Prime-to-p Isogenies, Component Groups, and Cohomological Inertia for Néron Models

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Abstract

We study Néron models in arithmetic families over regular integral bases of finite type over $S = \operatorname{Spec}\mathbb{Z}$, with $\operatorname{Spec}\mathbb{Z}$ as the basic motivating case.

(1) *Isogeny extension and control of component groups.* Prime-to-residue-characteristic isogenies are controlled on a fixed tame codimension-one locus. Their restrictions to identity components have finite étale kernels killed by the isogeny degree, and the induced morphisms on component groups have kernel and cokernel killed by a bounded power of that degree. Consequently, the associated Tamagawa-index defect is bounded in the stated finite-isogeny family.

(2) *Semicontinuity.* For $p \nmid m$, the function $t \mapsto v_p(\#\Phi_{A,t})$ is upper semicontinuous on a common dense open locus. Across degree-m prime-to-p isogenies, the induced component-group maps have kernel and cokernel killed by m; hence their orders divide a bounded power of m.

(3) *Cohomology and conductors.* On the tame locus, the inertia-invariant subspaces of $H^1_{\text{ét}}$ are invariant under prime-to-residue-characteristic isogeny. Since isogenous abelian varieties have isomorphic rational ℓ-adic Tate modules, the Artin conductor exponent itself is unchanged under the isogenies considered here. The component-group estimates in this paper control only the associated local geometric defect terms and Tamagawa-index data.

(4) *Hecke uniformity.* For integral models of modular curves $X_0(N)$, after restricting to the dense tame locus where the prime-to-p Hecke correspondence T_ℓ, $\ell \nmid Np$, extends through genuine prime-to-residue-characteristic isogenies, the induced maps are finite étale on identity components and act on component groups with kernel and cokernel annihilated by a bounded power of ℓ. Consequently, toric ranks, dimensions of inertia-invariant cohomology, and component-group defect terms are uniformly controlled along the corresponding Hecke orbit on this locus.

In the Hodge-type Shimura setting, the analogous statement is conditional on the chosen integral-model and extension hypotheses: the relevant prime-to-p Hecke correspondence must extend over a dense open as a finite correspondence whose induced maps on the abelian scheme are genuine prime-to-residue-characteristic isogenies with finite étale kernels. Under these hypotheses, the same component-group and inertia-invariant cohomology conclusions follow.

(5) *Heights.* On the semistable locus, for abelian schemes equipped with a fixed symmetric ample line bundle, the line of invariant differentials $\omega_{A/S}$ enters the standard Arakelov-theoretic decomposition of canonical heights. The paper records compatibility of these local height contributions with the tame geometric data controlled above. It does not claim a new uniform positive lower bound for all non-torsion points in an arbitrary prime-to-Σ isogeny class.

MSC 2020: Primary 14K15, 11G10; Secondary 14L15, 14F20, 11G50, 11F80, 14G35, 14G40

Keywords: Néron model, component group, Tamagawa number, prime-to-p isogeny, semicontinuity, étale cohomology, inertia invariants, Artin conductor, Hecke correspondence (modular curves; Hodge-type under hypotheses), modular curve, Shimura variety

1 Introduction and Main Results

Motivation

Néron models sit at the interface of algebraic geometry and arithmetic: they extend abelian varieties across nonarchimedean places while preserving the universal mapping property that governs integral points and specialization. Over the arithmetic base $S = \operatorname{Spec}(\mathbb{Z})$ (see Notation 2.1), they encode reduction data through the identity component A^0, the component groups Φ_p at primes p, and the invariant differential line $\omega_{A/S}$ (Definition 2.6 and construction 2.10). These geometric pieces control the shape of local Euler factors, conductor exponents, and Tamagawa numbers (Definition 2.13 and proposition 2.14), and interact with global height theory, the Mordell–Weil group $A(\mathbb{Q})$, and rational points on curves mapping to A. While the foundational existence and formal properties are classical—see [3] for Néron models and component groups, [5] for vanishing cycles and monodromy formalism, [9] for degenerations of abelian varieties, [10] for standard facts on abelian varieties, [11] for good reduction phenomena, [12] for the basic local harmonic-analysis background used in Euler factors, and [13] for a detailed modern discussion of Néron-model techniques—arithmetic applications often require carefully organized tame prime-to-residue-characteristic control across selected arithmetic families—especially along finite isogeny and Hecke correspondences on integral models of modular/Shimura varieties. This paper records such control on explicit dense opens and separates the resulting component-group and inertia-invariant consequences from the standard background input on Néron models, Tate modules, and height functoriality.

Global framing and scope. Although $\operatorname{Spec}\mathbb{Z}$ serves as the basic arithmetic base and motivating example, all results asserting variation, semicontinuity, or uniform bounds are formulated for abelian schemes or Néron models varying in arithmetic families A/T with T a regular integral scheme of finite type over $\operatorname{Spec}\mathbb{Z}$. No theorem claims the existence of infinitely many additive fibers for a fixed abelian variety over $\mathbb{Q}$.

Contributions. The paper isolates a tame prime-to-residue-characteristic locus on which isogenies of Néron models admit uniform control on identity components and component groups. The main contributions are: (i) extension and uniqueness of prime-to-p isogenies in the stated codimension-one/Dedekind sense; (ii) uniform m-torsion control of kernels and cokernels on component groups for a fixed finite class of degree-m isogenies; (iii) compatibility of inertia-invariant ℓ-adic cohomology with this tame isogeny control; (iv) conditional Hecke-orbit uniformity on integral modular and Hodge-type loci where the relevant correspondences extend as genuine prime-to-p isogenies; and (v) compatibility, but not a new global height gap, for the standard local height contributions attached to $\omega_{A/S}$.

Outline of results

The logical flow is as follows; each entry records the downstream arithmetic consequences and a representative example.

- **Existence/uniqueness in arithmetic families.** *Theorem 3.1 packages the standard codimension-one Néron-model extension and functoriality facts needed later for prime-to-p isogenies. Its role is to fix the precise tame/Dedekind scope in which component-group maps are controlled; it is not used as a new global Néron-model theorem over higher-dimensional bases.*
- **Component groups and upper semicontinuity.** Theorem 3.5 constructs a common tame open $U \subseteq T^{\circ}$ on which the relevant component-group sheaf is finite étale, so that $t \mapsto v_q(\#\Phi_{A,t})$ is locally constant and hence formally upper semicontinuous. The real content is the common-open and kernel–cokernel control for the fixed finite class of prime-to-residue-characteristic isogenies. Lemma 3.8 records only constructibility beyond this tame open.
- **Cohomological invariants under base change and isogeny.** *In Theorem 4.1 we compare $H^1_{\text{ét}}(\cdot)^{I_p}$ across the family and along isogenies (Items (i) to (iii)), implying the Euler quotient bound Proposition 6.1. This is illustrated by Examples 4.4 and 4.5.*
- **Hecke uniformity (precise scope).** For modular curves, T_ℓ with $\ell \nmid Np$ acts by ℓ–power, prime-to-p isogenies on Néron models over a dense open with uniform control of identity components, component groups, and local factor shapes (Theorem 5.3 and corollary 5.5). For Hodge-type Shimura varieties, an analogous conclusion holds only under the explicit integral-model hypotheses and after removing primes where the level is ramified; see Theorem 5.10 and remark 5.4 for the exact conditions.
- **Heights.** Propositions 4.6 and 6.5 record compatibility of the standard semistable local height contributions, including those involving $\omega_{A/S}$, with the tame geometric data controlled earlier. No new global uniform positive lower bound for canonical heights across arbitrary isogeny classes is asserted.

2 Preliminaries

Throughout, we work over the base scheme $\operatorname{Spec}\mathbb{Z}$. Unless otherwise indicated, all schemes are separated and of finite type over their base. For general references: we use [7] for scheme-theoretic conventions; [6] for étale cohomology; [4, 5] for the vanishing-cycles and monodromy formalism; [3] for Néron models, component groups, and smoothening; and [9, 10] for background on abelian varieties and their degenerations.

All background results stated in this section are standard; any novel contributions begin in later sections.

2.1 Schemes and the arithmetic base $\operatorname{Spec}(\mathbb{Z})$

Notation 2.1 (Arithmetic base)**.** We denote by $S = \operatorname{Spec}\mathbb{Z}$ the base scheme. For a prime p we write $S_p = \operatorname{Spec}\mathbb{Z}_{(p)}$ and $\kappa(p) = \mathbb{F}_p$ for its residue field. The generic point is $\eta = \operatorname{Spec}\mathbb{Q}$. If X/S is a scheme, we denote by X_η the generic fiber and by X_p the special fiber over $\kappa(p)$.

Notation 2.2 (Indices for points and primes)**.** Throughout the paper we *reserve* the following symbols:

- t denotes a *closed point* of the base (e.g. $t \in S$ or $t \in T$); its residue characteristic is $p = \operatorname{char}\kappa(t)$.
- p denotes the *residue characteristic* attached to a closed point t as above.
- q denotes an *auxiliary rational prime* used only as a valuation index in expressions like $v_q(\#\Phi_{A,t})$.

This matches the usage in Theorem 3.5 and avoids conflicts where the letter p appears simultaneously as a closed point and as a residue characteristic.

Remark 2.3 (Standing scope and hypothesis convention). Throughout the paper we repeatedly shrink to dense opens. To make every later statement checkable at the level of hypotheses, we adopt the following convention.

(S1) Codimension-1 interpretation when $\dim T > 1$**.** If T is regular of dimension > 1, any invocation of the Néron mapping property and any "extension to a morphism of Néron models" is to be read *fiberwise over codimension-*1 *points* (i.e. after restricting to the strict henselization of the DVR at each codimension-1 point).

(S2) Fixed tame/étale locus for a chosen degree m**.** Whenever an integer $m \geq 1$ is fixed, we shrink to a dense open on which: (i) m is invertible, and (ii) the Néron model is semiabelian (semistable) so that vanishing-cycle formalism applies. We then further shrink to a dense open $U = U(m)$ on which every degree-m prime-to-residue-characteristic isogeny considered has kernel extending to a *finite étale* subgroup scheme (equivalently, all such kernels are finite étale on U).

(S3) Quantifiers. Accordingly, phrases such as "for every degree-m isogeny" always mean: *for every degree-m isogeny whose kernel is finite étale over the fixed open $U = U(m)$.* Whenever U is not the same as the one fixed earlier, it will be explicitly renamed.

Remark 2.4 (Global vs. local arithmetic). The structure of X over $\operatorname{Spec}\mathbb{Z}$ reflects both global information (e.g. height pairings, discriminants) and local invariants (e.g. component groups at p). Connecting these perspectives is central for the arithmetic applications developed later.

Lemma 2.5 (Flat base change for smooth morphisms)**.** *Let $X \to S$ be smooth and $S' \to S$ flat. Then the base change $X \times_S S' \to S'$ is smooth, and formation of the sheaf of differentials $\Omega^1_{X/S}$ commutes with base change.*

Proof. This is standard, see [7]. ☐

2.2 Néron models: definitions and properties

Definition 2.6 (Néron model)**.** Let K be the function field of S and A/K an abelian variety. A *Néron model* of A over S is a smooth separated group scheme $\mathcal{A}/S$ of finite type such that $\mathcal{A}_\eta \cong A$ and satisfying the *Néron mapping property*: for every smooth S-scheme Y and every K-morphism $f_\eta : Y_\eta \to \mathcal{A}_\eta$, there exists a unique S-morphism $f : Y \to \mathcal{A}$ extending f_η.

Lemma 2.7 (Existence of Néron models)**.** *If A/K is an abelian variety, then a Néron model $\mathcal{A}/S$ exists and is unique up to unique isomorphism.*

Proof. This is classical, see [3]. ☐

Proposition 2.8 (Component groups)**.** *Let $\mathcal{A}/S$ be a Néron model of an abelian variety A/K. For each closed point $p \in S$, denote by $\Phi_p = \mathcal{A}_p/\mathcal{A}_p^0$ the component group of the special fiber. Then Φ_p is a finite étale group scheme over $\kappa(p)$, and the Tamagawa number at p is given by $c_p = \#\Phi_p(\kappa(p))$.*

Proof. See [3]. ☐

Remark 2.9 (Relation to reduction type). Good reduction corresponds to Φ_p trivial, while semistable reduction means that the identity component $(\mathcal{A}_p)^0$ of the special fiber is *semiabelian* (in particular, it contains a toric part). The component group $\Phi_p := \mathcal{A}_p/(\mathcal{A}_p)^0$ is always a finite étale group scheme over $\kappa(p)$, and remains finite étale in both good and semistable reduction. Thus:

$$\text{Good reduction} \;\Rightarrow\; \Phi_p \text{ trivial},$$

and

$$\text{Semistable reduction} \;\Rightarrow\; (\mathcal{A}_p)^0 \text{ semiabelian (with torus part)},$$

$$\text{while } \Phi_p \text{ is finite étale.}$$

This corrects the common shorthand "Φ_p a torus," since the toric part lies in the identity component rather than in the component group.

Construction 2.10 (Sheaf of differentials)**.** Let $\mathcal{A}/S$ be a Néron model. Define $\omega_{\mathcal{A}/S} = e^*(\Omega^1_{\mathcal{A}/S})$, the pullback of the sheaf of differentials along the identity section $e : S \to \mathcal{A}$. This invertible sheaf controls height pairings and appears in Arakelov intersection theory.

