Tor-theoretic Krasner Constants and Torsion-Theoretic Approximation Defects over Henselian Valued Fields

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Abstract

Classical Krasner theory controls the stability of algebraic extensions through valuation inequalities: if an algebraic element is approximated sufficiently closely, then the generated extension stabilizes. This article develops a Tor-theoretic obstruction formalism associated with Krasner-type approximation over Henselian valued fields. Given algebraic elements $\alpha, \beta \in \overline{K}$ algebraic over K, we attach to the pair an approximation quotient module

$$A_{\alpha,\beta} := \frac{\mathcal{O}_K[\alpha,\beta]}{\mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta]}.$$

We define the Tor-theoretic Krasner defect modules

$$\mathrm{DKD}_i(\alpha,\beta) := \mathrm{Tor}_i^{\mathcal{O}_K}(A_{\alpha,\beta}, k_K).$$

In the discretely valued setting, this construction has homological amplitude at most one: DKD_0 records the residual approximation quotient, while DKD_1 records the π-torsion of $A_{\alpha,\beta}$. The paper therefore does not claim a higher obstruction theory over discrete valuation rings. Its main contribution is to isolate, after classical Krasner field-theoretic stabilization, the remaining integral saturation obstruction

$$\mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta] \subseteq \mathcal{O}_K[\alpha,\beta].$$

Tame defectless hypotheses alone are not used to prove flatness of $A_{\alpha,\beta}$; rather, flatness or saturation is stated as the precise additional integral condition under which the positive Tor-defect vanishes. Wild and inseparable examples are presented as sources of possible non-vanishing unless an explicit torsion class is exhibited.

Keywords: Krasner constant; Henselian valued fields; approximation; ramification; defect; Tor modules; tame extensions; wild ramification; Artin–Schreier extensions.

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1 Introduction

Classical Krasner theory is one of the central tools in the arithmetic of valued fields (see [9]; see [6]; compare also the approximation-maximality framework and distinguished approximation sequences developed in [8, Section 2, pp. 618–620]). In its usual form, it asserts that if an algebraic element β is sufficiently close to another algebraic element α, then the field generated by α is contained in, or coincides with, the field generated by β. The relevant threshold is encoded by Krasner's constant.

The classical philosophy is valuation-theoretic:

$v(\alpha - \beta) > \kappa(\alpha) \Longrightarrow K(\alpha) \subseteq K(\beta),$

see [6, Theorem 1.1];
see also [7, Introduction, pp. 1095–1096];
see also [13, Introduction, pp. 225–226];
compare also the main-invariant formalism and approximation principles developed in [9, Section 1];
see also [11, Theorem 1.1];
see also [3, pp. xiii–xiv].

This paper proposes a Tor-theoretic bookkeeping formalism attached to that valuation-theoretic philosophy. In the discretely valued case this formalism has homological amplitude at most one, so its main content is the separation between the residual quotient and the first torsion-detection term. Thus, instead of claiming a genuinely higher obstruction theory over DVRs, we study whether residual or first torsion-theoretic discrepancies remain in the integral structures attached to α and β.

1.1 Main idea

Let (K, v) be a Henselian valued field, let $\alpha \in \overline{K}$, and set $L = K(\alpha)$. For an approximating element $\beta \in \overline{K}$, we consider the quotient

$$A_{\alpha,\beta} := \frac{\mathcal{O}_K[\alpha, \beta]}{\mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta]}.$$

This module records discrepancies between the integral structures generated by α and β.

The Tor-theoretic Krasner defect modules are then defined by

$$\mathrm{DKD}_i(\alpha, \beta) := \mathrm{Tor}_i^{\mathcal{O}_K}(A_{\alpha,\beta}, k_K), \qquad i \geq 0.$$

The guiding principle is:

> Classical Krasner stability gives field-theoretic rigidity; the modules introduced here provide information about whether residual or torsion-theoretic discrepancies remain in the associated integral structures.

1.2 Novelty and contribution

The contribution of the paper is threefold.

(i) We attach explicit Tor-theoretic obstruction modules to Krasner-type approximation.

(ii) We isolate the genuinely integral input needed after classical Krasner stabilization: over a DVR, the positive Tor-defect is controlled by the torsion/saturation behaviour of the approximation quotient.

This should be contrasted with classical valuation-theoretic characterizations of Krasner constants, tame extensions, and trace-theoretic approximation phenomena developed in [12, §§1–2].

(iii) We isolate conditional mechanisms by which wild ramification and inseparable approximation may produce non-trivial Tor-theoretic defects, provided an explicit torsion or saturation witness is exhibited.

1.3 Structure of the paper

- Section 2 recalls Krasner constants and Henselian approximation.
- Section 3 introduces approximation quotient modules.
- Section 4 defines Tor-theoretic Krasner defect modules.
- Section 5 separates classical tame defectless Krasner stabilization from the additional integral flatness or saturation condition needed to force vanishing of the positive Tor-defect.
- Section 6 gives obstruction results in wild ramification.
- Section 7 develops explicit examples.
- Section 8 compares the Tor-theoretic framework with classical Krasner theory.

2 Krasner Constants and Henselian Approximation

2.1 Valued-field preliminaries

Let (K, v) be a valued field with valuation ring $\mathcal{O}_K$, maximal ideal $\mathfrak{m}_K$, residue field k_K, and value group vK. We fix an algebraic closure $\overline{K}$ and extend v to $\overline{K}$ (see also [1, Chapter I, §§1–2] for general valuation-theoretic preliminaries).

Definition 2.1 (Henselian valued field)**.** A valued field (K, v) is called Henselian if the valuation v extends uniquely to every algebraic extension of K, equivalently if Hensel's lemma holds for $\mathcal{O}_K$ (see [5, Chapter II, §§1–2, pp. 26–30], see also [6, pp. 233–234]).

Definition 2.2 (Krasner constant)**.** Let $\alpha \in \overline{K}$ be separable over K. The Krasner constant of α over K is

$$\kappa_K(\alpha) := \max_{\substack{\sigma: K(\alpha) \hookrightarrow \overline{K} \\ \sigma(\alpha) \neq \alpha}} v(\alpha - \sigma(\alpha)),$$

(compare the invariant-theoretic formulation developed in [11, Introduction and Theorem 1.1]; see also [5, Chapter II, §2, pp. 26–30]; see also [7, Eqs. (1)–(3), pp. 1095–1096]; compare [13, pp. 225–226]) where σ ranges over the K-embeddings of $K(\alpha)$ into $\overline{K}$ distinct from the distinguished embedding determined by the chosen element $\alpha \in \overline{K}$ (see [12, Introduction, pp. 2975–2976]; see also [6, Theorem 1.2]; compare also the valuation-theoretic invariant $\delta_K(\alpha)$ and its Krasner-type approximation behavior in [9, Section 1]).

Theorem 2.3 (Classical Krasner lemma)**.** *Let (K, v) be Henselian and let $\alpha, \beta \in \overline{K}$, with α separable over K. If*

$$v(\alpha - \beta) > \kappa_K(\alpha),$$

then

$$K(\alpha) \subseteq K(\beta).$$

(see [12, Introduction, p. 2976]; see also [5, Chapter II, §2, pp. 26–30]; see also [2, Chapter 4, §4.1]))

Proof. Let $L = K(\alpha)$. Since α is separable over K, the K-embeddings of L into $\overline{K}$ are distinct on α. For every K-embedding $\sigma : L \hookrightarrow \overline{K}$ with $\sigma(\alpha) \neq \alpha$, the definition of the Krasner constant gives

$$v(\alpha - \sigma(\alpha)) \leq \kappa_K(\alpha).$$

By hypothesis,

$$v(\alpha - \beta) > \kappa_K(\alpha),$$

and hence

$$v(\alpha-\beta) > v(\alpha-\sigma(\alpha)).$$

Using the ultrametric inequality (see [1, Chapter I, §1, pp. 5–6]; see also [2, Chapter 1, Proposition 1.1.1, pp. 5–6]), we obtain

$$\begin{aligned} v(\beta-\sigma(\alpha)) &= v\big((\beta-\alpha)+(\alpha-\sigma(\alpha))\big) \\ &= \min\{v(\beta-\alpha), v(\alpha-\sigma(\alpha))\} \\ &= v(\alpha-\sigma(\alpha)). \end{aligned}$$

Thus among all conjugates of α over K, the element α is the unique conjugate lying closest to β.

Now let

$$\tau \in \mathrm{Aut}_{v,K(\beta)}(\overline{K}),$$

where $\mathrm{Aut}_{v,K(\beta)}(\overline{K})$ denotes the subgroup of automorphisms of the valued field $\overline{K}$ that fix $K(\beta)$ pointwise and preserve the chosen extension of v. Since τ fixes K and fixes β, the element $\tau(\alpha)$ is again a K-conjugate of α. Moreover,

$$v(\tau(\alpha)-\beta) = v(\tau(\alpha)-\tau(\beta)) = v(\alpha-\beta).$$

By the uniqueness just proved, this forces

$$\tau(\alpha) = \alpha.$$

Hence α is fixed by every automorphism of $\overline{K}$ over $K(\beta)$. Therefore

$$\alpha \in K(\beta).$$

Consequently,

$$K(\alpha) \subseteq K(\beta),$$

as claimed. (see [6, Theorem 1.1]; Related valuation-theoretic refinements involving tame fields, defectless extensions, and approximation invariants were developed in [11, Theorems 1.1–1.3].) □

Further valuation-theoretic and trace-theoretic analyses of Krasner constants and tame extensions may be found in [10, 12]. For valuation-theoretic refinements involving tame extensions and equidistant conjugates, see [13, Theorem A and Theorem 1.1, pp. 226–227]. Related approximation-theoretic phenomena for sequences of algebraic elements over complete valued fields are studied in [8, Theorem 2.1, pp. 618–620].

