COHOMOLOGICAL MONODROMY, VANISHING CYCLES, AND COMPONENT–GROUP CORRECTIONS FOR ELLIPTIC CURVES WITH BAD REDUCTION: A UNIFORM ℓ–INDEPENDENT LEFSCHETZ–CONDUCTOR FRAMEWORK VIA NÉRON MODELS, NEARBY CYCLES, AND FROBENIUS GEOMETRY

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Abstract. We give a uniform reformulation of standard nearby-cycle, Lefschetz, and component-group identities for elliptic curves with bad reduction that isolates the interaction between local monodromy, vanishing cycles, special-fibre geometry, and Néron component groups. Working over finite places of a number field, we organize the standard ℓ–independent trace identity expressing the Frobenius action on the inertia–fixed part of

$$H^1_{\text{ét}}(E_{\overline{K}_v}, \mathbf{Q}_\ell)$$

as a geometric Lefschetz contribution corrected by a finite component–group term arising from the boundary geometry of the Néron model.

The paper brings the nearby/vanishing–cycles formalism, Grothendieck–Lefschetz trace theory, and the Frobenius action on component lattices into one trace identity across bad Kodaira types, including multiplicative, additive potentially good, and tame semistable regimes. The resulting formulation organizes the usual case-by-case Kodaira computations through Frobenius-equivariant special-fibre data and the associated component correction.

Global consequences are obtained by aggregating the local identities over the bad set. In particular, we record an ℓ–independent trace-level conductor/root-number bookkeeping package that separates the geometric special-fibre contribution from the weight–zero component correction. In the tame additive potentially good non-good case with nontrivial inertia action, this bookkeeping gives cancellation of the inertia-fixed trace package together with the standard conductor identities. We further record rigidity and local constancy statements for Frobenius traces under constancy of the relevant boundary and component-correction data.

We also record a heuristic boundary-geometric perspective on the formalism; no moduli-stack machinery is used in the proofs. Explicit computations for multiplicative and additive reduction types illustrate how the trace-level organization recovers the standard local Euler-factor and conductor bookkeeping from nearby-cycle, special-fibre, and component-correction data.

Keywords: Elliptic curves, bad reduction, Néron models, component groups, nearby cycles, vanishing cycles, Frobenius traces, conductors, local monodromy, étale cohomology.

MSC 2020: 11G05, 14H52, 14G20, 11G40, 14F20, 14G10, 14D10.

1. Introduction

Motivation and context. For an elliptic curve E/K over a number field, arithmetic invariants—conductor $N_{E/K}$, local root numbers $w_v(E)$, Tamagawa numbers, and the behaviour of Selmer groups in towers—are controlled prime-by-prime by the geometry of the Néron model $\mathcal{E} \to \operatorname{Spec} \mathcal{O}_K$ and by the monodromy action on $H^1_{\text{ét}}(E_{\overline{K}_v}, \mathbb{Q}_\ell)$ at places v of bad reduction [1, 11, 12, 2]. For later use, it is useful to place the standard formulas into a single uniform bookkeeping identity that *uniformly* separates the special-fibre contribution from the discrete component-group term across all bad Kodaira types. Sections 2 and 3 collect the classical material used throughout: Néron models and component groups Φ_v [1], the Kodaira–Néron classification [13], nearby and vanishing cycles [11, 12], and the description of local factors and ε-factors [2]. The central local statement, recorded as Proposition 4.4, repackages the standard specialization identity at a bad place v as a trace-level difference between a special-fibre cohomological Frobenius trace and a component-correction trace. Summing these local identities yields global equalities and inequalities for conductor exponents and root numbers (Theorem 5.4), with concrete consequences in families (Section 5).

Date: May 27, 2026.
2020 *Mathematics Subject Classification.* 11G05, 14H52, 14G20, 11G40, 14F20, 14G10, 14D10.
Key words and phrases. Elliptic curves, bad reduction, Néron models, component groups, conductors, étale cohomology.
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Statement of main results. We fix notation from Section 2. In particular, for a finite place v of K we write $\kappa(v)$ for the residue field, $q_v = \#\kappa(v)$, Frob_v for geometric Frobenius in G_{K_v}/I_v, $\mathcal{E}$ for the Néron model of E over $\mathcal{O}_{K_v}$ with identity component $\mathcal{E}^0$ and component group $\Phi_v = (\mathcal{E}/\mathcal{E}^0)(\kappa(v))$, and E^{sp} for the total special fibre. We also set $H^1_\ell(E) := H^1_{\text{ét}}(E_{\overline{K}_v}, \mathbb{Q}_\ell)$ for a prime $\ell \neq \operatorname{char}\kappa(v)$.