2.3 Cohomological background

Lemma 2.11 (Localization for abelian schemes; semistable Raynaud description)**.** *Let p be a rational prime, $K = \mathbb{Q}_p$, $\mathbb{Z}_p$ its ring of integers, and $\mathbb{F}_p$ its residue field.* (Proper smooth case.) *If $\mathcal{A}/\mathbb{Z}_p$ is an **abelian scheme** (i.e. proper and smooth), then for every prime $\ell \neq p$ and every $i \geq 0$ there is a canonical specialization isomorphism*

$$H^i_{\text{ét}}(\mathcal{A}_{\overline{K}}, \mathbb{Q}_\ell) \;\cong\; H^i_{\text{ét}}(\mathcal{A}_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell),$$

functorial and compatible with G_K. (Semistable case.) *If A/K has semistable reduction and $\mathcal{A}^0_{\overline{\mathbb{F}}_p}$ fits into the Raynaud extension*

$$0 \to T \to \mathcal{A}^0_{\overline{\mathbb{F}}_p} \to B \to 0,$$

with T a torus of rank τ_p and B an abelian variety of dimension a, then

$$\dim_{\mathbb{Q}_\ell} V_\ell(A)^{I_p} = 2a + \tau_p = 2g - \tau_p.$$

This statement concerns the I_p-invariants of the generic-fiber Tate module. It should not be read as an unrestricted comparison theorem for étale cohomology of an arbitrary non-proper Néron model.

Proof. In the proper and smooth (abelian–scheme) case, the claimed specialization isomorphism is the standard one furnished by smooth and proper base change for $\mathcal{A}/\mathbb{Z}_p$ together with the Néron–Ogg–Shafarevich criterion for $\ell \neq p$ ([3], [4, 5]).

In the semistable case, the asserted formula is the standard Raynaud-extension description of the inertia invariants of the generic-fiber Tate module. More precisely, when the identity component of the special fiber fits into $0 \to T \to$

$\mathcal{A}^0_{\overline{\mathbb{F}}_p} \to B \to 0$, the invariant part has dimension $2\dim B + \operatorname{rank} T = 2g - \tau_p$. This is only a statement about $V_\ell(A)^{I_p}$ in the semistable/Raynaud-extension setting, not a nearby-cycle comparison theorem for arbitrary non-proper Néron models. □

Remark 2.12 (Scope clarification for Lemma 2.11). Lemma 2.11 combines proper–smooth specialization for abelian schemes with the standard semistable Raynaud-extension calculation of $V_\ell(A)^{I_p}$. It is not used as a general nearby-cycle identification for the étale cohomology of arbitrary non-proper Néron models.

Definition 2.13 (Conductor exponent)**.** Let A/K be an abelian variety with Néron model $\mathcal{A}/S$. For a prime p, the conductor exponent f_p is defined via the Artin conductor of the ℓ-adic Tate module $T_\ell(A)$. Equivalently, f_p measures the Swan conductor of the inertia action on $H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_\ell)$ [5].

Proposition 2.14 (Discriminant and conductor)**.** *For elliptic curves, the minimal discriminant Δ and the conductor N satisfy*

$$\operatorname{ord}_p(\Delta) \geq f_p,$$

with equality for semistable reduction.

Proof. This follows from Tate's algorithm and the analysis of Kodaira types [12]. □

Remark 2.15 (No unboundedness assertion for additive fibers). We do not use any assertion that infinitely many additive fibers force unbounded component groups. In small dimensions, especially for elliptic curves, additive Kodaira component groups are bounded in small finite sets. Any growth statement in this paper concerns explicitly specified families and explicitly tracked invariants, not a general unboundedness theorem for additive reduction.

3 Structural Properties of Néron Models

We fix the base and notation from Notation 2.1. All background on existence, identity components, and component groups is in Definition 2.6, lemma 2.7, proposition 2.8, and remark 2.9. In this section we establish structural results for Néron models in arithmetic families and under isogeny, and we record consequences for the invariants tracked in Definition 2.13 and proposition 2.14 and the explicitly tracked component-group and local-defect invariants.

3.1 Existence and uniqueness in arithmetic families

Theorem 3.1 (Codimension-one Néron-model control and tame isogeny functoriality)**.** *Let $A_{\eta_T}/K(T)$ be an abelian variety. After replacing T by a nonempty open subscheme T°, the following assertions hold in the Dedekind case globally, and in higher dimension only after restriction to codimension-one DVRs. No global Néron model over a higher-dimensional base is asserted.*

Scope for (i)–(iii). *The global Néron mapping property holds on T° only when $\dim T = 1$ (Dedekind/regular). For general regular T of higher dimension, statements* (i)–(iii) *are to be read* fiberwise over codimension-1 points *(equivalently, on strict henselizations of DVRs), i.e. as assertions about the Néron* lft-*model in codimension* 1.

(i) ***(Relative Néron mapping property; global in*** $\dim T = 1$***, fiberwise otherwise).***

- *If $\dim T = 1$ (Dedekind/regular), then for every smooth T°–scheme Y and every $K(T)$–morphism $Y_{\eta_T} \to A_{\eta_T}$ there is a unique T°–morphism $Y \to \mathcal{A}_{T^\circ}$ extending it; thus $\mathcal{A}_{T^\circ}$ is the Néron model of A_{η_T} over T°.*
- *For general regular T, the same mapping property holds* for each codimension-1 point *of T° (i.e. after restricting to the local Dedekind base); in particular $\mathcal{A}_{T^\circ}$ is a Néron* lft-*model in codimension* 1.

(ii) ***(Minimality among smooth separated models; codimension-one meaning in higher dimension)*** *If* $\dim T = 1$*, then for any smooth separated group scheme $G \to T^\circ$ whose generic fiber is isomorphic to A_{η_T}, there exists a unique T°-morphism $G \to \mathcal{A}_{T^\circ}$ restricting to the identity on the generic fiber. For general regular T of higher dimension, this assertion is made only after restriction to codimension-one DVRs; no global minimality statement over the higher-dimensional base T° is asserted.*

(iii) ***(Compatibility with base change in codimension*** ≥ 1***).*** *If* $\dim T = 1$*, then for any regular dominant $T' \to T^\circ$ the pullback $(\mathcal{A}_{T^\circ})_{T'}$ is the Néron model of $(A_{\eta_T})_{K(T')}$ over T'. For general regular T, this compatibility holds* after restriction to codimension-1 points *of T' (hence along all DVR bases).*

Hypotheses for (iv). *Fix an integer $m \geq 1$. After possibly shrinking T°, assume that*

- *(H1) every fiber of $\mathcal{A}_{T^\circ} \to T^\circ$ is semiabelian (i.e. A has semistable reduction on T°), and*
- *(H2) for the degree-m isogeny ϕ_{η_T} considered in* (iv)*, the kernel scheme $\ker(\phi_{\eta_T})$ is finite étale of order m (equivalently, m is invertible on T° and the kernel extends to a finite étale subgroup scheme on T°).*

Under (H1)–(H2)*, the vanishing-cycle formalism of [5] applies and the induced morphism on component groups is well defined.*

(iv) ***(Isogeny functoriality and component-group control on the tame prime-to-p locus)***

Assume (H1)–(H2) *and let $\phi_{\eta_T} : A_{\eta_T} \to B_{\eta_T}$ be an isogeny of degree m prime to all residue characteristics on T°.*

Scope. *The extension of an isogeny to a morphism of Néron models in (iv) is guaranteed only in* *the* prime-to-residue-characteristic, semistable range *and over* Dedekind *(dimension-one regular) bases, where the Néron mapping property of (i) applies. Over higher-dimensional regular schemes T, the mapping property holds only fiberwise on codimension-one points, so the extended morphism should be interpreted in that restricted sense. In particular, the control of kernels and cokernels on $\Phi_{A,t} \to \Phi_{B,t}$ depends on this semistable prime-to-p hypothesis; without it, vanishing-cycle arguments and the functoriality of the Néron mapping property are not available globally.*

Moreover, after shrinking T° as in Theorem 3.5, all conclusions above hold uniformly over the fixed open $U \subseteq T^\circ$ of Notation 3.6.

Let $\varphi_{\eta_T} : A_{\eta_T} \to B_{\eta_T}$ be an isogeny of degree m prime to all residue characteristics on T°.

Then, in the above Dedekind/codimension-one sense, φ_{η_T} extends uniquely to a morphism of the corresponding Néron models. For every closed point t in the fixed tame locus:

- *the induced morphism on identity components*

$$(A_t)^0 \longrightarrow (B_t)^0$$

  is an isogeny of semiabelian varieties whose kernel is finite étale and killed by m. In the abelian-scheme case its degree divides m.

- *the induced morphism on component groups*

$$\Phi_{A,t} \longrightarrow \Phi_{B,t}$$

  has kernel and cokernel killed by m, hence their orders divide a power of m. No assertion is made that their orders divide m, and no divisibility formula for $c_{B,t}/c_{A,t}$ is claimed without a separate finite-group calculation.

Remark 3.2 (Scope of Theorem 3.1). The global Néron mapping property is used only over Dedekind bases. For higher-dimensional regular bases, all invocations are understood after restriction to codimension-one DVRs, as fixed in Remark 2.3.

Scope. The ingredients used in this theorem are standard consequences of Néron-model functoriality, semistable reduction, and finite-étale prime-to-residue-characteristic isogeny theory. The contribution here is not a new existence theorem for Néron models over higher-dimensional bases, but the precise packaging of these standard inputs on a fixed tame codimension-one locus in a form suitable for the component-group, cohomological, and Hecke-uniformity applications below.

Proof. Choose a model of A_{η_T} over some open $U \subseteq T$ and apply the Néron smoothening process *fiberwise in codimension* 1. After shrinking to $T^\circ \subseteq U$, this yields a smooth separated group scheme $A_{T^\circ} \to T^\circ$ satisfying the Néron mapping property *globally if* $\dim T = 1$ *(Dedekind/regular), and otherwise fiberwise over the codimension-*1 *points of* T ([3]).

For any regular dominant $T' \to T^\circ$, flat base change for smooth morphisms (Lemma 2.5) preserves smoothness and, in the Dedekind case $\dim T = 1$, the global mapping property. In higher dimension, the same argument applies fiberwise in codimension 1, yielding (iii) in that sense.

For (iv), let φ_{η_T} be an isogeny of degree m with $(m, \operatorname{char} \kappa(t)) = 1$ for all $t \in T^\circ$. Functoriality of the Néron mapping property ([3]) yields a unique extension $\varphi\colon A_{T^\circ} \to B_{T^\circ}$. On the fixed tame locus the kernel on the identity component is finite étale and killed by m. Passing to the special fiber and using the standard component-group sequence for morphisms of semiabelian/Néron models over DVRs, one obtains an induced map

$$\Phi_{A,t} \longrightarrow \Phi_{B,t}$$

whose kernel and cokernel are killed by m. Therefore their orders divide some power of m, with exponent bounded by the ranks of the finite component groups occurring on the fixed tame family. We do not infer $\# \ker \mid m$, $\# \operatorname{coker} \mid m$, or a divisibility relation between $c_{A,t}$ and $c_{B,t}$.

More explicitly, over the strict henselization of the DVR at t, the exact sequence

$$0 \longrightarrow A_t^0 \longrightarrow A_t \longrightarrow \Phi_{A,t} \longrightarrow 0$$

and its analogue for B, together with the snake lemma applied to the finite-étale prime-to-p isogeny on identity components, gives that both the kernel and cokernel on geometric component groups are annihilated by m. This is the standard BLR/Raynaud component-group functoriality for morphisms of semiabelian Néron models over DVRs.

Note. Throughout, when $\dim T > 1$ the assertions above are understood *fiberwise in codimension* 1 (Néron lft–model sense; see [3]); no global mapping property over higher–dimensional regular bases is used. The extension of prime-to-p isogenies to Néron models and the finite–étale behavior on identity components are standard in the literature; see [3] for the Néron-model functoriality/extension formalism, and [**?**] for the specialization/criterion input used in the prime-to-p extension discussion. The control of component-group kernels and cokernels by the isogeny degree follows from [3] and the formalism of vanishing cycles therein. These references justify the structural claims of Theorem 3.1 (iv) and delimit the hypotheses (semiabelian reduction on an open dense locus, kernel scheme finite étale of order m).

Shrinking once to the locus where the m-torsion is finite étale and the fibers are semiabelian yields the fixed open U of Theorem 3.5.

(See also [5] and [3] for the prime-to-p semistable case of isogeny extension and component-group control.)

□

For each closed $t \in T^\circ$ in the prime-to-residue-characteristic semistable locus, the morphism

$$\Phi_{A,t} \to \Phi_{B,t}$$

has kernel and cokernel annihilated by m. Consequently, the associated component-group defect terms are uniformly controlled in the kernel–cokernel sense by the isogeny degree m. No divisibility formula for $c_{B,t}/c_{A,t}$ is asserted here.

The Artin-conductor issue is treated separately in Theorem 4.1; here the conclusion concerns component groups and Tamagawa-index data only.

This component-group control provides the geometric input used later in Theorem 4.1 and in the tame Hecke-uniformity statements of Section 5.

Example 3.3 (Wild additive prime and failure of the tame prime-to-p mechanism). Let $E/\mathbb{Q}$ admit a 2–isogeny

$$\varphi_\eta : E \longrightarrow E'$$

and suppose that E has additive reduction at $p = 2$ (for example Kodaira type II or III). Set

$$T^\circ = S \setminus \{2\}.$$

Then the hypotheses of Theorem 3.1(iv) hold on T°, but fail at the wild prime $p = 2$.

Geometric obstruction. Over $\mathbb{Z}_2$, the finite flat kernel scheme

$$\ker(\varphi)$$

has order 2 but need not remain étale on the special fiber. Instead, the special fiber may become connected and local, intersecting the identity component of the Néron model. After a suitable finite flat base change, the special fiber may become connected and non-étale rather than remaining an étale constant group scheme of $\underline{\mathbb{Z}/2\mathbb{Z}}$. Equivalently,

$$\ker(\varphi) \text{ is finite étale on } T^\circ \quad \text{but} \quad \ker(\varphi)\big|_{\operatorname{Spec}\mathbb{F}_2} \text{ is non-étale.}$$

Consequently, the induced morphism on identity components

$$E_2^0 \longrightarrow (E_2')^0$$

is no longer finite étale.

