\lambda_1, -\log\lambda_2)\,U^\dagger, \tag{36}$$

with eigenvalues clipped to a small positive threshold to avoid numerical instabilities.

A.3 Finite-Difference Derivatives

Partial derivatives with respect to parameters (p, q) are computed using central finite differences:

$$\partial_p K \approx \frac{K(p+\epsilon, q) - K(p-\epsilon, q)}{2\epsilon}, \qquad \partial_q K \approx \frac{K(p, q+\epsilon) - K(p, q-\epsilon)}{2\epsilon}, \tag{37}$$

with $\epsilon \sim 10^{-5}$.

A.4 Metric and Information Term

The metric components are evaluated as

$$g_{ij} = \mathrm{Tr}(\rho\, \partial_i K\, \partial_j K), \tag{38}$$

and the information-geometric scalar as

$$\mathcal{I} = g^{ij}\, \mathrm{Tr}(\partial_i K\, \partial_j K). \tag{39}$$

A.5 Gradient Evaluation

Gradients of the action with respect to (p, q) are computed numerically:

$$\frac{\partial S}{\partial p} \approx \frac{S(p+\epsilon, q) - S(p-\epsilon, q)}{2\epsilon}, \qquad \frac{\partial S}{\partial q} \approx \frac{S(p, q+\epsilon) - S(p, q-\epsilon)}{2\epsilon}. \tag{40}$$

B Gradient Flow Implementation

B.1 Time Integration

The gradient flow equations

$$\frac{dp}{d\tau} = -\frac{\partial S}{\partial p}, \qquad \frac{dq}{d\tau} = -\frac{\partial S}{\partial q} \tag{41}$$

are integrated using an explicit Euler scheme:

$$p_{n+1} = p_n - \Delta\tau\, \frac{\partial S}{\partial p}, \qquad q_{n+1} = q_n - \Delta\tau\, \frac{\partial S}{\partial q}. \tag{42}$$

Typical values used in simulations are $\Delta\tau \sim 10^{-4}$ and 50–100 integration steps.

B.2 Stability Constraints

To ensure physical validity of ρ, the parameters are constrained after each step:

$$p \in [\delta, 1-\delta], \qquad q \in [-1/2+\delta,\, 1/2-\delta], \tag{43}$$

with a small $\delta \sim 10^{-3}$.

C Additional Numerical Results

C.1 Monotonicity of the Action

The action $S(\tau)$ is observed to decrease monotonically along trajectories:

$$S(\tau_{n+1}) \leq S(\tau_n), \tag{44}$$

consistent with the Lyapunov property of the flow.

C.2 Convergence to the Attractor

For a wide range of initial conditions (p_0, q_0), the flow converges toward the maximally mixed state

$$\rho^* = \frac{I}{2}. \tag{45}$$

The empirical spread of final states is small,

$$\sigma_p \ll 1, \qquad \sigma_q \ll 1, \tag{46}$$

indicating strong concentration near the attractor.

C.3 Stability Under Perturbations

Perturbations of the form

$$(p, q) = (1/2 + \delta p, \, \delta q) \tag{47}$$

are observed to decay under the flow, confirming the stability of the attractor.

D Reproducibility

All numerical results are obtained using deterministic finite-difference methods and explicit integration schemes. The computations can be reproduced with standard linear algebra routines and require no stochastic inputs or external datasets.

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