Remark 2.4. The classical theorem is field-theoretic. It does not directly measure whether the integral models $\mathcal{O}_K[\alpha]$, $\mathcal{O}_K[\beta]$, and $\mathcal{O}_K[\alpha,\beta]$ agree, nor whether their residue and torsion structures behave stably (see [5, Chapter I, §§1–2, pp. 5–9]). The distinction between field-theoretic approximation and finer valuation-theoretic behavior is already visible in the analysis of conditions under which $D_K(a) = v_K(a)$; see [13, Theorem 1.1, pp. 226–230].

3 Approximation Quotient Modules

3.1 The basic construction

Remark 3.1 (Standing ambient convention). Throughout Sections 3–7, all approximation quotients are formed inside the fixed algebraic closure $\overline{K}$ chosen in Section 2. Thus α and β are treated as specified algebraic elements of $\overline{K}$, and the rings $\mathcal{O}_K[\alpha]$, $\mathcal{O}_K[\beta]$, and $\mathcal{O}_K[\alpha,\beta]$ are the corresponding $\mathcal{O}_K$-subalgebras of $\overline{K}$. No additional embedding choice is involved in the definition of $A_{\alpha,\beta}$. Whenever Krasner's constant or Krasner's lemma is used, the relevant element is assumed separable over K.

Throughout the paper, unless explicitly stated otherwise, all extensions are assumed finite and all approximation quotient modules are assumed finitely generated over $\mathcal{O}_K$.

Remark 3.2 (Scope of valuation-theoretic generality)*.* The construction of the approximation quotient and of the modules $\mathrm{DKD}_i(\alpha, \beta)$ makes sense over an arbitrary Henselian valuation ring. However, the structural vanishing and torsion-detection results in Sections 5–7 are intentionally stated under discrete valuation hypotheses, where $\mathcal{O}_K$ is Noetherian (see [5, Chapter I, §2, pp. 7–9]; see also [1, Chapter II, §§6–10]), the residue field has projective dimension one, and DKD_1 admits a concrete interpretation as π-torsion. Thus the general valuation-ring formalism is used only as the ambient definition, while the main stability theorems are DVR-level results.

Remark 3.3 (Non-automatic finiteness)*.* The finite-generation hypothesis in the preceding convention is a genuine assumption, not a consequence of algebraicity of α and β. If α or β is not integral over $\mathcal{O}_K$, the rings $\mathcal{O}_K[\alpha]$, $\mathcal{O}_K[\beta]$, and $\mathcal{O}_K[\alpha, \beta]$ may contain denominators and need not be finite $\mathcal{O}_K$-modules (see [5, Chapter I, §§1–3, pp. 5–10]). All later statements involving Tor-detection over a DVR are therefore to be read under this explicit finite-generation hypothesis.

Classical valuation-theoretic approximation sets and maximality phenomena in Henselian valued fields motivate the approximation quotient formalism introduced below; compare [6, Theorems 1.1–1.3]; compare also the approximation-maximality framework and distinguished approximation sequences developed in [8, Section 2, pp. 618–620]; also compare [8, Section 2 and Section 4, pp. 618–624].

Definition 3.4 (Approximation quotient)**.** Let (K, v) be Henselian, and let $\alpha, \beta \in \overline{K}$ be algebraic over K. Inside the fixed algebraic closure $\overline{K}$, define

$$A_{\alpha,\beta} := \frac{\mathcal{O}_K[\alpha, \beta]}{\mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta]}.$$

Here $\mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta]$ denotes the $\mathcal{O}_K$-submodule of $\mathcal{O}_K[\alpha, \beta]$ generated by the two subrings. We call $A_{\alpha,\beta}$ the approximation quotient module attached to the pair (α, β).

Lemma 3.5 (Independence of auxiliary realization)**.** *The module $A_{\alpha,\beta}$ depends only on the specified elements $\alpha, \beta \in \overline{K}$, and not on any auxiliary finite K-algebra used to realize the subrings.*

Proof. By the standing ambient convention, the rings

$$\mathcal{O}_K[\alpha], \qquad \mathcal{O}_K[\beta], \qquad \mathcal{O}_K[\alpha, \beta]$$

are subrings of the fixed algebraic closure $\overline{K}$. Hence the $\mathcal{O}_K$-submodule

$$\mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta] \subseteq \mathcal{O}_K[\alpha, \beta]$$

is also intrinsically determined inside $\overline{K}$.

Therefore the quotient

$$A_{\alpha,\beta} = \frac{\mathcal{O}_K[\alpha, \beta]}{\mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta]}$$

is formed from these fixed subrings and is independent of any auxiliary presentation or realization. This proves the claimed canonical $\mathcal{O}_K$-module independence. □

Remark 3.6*.* The quotient $A_{\alpha,\beta}$ compares the jointly generated $\mathcal{O}_K$-order $\mathcal{O}_K[\alpha, \beta]$ with the $\mathcal{O}_K$-submodule generated by the two separately generated $\mathcal{O}_K$-orders $\mathcal{O}_K[\alpha]$ and $\mathcal{O}_K[\beta]$. It should be regarded as a candidate obstruction module measuring this additive discrepancy, not as a canonical intersection-theoretic or deformation-theoretic invariant (see [6, Eq. (1.1), p. 234]).

Lemma 3.7 (Triviality criterion)**.** *If*

$$\mathcal{O}_K[\alpha,\beta] = \mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta]$$

as $\mathcal{O}_K$-submodules of the chosen common ambient algebra, then $A_{\alpha,\beta} = 0$.

Proof. By Definition 3.4,

$$A_{\alpha,\beta} = \frac{\mathcal{O}_K[\alpha,\beta]}{\mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta]}.$$

The hypothesis identifies the numerator with the denominator as $\mathcal{O}_K$-submodules. Hence

$$A_{\alpha,\beta} = \frac{\mathcal{O}_K[\alpha,\beta]}{\mathcal{O}_K[\alpha,\beta]} = 0.$$

□

3.2 Residue interpretation

Proposition 3.8 (Residue vanishing criterion)**.** *Assume that $A_{\alpha,\beta}$ is finitely generated over $\mathcal{O}_K$. Then*

$$A_{\alpha,\beta} \otimes_{\mathcal{O}_K} k_K = 0$$

if and only if

$$A_{\alpha,\beta} = \mathfrak{m}_K A_{\alpha,\beta}.$$

Consequently, if $\mathcal{O}_K$ is local and $A_{\alpha,\beta}$ is finitely generated, then

$$A_{\alpha,\beta} \otimes_{\mathcal{O}_K} k_K = 0 \quad \Longleftrightarrow \quad A_{\alpha,\beta} = 0.$$

Proof. Since $k_K = \mathcal{O}_K/\mathfrak{m}_K$, for any $\mathcal{O}_K$-module M there is a canonical isomorphism

$$M \otimes_{\mathcal{O}_K} k_K \cong M/\mathfrak{m}_K M.$$

Applying this to $M = A_{\alpha,\beta}$ gives

$$A_{\alpha,\beta} \otimes_{\mathcal{O}_K} k_K \cong A_{\alpha,\beta}/\mathfrak{m}_K A_{\alpha,\beta}.$$

Therefore

$$A_{\alpha,\beta} \otimes_{\mathcal{O}_K} k_K = 0$$

if and only if

$$A_{\alpha,\beta} = \mathfrak{m}_K A_{\alpha,\beta}.$$

If $A_{\alpha,\beta}$ is finitely generated over the local ring $\mathcal{O}_K$, Nakayama's lemma then implies (see [4, Theorem and discussion on Nakayama's lemma, §2, pp. 6–14], see also [5, Chapter I, §2, pp. 7–9])

$$A_{\alpha,\beta} = 0.$$

The converse is immediate. This proves the criterion. □

4 Tor-Theoretic Krasner Defect Modules

Definition 4.1 (Tor-theoretic Krasner defect)**.** Let (K, v) be Henselian and let $\alpha, \beta \in \overline{K}$. For $i \geq 0$, define

$$\mathrm{DKD}_i(\alpha, \beta) := \mathrm{Tor}_i^{\mathcal{O}_K}(A_{\alpha,\beta}, k_K).$$

(see [4, Appendix B, pp. 274–282] for background on Tor functors; The valuation-theoretic background underlying these obstruction modules is closely related to approximation-maximality phenomena studied in [6, Theorems 1.1–1.3]). We call $\mathrm{DKD}_i(\alpha, \beta)$ the i-th Tor-theoretic Krasner defect module.

The choice of Tor with the residue field is motivated by the fact that reduction modulo the maximal ideal detects whether approximation discrepancies survive on the special fibre. Over a DVR this detection has only two levels: the residual quotient and the first torsion term.

Remark 4.2. The module $\mathrm{DKD}_0(\alpha, \beta)$ is the residual reduction of the approximation quotient. In the discretely valued case, the only possible positive-degree contribution is $\mathrm{DKD}_1(\alpha, \beta)$, which records the π-torsion of $A_{\alpha,\beta}$ (see [5, Chapter I, §1, pp. 5–6]).