Failure of tame component-group control. Because the kernel is not étale at 2, the prime-to-residue-characteristic mechanism of Theorem 3.1(iv) does not apply. The exact sequence controlling component groups therefore loses the uniform "killed by 2" bound:

$$\Phi_{E,2} \longrightarrow \Phi_{E',2}$$

need not have kernel or cokernel annihilated by 2. Hence the divisibility relation

$$c_{E',2} \mid 2^r c_{E,2}$$

may fail. In geometric terms, additional wild finite-flat behavior appears on the special fiber outside the tame semistable locus considered in Theorems 3.1 and 3.5.

Cohomological consequence. For $\ell \neq 2$, wild inertia may act nontrivially on

$$H^1_{\text{ét}}(E_{\mathbb{Q}_2}, \mathbb{Q}_\ell),$$

so a nonzero Swan contribution can appear in the conductor formula. Thus the tame comparison statements of Theorem 4.1 and the uniform component-group bounds of Theorem 3.5 are not expected to persist at the wild additive prime.

Scope clarification. This example illustrates only the failure of the tame geometric control mechanism at primes dividing the isogeny degree. It should not be interpreted as asserting variation of the Artin conductor of the rational ℓ–adic Tate module under isogeny; compare Theorem 4.1(iii).

3.2 Uniform semi-continuity of component indices under isogeny

Lemma 3.4 (Finite-group control for tame component maps)**.** *Let R be a strict henselian DVR with residue characteristic p, and let $\varphi : \mathcal{A} \to \mathcal{B}$ be a morphism of Néron models induced by an isogeny of generic fibers of degree m, with $p \nmid m$. Assume that the induced morphism on identity components has finite étale kernel killed by m. Then the induced morphism on geometric component groups*

$$\Phi_{\mathcal{A},\bar{k}} \longrightarrow \Phi_{\mathcal{B},\bar{k}}$$

has kernel and cokernel annihilated by m. Consequently their orders divide a power of m.

This statement concerns geometric component groups over $\bar{k}$. Statements for k-rational component groups require taking Galois invariants and are not asserted without additional hypotheses on the residue field.

Theorem 3.5 (Constructibility and tame local constancy of component-group orders)**.** *In the setting of Theorem 3.1, fix an integer $m \geq 1$ and shrink T° so that:*

(H1) *m is invertible on T° (equivalently, all residue characteristics on T° are prime to m), and*

(H2) *$A_{T^\circ} \to T^\circ$ is semiabelian (semistable), so that component groups and vanishing-cycle bounds apply on T°.*

Let $\mathcal{I}_m$ be a fixed finite collection of degree-m isogenies

$$\varphi_{\eta_T} : A_{\eta_T} \to B_{\eta_T}$$

whose kernels extend, after possibly shrinking T°, to finite étale subgroup schemes. Then there exists a Zariski open dense subscheme

$$U = U(\mathcal{I}_m) \subseteq T^\circ$$

such that the following conclusions hold simultaneously for every isogeny $\varphi_{\eta_T} \in \mathcal{I}_m$ and every closed point $t \in U$:

1. *The induced morphism on component groups satisfies*

$$\ker\big(\Phi_{A,t} \to \Phi_{B,t}\big) \quad \textit{and} \quad \operatorname{coker}\big(\Phi_{A,t} \to \Phi_{B,t}\big)$$

are killed by m. In particular, their orders divide a power of m; no assertion is made that those orders divide m.

2. *Fix a rational prime $q \nmid m$, used only as a valuation index and distinct from the residue characteristic $p = \operatorname{char} \kappa(t)$. On the open locus U, the sheaf of connected components is finite étale, and hence the function*

$$t \longmapsto v_q\big(\#\Phi_{A,t}\big)$$

is locally constant. In particular, it is upper semi-continuous on U.

Notation 3.6 (Fixed tame locus)**.** Fix once and for all the dense open $U \subseteq T^\circ$ provided by Theorem 3.5 for the chosen degree m prime to all residue characteristics on T°. All statements in Sections 4–6 that refer to "on a dense open" are henceforth to be read over this fixed U, unless explicitly stated otherwise.

N.B. On the finite-étale locus, the component-group order is locally constant; there the upper semicontinuity statement is only formal. The actual content is the construction of a common tame open U on which the chosen finite family of prime-to-residue-characteristic isogenies has uniformly controlled component-group kernel and cokernel.

Scope. On the finite-étale locus the component-group order is locally constant, so the semicontinuity assertion is formal. The content of the theorem is the construction of a common tame open for the fixed finite family of prime-to-residue-characteristic isogenies and the resulting uniform kernel–cokernel control of component-group maps.

Proof. By Lemma 2.5 we may shrink T° so that every fiber of $A_{T^\circ} \to T^\circ$ is semiabelian and the identity components form a semiabelian scheme. Let m be prime to all residue characteristics on T°. For any degree-m isogeny φ_{η_T}, its kernel on the generic fiber is a finite flat group scheme of order m. Because $(m, \operatorname{char}\kappa(t)) = 1$ for all $t \in T^\circ$, there exists a dense open $U \subseteq T^\circ$ on which this kernel extends to a finite *étale* subgroup of $A[m]$, itself finite étale over U. Over each codimension-one DVR in the fixed tame locus, the isogeny restricts to an isogeny of semiabelian identity components with finite étale kernel killed by m. Applying Lemma 3.4 over the strict henselization of the corresponding DVR gives the asserted kernel–cokernel control on geometric component groups. This gives (i).

Before invoking semicontinuity, we note that $\pi_0(A_U/U)$ is in fact a finite étale group sheaf on a dense open subset of U. Indeed, for a smooth separated group scheme A_U/U of finite type, the identity component $(A_U)^0$ is open and of finite index, and the quotient sheaf

$$\pi_0(A_U/U) := A_U/(A_U)^0$$

is representable by a finite étale U–group scheme (see [3]). Shrinking U if necessary, we may assume that this quotient is finite étale; hence its fiber lengths are locally constant and define a constructible function on U.

For the full base T°, the closedness of the loci $\{t : v_q(\#\Phi_{A,t}) \geq r\}$ is recorded in Lemma 3.8.

Remark 3.7 (Scope clarification for semicontinuity). On the locus where $\pi_0(A_U/U)$ is finite étale, the function $t \mapsto \#\Phi_{A,t}$ is locally constant, so upper semicontinuity is automatic *there*. The content of Theorem 3.5 is that one can (after shrinking to a *single* dense tame open U) control component-group indices *uniformly for all degree-m prime-to-residue isogenies*, and moreover identify/parameterize the possible jump loci when π_0 ceases to be étale (e.g. at additive or wild fibers), via the closedness statement Lemma 3.8 and the quantitative bounds supplied later by vanishing cycles (Theorem 4.1(iii)). Thus Theorem 3.5 should be read as describing a maximal tame open together with effective control of boundary variation.

Lemma 3.8 (Constructibility of component-group orders)**.** *Let T° and $A_{T^\circ} \to T^\circ$ be as above, and fix a rational prime $q \nmid m$. After stratifying T° into finitely many locally closed subschemes, the function*

$$t \longmapsto \#\Phi_{A,t}$$

is locally constant on each stratum. Consequently,

$$t \longmapsto v_q(\#\Phi_{A,t})$$

is constructible. On the dense open $U \subseteq T^\circ$ where $\pi_0(A_U/U)$ is finite étale, this function is locally constant.

Proof. The sheaf of connected components

$$\pi_0(A_{T^\circ}/T^\circ)$$

is a constructible étale group sheaf whose geometric fiber at a point t is $\Phi_{A,t}$; see [3]. Hence the fiber cardinality

$$t \longmapsto \#\Phi_{A,t}$$

is a constructible function on T°. Equivalently, after a finite stratification of T° into locally closed subschemes, this function is locally constant on each stratum. Applying v_q to these locally constant values gives the constructibility of $t \mapsto v_q(\#\Phi_{A,t})$.

On the dense open U where $\pi_0(A_U/U)$ is finite étale, the fibers have locally constant cardinality, so $t \mapsto v_q(\#\Phi_{A,t})$ is locally constant on U. ☐

☐

On U, the local Tamagawa numbers

$$c_{A,t} = \#\Phi_{A,t}(\kappa(t))$$

are locally constant on finite étale strata and are controlled under the fixed finite collection of prime-to-residue-characteristic isogenies by the kernel/cokernel bounds above. This gives uniform control of component-group indices and auxiliary local defect terms. It does not imply a conductor variation inequality; Artin conductor exponents are unchanged under the isogenies considered here.

This theorem supplies the geometric input for the cohomological invariance (Theorem 4.1) by ensuring that all wild variation of inertia invariants is confined to a finite set of excluded primes. It also furnishes the analytic base for conductor stability (Corollary 6.2) and Hecke-orbit uniformity (Theorems 5.3 and 5.10), thereby linking the geometric control of component groups to the corresponding arithmetic consequences.

Example 3.9 (Boundary failure of the finite–étale kernel hypothesis). Retain the curves $E\colon y^2 = x^3 - x$ and $E'\colon y^2 = x^3 + 4x$. At $p = 2$, the degree of the isogeny is not prime to the residue characteristic, and the kernel need not extend as a finite étale subgroup scheme. Hence the prime-to-p conclusions of Theorem 3.5 do not apply at this prime.

This example should not be read as asserting that the Artin conductor of the rational ℓ-adic Tate module changes under an isogeny. Rather, it illustrates only the failure of the geometric hypotheses used to control identity components and component-group maps. The local Néron-model geometry may fail to be finite étale, but the rational Tate modules of isogenous abelian varieties remain isomorphic.

3.3 Cohomological consequences

By Lemma 2.11 and Definition 2.13, the inertia action on $H^1_{\text{ét}}(A_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ detects the reduction type and the conductor exponent f_p. Theorems 3.1 and 3.5 control the étale component groups Φ_p and hence the local Tamagawa factors c_p (Proposition 2.8) under base change and isogeny across arithmetic families. Combined with Construction 2.10 and the positivity statements used later, this provides the geometric input for the cohomological comparison results in the next section.

Section 4 will use the structural controls above to compare cohomology under base change and derive arithmetic consequences for local factors and heights.

4 Cohomological Invariants and Applications

Standing convention for this section. We work over the fixed open $U \subseteq T^\circ$ from Theorem 3.5 (Notation 3.6); in particular, all degree-m isogenies considered below have kernel finite étale over U, and the component-group control is uniform on U. All uses of "extension to Néron models" and all quantifiers over degree-m isogenies are understood in the sense of Remark 2.3.
We keep the base $S = \operatorname{Spec} \mathbb{Z}$ and the notation from Notation 2.1. Background on Néron models and component groups was set in Definition 2.6, lemma 2.7, proposition 2.8, remark 2.9, and construction 2.10, while conductor and cohomological inertia were fixed in Lemma 2.11, definition 2.13, and proposition 2.14. Structural properties in families and under prime-to-residue-characteristic isogenies were established in Theorems 3.1 and 3.5 and illustrated by Example 7.1 & the preceding tame-isogeny examples. Here we quantify how ℓ–adic cohomology and inertia invariants vary in arithmetic families and under isogeny, and we record consequences for local factors and heights.

4.1 Uniform comparison in families and under isogeny

Theorem 4.1 (Refined comparison of cohomology under base change and prime-to-p isogeny)**.** *Let T be a regular integral scheme of finite type over S, with generic point η_T and function field $K(T)$. Let $A_{\eta_T}/K(T)$ be an abelian variety of dimension g, and let $A_{T^\circ} \to T^\circ$ be its Néron model on a nonempty open $T^\circ \subseteq T$ as in Theorem 3.1. Fix a prime ℓ. After possibly shrinking to a dense open $U \subseteq T^\circ$, the following hold for every closed point $t \in U$ with residue characteristic $p = \operatorname{char} \kappa(t)$:*

(i) ***(Base–change invariance on the abelian–scheme locus; nearby–cycles on the semistable locus)*** *On the locus where A_U/U is an abelian scheme, proper and smooth base change yields a canonical isomorphism*

$$H^i_{\text{ét}}\big((A_U)_{\eta_T}, \mathbb{Q}_\ell\big) \xrightarrow{\sim} H^i_{\text{ét}}\big((A_U)_{K(t)}, \mathbb{Q}_\ell\big)^{I_t}.$$

On the semistable (tame) locus for a general Néron model, nearby cycles identify $H^1_{\text{ét}}\big(A_{K(t)}, \mathbb{Q}_\ell\big)^{I_t}$ with the cohomology of the semiabelian part of the special fiber, hence for $\ell \neq p$,

$$\dim_{\mathbb{Q}_\ell} H^1_{\text{ét}}\big(A_{K(t)}, \mathbb{Q}_\ell\big)^{I_t} = 2g - \tau_t.$$

(ii) ***(Isogeny invariance of rational inertia invariants)*** *Let $\varphi_{\eta_T} : A_{\eta_T} \to B_{\eta_T}$ be an isogeny of degree m with $(m, p) = 1$ on U, and let $\varphi : A_U \to B_U$ be its extension from Theorem 3.1. Then for $\ell \neq p$ the induced map on I_t–invariants*

$$H^1_{\text{ét}}\big(A_{\overline{K(t)}}, \mathbb{Q}_\ell\big)^{I_t} \xrightarrow{\sim} H^1_{\text{ét}}\big(B_{\overline{K(t)}}, \mathbb{Q}_\ell\big)^{I_t}$$

is an isomorphism. Equivalently, the function $t \mapsto \dim_{\mathbb{Q}_\ell} H^1_{\text{ét}}\big(A_{\overline{K(t)}}, \mathbb{Q}_\ell\big)^{I_t}$ is constructible on the fixed tame semistable locus and invariant under all prime-to-p isogenies of fixed degree m.

(iii) ***(Artin conductor identity and isogeny invariance)*** *For $\ell \neq p$, the Artin conductor is given by*

$$f_{A,t} = \tau_t + \operatorname{Swan}\big(H^1_{\text{ét}}(A_{K(t)}, \mathbb{Q}_\ell)\big).$$

On the semistable tame locus the Swan term vanishes. If $\varphi : A_{\eta_T} \to B_{\eta_T}$ is one of the prime-to-residue-characteristic isogenies considered on U, then the rational ℓ-adic Tate modules of A and B are isomorphic. Hence

$$f_{A,t} = f_{B,t}.$$

The component-group estimates of Theorems 3.1 and 3.5 therefore control only the auxiliary component-group and Tamagawa-index defect terms, not a variation of the Artin conductor.

Lemma 4.2 (Prime-to-p isogenies: tame data and Artin conductors)**.** *Let $t \in U$ be as in Theorem 4.1, and let $\varphi : A_{\eta_T} \to B_{\eta_T}$ be a prime-to-residue-characteristic isogeny whose extension over U is finite étale on identity components. Then:*

(a) *The tame toric contribution is preserved:*

$$\tau_t(A) = \tau_t(B).$$

(b) *The rational ℓ-adic Tate modules are isomorphic for every $\ell \neq p$, and hence the Artin conductor exponents agree:*

$$f_{A,t} = f_{B,t}.$$

(c) *The component-group map*

$$\Phi_{A,t} \to \Phi_{B,t}$$

has kernel and cokernel annihilated by the isogeny degree. This gives uniform control of Tamagawa-index and local geometric defect terms, but not a conductor-variation inequality.

Proof. **Scope.** We first separate the two comparison statements that are used in the argument. On the abelian-scheme locus, A_U/U is proper and smooth, so smooth and proper base change applies directly and gives the usual specialization isomorphisms. On the semistable locus for a Néron model, however, A_U is smooth and separated but need not be proper. Thus we do not use smooth and proper base change as a cohomology comparison theorem for the non-proper Néron model itself. Instead, we use the standard Raynaud-extension description of the inertia invariants of the generic-fiber Tate module.

More precisely, after restricting to the strict henselization at a codimension-one point t, let the identity component of the special fiber have Raynaud extension

$$0 \longrightarrow T_t \longrightarrow (A_t)^0 \longrightarrow B_t \longrightarrow 0,$$

where T_t is a torus of rank τ_t and B_t is an abelian variety. Then, for $\ell \neq p = \operatorname{char} \kappa(t)$,

$$\dim_{\mathbb{Q}_\ell} V_\ell(A)^{I_t} = 2g - \tau_t.$$

This is a statement about the rational ℓ-adic Tate module of the generic abelian variety and its inertia action. It is not a statement about the full étale cohomology of an arbitrary non-proper Néron model.

For Item (i), on the good-reduction locus A_U/U is an abelian scheme, and smooth and proper base change gives the asserted specialization isomorphism. On the semistable tame locus, the monodromy/Raynaud formalism [4, 5] gives the displayed formula for $V_\ell(A)^{I_t}$, equivalently for

$$H^1_{\text{ét}}\big(A_{\overline{K(t)}}, \mathbb{Q}_\ell\big)^{I_t}.$$

After shrinking U, the reduction type is constant on the relevant strata, so the function

$$t \longmapsto \dim_{\mathbb{Q}_\ell} H^1_{\text{ét}}\big(A_{\overline{K(t)}}, \mathbb{Q}_\ell\big)^{I_t}$$

is constructible on the fixed semistable tame locus.

For Item (ii), let $\varphi_{\eta_T} : A_{\eta_T} \to B_{\eta_T}$ be one of the prime-to-residue-characteristic isogenies considered on the fixed tame open U. By Theorem 3.1, interpreted in the Dedekind/codimension-one sense fixed in Remark 2.3, it extends to the corresponding morphism of Néron models on the tame locus. The extension and the finite-$\acute{e}tale$ kernel hypothesis are needed for the identity-component and component-group control proved in Theorems 3.1 and 3.5. For the rational cohomological statement itself, the essential input is the classical fact that an isogeny of abelian varieties induces an isomorphism of rational Tate modules:

$$V_\ell(A) := T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \xrightarrow{\sim} V_\ell(B) := T_\ell(B) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

as $G_{K(t)}$-representations. Taking I_t-invariants of this isomorphism gives

$$V_\ell(A)^{I_t} \xrightarrow{\sim} V_\ell(B)^{I_t}.$$

Equivalently,

$$H^1_{\text{ét}}\big(A_{\overline{K(t)}}, \mathbb{Q}_\ell\big)^{I_t} \xrightarrow{\sim} H^1_{\text{ét}}\big(B_{\overline{K(t)}}, \mathbb{Q}_\ell\big)^{I_t}.$$

Thus the dimension of the inertia-invariant rational cohomology is preserved under the isogenies considered here. In particular, on the fixed tame semistable locus, the toric-rank contribution appearing in the Raynaud formula is the same for A and B.

For Item (iii), the Artin conductor exponent is computed from the rational ℓ-adic Galois representation

$$H^1_{\text{ét}}\big(A_{\overline{K(t)}}, \mathbb{Q}_\ell\big) \cong V_\ell(A)^\vee.$$

Equivalently, it has the usual tame-plus-wild form

$$f_{A,t} = \tau_t + \operatorname{Swan}\Big(H^1_{\text{ét}}\big(A_{\overline{K(t)}}, \mathbb{Q}_\ell\big)\Big),$$

with the Swan term equal to 0 on the tame semistable locus. Since φ_{η_T} induces an isomorphism of rational ℓ-adic Galois representations for A and B, their Artin conductor exponents agree:

$$f_{A,t} = f_{B,t}.$$

The component-group estimates proved earlier therefore control only the Tamagawa-index and local geometric defect terms. They should not be read as a conductor-variation inequality. □

Analytically, Theorem 4.1(ii) ensures that the tame pole order of the local L–factor $L_t(A, s)$ remains constant under all prime-to-p isogenies. The component-group estimates above control only the associated Tamagawa-index and local geometric defect terms.

Corollary 4.3 (Invariant local factor on the semistable locus)**.** *Under the hypotheses of Theorem 4.1 with $\ell \neq p$, the invariant local factor is*

$$P^I_{A,p}(X) = \det\big(1 - X \operatorname{Frob}_p \mid V_\ell(A)^{I_p}\big).$$

It has degree $2g - \tau_p$ and is invariant under the prime-to-p isogenies considered on U. If the torus in the Raynaud extension is split, then a factor $(1 - p^{-s})^{\tau_p}$ occurs in the corresponding local factor. In the non-split toric case the toric factor is instead determined by Frobenius on the character lattice of the torus.

Example 4.4 (Bad reduction and inertia action). Let $E/\mathbb{Q}$ be an elliptic curve and fix $\ell \neq p$.

Good reduction. If E has good reduction at p, then the special fiber E_p is an abelian variety and I_p acts trivially on $T_\ell(E)$; equivalently,

$$H^1_{\text{ét}}(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)^{I_p} \cong H^1_{\text{ét}}(E_{\overline{\mathbb{F}_p}}, \mathbb{Q}_\ell), \qquad \dim_{\mathbb{Q}_\ell} H^1_{\text{ét}}(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)^{I_p} = 2.$$

The local Euler factor is

$$L_p(E, s)^{-1} = 1 - a_p\, p^{-s} + p^{1-2s},$$

with $a_p = p + 1 - \#E(\mathbb{F}_p)$ (trace of Frobenius on H^1). Here $\tau_p = 0$ and the tame factor $(1 - p^{-s})^{\tau_p}$ is absent, in agreement with Corollary 4.3.

Split multiplicative reduction. Assume E has split multiplicative reduction at p. Then the identity component of the Néron special fiber is a torus,

$$(E_p)^0 \simeq \mathbb{G}_m, \qquad \tau_p = 1.$$

The I_p–action on $H^1_{\text{ét}}(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ is unipotent with a single invariant line coming from the toric part, hence

$$\dim_{\mathbb{Q}_\ell} H^1_{\text{ét}}(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)^{I_p} = 1.$$

The local factor splits as

$$L_p(E, s)^{-1} = (1 - p^{-s}) \cdot (1 - \alpha\, p^{-s}), \qquad |\alpha| = 1,$$

where the factor $(1 - p^{-s})$ is the tame toric contribution predicted by Corollary 4.3 with $\tau_p = 1$.

Additive potentially good reduction. If E has additive potentially good reduction at p, then wild inertia may act nontrivially on $H^1_{\text{ét}}(E_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_\ell)$, and the Swan term can be positive. A tame extension does not, in general, kill wild inertia. After a suitable finite extension, which may be wildly ramified depending on p, the curve may acquire good reduction. This case therefore lies outside the tame semistable mechanism used in Theorem 4.1 and corollary 4.3.

Example 4.5 (Counterexample illustrating necessity of the prime-to-p hypothesis). Let $E/\mathbb{Q}$ admit a 2–isogeny $\varphi_\eta : E \to E'$ and suppose E has *additive* reduction at $p = 2$.

Geometric failure on the special fiber. Extend φ_η to a morphism of Néron models over $\mathbb{Z}_2$. Because the residue characteristic divides the isogeny degree, the kernel scheme $\ker(\varphi)$ is finite flat of order 2 but *need not be étale* on the special fiber: it can specialize to a connected local group (e.g. α_2) meeting $(E_2)^0$. Consequently, the induced map on identity components $(E_2)^0 \to (E'_2)^0$ is not finite étale, and the exact sequence controlling component groups loses the "killed by 2" bounds.

Galois-theoretic clarification. Although the finite flat kernel may fail to be étale at $p = 2$, this does not imply that the rational ℓ-adic cohomology changes under the isogeny. For every $\ell \neq 2$, the isogeny induces an isomorphism of rational Galois representations

$$V_\ell(E) \xrightarrow{\sim} V_\ell(E'),$$

and hence also an isomorphism on inertia-invariant subspaces

$$H^1_{\text{ét}}(E_{\overline{\mathbb{Q}}_2}, \mathbb{Q}_\ell)^{I_2} \xrightarrow{\sim} H^1_{\text{ét}}(E'_{\overline{\mathbb{Q}}_2}, \mathbb{Q}_\ell)^{I_2}.$$

Thus the failure at $p = 2$ is not a failure of rational Tate-module isogeny invariance. It is only a failure of the tame finite-étale kernel mechanism used to control identity components and component groups.

Local geometric implication. The example shows that outside the prime-to-p range the tame component-group argument no longer applies. One may encounter non-étale finite-flat kernels and wild geometric behavior on the special fiber. This is a failure of the tame geometric control mechanism, not a claim that the Artin conductor of the rational Tate module varies under isogeny.

4.2 Heights and invariant differentials

Proposition 4.6 (Compatibility of canonical heights with semistable local data)**.** *Let A/S be a Néron model and let L be a symmetric ample line bundle on the generic fiber A_η. Assume semistable reduction over a dense open $U \subseteq S$, and write τ_p for the toric rank at a prime $p \in U$.*

Then the Néron–Tate height $\widehat{h}_L$ admits a decomposition into local contributions compatible with the semistable geometry on U. In particular, the local terms arising from the toric part and from the invariant differential sheaf $\omega_{A/S}$ vary functorially under the prime-to-residue-characteristic isogenies considered earlier in this paper.

More precisely, on the fixed semistable locus U, the decomposition of local Néron symbols and the associated Arakelov-theoretic contributions are compatible with:

1. *the toric-rank terms τ_p,*
2. *the invariant differential sheaf $\omega_{A/S}$, and*
3. *finite étale prime-to-p isogenies on identity components.*

In particular, the results of Sections 3–4 provide uniform control of the local geometric contributions *appearing in the canonical-height decomposition on the fixed tame locus U.*

No claim is made here of a new global uniform positive lower bound for all non-torsion points in an arbitrary prime-to-Σ isogeny class.

Justification and scope. The proposition records only the compatibility between the semistable local geometry and the standard decomposition of canonical heights. It should not be interpreted as establishing a new global height-gap theorem.

The relevant inputs are classical:

- the Faltings–Hriljac description of Néron–Tate heights ([14], [15]),
- the interpretation of $\omega_{A/S}$ in Arakelov geometry,
- the standard decomposition of local Néron symbols on semistable models, and
- functoriality of canonical heights under isogeny.

The results proved earlier in this paper contribute only the uniform control of the semistable local geometric data on the fixed tame open U. Any genuine uniform positive lower bound for canonical heights across entire isogeny classes would require substantially stronger global Arakelov-theoretic input and is not claimed here.

Proof. On the semistable locus U, the canonical height decomposes into local Néron-symbol contributions together with archimedean terms; see [14], [15], and [16]. The toric-rank terms and the invariant differential sheaf $\omega_{A/S}$ contribute functorially under the finite étale prime-to-residue-characteristic isogenies considered in Sections 3–4.

The component-group and semistable-control results established earlier therefore imply compatibility of the corresponding local geometric contributions on the fixed tame locus U. No additional global lower-bound statement is asserted. □

Since τ_p is isogeny-invariant on the dense open U by Theorem 4.1(ii), the semistable local contributions appearing in the standard height decomposition are compatible with the prime-to-p isogenies considered here.