Proposition 4.3 (Conditional comparison under composition-compatible approximations)**.** *Let (K, v) be Henselian, and let $\alpha, \beta, \gamma \in \overline{K}$ be algebraic over K, all viewed inside the fixed algebraic closure $\overline{K}$ of the standing ambient convention. Assume that the triple is composition-compatible in the following sense:*

$$\mathcal{O}_K[\alpha, \beta] + \mathcal{O}_K[\beta, \gamma] \subseteq \mathcal{O}_K[\alpha, \gamma]$$

and

$$(\mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta]) + (\mathcal{O}_K[\beta] + \mathcal{O}_K[\gamma]) \subseteq \mathcal{O}_K[\alpha] + \mathcal{O}_K[\gamma].$$

These hypotheses are not automatic functoriality assumptions. They impose a strong compatibility condition ensuring that addition is well defined on the chosen approximation quotients. Then addition induces a natural $\mathcal{O}_K$-linear morphism

$$A_{\alpha,\beta} \oplus A_{\beta,\gamma} \longrightarrow A_{\alpha,\gamma}.$$

Consequently, for every $i \geq 0$, there is a natural comparison morphism

$$\mathrm{DKD}_i(\alpha, \beta) \oplus \mathrm{DKD}_i(\beta, \gamma) \longrightarrow \mathrm{DKD}_i(\alpha, \gamma).$$

Moreover, for composition-compatible quadruples $\alpha, \beta, \gamma, \delta$, these morphisms are compatible with successive composition.

Proof. By definition,

$$A_{\alpha,\beta} = \frac{\mathcal{O}_K[\alpha, \beta]}{\mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta]}, \quad A_{\beta,\gamma} = \frac{\mathcal{O}_K[\beta, \gamma]}{\mathcal{O}_K[\beta] + \mathcal{O}_K[\gamma]}, \quad A_{\alpha,\gamma} = \frac{\mathcal{O}_K[\alpha, \gamma]}{\mathcal{O}_K[\alpha] + \mathcal{O}_K[\gamma]}.$$

The assumed inclusion

$$\mathcal{O}_K[\alpha, \beta] + \mathcal{O}_K[\beta, \gamma] \subseteq \mathcal{O}_K[\alpha, \gamma]$$

allows addition to define an $\mathcal{O}_K$-linear map

$$\mathcal{O}_K[\alpha, \beta] \oplus \mathcal{O}_K[\beta, \gamma] \longrightarrow \mathcal{O}_K[\alpha, \gamma], \qquad (x, y) \longmapsto x + y.$$

The second compatibility assumption ensures that if

$$x \in \mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta] \quad \text{and} \quad y \in \mathcal{O}_K[\beta] + \mathcal{O}_K[\gamma],$$

then

$$x + y \in \mathcal{O}_K[\alpha] + \mathcal{O}_K[\gamma].$$

Hence the above map descends to the quotients and gives

$$A_{\alpha,\beta} \oplus A_{\beta,\gamma} \longrightarrow A_{\alpha,\gamma}.$$

Applying the functor

$$\mathrm{Tor}_i^{\mathcal{O}_K}(-, k_K)$$

and using its additivity on finite direct sums gives natural maps

$$\mathrm{Tor}_i^{\mathcal{O}_K}(A_{\alpha,\beta}, k_K) \oplus \mathrm{Tor}_i^{\mathcal{O}_K}(A_{\beta,\gamma}, k_K) \longrightarrow \mathrm{Tor}_i^{\mathcal{O}_K}(A_{\alpha,\gamma}, k_K).$$

By Definition 4.1, these are precisely

$$\mathrm{DKD}_i(\alpha, \beta) \oplus \mathrm{DKD}_i(\beta, \gamma) \longrightarrow \mathrm{DKD}_i(\alpha, \gamma).$$

For a composition-compatible quadruple $\alpha, \beta, \gamma, \delta$, the assertion follows from associativity of addition before passing to quotients. Thus the comparison morphisms are compatible with successive composition. □

Proposition 4.4 (Flatness criterion)**.** *If $A_{\alpha,\beta}$ is flat over $\mathcal{O}_K$ (see [5, Chapter I, §§1–2, pp. 5–9]), then*

$$\mathrm{DKD}_i(\alpha, \beta) = 0 \qquad \textit{for all } i \geq 1.$$

Proof. Recall from Definition 4.1 that

$$\mathrm{DKD}_i(\alpha, \beta) := \mathrm{Tor}_i^{\mathcal{O}_K}(A_{\alpha,\beta}, k_K), \qquad i \geq 0.$$

Assume that $A_{\alpha,\beta}$ is flat as an $\mathcal{O}_K$-module. By the standard characterization of flatness via vanishing of higher Tor-groups (see [4, §7, pp. 45–53 and Appendix B]), the functor

$$A_{\alpha,\beta} \otimes_{\mathcal{O}_K} (-)$$

is exact on the category of $\mathcal{O}_K$-modules. Equivalently, for every $\mathcal{O}_K$-module M,

$$\mathrm{Tor}_i^{\mathcal{O}_K}(A_{\alpha,\beta}, M) = 0 \qquad (i \geq 1).$$

Applying this to the residue field $M = k_K$, we obtain

$$\mathrm{Tor}_i^{\mathcal{O}_K}(A_{\alpha,\beta}, k_K) = 0 \qquad (i \geq 1).$$

Hence

$$\mathrm{DKD}_i(\alpha, \beta) = 0 \qquad \text{for all } i \geq 1.$$

Thus flatness of the approximation quotient eliminates all positive-degree Tor terms after passage to the residue field. □

We now record the point at which the general formalism becomes genuinely discrete. The following identification uses a uniformizer and the length-one free resolution of the residue field, and therefore belongs to the DVR specialization rather than to arbitrary Henselian valuation rings.

Proposition 4.5 (Torsion detection over a DVR)**.** *Assume that $\mathcal{O}_K$ is a discrete valuation ring with uniformizer π, and that $A_{\alpha,\beta}$ is finitely generated over $\mathcal{O}_K$. Then there is a natural identification*

$$\mathrm{DKD}_1(\alpha, \beta) \simeq \{x \in A_{\alpha,\beta} : \pi x = 0\}.$$

In particular, if $A_{\alpha,\beta}$ has non-trivial $\mathcal{O}_K$-torsion, then

$$\mathrm{DKD}_1(\alpha, \beta) \neq 0.$$

Proof. Since $\mathcal{O}_K$ is a discrete valuation ring, the residue field k_K has the standard free resolution

$$0 \longrightarrow \mathcal{O}_K \xrightarrow{\cdot\pi} \mathcal{O}_K \longrightarrow k_K \longrightarrow 0.$$

([4, §11, pp. 78–85 and Appendix B]) Tensoring this resolution with $A_{\alpha,\beta}$ gives

$$0 \longrightarrow \mathrm{Tor}_1^{\mathcal{O}_K}(A_{\alpha,\beta}, k_K) \longrightarrow A_{\alpha,\beta} \xrightarrow{\cdot\pi} A_{\alpha,\beta} \longrightarrow A_{\alpha,\beta} \otimes_{\mathcal{O}_K} k_K \longrightarrow 0.$$

Hence

$$\mathrm{DKD}_1(\alpha, \beta) = \mathrm{Tor}_1^{\mathcal{O}_K}(A_{\alpha,\beta}, k_K) \simeq \ker\Big(A_{\alpha,\beta} \xrightarrow{\cdot\pi} A_{\alpha,\beta}\Big) = \{x \in A_{\alpha,\beta} : \pi x = 0\}.$$

If $A_{\alpha,\beta}$ has non-trivial $\mathcal{O}_K$-torsion, then some non-zero element is killed by a power of π. Choosing such an element with minimal annihilator exponent, one obtains a non-zero element killed by π. Therefore $\mathrm{DKD}_1(\alpha, \beta) \neq 0$. □

5 Tor-Theoretic Krasner Stability in the Tame Defectless Case

Remark 5.1 (Specialization to the discretely valued case)*.* From this section onward, the main structural results are formulated in the discretely valued setting unless explicitly stated otherwise. This specialization is not accidental: the proofs use the DVR resolution $0 \to \mathcal{O}_K \xrightarrow{\pi} \mathcal{O}_K \to k_K \to 0$, the Noetherian finiteness of $\mathcal{O}_K$, and the resulting identification of DKD_1 with the π-torsion of $A_{\alpha,\beta}$. The preceding general valuation-ring definitions provide the ambient formalism, but the tame stability theory proved below is a DVR theory.

Definition 5.2 (Tame defectless extension)**.** A finite extension L/K of Henselian valued fields is called tame defectless if it is defectless, its ramification index is prime to the residue characteristic, and the residue field extension is separable (This is the standard notion of a finite tame extension in valuation theory; see [5, Chapter III, §§3–7, pp. 50–59] and [12, p. 2977]; see also [13, Introduction and Theorem A, pp. 225–226]; For related characterizations of defectless algebraic extensions through approximation invariants and Krasner-type principles, see [9, Theorem 1.1]).

Lemma 5.3 (Fitting support control for the first Tor-theoretic defect)**.** *Assume that $\mathcal{O}_K$ is a discrete valuation ring and that $A_{\alpha,\beta}$ is a finitely generated $\mathcal{O}_K$-module. Then*

$$\mathrm{Supp}_{\mathcal{O}_K}\big(\mathrm{DKD}_1(\alpha, \beta)\big) \subseteq \mathrm{Supp}_{\mathcal{O}_K}(A_{\alpha,\beta}) = V\big(\mathrm{Fitt}_0^{\mathcal{O}_K}(A_{\alpha,\beta})\big).$$

In particular, the first Tor-theoretic Krasner defect is supported on the Fitting support of the approximation quotient.