4.3 Link to moduli-theoretic applications

The functions $t \mapsto \tau_t$ and $t \mapsto c_{A,t}$ are constructible and locally constant on the fixed tame open U. In the next section this control is applied to Hecke and isogeny orbits on integral models of modular curves and, under stated extension hypotheses, on selected Hodge-type Shimura varieties.

5 Moduli-Theoretic Aspects

We retain the base and notation from Notation 2.1. Theorems 3.1 and 3.5 control Néron models in arithmetic families and under prime-to-residue-characteristic isogenies; Theorem 4.1 describes the corresponding cohomological invariants and conductor behavior. We now work over integral moduli (modular curves and selected Shimura varieties) and extract uniform statements along Hecke orbits.

Standing hypotheses for Section 5. Whenever we assert Hecke-orbit uniformity over an integral model, the statement is understood on a dense open locus where (i) the relevant Hecke correspondence extends integrally and factors through prime-to-residue-characteristic isogenies, and (ii) we have shrunk to the tame open U of Theorem 3.5 so that the component-group control of Theorems 3.1 and 3.5 applies. This separates the moduli-theoretic input from the Néron-model mechanism proved in this paper (cf. Remark 5.4).

5.1 Néron models and integral models of modular curves

Notation 5.1 (Integral modular curves and Jacobians)**.** Fix a positive integer $N \geq 1$. Let $\mathcal{X}_0(N)$ denote an integral regular model of the modular curve $X_0(N)$ over $\operatorname{Spec}\mathbb{Z}$ away from a finite set of primes (Deligne–Rapoport and Katz–Mazur). Write $J_0(N) = \operatorname{Pic}^0(X_0(N))$ for the Jacobian over $\mathbb{Q}$, and J/S for its Néron model (Definition 2.6 and lemma 2.7). For a prime p, denote by $\Phi_{J,p}$ the component group and by $c_{J,p} = \#\Phi_{J,p}(\mathbb{F}_p)$ its local Tamagawa number (Proposition 2.8).

Remark 5.2 (Hecke correspondences on integral models). For a prime $\ell \nmid N$, the classical Hecke correspondence T_ℓ extends to a finite étale correspondence on $\mathcal{X}_0(N)$ over $\mathbb{Z}[1/N\ell]$ (Deligne–Rapoport, Katz–Mazur), hence induces an algebraic correspondence on $J_0(N)$ over $\mathbb{Z}[1/N\ell]$ which is a composition of isogenies of degree a power of ℓ.

Theorem 5.3 (Conditional Hecke-uniformity of components and inertia on $J_0(N)$)**.** *Let $N \geq 1$, and let $U \subseteq \operatorname{Spec}\mathbb{Z}$ be the open subscheme obtained by removing a finite set of primes containing those dividing N. Fix a prime $\ell \nmid N$.*

Assume *that, after shrinking to a dense open $U_\ell \subseteq U$, the correspondence T_ℓ decomposes over U_ℓ into finitely many morphisms of the relevant Néron models whose generic fibers are ℓ-power isogenies and whose kernels are finite étale on identity components.*

Then the following holds:

For each $p \in U_\ell$ and for every T_ℓ–translate of a rational point in the Hecke orbit on $J_0(N)$, the induced correspondence on Néron models

$$J_{U_\ell} \dashrightarrow J_{U_\ell}$$

restricts on the fiber over p to a prime-to-p isogeny on the identity components and induces a morphism on component groups

$$\Phi_{J,p} \longrightarrow \Phi_{J,p}$$

whose kernel and cokernel are annihilated by a power of ℓ. Consequently:

(i) the associated Tamagawa-index defect is controlled by finite groups killed by a power of ℓ; equivalently, the possible discrepancy is ℓ-primary with bounded exponent in the stated finite correspondence;

(ii) the toric rank $\tau_p = \operatorname{rank}_{\mathbb{Z}} T_p((J_p)^0)$ of the connected special fiber is constant along the T_ℓ–orbit, hence

$$\dim_{\mathbb{Q}_\ell} H^1_{\text{ét}}(J_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)^{I_p} = 2g - \tau_p$$

is Hecke-invariant for $\ell \neq p$ by Theorem 4.1(i);

(iii) the Artin conductor exponent attached to the rational ℓ-adic Tate representation is unchanged under the prime-to-p isogenies considered here; the uniform bounds concern only component-group, Tamagawa-index, and auxiliary local geometric defect terms.

Comment on scope and novelty. This theorem globalizes the local control of Theorems 3.1, 3.5 and 4.1 from a single isogeny to the entire Hecke orbit. Whereas the classical references treat T_ℓ primarily on the generic fiber of $J_0(N)$, the present conditional statement works on a common dense open U_ℓ where the correspondence acts through prime-to-p isogenies of Néron models. This gives uniform control of identity components, component groups, and tame local geometric defect terms across the corresponding Hecke translates.

This "Hecke-uniformity locus" links the geometric Néron theory to the arithmetic of local L-factors, yielding orbitwise constancy of the tame exponent τ_p. Since the rational ℓ-adic Tate modules are isomorphic under the isogenies considered here, the Artin conductor exponent itself is unchanged; no variation bound for $f_{J,p}$ is claimed.

Proof. By Remark 5.2, over $\mathbb{Z}[1/N\ell]$ the correspondence T_ℓ decomposes into finitely many degeneracy maps whose induced endomorphisms on $J_0(N)$ factor as isogenies of ℓ-power degree. Shrinking to a dense open $U_\ell \subseteq U$, these isogenies extend to morphisms of Néron models by Theorem 3.1. For every $p \in U_\ell$ each factor has degree prime to p, so the morphisms on identity components are finite étale and the induced maps on $\Phi_{J,p}$ have kernel and cokernel annihilated by a power of ℓ by Theorem 3.5.

Item (ii) follows from Theorem 4.1(ii), giving Hecke-invariance of the dimension of inertia invariants. Item (iii) follows from the rational Tate-module isomorphism in Theorem 4.1(iii); the component-group bounds concern only Tamagawa-index and local geometric defect terms. □

Remark 5.4 (Scope and justification of Theorem 5.3). The statement above should be interpreted in the following conditional sense. On modular curves and Hodge-type Shimura varieties, prime-to-p Hecke correspondences extend to the integral models on a dense open locus only after possibly removing the primes where the level structure is ramified. In these cases, the extension acts by prime-to-p isogenies on the Néron models of the universal Jacobians, and the induced maps on identity components and component groups are controlled by Theorems 3.1 and 3.5. For full justification one must appeal to the integral moduli constructions of Deligne–Rapoport and Katz–Mazur for modular curves, and to their analogues for Hodge-type Shimura varieties (see [1], [2]). In the modular-curve case, the required extension properties are standard consequences of the integral moduli constructions of Deligne–Rapoport and Katz–Mazur (see [1, 2]), together with the functoriality results of Theorems 3.1 and 3.5 on the tame open. We record the statement in the conditional form above to separate the moduli-theoretic input from the Néron-model and component-group uniformity mechanisms proved in this paper.

The correspondence T_ℓ acts on J_{U_ℓ}, on the stated tame open and under the stated extension hypotheses, through prime-to-p isogenies whose geometric kernels are finite *é*tale. Hence, by the functorial exact sequence of component groups,

$$\ker(\Phi_{J,p} \to \Phi_{J,p}) \quad \text{and} \quad \operatorname{coker}(\Phi_{J,p} \to \Phi_{J,p})$$

are annihilated by a bounded power of ℓ.

Moreover, the rational ℓ-adic Tate modules are isomorphic along the corresponding isogenies. Therefore the inertia-invariant local factor

$$P^I_{J,p}(X) = \det\big(1 - X\,\mathrm{Frob}_p \mid V_\ell(J_0(N))^{I_p}\big)$$

is unchanged along the correspondence. If the toric part of the Raynaud extension is split at p, then

$$L_p(J_0(N), s)^{-1} = (1 - p^{-s})^{\tau_p} Q_p(p^{-s}),$$

with Q_p determined by the non-toric invariant part. In the non-split toric case, the toric factor is governed by Frobenius on the character lattice of the torus, and no factor $(1 - p^{-s})^{\tau_p}$ is asserted.

Thus, on the stated tame open and under the stated extension hypotheses, the Hecke correspondence preserves the rational inertia-invariant representation and gives bounded component-group defect terms.

Corollary 5.5 (Invariant local factor along Hecke orbits). *For $p \in U_\ell$ and $\ell \neq p$, the inertia-invariant local factor*

$$P^I_{J,p}(X) = \det\big(1 - X\,\mathrm{Frob}_p \mid V_\ell(J_0(N))^{I_p}\big)$$

is independent of the point in the T_ℓ-orbit and has degree $2g - \tau_p$. Equivalently,

$$L_p(J_0(N), s)^{-1} = P^I_{J,p}(p^{-s})$$

on the invariant part.

If the toric part of the Raynaud extension is split at p, then this invariant factor further decomposes as

$$P^I_{J,p}(p^{-s}) = (1 - p^{-s})^{\tau_p} Q_p(p^{-s}),$$

with $\deg Q_p = 2g - 2\tau_p$. In the non-split toric case, the toric factor is governed by the Frobenius action on the character lattice of the torus, and no factor $(1 - p^{-s})^{\tau_p}$ is asserted.

Proof. This is exactly the invariant local-factor statement of Corollary 4.3, applied uniformly along the Hecke orbit using Theorem 5.3. The additional displayed factorization holds only in the split toric case. □

Example 5.6 (Hecke invariance away from the level). Let $N = 11$ and $p \neq 11$. Over $U = \operatorname{Spec}\mathbb{Z} \setminus \{11\}$, the integral model $\mathcal{X}_0(11)$ is regular (Deligne–Rapoport), and its Jacobian $J_0(11)$ is an elliptic curve over $\mathbb{Q}$. For any prime $\ell \neq 11, p$, the Hecke correspondence T_ℓ extends over $\mathbb{Z}[1/11\ell]$ as a finite correspondence whose induced endomorphism on $J_0(11)$ is an isogeny of ℓ–power degree. By Theorem 5.3, for such p the following hold simultaneously:

- the toric rank τ_p of $(J_p)^0$ is constant along the T_ℓ–orbit;
- $\dim_{\mathbb{Q}_\ell} H^1_{\text{ét}}(J_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)^{I_p} = 2g - \tau_p$ is Hecke–invariant by Theorem 4.1;
- the Tamagawa number $c_{J,p} = \#\Phi_{J,p}(\mathbb{F}_p)$ varies only by a factor dividing a power of ℓ.

Geometric interpretation. The Hecke correspondence T_ℓ on the modular curve $X_0(11)$ is represented by two finite maps

$$\pi_1, \pi_2 : X_0(11\ell) \longrightarrow X_0(11),$$

each étale over U, inducing via the Picard functor a pair of morphisms on Jacobians

$$\pi_{1,*}, \pi_2^* : J_0(11) \longrightarrow J_0(11\ell) \longrightarrow J_0(11),$$

whose composition is the Hecke operator T_ℓ acting on $J_0(11)$ by an isogeny of degree ℓ^2. After removing finitely many bad primes (those dividing 11ℓ), both π_1, π_2 and the resulting isogeny extend to morphisms of Néron models, remaining finite étale on identity components and inducing maps $\Phi_{J,p} \to \Phi_{J,p}$ with kernel and cokernel killed by ℓ^2. Hence, by Theorem 5.3, all local invariants $(\tau_p, c_{J,p}, f_{J,p})$ remain uniform across the Hecke orbit for all $p \nmid 11\ell$.

Arithmetic verification. Since $J_0(11)$ is the elliptic curve $E\colon y^2 + y = x^3 - x^2 - 10x - 20$ of conductor 11, its reduction is good at all $p \neq 11$. Therefore $\tau_p = 0$ and $\Phi_{J,p}$ is trivial for all such p, implying that the Hecke orbit is arithmetically rigid on U. The local Euler factor is

$$L_p(J_0(11), s)^{-1} = 1 - a_p p^{-s} + p^{1-2s},$$

and remains identical for all T_ℓ–translates; in particular, a_p (the trace of Frobenius on $H^1_{\text{ét}}$) is invariant under T_ℓ.

Analytic consequence. For $p \neq 11, \ell$, we have

$$L_p(J_0(11), s)^{-1} = (1-p^{-s})^{\tau_p} P_p(p^{-s}), \qquad \deg P_p = 2-\tau_p,$$

and both τ_p and P_p are invariant on the Hecke orbit. The constancy of $\tau_p = 0$ confirms that the Néron model of $J_0(11)$ remains abelian with good reduction everywhere on U, while Theorem 5.3 shows that this behavior persists under all prime-to-p Hecke correspondences.

Example 5.7 (Necessity of the prime-to-p condition). Let $p \mid N$ or take $\ell = p$. Then the Hecke correspondence T_ℓ on $\mathcal{X}_0(N)$ no longer extends to a finite étale correspondence on the integral model at p; its degeneracy maps π_1, π_2 become inseparable at the special fiber, and the induced morphism on identity components of $J_0(N)$ at p may fail to be finite étale. Consequently:

- the equality of toric ranks τ_p along the T_ℓ–orbit can break down;
- the map $\Phi_{J,p} \to \Phi_{J,p}$ may acquire non–étale kernel or cokernel not annihilated by any power of ℓ;
- the bound on conductor exponents in Theorem 5.3(iii) can fail because wild inertia contributes to the Swan term.

Mechanism of failure. At primes $p \mid N$, the modular curve $\mathcal{X}_0(N)$ has a non–semistable special fiber: the cuspidal components intersect with multiplicity p, producing vertical components whose intersection points are inseparable under π_1, π_2. Hence, the morphisms induced on Néron identity components $(J_p)^0$ are not finite étale; the corresponding finite flat group schemes on the special fiber have connected components (e.g. α_p) rather than étale $\mathbb{Z}/p\mathbb{Z}$ parts.

Arithmetic manifestation. For $\ell = p$, wild inertia $P_p \trianglelefteq I_p$ acts nontrivially on $H^1_{\text{ét}}(J_{\mathbb{Q}_p}, \mathbb{Q}_\ell)$, producing a positive Swan term. Then

$$f_{T_p(J),p} - f_{J,p} = \left(\tau'_p - \tau_p\right) + \left(\mathrm{Swan}(H^1_{\text{ét}}(J'_{\mathbb{Q}_p}, \mathbb{Q}_\ell)) - \mathrm{Swan}(H^1_{\text{ét}}(J_{\mathbb{Q}_p}, \mathbb{Q}_\ell))\right).$$

may exceed 1 in absolute value, violating the tame bound from Theorem 5.3(iii). Thus, the hypothesis $\ell \neq p$ in Theorem 5.3 is essential: it excludes the primes where inseparability or wild inertia destroys the étaleness on identity components and uniformity of conductor behavior.

By Theorem 5.3 and Corollary 5.5, along T_ℓ–orbits with $\ell \neq p$ the τ_p–exponent of $(1-p^{-s})$ in $L_p(J_0(N), s)^{-1}$ is constant and $f_{J,p}$ varies in a range controlled by ℓ. Hence global $L(J_0(N), s)$ exhibits orbitwise uniformity of local factors outside $\{p, \ell\}$, and height bounds via Proposition 4.6 are stable along these orbits.

5.2 Relation with Shimura varieties

Notation 5.8 (Integral models and universal abelian schemes)**.** Let (G, X) be a Shimura datum of Hodge type with reflex field E. Fix a compact open $K = K^p K_p \subset G(\mathbb{A}_f)$ such that K_p is hyperspecial. For $v \nmid p$ a finite place of E where K_p is hyperspecial, let $\mathcal{S}_K$ denote the integral canonical model over $\mathcal{O}_{E,(v)}$ (Kisin; Faltings–Chai for PEL). There exists an abelian scheme $\mathcal{A} \to \mathcal{S}_K$ with additional structure realizing the Shimura data (Hodge type case), whose generic fiber is of dimension g.

Remark 5.9 (Prime-to-p Hecke action). For $h \in G(\mathbb{A}_f^p)$, the prime-to-p Hecke correspondence $[h] : \mathcal{S}_K \dashrightarrow \mathcal{S}_K$ is finite and induces an isogeny of $\mathcal{A}$ of degree dividing a fixed power of the common denominator of h; this is prime to p on the hyperspecial locus.

Theorem 5.10 (Conditional Hecke-uniformity on Hodge-type integral models)**.** *In the setting of Notation 5.8, assume that, after removing finitely many primes of bad reduction, the chosen Hodge-type integral model carries the relevant abelian scheme*

$$\mathcal{A} \longrightarrow \mathscr{S}$$

over a dense open

$$U \subseteq \operatorname{Spec} \mathcal{O}_{E,(v)}.$$

Fix a prime-to-p Hecke correspondence $[h]$. Assume further that there exists a dense open

$$U_h \subseteq U$$

on which $[h]$ extends as a finite correspondence whose induced maps on the relevant abelian scheme are genuine prime-to-residue-characteristic isogenies with finite étale kernels.

Then, for every closed point $t \in U_h$, the following conclusions hold:

(i) The induced isogenies on the fibers of $\mathcal{A}$ have degree prime to the residue characteristic. Hence, on identity components, the corresponding maps are finite étale, and the induced morphisms on component groups

$$\Phi_{\mathcal{A},t} \longrightarrow \Phi_{\mathcal{A},t'}$$

for points $t, t' \in U_h$ related by the correspondence have kernel and cokernel annihilated by a number depending only on the degree of $[h]$.

(ii) For every $\ell \neq \mathrm{char}(\kappa(t))$, the induced maps preserve inertia-invariant cohomology:

$$H^1_{\text{ét}}(\mathcal{A}_{\overline{K(t)}}, \mathbb{Q}_\ell)^{I_t} \xrightarrow{\sim} H^1_{\text{ét}}(\mathcal{A}_{\overline{K(t')}}, \mathbb{Q}_\ell)^{I_{t'}}.$$

In particular, on the fixed tame open U_h, the dimension of the inertia-invariant part of $H^1_{\text{ét}}$ and the toric rank are invariant along the prime-to-residue-characteristic isogenies induced by $[h]$.

(iii) The Artin conductor of the rational ℓ-adic Tate representation is unchanged under these isogenies:

$$f_{\mathcal{A},t} = f_{\mathcal{A},t'}.$$

The component-group defect terms, however, are uniformly controlled by the degree of the correspondence. Thus the statement gives uniform control of tame local geometric data and component-group indices along the $[h]$-correspondence, not a new conductor-variation inequality.

Novelty and relation to the global theory. This theorem records the formal consequence of the moduli-theoretic extension hypotheses imposed above. It should not be read as proving, from scratch, that every Hodge-type Hecke correspondence extends integrally with the required finite-étale kernel property.

Under those hypotheses, the prime-to-residue-characteristic isogeny-control of Theorem 3.1 and the finite-set uniformity of Theorem 3.5 apply on the dense open U_h. Consequently, the correspondence preserves the relevant tame inertia-invariant cohomology and controls the component-group defect terms along the induced isogenies. The resulting statement is a conditional Hecke-uniformity result on the chosen tame locus.

Proof. By hypothesis, the fixed prime-to-p Hecke correspondence $[h]$ extends over the dense open U_h as a finite correspondence whose induced maps on the relevant abelian scheme are prime-to-residue-characteristic isogenies with finite étale kernels. Therefore the hypotheses of Theorem 3.1 apply to each such isogeny on U_h.

Applying Theorem 3.1 gives finite-étale behavior on identity components, while Theorem 3.5 gives uniform control of the induced maps on component groups, with bounds depending only on the degree of the fixed correspondence $[h]$. The isomorphism on inertia invariants follows from Theorem 4.1(ii).

Finally, since the rational ℓ-adic Tate modules of isogenous abelian varieties are isomorphic, the Artin conductor exponent is unchanged under the isogenies considered here. Thus the theorem asserts uniform control of the tame local geometric and component-group data, rather than a conductor-variation inequality. □

Corollary 5.11 (Invariant local factors on Hecke orbits for $\mathcal{A}\to\mathcal{S}_K$)**.** *On U_h and for $\ell \neq \mathrm{char}(\kappa(t))$, the inertia-invariant local factor is*

$$P^I_{\mathcal{A},t}(X) = \det\bigl(1 - X\mathrm{Frob}_t \mid V_\ell(\mathcal{A})^{I_t}\bigr)\,.$$

It has degree $2g - \tau_t$ and is constant along the $[h]$-orbit. If the torus in the Raynaud extension is split, then the corresponding local factor contains the factor $(1 - \mathbf{N}t^{-s})^{\tau_t}$. In the non-split toric case, the toric factor is governed by Frobenius on the character lattice of the torus, and no such split factor is asserted.

Proof. Immediate from Corollary 4.3 applied in the Hodge-type context together with Theorem 5.10(ii). □

Example 5.12 (Siegel modular case). Take $G = \mathrm{GSp}_{2g}$ with hyperspecial K_p; then $\mathcal{S}_K$ is the Siegel moduli scheme parametrizing principally polarized abelian schemes $\mathcal{A}\to\mathcal{S}_K$ of relative dimension g. For every rational prime $\ell \neq p$, the ℓ-adic Tate module $T_\ell(\mathcal{A})$ is equipped with the standard symplectic form preserved by G, and the prime-to-p Hecke algebra acts, on the stated integral locus, through correspondences induced by prime-to-p isogenies preserving this symplectic structure up to the usual similitude factor.

Geometric picture. On a dense open $U_h \subseteq \operatorname{Spec}\mathcal{O}_{E,(v)}$ (obtained by removing finitely many bad primes), the Hecke correspondence $[h]$ induces a prime-to-p isogeny

$$[h]\colon (\mathcal{A}_t, \lambda_t) \longrightarrow (\mathcal{A}'_t, \lambda'_t)$$

between principally polarized abelian varieties, with kernel finite étale and killed by the Hecke degree $\deg([h])$. By Theorem 5.10(i), this isogeny extends, in the stated Dedekind/codimension-one sense, to a morphism on the relevant Néron models whose restriction to identity components is finite étale, and it induces on component groups

$$\Phi_{\mathcal{A},t} \longrightarrow \Phi_{\mathcal{A}',t}$$

a morphism whose kernel and cokernel are annihilated by $\deg([h])$. Thus the component indices are controlled in the kernel–cokernel sense; no divisibility formula for $c_{\mathcal{A}',t}/c_{\mathcal{A},t}$ is asserted without the corresponding finite-group calculation.

Cohomological and arithmetic consequences. For $\ell \neq p$ the induced map on inertia-invariants,

$$H^1_{\text{ét}}(\mathcal{A}_{\overline{K(t)}}, \mathbb{Q}_\ell)^{I_t} \xrightarrow{\simeq} H^1_{\text{ét}}(\mathcal{A}'_{\overline{K(t)}}, \mathbb{Q}_\ell)^{I_t},$$

is an isomorphism by Theorem 5.10(ii). Hence

$$\dim_{\mathbb{Q}_\ell} H^1_{\text{ét}}(\cdot)^{I_t} = 2g - \tau_t,$$

and τ_t is constant on the Hecke orbit. The component-group maps have kernel and cokernel killed by $\deg([h])$, so the associated Tamagawa-index and local geometric defect terms are uniformly controlled. The Artin conductor exponent itself is unchanged under the isogeny:

$$f_{\mathcal{A}',t} = f_{\mathcal{A},t}.$$

Equivalently, the inertia-invariant local factor

$$P^I_{\mathcal{A},t}(X) = \det\bigl(1 - X\mathrm{Frob}_t \mid V_\ell(\mathcal{A})^{I_t}\bigr)$$

is Hecke-invariant on U_h. Only when the torus in the Raynaud extension is split does the corresponding local factor contain the explicit factor

$$(1 - \mathbf{N}t^{-s})^{\tau_t}.$$

In the non-split toric case, the toric factor is governed by Frobenius on the character lattice of the torus, and no split-torus factorization is asserted. This realizes, in the Siegel case, the geometric-to-arithmetic uniformity asserted in Theorem 5.10, with the Artin conductor treated as isogeny-invariant rather than bounded by component-group variation.

Analytic viewpoint. Since each prime-to-p Hecke operator is an isogeny of degree prime to all residue characteristics on U_h, the inertia-invariant local factors

$$P^I_{\mathcal{A},t}(X)$$

form a single Hecke-invariant family on each Hecke orbit. The invariant factors coincide; explicit toric factors are asserted only in the split-torus case. Thus the Hodge-type integral model behaves, on the fixed tame locus and under the stated extension hypotheses, as a single prime-to-p isogeny class for the purposes of inertia-invariant cohomology, component-group control, and local geometric defect terms.

Example 5.13 (Failure at parahoric level). If K_p is parahoric (non-hyperspecial) or if the Hecke correspondence $[h]$ has degree divisible by the residue characteristic, the morphism on identity components in Theorem 5.10 need *not* be finite étale, and the uniform bounds on $\Phi_{\mathcal{A},t}$ and $f_{\mathcal{A},t}$ can fail.

Geometric mechanism. At such primes the local model of $\mathcal{S}_K$ is singular, with non-semistable vertical components in the special fiber. The kernel of the Hecke correspondence $[h]$ on $\mathcal{A}$ then meets the identity component nontrivially, so its special fiber contains a connected local subgroup (e.g. α_p) rather than a finite étale one. Consequently,

$$(\mathcal{A}_t)^0 \longrightarrow (\mathcal{A}'_t)^0$$

fails to be finite étale, and the induced map on component groups $\Phi_{\mathcal{A},t} \to \Phi_{\mathcal{A}',t}$ may have kernel or cokernel not killed by any power of p.

Cohomological consequence. On the Galois side, wild inertia $P_t \trianglelefteq I_t$ acts nontrivially on $H^1_{\text{ét}}$, creating a positive Swan term:

$$f_{\mathcal{A}',t} - f_{\mathcal{A},t} = \left(\tau'_t - \tau_t\right) + \left(\mathrm{Swan}(H^1_{\text{ét}}(\mathcal{A}'_{K(t)}, \mathbb{Q}_\ell)) - \mathrm{Swan}(H^1_{\text{ét}}(\mathcal{A}_{K(t)}, \mathbb{Q}_\ell))\right) > 0.$$

Hence the tame conductor control of Theorem 5.10(iii) breaks down.

Analytic manifestation. The tame Euler-factor bookkeeping used above is no longer available at such a wild boundary point. Any contribution here should be treated as an auxiliary wild local defect term, not as a bounded conductor-jump statement.

On U_h, Corollary 5.11 shows invariance of the tame factor $(1 - \mathbf{N}t^{-s})^{\tau_t}$ and uniform control of $f_{\mathcal{A},t}$ along Hecke orbits. Hence the global L-function of the family exhibits orbit-wise stability outside a finite set of primes, and height lower bounds derived from $\omega_{\mathcal{A}/\mathcal{S}_K}$ (cf. Construction 2.10 and Proposition 4.6) are preserved across prime-to-p Hecke translates.

5.3 Forward linkage to arithmetic applications

Theorems 5.3 and 5.10 provide the moduli-level input for the number-theoretic statements of the next section: uniformity of tame local Euler-factor shapes, preservation of inertia-invariant cohomology, and uniform control of component-group defect terms along the Hecke correspondences within their stated hypotheses. Section 6 uses these facts only on the fixed tame opens where the relevant Hecke correspondences induce genuine prime-to-residue-characteristic isogenies with finite étale kernels.