Proof. By Proposition 4.5, the module $\mathrm{DKD}_1(\alpha, \beta)$ identifies with the π-torsion submodule of $A_{\alpha,\beta}$. Hence it is a submodule of $A_{\alpha,\beta}$, and therefore

$$\mathrm{Supp}_{\mathcal{O}_K}\big(\mathrm{DKD}_1(\alpha, \beta)\big) \subseteq \mathrm{Supp}_{\mathcal{O}_K}(A_{\alpha,\beta}).$$

For a finitely generated module over a Noetherian ring, the support is defined by the zeroth Fitting ideal ([4, Chapter 1 and Appendix B]):

$$\mathrm{Supp}_{\mathcal{O}_K}(A_{\alpha,\beta}) = V\big(\mathrm{Fitt}_0^{\mathcal{O}_K}(A_{\alpha,\beta})\big).$$

Combining these two facts gives the desired inclusion. □

Theorem 5.4 (Torsion-freeness criterion for the first defect)**.** *Assume that $\mathcal{O}_K$ is a discrete valuation ring and that $A_{\alpha,\beta}$ is a finitely generated $\mathcal{O}_K$-module. If*

$$\mathrm{DKD}_1(\alpha, \beta) = 0,$$

then $A_{\alpha,\beta}$ is torsion-free over $\mathcal{O}_K$. If, moreover, $A_{\alpha,\beta}$ is finitely presented, then $A_{\alpha,\beta}$ is flat over $\mathcal{O}_K$.

Proof. Let π be a uniformizer of $\mathcal{O}_K$. By Proposition 4.5, one has

$$\mathrm{DKD}_1(\alpha, \beta) \simeq \{x \in A_{\alpha,\beta} : \pi x = 0\}.$$

Thus $\mathrm{DKD}_1(\alpha, \beta) = 0$ means that multiplication by π is injective on $A_{\alpha,\beta}$.

Suppose that $A_{\alpha,\beta}$ had a non-zero torsion element x. Since $\mathcal{O}_K$ is a discrete valuation ring, some power π^n annihilates x. Choosing n minimal, the element $\pi^{n-1}x$ is non-zero and is killed by π, contradicting the injectivity of multiplication by π. Hence $A_{\alpha,\beta}$ is torsion-free over $\mathcal{O}_K$.

Finally, over a discrete valuation ring, every finitely presented torsion-free module is flat ([4, §7 and §11, pp. 45–53, 78–85]). Therefore, if $A_{\alpha,\beta}$ is finitely presented, it is flat over $\mathcal{O}_K$. □

Theorem 5.5 (Finite-length and amplitude bound)**.** *Let (K, v) be a Henselian discretely valued field with valuation ring O_K and residue field k_K. Assume that the approximation quotient module*

$$A_{\alpha,\beta} = \frac{O_K[\alpha, \beta]}{O_K[\alpha] + O_K[\beta]}$$

is finitely generated over O_K.

Then:

(i) *For every $i \geq 0$, the Tor-theoretic Krasner defect module*

$$DKD_i(\alpha, \beta) = \mathrm{Tor}_i^{O_K}(A_{\alpha,\beta}, k_K)$$

is a finite-dimensional k_K-vector space.

(ii) *One has*

$$DKD_i(\alpha, \beta) = 0 \qquad (i > 1).$$

(iii) *If $A_{\alpha,\beta}$ is torsion-free over O_K, then*

$$DKD_i(\alpha, \beta) = 0 \qquad (i \geq 1).$$

Equivalently, only the degree-zero defect

$$DKD_0(\alpha, \beta) = A_{\alpha,\beta} \otimes_{O_K} k_K$$

may survive.

Proof. Since O_K is a discrete valuation ring, the residue field k_K admits the free resolution

$$0 \longrightarrow O_K \xrightarrow{\cdot \pi} O_K \longrightarrow k_K \longrightarrow 0,$$

where π is a uniformizer of O_K. Thus the projective dimension of k_K over O_K equals one (see [4, Appendix B, pp. 274–282]).

Tensoring this resolution with the finitely generated O_K-module $A_{\alpha,\beta}$ yields

$$0 \longrightarrow \mathrm{Tor}_1^{O_K}(A_{\alpha,\beta}, k_K) \longrightarrow A_{\alpha,\beta} \xrightarrow{\cdot \pi} A_{\alpha,\beta} \longrightarrow A_{\alpha,\beta} \otimes_{O_K} k_K \longrightarrow 0.$$

Because the resolution has length one, all higher Tor-groups vanish:

$$\mathrm{Tor}_i^{O_K}(A_{\alpha,\beta}, k_K) = 0 \qquad (i > 1).$$

Hence

$$DKD_i(\alpha, \beta) = 0 \qquad (i > 1).$$

Moreover,

$$DKD_0(\alpha, \beta) = A_{\alpha,\beta} \otimes_{O_K} k_K$$

and

$$DKD_1(\alpha,\beta) = \ker\Big(A_{\alpha,\beta} \xrightarrow{\cdot\pi} A_{\alpha,\beta}\Big).$$

Since $A_{\alpha,\beta}$ is finitely generated over the DVR O_K, the quotient $A_{\alpha,\beta}/\pi A_{\alpha,\beta}$ is a finitely generated k_K-vector space. Moreover, $\ker(\pi : A_{\alpha,\beta} \to A_{\alpha,\beta})$ is annihilated by π and is a submodule of the finitely generated O_K-module $A_{\alpha,\beta}$; hence it is also finite-dimensional over k_K.

Finally, if $A_{\alpha,\beta}$ is torsion-free over O_K, multiplication by π is injective on $A_{\alpha,\beta}$. Therefore

$$DKD_1(\alpha,\beta) = 0.$$

Combined with the vanishing for $i > 1$, this shows that only $DKD_0(\alpha,\beta)$ may survive. □

Remark 5.6 (DVR amplitude). Over a discrete valuation ring the Tor-theoretic Krasner defect has homological amplitude at most one. Thus, in the discretely valued case, the Tor formalism has only two possible contributions: the residual term $\mathrm{DKD}_0(\alpha,\beta)$ and the first torsion-detection term $\mathrm{DKD}_1(\alpha,\beta)$. The point of the formalism is therefore not the existence of arbitrarily high Tor groups over DVRs, but the separation between residue-level discrepancy and first Tor torsion obstruction.

Proposition 5.7 (Formal flatness reduction after Krasner stabilization). *Let (K, v) be Henselian, let $\alpha \in \overline{K}$, and set $L = K(\alpha)$. Assume that:*

(i) L/K is finite separable;

(ii) $\beta \in \overline{K}$ satisfies

$$v(\alpha - \beta) > \kappa_K(\alpha);$$

(iii) the approximation quotient $A_{\alpha,\beta}$ is flat over $\mathcal{O}_K$.

Then

$$\mathrm{DKD}_i(\alpha,\beta) = 0 \qquad \textit{for all } i \geq 1.$$

Proof. By the classical Krasner lemma, the inequality

$$v(\alpha - \beta) > \kappa_K(\alpha)$$

implies

$$K(\alpha) \subseteq K(\beta).$$

Thus the approximation has crossed the classical Krasner threshold: the generated field $K(\alpha)$ is already contained in the field generated by β.

The remaining question is not field-theoretic but torsion-theoretic. Namely, one must determine whether the integral approximation quotient

$$A_{\alpha,\beta} = \frac{\mathcal{O}_K[\alpha,\beta]}{\mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta]}$$

still carries Tor-theoretic obstructions after reduction modulo $\mathfrak{m}_K$.

By hypothesis, $A_{\alpha,\beta}$ is flat over $\mathcal{O}_K$. Therefore the functor

$$A_{\alpha,\beta} \otimes_{\mathcal{O}_K} (-)$$

is exact ([4, §7, pp. 45–53]). Equivalently,

$$\mathrm{Tor}_i^{\mathcal{O}_K}(A_{\alpha,\beta}, M) = 0 \qquad \text{for all } i \geq 1$$

and for every $\mathcal{O}_K$-module M. Taking $M = k_K$, we obtain

$$\mathrm{Tor}_i^{\mathcal{O}_K}(A_{\alpha,\beta}, k_K) = 0 \qquad (i \geq 1).$$

By definition of the Tor-theoretic Krasner defect modules,

$$\mathrm{DKD}_i(\alpha, \beta) = \mathrm{Tor}_i^{\mathcal{O}_K}(A_{\alpha,\beta}, k_K).$$

Hence

$$\mathrm{DKD}_i(\alpha, \beta) = 0 \qquad \text{for all } i \geq 1.$$

Thus the substantive input in this statement is the separation of the classical Krasner field-theoretic stabilization from the additional flatness condition on $A_{\alpha,\beta}$. The proposition should be read as a reduction step: once flatness of the approximation quotient has been verified by independent integral methods, the positive Tor-defects vanish formally. □

Remark 5.8. The preceding proposition is intentionally formal. Its role is not to derive flatness of $A_{\alpha,\beta}$ from tame defectlessness, but to record the exact homological consequence of having already established that flatness. The genuinely arithmetic problem is therefore shifted to proving torsion-freeness or flatness of $A_{\alpha,\beta}$ from concrete integral compatibility conditions, as in the next result.

The valuation-theoretic role of defectless extensions, approximation maxima, and distinguished approximation chains in Henselian fields was analyzed in [8, Theorem 2.2 and Section 4, pp. 618–623]; the theorem below develops a torsion-theoretic refinement at the level of integral approximation quotients.

Theorem 5.9 (Saturation criterion after Krasner stabilization)**.** *Let (K, v) be a Henselian discretely valued field with valuation ring $\mathcal{O}_K$ and uniformizer π. Let*

$$L = K(\alpha)/K$$

be finite tame defectless (in the classical valuation-theoretic sense discussed in [5, Chapter III, §§3–7, pp. 50–59] and [12, p. 2977]; see also [6, Theorem 1.3]), and assume that

$$v(\alpha - \beta) > \kappa_K(\alpha).$$

The tame defectless hypothesis is used here only to place the statement in the classical arithmetic Krasner-stability setting; the equivalence below is a purely integral DVR saturation criterion and does not use tame defectlessness in its proof (compare the valuation-theoretic characterization of the equality $D_K(a) = v_K(a)$ in [13, Theorem 1.1, pp. 226–230]; compare also the defectless approximation framework and main-invariant stabilization results of [9, Theorem 1.1 and Theorem A]). Set

$$N = \mathcal{O}_K[\alpha, \beta], \qquad M = \mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta].$$

Assume that $A_{\alpha,\beta} = N/M$ is finitely generated over $\mathcal{O}_K$. Then the following are equivalent:

(i) $DKD_1(\alpha, \beta) = 0$*;*

(ii) $A_{\alpha,\beta}$ *is torsion-free over* $\mathcal{O}_K$*;*

(iii) M *is* π*-saturated in* N*, i.e.*

$$x \in N,\ \pi x \in M \quad \Longrightarrow \quad x \in M.$$

Consequently, after the classical Krasner threshold has been crossed, the remaining first Tor-theoretic obstruction is precisely the failure of saturation of

$$\mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta] \subseteq \mathcal{O}_K[\alpha, \beta].$$

If, in addition, $A_{\alpha,\beta} \otimes_{\mathcal{O}_K} k_K = 0$, then $A_{\alpha,\beta} = 0$, and hence

$$DKD_i(\alpha, \beta) = 0 \qquad (i \geq 1).$$

Proof. The Krasner inequality gives the classical field-theoretic stabilization

$$K(\alpha) \subseteq K(\beta).$$

This does not by itself force equality of the integral structures $\mathcal{O}_K[\alpha]$, $\mathcal{O}_K[\beta]$, and $\mathcal{O}_K[\alpha, \beta]$. The remaining question is whether the submodule

$$M = \mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta]$$

is saturated inside

$$N = \mathcal{O}_K[\alpha, \beta].$$

Since $\mathcal{O}_K$ is a discrete valuation ring, Proposition 4.5 gives

$$DKD_1(\alpha, \beta) \simeq \{\, \bar{x} \in N/M : \pi\bar{x} = 0 \,\}.$$

Thus $DKD_1(\alpha, \beta) = 0$ if and only if N/M has no non-zero element killed by π. Over a discrete valuation ring this is equivalent to N/M being torsion-free ([4, §11, pp. 78–85]).

Finally, N/M has no non-zero element killed by π precisely when

$$x \in N, \ \pi x \in M \quad \Longrightarrow \quad x \in M,$$

which is exactly the π-saturation of M in N.

If moreover

$$A_{\alpha,\beta} \otimes_{\mathcal{O}_K} k_K = 0,$$

then Nakayama's lemma gives $A_{\alpha,\beta} = 0$. Therefore all higher Tor groups

$$DKD_i(\alpha, \beta) = \mathrm{Tor}_i^{\mathcal{O}_K}(A_{\alpha,\beta}, k_K)$$

vanish for $i \geq 1$. □

Remark 5.10. The point of Theorem 5.9 is not to assume that the two integral orders are already equal. Rather, after Krasner's field-theoretic stabilization, it identifies the genuine remaining integral obstruction: the possible failure of saturation of $\mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta]$ inside $\mathcal{O}_K[\alpha, \beta]$. Hence the theorem is no longer tautological.

Proposition 5.11 (Different-controlled first Tor-theoretic defect)**.** *Let (K, v) be a Henselian discretely valued field, and let $L = K(\alpha)/K$ be finite separable (compare the valuation-theoretic discussion of the different and ramification in [5, Chapter III, §§2–7, pp. 48–59], and the trace-theoretic analysis surrounding Krasner constants in [12, Lemma 2.3 and Remark 2.4]), with valuation ring $\mathcal{O}_L$. Assume that β satisfies*

$$v(\alpha - \beta) > \kappa_K(\alpha),$$

so that $K(\alpha) \subseteq K(\beta)$. Let

$$\mathfrak{d}_{L/K} := \mathfrak{D}_{L/K} \cap \mathcal{O}_K$$

be the contraction of the different ideal to $\mathcal{O}_K$. Assume that the $\mathcal{O}_K$-torsion submodule of $A_{\alpha,\beta}$ is annihilated by a power of the contracted ideal $\mathfrak{d}_{L/K}$. Then

$$\mathrm{DKD}_1(\alpha, \beta)$$

is annihilated by a power of $\mathfrak{d}_{L/K}$ (see also the discussion of tame approximation and Krasner constants in [13, pp. 225–230]).

Proof. By Krasner's lemma, the inequality

$$v(\alpha - \beta) > \kappa_K(\alpha)$$

implies

$$K(\alpha) \subseteq K(\beta).$$

Thus the valuation-theoretic approximation has already forced the expected field-theoretic stabilization.

Since $\mathcal{O}_K$ is a discrete valuation ring, Proposition 4.5 identifies the first Tor-theoretic defect with the π-torsion of the approximation quotient:

$$\mathrm{DKD}_1(\alpha, \beta) \simeq \{x \in A_{\alpha,\beta} : \pi x = 0\}.$$

In particular, $\mathrm{DKD}_1(\alpha, \beta)$ is a submodule of the torsion submodule of $A_{\alpha,\beta}$.

By the additional hypothesis, the torsion submodule of $A_{\alpha,\beta}$ is annihilated by some power of the contracted different ideal

$$\mathfrak{d}_{L/K} := \mathfrak{D}_{L/K} \cap \mathcal{O}_K.$$

Therefore the submodule

$$\mathrm{DKD}_1(\alpha, \beta) \subseteq (A_{\alpha,\beta})_{\mathrm{tors}}$$

is also annihilated by the same power of $\mathfrak{d}_{L/K}$. This proves the claim. □

Corollary 5.12 (Classical stability plus flatness implies Tor-theoretic stability)**.** *Under the hypotheses of Proposition 5.7, one has*

$$\mathrm{DKD}_i(\alpha, \beta) = 0 \qquad (i \geq 1).$$

Proof. This is exactly the conclusion of Proposition 5.7. The Krasner inequality gives the field-theoretic inclusion $K(\alpha) \subseteq K(\beta)$, while the flatness hypothesis on $A_{\alpha,\beta}$ eliminates the positive-degree Tor terms. □

6 Wild Approximation Obstructions

Immediate extensions, approximation maxima, and distinguished approximation sequences in Henselian valued fields are closely connected through the valuation-theoretic criteria developed in [8, Section 4 and Theorem 2.2, pp. 621–624] (compare also the approximation-maximality phenomena and main-invariant methods developed in [9, Sections 1–4]).

Proposition 6.1 (Wild approximation obstruction)**.** *Let (K, v) be Henselian discretely valued of residue characteristic $p > 0$ [5, Chapter IV, §§1–4, pp. 61–77], with valuation ring $\mathcal{O}_K$ and uniformizer π, and let (β_n) be an approximation family with $v(\alpha - \beta_n) > \kappa_K(\alpha)$ for all sufficiently large n. Assume that for some such n, the quotient*

$$A_{\alpha,\beta_n} = \frac{\mathcal{O}_K[\alpha, \beta_n]}{\mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta_n]}$$

contains a non-zero π-torsion class. Then

$$\mathrm{DKD}_1(\alpha, \beta_n) \neq 0.$$

Proof. For sufficiently large n, the Krasner inequality gives

$$v(\alpha - \beta_n) > \kappa_K(\alpha),$$

and hence the classical Krasner lemma implies

$$K(\alpha) \subseteq K(\beta_n).$$

Thus the field-theoretic part of the approximation has stabilized (compare the tame stabilization framework of [13, Theorem A, pp. 225–226]).

The remaining question is integral rather than field-theoretic. By the hypothesis, A_{α,β_n} contains a non-zero class $\overline{x}$ satisfying $\pi\overline{x} = 0$. Since $\mathcal{O}_K$ is a discrete valuation ring, Proposition 4.5 identifies

$$\mathrm{DKD}_1(\alpha, \beta_n) \simeq \{x \in A_{\alpha,\beta_n} : \pi x = 0\}.$$

Therefore $\overline{x}$ defines a non-zero element of $\mathrm{DKD}_1(\alpha, \beta_n)$, and hence

$$\mathrm{DKD}_1(\alpha, \beta_n) \neq 0.$$

□

Remark 6.2. This proposition records the precise obstruction mechanism: classical Krasner constants control field-theoretic stability (see [5, Chapter IV, §3, pp. 73–76]), while Tor-theoretic Krasner defects provide information about residual and torsion-theoretic instability.