6 Applications to Number Theory

We keep the base and notation from **??**. Structural controls for Néron models in families and under prime-to-residue-characteristic isogenies were established in Theorems 3.1 and 3.5. The cohomological input is the preservation of inertia-invariant $H^1_{\text{ét}}$ on the fixed tame locus, as in Theorem 4.1. The applications below concern tame local Euler-factor shapes, component-group defect terms, and compatibility of local height decompositions. They do not assert conductor variation under isogeny and do not prove a new uniform positive height gap.

Standing hypotheses for Section 6. All local and global bounds below are invoked only on the dense tame locus U from Theorem 3.5; in particular, every prime-to-p isogeny/Hecke map is used only where its kernel is finite étale and the induced morphism on component groups has uniformly bounded kernel/cokernel (Theorems 3.1 and 3.5). Any Hecke-orbit input is taken from Section 5 in its stated scope.

6.1 Zeta functions and L-functions

Proposition 6.1 (Finite Euler quotient in prime-to-p isogeny/Hecke families)**.** *Let $A/\mathbb{Q}$ be an abelian variety with Néron model A/S, and let $\varphi : A \to B$ be an isogeny of degree m with $(m,p) = 1$ for all but finitely many primes. Then there exists a finite set of primes Σ (containing those dividing m and the finitely many primes excluded in Theorems 3.1 and 3.5) such that*

$$\frac{L(B,s)}{L(A,s)} = \prod_{p\in\Sigma} R_p(p^{-s}),$$

where each $R_p(X) \in \mathbb{Q}(X)$ is a rational function whose denominator divides a power of m. Moreover, for $p \notin \Sigma$ one has

$$L_p(B,s)^{-1} = (1-p^{-s})^{\tau_p}\, P_{A,p}(p^{-s}),$$

and

$$L_p(A,s)^{-1} = (1-p^{-s})^{\tau_p}\, P_{A,p}(p^{-s}).$$

with the same τ_p and polynomial $P_{A,p}$ as in Corollary 4.3.

Proof. By Theorems 3.1 and 3.5, after removing finitely many primes the isogeny extends over S with prime-to-p degree and induces maps on component groups whose kernel and cokernel are killed by m. Then Theorem 4.1(ii) gives an isomorphism on I_p–invariants of $H^1_{\text{ét}}$ for $\ell \neq p$, and Corollary 4.3 fixes the local shape. Thus, outside a finite set Σ, the local factors of A and B coincide; at $p \in \Sigma$ their ratio is a rational function with denominator dividing a power of m by Theorems 3.1 and 3.5. □

Corollary 6.2 (Local conductor stability)**.** *Under the hypotheses of Proposition 6.1, the local Artin conductor exponents of A and B agree at every prime where the comparison is made:*

$$f_{B,p} = f_{A,p}.$$

The component-group estimates used above control only Tamagawa-index and local geometric defect terms; they do not give, and are not used to give, a conductor-variation inequality.

Proof. The rational ℓ-adic Tate modules of isogenous abelian varieties are isomorphic as Galois representations. Hence their Artin conductor exponents agree. The component-group bounds from Theorems 3.1 and 3.5 are separate Tamagawa-index estimates and do not measure variation of the Artin conductor. □

Example 6.3 (Modular Jacobians along Hecke orbits). For $J_0(N)$ over $\mathbb{Q}$ and a prime $\ell \nmid N$, Theorem 5.3 yields a dense open U_ℓ where each T_ℓ–translate acts by an ℓ–power isogeny prime to p for $p \in U_\ell$. Hence Proposition 6.1 and Corollary 5.5 give

$$\frac{L(T_\ell(J_0(N)), s)}{L(J_0(N), s)} = \prod_{p \in \Sigma_\ell} R_{p,\ell}(p^{-s})$$

with $\Sigma_\ell \subset \{p : p \mid N\ell\} \cup$ (finitely many), and $R_{p,\ell}$ having denominators dividing a power of ℓ. In particular, outside Σ_ℓ the tame exponent τ_p and the polynomial part P_p remain unchanged along the T_ℓ–orbit.

Example 6.4 (Necessity of the prime-to-p condition). Let $E/\mathbb{Q}$ admit a p–isogeny with additive reduction at p. Then $\ker(\varphi)$ is not étale on the special fiber and Theorem 4.1(ii) need not hold for I_p–invariants. The quotient $L(\varphi(E), s)/L(E, s)$ may acquire a wild factor at p not controlled by powers of p alone, demonstrating the necessity of excluding such primes in Proposition 6.1.

By Proposition 6.1 and Corollary 6.2, isogenies and prime-to-p Hecke correspondences alter global L–functions by a finite Euler quotient whose denominators are uniformly bounded in terms of the isogeny degree, while preserving the tame exponent τ_p off a finite set. This gives orbitwise stability of local factors and conductors on the loci from Theorems 5.3 and 5.10.

6.2 Height theory and rational points

Proposition 6.5 (Recorded compatibility of local height decompositions under tame isogenies)**.** *Let $A/\mathbb{Q}$ be an abelian variety with Néron model $\mathcal{A}/S$, and let L be a fixed symmetric ample line bundle on A. Then there exists a finite set of primes Σ such that, on the fixed semistable tame locus outside Σ, the local terms appearing in the standard canonical-height decomposition are compatible with the prime-to-residue-characteristic isogenies considered in this paper.*

More precisely, if B is prime-to-Σ isogenous to A and

$$\varphi : A \to B$$

is a prime-to-Σ isogeny, then:

(i) The canonical heights satisfy the standard functoriality relation

$$\widehat{h}_{\varphi^* L}(P) = \widehat{h}_L(\varphi(P)) \qquad \text{for all } P \in A(\mathbb{Q}).$$

The comparison of canonical heights on $B(\mathbb{Q})$ is understood only through the standard functoriality and comparison formulas for canonical heights under isogeny. In particular, we do not assert here any new uniform additive bound, global height gap, or positive lower bound across an arbitrary prime-to-Σ isogeny class.

(ii) On the fixed semistable tame open, the toric local terms, the contribution of $\omega_{\mathcal{A}/S}$, and the component-group defect terms appearing in the standard local height decomposition remain uniformly controlled under the prime-to-Σ isogenies considered here.

This proposition is a compatibility statement only. It does not assert a new uniform positive lower bound for all non-torsion points in a prime-to-Σ isogeny class.

Proof. Item (i) is the standard functoriality of the Néron–Tate height under isogeny; see, for example, [10] and the standard references on canonical heights on abelian varieties.

On the fixed semistable tame locus, Theorem 4.1(ii) provides the preservation of inertia-invariant cohomology and toric-rank data under the prime-to-residue-characteristic isogenies considered here. Together with the identification of the tame local factors in Corollary 4.3 and the role of the invariant differential sheaf $\omega_{\mathcal{A}/S}$ from Construction 2.10, this yields compatibility of the corresponding local geometric terms in the standard canonical-height decomposition.

No global height-gap theorem or new uniform positive lower bound is claimed. □

Corollary 6.6 (Finiteness criterion for rational points)**.** *Let $C/\mathbb{Q}$ be a smooth projective curve and let $\phi : C \to A$ be a nonconstant morphism to an abelian variety $A/\mathbb{Q}$ with Néron model $\mathcal{A}/S$. Assume:*

(a) $A(\mathbb{Q})$ has rank 0;

(b) the map ϕ has finite fibers on $C(\mathbb{Q})$ (equivalently: no positive-dimensional translate of an abelian subvariety is contained in $\phi(C)$).

Then $C(\mathbb{Q})$ is finite. (Compare the general Mordell–Lang/Faltings finiteness perspective in [8].) Moreover, if A varies within a prime-to-Σ isogeny class for some finite Σ, and (a)–(b) hold for one representative, then they hold uniformly across the class and the finiteness of $C(\mathbb{Q})$ persists.

Proof. By (a), $A(\mathbb{Q})$ is finite. Thus $\phi(C(\mathbb{Q}))$ is finite. By (b), each rational point in the image has finitely many preimages, hence $C(\mathbb{Q})$ is finite. For the uniformity statement, use Proposition 6.5(i) to transport rank 0 across the prime-to-Σ isogeny class and note that (b) is unaffected by replacing A with an isogenous abelian variety and composing ϕ with an isogeny. □

Example 6.7 (Application to modular curves via rank zero). Let X be a modular curve (e.g. a quotient of $X_0(N)$) with Jacobian J and Abel–Jacobi map $\phi : X \to J$. If $J(\mathbb{Q})$ has rank 0 (this occurs for several small levels), then Corollary 6.6 gives $X(\mathbb{Q})$ finite. Under prime-to-p Hecke correspondences on the dense opens of Theorem 5.3, the conclusion persists across the orbit.

Example 6.8 (Necessity of the rank hypothesis). Let $E/\mathbb{Q}$ be an elliptic curve of positive rank and take $C = E$ with $\phi = \mathrm{id}_E$. Then $C(\mathbb{Q})$ is infinite, so Corollary 6.6 fails without (a).

Proposition 6.5 shows that tame local contributions and the differential line $\omega_{A/S}$ yield uniform height information across prime-to-p isogeny classes and Hecke orbits on the loci of Theorems 5.3 and 5.10. Combined with Northcott's property for canonical heights and the rank-0 hypothesis, Corollary 6.6 gives concrete finiteness statements for rational points on curves mapping to such abelian varieties.

6.3 Link to examples

The orbitwise stability of tame local factors and the compatibility of local height contributions provide the input for the explicit computations in Section 7.

7 Examples and Computations

We illustrate the structural and cohomological statements from Theorems 3.1, 3.5 and 4.1 and the arithmetic consequences from Corollaries 4.3, 6.2 and 6.6 and propositions 6.1 and 6.5 by working out two standard testbeds: an elliptic curve with CM and a modular Jacobian. Throughout we keep the notation from Notation 2.1, definitions 2.6 and 2.13, and construction 2.10.

Example 7.1 (The curve $y^2 = x^3 - x$: tame isogeny control, component groups, and local factors). Let

$$E/\mathbb{Q}: \quad y^2 = x^3 - x.$$

This curve has $j(E) = 1728$, admits the 2-isogeny

$$\varphi_\eta : E \longrightarrow E', \qquad \varphi_\eta(x,y) = \left(\frac{x^2+1}{x}, \frac{y(x^2-1)}{x^2}\right),$$

where

$$E': \quad y^2 = x^3 + 4x.$$

The discriminants are

$$\Delta_E = -64, \qquad \Delta_{E'} = -4096.$$

Set

$$T = S = \operatorname{Spec}\mathbb{Z}, \qquad U = T^\circ = \operatorname{Spec}\mathbb{Z} \setminus \{2\}.$$

Then the isogeny degree is invertible on U, and the kernel of φ_η extends over U to a finite étale subgroup of $E[2]$. Hence the prime-to-residue-characteristic conclusions of Theorem 3.1 and Theorem 3.5 apply on U.

Good-reduction and component-group computation on U. For every odd prime $p \neq 2$, both E and E' have good reduction. Thus

$$\Phi_{E,p} = \Phi_{E',p} = 0, \qquad c_{E,p} = c_{E',p} = 1.$$

The induced morphism on component groups

$$\Phi_{E,p} \longrightarrow \Phi_{E',p}$$

has kernel and cokernel killed by 2, trivially in this good-reduction case. Therefore, for every rational prime $q \neq 2$, the function

$$p \longmapsto v_q(\#\Phi_{E,p})$$

is constant on U, hence upper semi-continuous. This gives a concrete instance of the fixed-tame-open control asserted in Theorem 3.5.

General multiplicative comparison. More generally, if $F/\mathbb{Q}$ is an elliptic curve admitting a 2-isogeny $F \to F'$, and $p \neq 2$ is a split multiplicative prime lying in the chosen tame locus, then

$$F_p^0 \simeq \mathbb{G}_m, \qquad \Phi_{F,p} \simeq \mathbb{Z}/\operatorname{ord}_p(\Delta_F)\mathbb{Z}.$$

The extended isogeny is finite étale on identity components, and

$$\ker(\Phi_{F,p} \to \Phi_{F',p}), \quad \operatorname{coker}(\Phi_{F,p} \to \Phi_{F',p})$$

are killed by 2. Consequently the associated Tamagawa-index defect is controlled by the isogeny degree; in the elliptic 2-isogeny case one obtains only 2-power variation in the component-group quotient.

Local factors and inertia on the good-reduction locus. For $p \neq 2$, the local Euler factor of E has the usual good reduction form

$$L_p(E,s)^{-1} = 1 - a_p p^{-s} + p^{1-2s},$$

and for $\ell \neq p$,

$$\dim_{\mathbb{Q}_\ell} H^1_{\text{ét}}(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)^{I_p} = 2.$$

Since E and E' are isogenous, their rational ℓ-adic Tate modules are isomorphic. Thus the Artin conductor exponents agree:

$$f_{E,p} = f_{E',p}.$$

On U, in fact,

$$\frac{L_p(E',s)}{L_p(E,s)} = 1, \qquad f_{E',p} - f_{E,p} = 0.$$

This is the cohomological invariance predicted by Theorem 4.1, not a new conductor-variation estimate.

The excluded prime $p = 2$. At $p = 2$, Tate's algorithm gives additive reduction for E (of Kodaira type III). Hence 2 is excluded from the tame open U. The 2-isogeny is no longer prime to the residue characteristic, and its kernel need not extend as a finite étale subgroup scheme over $\mathbb{Z}_2$. Thus the prime-to-p conclusions of Theorem 3.1 and Theorem 3.5 do not apply at 2.

Differentials and height compatibility. Let $\mathcal{E}/S$ be the Néron model of E, and let

$$\omega = \frac{dx}{2y}$$

be the invariant differential. The contribution of $\omega_{\mathcal{E}/S}$ to the standard Arakelov–Néron height decomposition is compatible with the tame local geometric data controlled above. This is only a compatibility statement, in the sense of Proposition 4.6; it is not a new uniform positive lower bound for all non-torsion points in arbitrary isogeny classes.

Example 7.2 (Jacobians of modular curves). We detail the case $J_0(11)$, which is an elliptic curve $E_{11}/\mathbb{Q}$ (the Jacobian of $X_0(11)$), and then indicate the general pattern.

Local structure at $p = 11$. On the regular integral model of $X_0(11)$ (Deligne–Rapoport; Katz–Mazur), the Néron model of $J_0(11)$ over $\mathbb{Z}$ has split multiplicative reduction at $p = 11$ of type I_5. Hence

$$\tau_{11} = 1, \qquad c_{11} = 5, \qquad f_{11} = 1,$$

and for $p \neq 11$ one has good reduction with $c_p = 1$ and $f_p = 0$. This fits the template from Corollary 4.3:

$$L_{11}(J_0(11), s)^{-1} = (1-11^{-s}) \cdot P_{11}(11^{-s}), \qquad \deg P_{11} = 1.$$

Hecke action and uniformity. Let $\ell \neq 11$ be prime. Over $\mathbb{Z}[1/11\ell]$ the Hecke correspondence T_ℓ induces an ℓ–power isogeny of $J_0(11)$ (Remark 5.2). By Theorem 5.3 there exists a dense open $U_\ell \subseteq \operatorname{Spec}\mathbb{Z}$ such that for every $p \in U_\ell$ the induced map on the special fiber is prime-to-p on identity components and

$$\Phi_{J,p} \longrightarrow \Phi_{J,p} \quad \text{has kernel/cokernel killed by a power of } \ell,$$

so τ_p and $\dim H^1_{\text{ét}}(\cdot)^{I_p}$ are constant along the T_ℓ–orbit (Theorem 4.1(ii)) and $f_{J,p}$ varies within the bound of Theorem 4.1(iii). In particular,

$$\frac{L(T_\ell(J_0(11)), s)}{L(J_0(11), s)} = \prod_{p \in \Sigma_\ell} R_{p,\ell}(p^{-s})$$

for a finite set Σ_ℓ and $R_{p,\ell}$ with denominators dividing a power of ℓ (Proposition 6.1).

Global consequences. Since $J_0(11)$ has Mordell–Weil rank 0 over $\mathbb{Q}$ and the Abel–Jacobi map $X_0(11) \to J_0(11)$ has finite fibers on rational points, Corollary 6.6 applies to conclude $X_0(11)(\mathbb{Q})$ is finite; the conclusion persists along prime-to-ℓ Hecke translates on the open loci of Theorem 5.3. The same mechanism applies to higher level modular curves $X_0(N)$ with Jacobians of rank 0; the local invariants τ_p, $c_{J,p}$, and $f_{J,p}$ behave uniformly along T_ℓ–orbits away from finitely many primes by Theorem 5.3 and corollary 5.5.

Uniform control of $\Phi_{J,p}$ and τ_p along T_ℓ–orbits yields stability of the tame factor in local Euler-factor bookkeeping and controls Tamagawa-index/local geometric defect terms; coupled with height compatibility (Proposition 6.5) this gives the stated finiteness consequences.

Example 7.3 (Tate curve with split multiplicative reduction). Fix a prime p and take a parameter $q \in p\mathbb{Z}_p$ with $n := v_p(q) \geq 1$. The Tate curve $E_q/\mathbb{Q}_p$ satisfies

$$E_q(\mathbb{Q}_p) \simeq \mathbb{Q}_p^\times/q^{\mathbb{Z}}, \qquad \text{Kodaira type } \mathrm{I}_n, \qquad \tau_p = 1,$$

and the Néron special fiber has $(E_q)_p^0 \simeq \mathbb{G}_m$ and $\Phi_{E_q,p} \simeq \mathbb{Z}/n\mathbb{Z}$. Hence by Proposition 2.8 the Tamagawa number is $c_p = n$, and by Corollary 4.3

$$L_p(E_q, s)^{-1} = (1 - p^{-s}) \cdot P_p(p^{-s}) \quad \text{with} \quad \deg P_p = 1.$$

Here the toric rank $\tau_p = 1$ forces the $(1 - p^{-s})$ factor and determines $\deg P_p = 1$; the explicit $\#\Phi_{E_q,p} = n$ realises the template of Corollary 4.3 and Proposition 2.8.

Example 7.4 (Potentially good, wild at p). Let $E/\mathbb{Q}$ be an elliptic curve with additive, potentially good reduction at a prime p (e.g. CM $j = 0$ or 1728 at certain p). Then $(E_p)^0$ is abelian, so $\tau_p = 0$ and

$$\dim_{\mathbb{Q}_\ell} H^1_{\text{ét}}(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)^{I_p} = 2 \quad \text{after killing the wild part,}$$

but over $\mathbb{Q}_p$ one may have $\operatorname{Swan}\big(H^1_{\text{ét}}(E_{\mathbb{Q}_p}, \mathbb{Q}_\ell)\big) > 0$, so

$$f_{E,p} = \tau_p + \operatorname{Swan} = \operatorname{Swan} > 0.$$

This shows that the tame bounds in Theorem 4.1(*iii*) hinge on excluding wild inertia.

Even with $\tau_p = 0$ (no torus, no $(1 - p^{-s})$), wild inertia can raise the conductor; this is exactly the "wild term" appearing in Theorem 4.1(*iii*) and the discussion following Corollary 4.3.

Example 7.5 (Product surface with mixed reduction). Let $A = E_1 \times E_2$ over $\mathbb{Q}$ with Néron model $\mathcal{A}/S$. Suppose at a prime p that E_1 has split multiplicative reduction of type I_{n_1} and E_2 has good reduction. Then

$$\tau_p(A) = \tau_p(E_1) + \tau_p(E_2) = 1 + 0 = 1,$$

and

$$\Phi_{A,p} \simeq \Phi_{E_1,p} \times \Phi_{E_2,p} \simeq \mathbb{Z}/n_1\mathbb{Z}.$$

hence

$$L_p(A, s)^{-1} = (1-p^{-s})^{\tau_p} \cdot P_p(p^{-s}), \quad \deg P_p = 2 \cdot 2 - \tau_p = 3,$$

in accordance with Corollary 4.3. If both E_i are split multiplicative with types I_{n_i}, then $\tau_p(A) = 2$ and $\Phi_{A,p} \simeq \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z}$; correspondingly $\deg P_p = 2 \cdot 2 - \tau_p = 2$.

The toric rank adds and forces $(1 - p^{-s})^{\tau_p}$; the degree drop of P_p matches $2g - \tau_p$ (Corollary 4.3), while $c_{A,p} = \#\Phi_{A,p}$ multiplies (Proposition 2.8).

Example 7.6 (Hecke-uniformity away from the level). Let $N \geq 1$ and consider $J_0(N)$ with Néron model over S. For a prime $\ell \nmid N$ there exists a dense open $U_\ell \subseteq \operatorname{Spec}\mathbb{Z}$ such that, for every $p \in U_\ell$ with $p \neq \ell$, the Hecke correspondence T_ℓ induces a prime-to-p isogeny on identity components and a map

$$\Phi_{J,p} \longrightarrow \Phi_{J,p} \quad \text{with kernel/cokernel killed by a power of } \ell.$$

Therefore τ_p and $\dim H^1_{\text{ét}}(J_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)^{I_p}$ are constant along the T_ℓ–orbit, and

$$L_p(T_\ell J_0(N), s)/L_p(J_0(N), s) = R_{p,\ell}(p^{-s})$$

has denominator dividing a power of ℓ (cf. Theorem 5.3 and its corollaries).

Counterexample (sharpness at $\ell = p$*).* If $\ell = p$ or $p \mid N$, the degeneracy maps are not finite étale on the special fiber; kernels on identity components may become connected (e.g. α_p), so the tame bounds fail and wild Swan terms can appear. Thus Theorem 5.3 genuinely requires $\ell \neq p$ and $p \nmid N$.

On U_ℓ the tame exponent τ_p and P_p in $L_p(J_0(N), s)^{-1} = (1-p^{-s})^{\tau_p} P_p(p^{-s})$ are Hecke-invariant, while $f_{J,p}$ can change only within the explicit ℓ–power bound; for $\ell = p$ this uniformity breaks.

The preceding examples show where the tame hypotheses are used and where they fail. They also illustrate how the component-group bounds of Theorem 3.5 enter the local-factor comparisons.

8 Conclusion and Outlook

Synthesis. Working over the arithmetic base from Notation 2.1, the paper established a geometric–cohomological control principle for Néron models and traced its arithmetic fallout along three axes.

Chain I (existence/rigidity ⇒ component groups ⇒ local factors). By Theorem 3.1 the prime-to-residue-characteristic extension and uniqueness of isogenies in arithmetic families hold on a dense open. Upper semicontinuity of component groups in fibers (Theorem 3.5) then yields the local factor template (Corollary 4.3), which identifies the tame exponent τ_p and constrains $L_p(A, s)$. The mechanism appears concretely in Example 7.1 and in the Hecke setting of Example 6.3.

Chain II (cohomology under base change ⇒ Euler quotients ⇒ conductor stability). The comparison Theorem 4.1 (items (i)–(iii)) proves invariance of I_p-invariants under prime-to-p isogenies on suitable opens, and quantifies possible jumps. This feeds directly into the finite Euler quotient statement (Proposition 6.1) and its conductor control (Corollary 6.2). Example 6.4 shows the sharpness of the prime-to-p hypothesis.

Chain III (differentials/heights ⇒ local height compatibility ⇒ conditional finiteness). The differential line bundle from Construction 2.10 enters the standard local height decomposition recorded in Proposition 4.6, and Proposition 6.5 records compatibility of those local terms under the tame prime-to-p isogenies considered here. Under the separate rank-0 and finite-fiber hypotheses, this gives the conditional finiteness criterion for rational points in Corollary 6.6, illustrated by Example 6.7 and contrasted by Example 6.8.

Outlook

(A) Higher-dimensional generalizations. Two concrete directions arise naturally.

- *Semi-abelian extensions over parahoric level.* Replacing abelian by semi-abelian schemes on integral models with parahoric level structure (cf. Theorem 5.10) should keep the tame exponent τ_p stable on the same open loci and refine conductor bounds for boundary degenerations; see also the parahoric counterexamples in Example 5.13 for necessary hypotheses.

- *Families over Dedekind subschemes.* Passing from $S = \operatorname{Spec}\mathbb{Z}$ to Dedekind subschemes of number fields does not alter the proofs in Theorems 3.1, 3.5 and 4.1; the local statements transport verbatim after replacing p by nonarchimedean places and tracking ramification indices.

(B) Derived and perverse sheaf refinements. The comparison in Theorem 4.1 can be lifted from invariants of $H^1_{\text{ét}}$ to a functorial statement for truncations of the vanishing-cycles complex $\mathcal{R}\Phi$ on the special fiber (cf. [5]). A derived-level control of the unipotent monodromy filtration would turn the inequalities in (iii) into equalities under test conditions (e.g. semistability plus a weight-monodromy constraint), providing sharper conductor formulas at wild primes.

(C) Motivic cohomology and regulators. The line $\omega_{A/S}$ in Construction 2.10 admits an interpretation via Deligne–Beilinson realizations. A comparison of the resulting regulators with the canonical heights used in Proposition 6.5 would clarify how far the finiteness mechanism in Corollary 6.6 extends beyond rank 0, for instance to situations with controlled positive rank but auxiliary height gaps on images of curves.

(D) Equidistribution along Hecke orbits. On the open loci of Theorem 5.3, the component-group and tame-exponent stability suggests Sato–Tate–type equidistribution for local factors along prime-to-p Hecke orbits (outside finite sets), with error terms depending only on the orbit's isogeny parameters. Example 5.6 and Example 5.12 provide natural testbeds.

(E) Effective and computational aspects. The explicit cases in Examples 7.1 and 7.2 can be extended to large families, supplying numerical evidence for the bounds in Corollary 6.2 and for the persistence of finiteness in Corollary 6.6. A systematic implementation of the local shape in Corollary 4.3 alongside height lower bounds from Proposition 6.5 could yield practical algorithms for certifying finiteness of $C(\mathbb{Q})$ when C maps to low-rank Jacobians.

(F) Open questions.

- *Hecke-uniform conductors.* Under the hypotheses of Theorem 5.3, is the set $\{f_{A,p}\}_{p\nmid m}$ constant on prime-to-m Hecke orbits for a Zariski-dense set of parameters?

- *Derived conductor formula.* Can the quantitative bound in Theorem 4.1(iii) be upgraded to an identity expressed via ranks of the graded pieces of the monodromy filtration on $\mathcal{R}\Phi$?

- *Regulator comparison.* For maps $\phi : C \to A$ as in Corollary 6.6, does a regulator lower bound on $\phi(C)$ suffice to deduce finiteness of $C(\mathbb{Q})$ without assuming rank 0?

Closing remark. The results above show that, on a common dense tame locus, prime-to-residue-characteristic isogenies and the associated control of component groups and inertia invariants impose uniform constraints on local Euler factors, conductors, and the variation of height-theoretic data (via $\omega_{A/S}$). The computations in Examples 7.1 and 7.2 verify the quantitative bounds in concrete families and indicate directions for extending the methods to higher-dimensional bases and higher-rank settings.

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