Proposition 6.3 (Explicit first-defect non-vanishing criterion)**.** *Assume that $\mathcal{O}_K$ is a discrete valuation ring with uniformizer π. Let $\alpha, \beta \in \overline{K}$ be such that the approximation quotient $A_{\alpha,\beta}$ is finitely generated over $\mathcal{O}_K$. Suppose that there exists an element*

$$0 \neq \overline{x} \in A_{\alpha,\beta}$$

satisfying

$$\pi\overline{x} = 0.$$

Then

$$\mathrm{DKD}_1(\alpha, \beta) \neq 0.$$

In particular, any explicit wild approximation family (β_n) for which A_{α,β_n} contains a non-zero π-torsion class gives an explicit non-vanishing example:

$$\mathrm{DKD}_1(\alpha, \beta_n) \neq 0.$$

Proof. By Proposition 4.5, over the discrete valuation ring $\mathcal{O}_K$ there is a natural identification

$$\mathrm{DKD}_1(\alpha, \beta) \simeq \{x \in A_{\alpha,\beta} : \pi x = 0\}.$$

The assumed element $0 \neq \overline{x} \in A_{\alpha,\beta}$ killed by π therefore defines a non-zero element of $\mathrm{DKD}_1(\alpha, \beta)$. Hence

$$\mathrm{DKD}_1(\alpha, \beta) \neq 0.$$

□

6.1 A saturation mechanism for first-defect non-vanishing

Example 6.4 (First-defect non-vanishing from an explicit saturation witness)**.** Let $\mathcal{O}_K$ be a discrete valuation ring with uniformizer π, fraction field K, and residue field k_K. Let $\alpha, \beta \in \overline{K}$ be algebraic over K, and set

$$N = \mathcal{O}_K[\alpha, \beta], \qquad M = \mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta].$$

Assume that the approximation quotient

$$A_{\alpha,\beta} = N/M$$

is finitely generated over $\mathcal{O}_K$.

Suppose there exists an element
$$x \in N$$
satisfying
$$x \notin M, \qquad \pi x \in M.$$
Then the image
$$\overline{x} \in A_{\alpha,\beta} = N/M$$
is non-zero, since $x \notin M$, while
$$\pi \overline{x} = 0,$$
because $\pi x \in M$. Consequently,
$$0 \neq \overline{x} \in \{y \in A_{\alpha,\beta} : \pi y = 0\}.$$

By Proposition 4.5,
$$\mathrm{DKD}_1(\alpha, \beta) \simeq \{y \in A_{\alpha,\beta} : \pi y = 0\}.$$
Hence the class $\overline{x}$ determines a non-zero element of $\mathrm{DKD}_1(\alpha, \beta)$, and therefore
$$\mathrm{DKD}_1(\alpha, \beta) \neq 0.$$

This example is a saturation-witness mechanism underlying the later wild and non-saturated approximation phenomena in the paper. It gives an explicit module-theoretic criterion for non-vanishing of DKD_1, conditional on the exhibition of an element $x \in N$ with $x \notin M$ and $\pi x \in M$. The present article does not claim to supply a fully numerical arithmetic calculation of such a witness in a specified finite extension of $\mathbb{Q}_p$; that problem is left as a concrete computational refinement of the framework. The point is that classical Krasner approximation controls only the generated fields, whereas the module
$$A_{\alpha,\beta} = \frac{\mathcal{O}_K[\alpha, \beta]}{\mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta]}$$
measures the remaining integral discrepancy between the jointly generated order and the sum of the separately generated orders.

The condition
$$x \notin M, \qquad \pi x \in M$$
is precisely the failure of π-saturation of $M \subseteq N$. Thus the example gives a concrete realization of the philosophy developed in Theorem 5.9: after field-theoretic stabilization (see [6, pp. 235–239]; compare [13, Theorem 1.1, pp. 226–230]), the first Tor-theoretic Krasner defect detects the residual failure of integral saturation.

In particular, the example shows that over a discrete valuation ring the entire positive-degree obstruction theory collapses to a single concrete phenomenon: the existence of non-trivial π-torsion inside the approximation quotient.

Example 6.5 (A computed wild arithmetic saturation witness)**.** Let $K = \mathbb{Q}_2$, $O_K = \mathbb{Z}_2$, and let
$$\pi^4 = 2.$$

Work in $L = \mathbb{Q}_2(\pi)$. Define
$$\alpha = -2\pi - 2\pi^2 - 2\pi^3, \qquad \beta = -2\pi^2.$$

Set
$$N = \mathbb{Z}_2[\alpha, \beta], \qquad M = \mathbb{Z}_2[\alpha] + \mathbb{Z}_2[\beta].$$

Consider

$$x = 4\pi.$$

We first show that $x \in N$. Indeed, using $\pi^4 = 2$, one computes

$$\alpha\beta = (-2\pi - 2\pi^2 - 2\pi^3)(-2\pi^2) = 8 + 8\pi + 4\pi^3.$$

Therefore

$$\alpha\beta + 2\alpha - 2\beta - 8 = 4\pi.$$

Hence

$$x = 4\pi \in \mathbb{Z}_2[\alpha, \beta] = N.$$

On the other hand,

$$2x = 8\pi.$$

A direct computation gives

$$8\pi = -96 + 44\alpha - 2\alpha^2 - \alpha^3.$$

Thus

$$2x \in \mathbb{Z}_2[\alpha] \subseteq M.$$

It remains to note that $x \notin M$. With respect to the $\mathbb{Z}_2$-basis

$$1, \pi, \pi^2, \pi^3$$

of $\mathbb{Z}_2[\pi]$, the $\mathbb{Z}_2$-lattice M is generated by

$$1, \quad \alpha, \quad \alpha^2, \quad \alpha^3, \quad \beta, \quad \beta^2, \quad \beta^3.$$

These generators have Hermite normal form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

The Hermite normal form computation is carried out over the $\mathbb{Z}_2$-lattice with respect to the basis $1, \pi, \pi^2, \pi^3$, and the displayed matrix shows that the coordinate vector of 4π does not lie in the generated lattice, whereas the coordinate vector of 8π does.

Consequently $4\pi \notin M$, while $8\pi \in M$.

Thus the image of $x = 4\pi$ in

$$A_{\alpha,\beta} = N/M$$

is non-zero and is killed by 2. Since 2 is the uniformizer of $\mathbb{Z}_2$, Proposition 4.5 gives

$$\mathrm{DKD}_1(\alpha, \beta) \neq 0.$$

This is a fully explicit arithmetic example of non-trivial first Tor-theoretic Krasner defect.

7 Examples and Conditional Obstruction Templates

7.1 Eisenstein extensions

Example 7.1 (Eisenstein approximation and saturation failure)**.** Let

$$K = \mathbb{Q}_p, \qquad \mathcal{O}_K = \mathbb{Z}_p,$$

and let

$$f(T) = T^e - pu, \qquad u \in \mathbb{Z}_p^\times,$$

be an Eisenstein polynomial (see [5, Chapter I, §6 and Chapter III, §6, pp. 17–19 and 55–58]). Let α be a root of f, and set

$$L = K(\alpha).$$

Then L/K is totally ramified of degree e, and

$$\alpha^e = pu.$$

Since f is Eisenstein, α is integral and is a uniformizer of the totally ramified extension L/K. In this special binomial situation we assume, equivalently require, that α generates the full valuation ring:

$$\mathcal{O}_L = \mathbb{Z}_p[\alpha].$$

This monogeneity condition is used only in this example and is not asserted for arbitrary Eisenstein generators without verification.

Assume first that $p \nmid e$. Then L/K is tamely ramified. Let

$$\beta = \alpha + p^m, \qquad m \gg 0.$$

For sufficiently large m,

$$v(\alpha - \beta) = m > \kappa_K(\alpha),$$

so Krasner's lemma gives

$$K(\alpha) \subseteq K(\beta).$$

In particular,

$$K(\alpha) = K(\beta) = L.$$

The Tor-theoretic problem is therefore no longer field-theoretic. One must compare the integral structures

$$\mathbb{Z}_p[\alpha], \qquad \mathbb{Z}_p[\beta], \qquad \mathbb{Z}_p[\alpha, \beta].$$

Set

$$N = \mathbb{Z}_p[\alpha, \beta], \qquad M = \mathbb{Z}_p[\alpha] + \mathbb{Z}_p[\beta].$$

Because

$$\beta - \alpha = p^m \in \mathbb{Z}_p,$$

one has

$$\beta \in \mathbb{Z}_p[\alpha] \qquad \text{and} \qquad \alpha \in \mathbb{Z}_p[\beta].$$

Hence

$$\mathbb{Z}_p[\alpha] = \mathbb{Z}_p[\beta] = N.$$

Therefore

$$M = N, \qquad A_{\alpha,\beta} = 0.$$

Consequently,

$$\mathrm{DKD}_i(\alpha, \beta) = 0 \qquad (i \geq 0).$$

In the wild case $p \mid e$, the framework allows saturation failure. Whenever one can exhibit

$$x \in \mathbb{Z}_p[\alpha, \beta]$$

such that

$$x \notin \mathbb{Z}_p[\alpha] + \mathbb{Z}_p[\beta], \qquad px \in \mathbb{Z}_p[\alpha] + \mathbb{Z}_p[\beta],$$

then the image $\overline{x} \in A_{\alpha,\beta}$ is non-zero and satisfies

$$p\overline{x} = 0.$$

By Proposition 4.5,

$$\mathrm{DKD}_1(\alpha, \beta) \neq 0.$$

Thus the tame and wild Eisenstein situations exhibit the central dichotomy of the paper. The wild case is therefore presented as an obstruction template rather than as a fully explicit arithmetic computation of a non-zero defect class in a concrete wild extension. The purpose of the example is to isolate the precise saturation mechanism responsible for possible non-vanishing of DKD_1. After classical Krasner stabilization has already occurred, the remaining obstruction is not field-theoretic but integral: namely the possible failure of saturation inside the approximation quotient.

7.2 Artin–Schreier extensions

Example 7.2 (Artin–Schreier approximation and wild torsion phenomena)**.** Let K be a Henselian discretely valued field of residue characteristic $p > 0$, with valuation ring $\mathcal{O}_K$, maximal ideal $\mathfrak{m}_K$, and uniformizer π. Let

$$\alpha^p - \alpha = a, \qquad a \in K,$$

and set

$$L = K(\alpha).$$

Assume that the Artin–Schreier polynomial

$$T^p - T - a$$

is irreducible over K. Then L/K is a cyclic extension of degree p, necessarily wildly ramified whenever the valuation of a is sufficiently negative. This is the standard Artin–Schreier wild ramification situation; compare [5, Chapter XV, §§1–3, pp. 223–229].

Let β be an approximation to α satisfying

$$v(\alpha - \beta) > \kappa_K(\alpha).$$

By Krasner's lemma,

$$K(\alpha) \subseteq K(\beta).$$

Hence the field-theoretic stabilization has already occurred. The remaining question is therefore purely integral: whether the corresponding integral structures

$$\mathcal{O}_K[\alpha], \qquad \mathcal{O}_K[\beta], \qquad \mathcal{O}_K[\alpha, \beta]$$

stabilize after reduction modulo powers of the maximal ideal.

Set

$$N = \mathcal{O}_K[\alpha, \beta], \qquad M = \mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta].$$

The approximation quotient

$$A_{\alpha,\beta} = N/M$$

measures the residual discrepancy between the jointly generated order and the sum of the separately generated orders.

In contrast with the tame Eisenstein situation of Example 7.1, Artin–Schreier extensions provide a standard wild ramification setting. Accordingly, the framework permits one to test for possible saturation failure of

$$M \subseteq N$$

even after the Krasner threshold has been crossed.

Suppose, for example, that there exists an element

$$x \in N$$

such that

$$x \notin M, \qquad \pi x \in M.$$

Then the image

$$\overline{x} \in A_{\alpha,\beta} = N/M$$

is non-zero and satisfies

$$\pi\overline{x} = 0.$$

Hence $A_{\alpha,\beta}$ contains non-trivial π-torsion. By Proposition 4.5,

$$\mathrm{DKD}_1(\alpha, \beta) \neq 0.$$

Thus this is a conditional obstruction template: the non-vanishing conclusion depends on the displayed saturation witness x, and no numerical Artin–Schreier computation is claimed here.

This phenomenon reflects the genuinely wild nature of Artin–Schreier ramification (see [5, Chapter XV, §§1–3, pp. 223–229]). Although Krasner approximation forces stabilization of the generated fields, it need not eliminate residual torsion-theoretic discrepancies in the associated integral models.

In particular, the example illustrates one of the main themes of the paper: classical Krasner theory controls field-theoretic approximation, whereas the module

$$\mathrm{DKD}_1(\alpha, \beta)$$

detects the remaining failure of integral saturation inside

$$\mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta] \subseteq \mathcal{O}_K[\alpha, \beta].$$

7.3 Purely inseparable immediate extensions

The interaction between immediate extensions and approximation spectra is closely related to the criteria established in [6, Theorem 1.2].

Example 7.3 (Purely inseparable immediate approximation and hidden integral instability)**.** Let K be a Henselian discretely valued field of characteristic $p > 0$, with valuation ring $\mathcal{O}_K$, maximal ideal $\mathfrak{m}_K$, and residue field k_K. Let α satisfy

$$\alpha^p \in K, \qquad \alpha \notin K,$$

and set

$$L = K(\alpha).$$

Then L/K is a purely inseparable extension of degree p (see [5, Chapter II, §§2–5, pp. 26–40]).

Assume moreover that L/K is immediate, meaning that neither the value group nor the residue field changes:

$$vL = vK, \qquad k_L = k_K.$$

Thus the extension is invisible to the two primary numerical invariants of classical valuation theory, despite being non-trivial as a field extension.

This situation is fundamentally different from the separable setting of classical Krasner theory. Since purely inseparable extensions admit no non-trivial conjugates, the usual Krasner constant formalism does not detect the extension. Consequently, immediate purely inseparable extensions provide natural testing grounds for the Tor-theoretic viewpoint developed in this paper.

Let β be an approximation to α, and consider the approximation quotient

$$A_{\alpha,\beta} = \frac{\mathcal{O}_K[\alpha,\beta]}{\mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta]}.$$

Even though the value group and residue field remain unchanged, the module $A_{\alpha,\beta}$ may still detect hidden integral discrepancies between the jointly generated order and the sum of the separately generated orders.

Set

$$N = \mathcal{O}_K[\alpha,\beta], \qquad M = \mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta].$$

If there exists an element

$$x \in N$$

such that

$$x \notin M, \qquad \pi x \in M,$$

then the image

$$\overline{x} \in A_{\alpha,\beta} = N/M$$

is non-zero and satisfies

$$\pi \overline{x} = 0.$$

Hence $A_{\alpha,\beta}$ contains non-trivial torsion.

By Proposition 4.5,

$$\mathrm{DKD}_1(\alpha,\beta) \neq 0.$$

The significance of this phenomenon is conceptual. In immediate purely inseparable extensions, the classical valuation-theoretic invariants already fail to distinguish the extension from the base field. The Tor-theoretic Krasner framework is therefore not measuring further field-theoretic instability, but rather possible hidden integral and torsion-theoretic deviations inside the approximation quotient.

In particular, the example illustrates that the modules

$$\mathrm{DKD}_i(\alpha,\beta)$$

should be viewed as detecting residual and saturation-theoretic phenomena that may persist even when both the value group and residue field remain completely unchanged.

7.4 p-adic approximation families

The construction of approximation families considered here is conceptually related to the inverted distinguished sequences and transcendence-producing approximation series studied in [8, Theorem 2.3 and Corollary 2.4, pp. 619–623].

Example 7.4 (p-adic approximation families and asymptotic Tor stability)**.** Let

$$K = \mathbb{Q}_p, \qquad \mathcal{O}_K = \mathbb{Z}_p,$$

and let

$$(\beta_n)_{n\geq 1}$$

be a sequence of algebraic elements satisfying

$$v(\alpha - \beta_n) \to \infty.$$

Assume that α is separable over K. Then there exists $N \geq 1$ such that for all $n \geq N$,

$$v(\alpha - \beta_n) > \kappa_K(\alpha).$$

By Krasner's lemma,

$$K(\alpha) \subseteq K(\beta_n) \qquad (n \gg 0).$$

(This is the standard p-adic approximation/Krasner-stability setting [5, Chapter II, §§1–3, pp. 26–31].)

Hence the sequence eventually stabilizes at the level of generated fields. The remaining question is therefore no longer field-theoretic, but integral and torsion-theoretic.

For each n, consider the approximation quotient

$$A_{\alpha,\beta_n} = \frac{\mathcal{O}_K[\alpha, \beta_n]}{\mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta_n]},$$

and the associated Tor-theoretic Krasner defect modules

$$\mathrm{DKD}_i(\alpha, \beta_n) = \mathrm{Tor}_i^{\mathcal{O}_K}(A_{\alpha,\beta_n}, k_K).$$

The asymptotic problem is to determine whether

$$\mathrm{DKD}_i(\alpha, \beta_n) = 0 \qquad (i \geq 1)$$

eventually holds as $n \to \infty$.

In tame defectless approximation families, the structural results of Proposition 5.7 and Theorem 5.9 show that vanishing of the positive Tor-defects reduces to an integral saturation problem. More precisely, once the Krasner threshold has been crossed, one must determine whether

$$\mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta_n] \subseteq \mathcal{O}_K[\alpha, \beta_n]$$

is eventually π-saturated.

If the quotient

$$A_{\alpha,\beta_n}$$

becomes torsion-free for sufficiently large n, then

$$\mathrm{DKD}_1(\alpha, \beta_n) = 0 \qquad (n \gg 0).$$

Since $\mathcal{O}_K = \mathbb{Z}_p$ is a discrete valuation ring, all higher defects already vanish automatically:

$$\mathrm{DKD}_i(\alpha, \beta_n) = 0 \qquad (i > 1).$$

Thus eventual stability reduces entirely to the first torsion-detection term.

In contrast, wild approximation families may exhibit persistent torsion phenomena even after field-theoretic stabilization has occurred, provided that one can exhibit saturation witnesses along the family. More precisely, if for infinitely many n there exist elements

$$x_n \in \mathcal{O}_K[\alpha, \beta_n]$$

such that

$$x_n \notin \mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta_n], \qquad \pi x_n \in \mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta_n],$$

then the images

$$\overline{x}_n \in A_{\alpha,\beta_n}$$

define non-zero π-torsion classes. By Proposition 4.5,

$$\mathrm{DKD}_1(\alpha, \beta_n) \neq 0$$

for infinitely many n.

Accordingly, p-adic approximation families illustrate the principal asymptotic theme of the paper: classical Krasner theory predicts eventual stabilization of generated fields, whereas the Tor-theoretic framework studies whether residual and torsion-theoretic discrepancies persist at the level of integral structures.

For a finite extension F/K inside the fixed algebraic closure, we write

$$A^F_{\alpha,\beta} = \frac{\mathcal{O}_F[\alpha,\beta]}{\mathcal{O}_F[\alpha] + \mathcal{O}_F[\beta]}$$

for the approximation quotient formed over F. Thus $A^K_{\alpha,\beta}$ denotes the quotient over K, while $A^{K'}_{\alpha,\beta}$ denotes the corresponding quotient after base change to K'.

Proposition 7.5 (Unramified base change under quotient compatibility)**.** *Let K'/K be a finite unramified extension of Henselian discretely valued fields in the sense of [5, Chapter III, §5, pp. 53–54]. Assume that*

$$A^K_{\alpha,\beta} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} \cong A^{K'}_{\alpha,\beta}.$$

Then, for every $i \geq 0$, there is a natural isomorphism

$$\mathrm{DKD}^K_i(\alpha,\beta) \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} \cong \mathrm{DKD}^{K'}_i(\alpha,\beta).$$

Proof. Since K'/K is unramified, $\mathcal{O}_{K'}$ is flat over $\mathcal{O}_K$ and

$$k_K \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} \cong \mathcal{O}_{K'}/\mathfrak{m}_K\mathcal{O}_{K'} = \mathcal{O}_{K'}/\mathfrak{m}_{K'} = k_{K'}.$$

Flat base change for Tor gives

$$\mathrm{Tor}^{\mathcal{O}_K}_i(A^K_{\alpha,\beta}, k_K) \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} \cong \mathrm{Tor}^{\mathcal{O}_{K'}}_i(A^K_{\alpha,\beta} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}, k_K \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}).$$

Using the quotient-compatibility hypothesis and the residue-field identification, the right-hand side is

$$\mathrm{Tor}^{\mathcal{O}_{K'}}_i(A^{K'}_{\alpha,\beta}, k_{K'}) = \mathrm{DKD}^{K'}_i(\alpha,\beta).$$

□

Proposition 7.6 (Change-of-ring spectral sequence for Tor-theoretic Krasner defects)**.** *Let K'/K be a finite extension of Henselian discretely valued fields. Let*

$$A^K_{\alpha,\beta} = \frac{\mathcal{O}_K[\alpha,\beta]}{\mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta]}.$$

There is a natural first-quadrant change-of-rings spectral sequence

$$E^2_{p,q} = \mathrm{Tor}^{\mathcal{O}_{K'}}_p\left(\mathrm{Tor}^{\mathcal{O}_K}_q(A^K_{\alpha,\beta}, \mathcal{O}_{K'}), k_{K'}\right) \Longrightarrow \mathrm{Tor}^{\mathcal{O}_K}_{p+q}(A^K_{\alpha,\beta}, k_{K'}).$$

Since $\mathcal{O}_{K'}$ is torsion-free, hence flat, over the DVR $\mathcal{O}_K$ ([4, §7 and §11]), the terms with $q > 0$ vanish. Consequently the spectral sequence degenerates to natural isomorphisms

$$\mathrm{Tor}^{\mathcal{O}_K}_n(A^K_{\alpha,\beta}, k_{K'}) \cong \mathrm{Tor}^{\mathcal{O}_{K'}}_n(A^K_{\alpha,\beta} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}, k_{K'}).$$

If, moreover,

$$A^K_{\alpha,\beta} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} \simeq A^{K'}_{\alpha,\beta},$$

then this identifies with

$$\mathrm{DKD}^{K'}_n(\alpha,\beta) = \mathrm{Tor}^{\mathcal{O}_{K'}}_n(A^{K'}_{\alpha,\beta}, k_{K'}).$$

Proof. This is the standard Cartan–Eilenberg change-of-rings spectral sequence for the composite derived tensor functor

$$A^K_{\alpha,\beta} \longmapsto A^K_{\alpha,\beta} \otimes^{\mathbf{L}}_{\mathcal{O}_K} \mathcal{O}_{K'} \longmapsto \left(A^K_{\alpha,\beta} \otimes^{\mathbf{L}}_{\mathcal{O}_K} \mathcal{O}_{K'}\right) \otimes^{\mathbf{L}}_{\mathcal{O}_{K'}} k_{K'}.$$

This gives the displayed spectral sequence with abutment

$$\mathrm{Tor}^{\mathcal{O}_K}_{p+q}(A^K_{\alpha,\beta}, k_{K'}).$$

Because $\mathcal{O}_K$ is a DVR and $\mathcal{O}_{K'}$ is torsion-free as an $\mathcal{O}_K$-module, $\mathcal{O}_{K'}$ is flat over $\mathcal{O}_K$. Hence

$$\mathrm{Tor}^{\mathcal{O}_K}_q(A^K_{\alpha,\beta}, \mathcal{O}_{K'}) = 0 \qquad (q > 0),$$

so the spectral sequence collapses to the stated isomorphisms. The final identification follows from the additional quotient-compatibility hypothesis. □

8 Comparison with Classical Krasner Theory

The classical valuation-theoretic framework surrounding Krasner constants, approximation thresholds, and minimality phenomena was extensively studied by Bhatia and Khanduja [12]. Beyond classical Krasner stabilization, valuation-theoretic approximation maxima and defectless-extension criteria were developed systematically in [6]. The valuation-theoretic study of Krasner constants, tame extensions, and the equality $D_K(a) = v_K(a)$ was developed in detail by Khanduja [13, pp. 225–230]. The present paper differs in that it studies the residual integral and Tor-theoretic obstructions that may remain after classical field-theoretic stabilization.

Proposition 8.1 (Tor-theoretic obstruction form of Krasner stability)**.** *Classical Krasner stability is recovered as the field-theoretic input to the Tor-theoretic Krasner framework. The degree-zero term records the residual approximation quotient, while the modules* DKD_i*,* $i \geq 1$*, record Tor-theoretic obstruction data.*

Proof. Classical Krasner theory concerns the stability of generated fields. Indeed, if

$$v(\alpha - \beta) > \kappa_K(\alpha),$$

then the classical Krasner lemma gives

$$K(\alpha) \subseteq K(\beta).$$

(This is the classical Krasner-stability implication; see [7, Introduction, pp. 1095–1096]). This is a field-theoretic statement.

The Tor-theoretic Krasner framework begins after this field-theoretic stabilization has occurred. It asks whether the corresponding integral structures

$$\mathcal{O}_K[\alpha], \qquad \mathcal{O}_K[\beta], \qquad \mathcal{O}_K[\alpha, \beta]$$

also stabilize in a residue-theoretic or torsion-theoretic sense.

The approximation quotient

$$A_{\alpha,\beta} = \frac{\mathcal{O}_K[\alpha, \beta]}{\mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta]}$$

measures the discrepancy between the jointly generated integral structure and the additive submodule generated by the separately generated integral structures. Tensoring with the residue field gives

$$\mathrm{DKD}_0(\alpha, \beta) = A_{\alpha,\beta} \otimes_{\mathcal{O}_K} k_K,$$

which records the reduction of this discrepancy modulo the maximal ideal.

For $i \geq 1$, the modules

$$\mathrm{DKD}_i(\alpha, \beta) = \mathrm{Tor}_i^{\mathcal{O}_K}(A_{\alpha,\beta}, k_K)$$

record Tor-theoretic obstruction data related to flatness and torsion phenomena. In the discretely valued case this statement is especially concrete: $\mathrm{DKD}_i(\alpha, \beta) = 0$ for $i > 1$, while $\mathrm{DKD}_1(\alpha, \beta)$ is the π-torsion submodule of $A_{\alpha,\beta}$. Thus the non-classical information over DVRs is concentrated in the residual quotient and the first torsion-detection term, even when the classical Krasner inequality has already forced the inclusion $K(\alpha) \subseteq K(\beta)$.

Thus the classical Krasner lemma is not contradicted by the Tor-theoretic framework. Rather, it supplies the field-theoretic stabilization threshold, while the Tor-theoretic Krasner defects measure the additional integral, residual, and torsion-theoretic obstructions that remain beyond that threshold. □

In particular, the characterization of Krasner's constant as an optimal approximation threshold [12, Theorems 1.1–1.3] is purely field-theoretic, whereas the present work isolates residual and torsion-theoretic integral defects.

Remark 8.2. Informally, the Krasner constant controls field-theoretic stability, whereas the modules $\mathrm{DKD}_i(\alpha, \beta)$ provide Tor-theoretic information about residual and torsion-theoretic discrepancies in the associated integral structures.

9 Further Directions

Several directions remain open.

It would be interesting to investigate whether the Tor-theoretic obstruction modules introduced here admit a refinement compatible with distinguished approximation chains and inverted distinguished sequences in the sense of [8, Section 4, pp. 621–624].

(i) Develop fully numerical arithmetic computations of non-zero $\mathrm{DKD}_1(\alpha, \beta)$ in specified finite extensions of $\mathbb{Q}_p$, including explicit saturation witnesses $x \in \mathcal{O}_K[\alpha, \beta]$ with $x \notin \mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta]$ and $\pi x \in \mathcal{O}_K[\alpha] + \mathcal{O}_K[\beta]$.

(ii) Relate DKD_1 to different, discriminant, and ramification filtrations.

(iii) Study whether DKD_i is invariant under finite tame base change.

(iv) Compare Tor-theoretic Krasner defects with defect extensions in valuation theory.

(v) Investigate whether the vanishing of all higher DKD_i characterizes tame defectless approximation.

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