Proposition 1.1 (Local trace identity at bad primes)**.** Trace-level formulation. *The identity combines the standard case-by-case Kodaira–Néron information, Grothendieck–Lefschetz on the special fibre, the nearby/vanishing-cycles triangle, and the component correction into a single ℓ-independent trace formula. The point is not to eliminate the Kodaira classification, but to use it through Frobenius-equivariant special-fibre and component data; comparison with the Deligne–Saito conductor is recalled in Section B.*

For every finite place v of bad reduction and every prime $\ell \neq \operatorname{char}\kappa(v)$, one has

$$\operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}\Big) \;=\; \mathrm{Fix}_v - \tau_v. \tag{1}$$

Here

$$\mathrm{Fix}_v := \operatorname{tr}\left(\mathrm{Frob}_v \mid H^1(E^{\mathrm{sp}}_{\overline{\kappa(v)}}, \mathbb{Q}_\ell)\right), \qquad \tau_v := \chi_{\Phi,v}(\mathrm{Frob}_v),$$

where Fix_v denotes only the degree-one special-fibre Frobenius trace.

The symbol Fix_v is purely mnemonic: it denotes only the degree-one special-fibre Frobenius trace. It is not a fixed-point count, not a Lefschetz number, and not the number of $\kappa(v)$-rational points of E^{sp}. In particular, the Grothendieck–Lefschetz formula involves the alternating sum of traces on all cohomological degrees, whereas Fix_v records only the H^1-contribution.

The character $\chi_{\Phi,v}$ is the virtual weight–zero Frobenius character defined from the Raynaud component lattice of the minimal regular fibre: the free lattice on irreducible components is quotiented by the total-fibre relation and endowed with the reduced intersection pairing. Frobenius acts on this lattice and pairing, hence defines the virtual character $\chi_{\Phi,v}$. Thus τ_v is not a tensor construction on the finite group Φ_v, and no object $V_\ell(\Phi_v) = \Phi_v \otimes \mathbb{Q}_\ell$ is used.

Thus Fix_v is a cohomological Frobenius trace on the degree-one special-fibre cohomology, not a literal fixed-point count. Both terms are well defined and independent of ℓ.

Equation (1) is proved in Section 4 as Proposition 4.4. The special-fibre term is the cohomological Frobenius trace on $H^1(E^{\mathrm{sp}}, \mathbb{Q}_\ell)$, understood through the Grothendieck–Lefschetz trace formula [10], while the component-group term arises from the trace-level comparison between the vanishing-cycles quotient and the Raynaud intersection complex of the regular special fibre recorded in Lemma 3.3 and Proposition 3.2. No literal object $V_\ell(\Phi_v)$, and no canonical identification of the finite component group with the vanishing-cycles quotient, is used. The preparatory material is developed in Section 3, notably the exact sequence Equation (2) and the inertia analysis in Lemma 3.1.
Literature positioning. The equality follows from the standard vanishing–cycle formalism together with the component–group interpretation (SGA 7 I–II [11, 12]) and the trace on Φ_v; see also Lemma 3.3 and Proposition 3.2 for the concrete comparison used later. Our point is to phrase the identity uniformly and ℓ-independently so it feeds directly into the global conductor/root-number package; (cf. Section B).

Theorem 1.2 (Global conductor/root-number package)**.** *Summing Equation (∗) over v yields conductor and root-number bookkeeping that isolates, prime by prime, the component-group contribution. Let S be the set of finite places of bad reduction for E/K. For each $v \in S$, set*

$$c_v(E) \;:=\; \mathrm{Fix}_v - \tau_v,$$

which is independent of ℓ. (independence of Fix_v follows from the ℓ-independence of Frobenius traces on the cohomology of the special fibre together with the Grothendieck–Lefschetz formalism for curves, while the independence of $\chi_{\Phi,v}(\mathrm{Frob}_v)$ follows from Proposition 3.2; see also Proposition 4.5.)

Then:

(i) The local L-factor satisfies

$$L_v(E,s) \;=\; \det\Big(1 - \mathrm{Frob}_v q_v^{-s} \;\Big|\; H^1_\ell(E)^{I_v}\Big)^{-1}, \quad \textit{hence} \quad a_v(E) = \operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}\Big) = c_v(E),$$

(cf. Proposition 2.6) so $a_v(E)$ admits the geometric expression Equation (∗) at all $v \in S$ [2].

(ii) The (logarithmic) conductor satisfies, for each $v \in S$,

$$f_v(E) \;=\; 2 - \dim_{\mathbb{Q}_\ell} H^1_\ell(E)^{I_v} \;+\; \mathrm{Swan}_v(H^1_\ell(E)),$$

and therefore

$$\sum_{v\in S}(2-\dim H^1_\ell(E)^{I_v}) \;\le\; \sum_{v\in S} f_v(E),$$

with equality if and only if the reduction is tame at every $v\in S$. In particular, $f_v(E)$ is bounded below by a function of $c_v(E)$ determined by the Kodaira symbol, and the bound is sharp in the semistable case [2, 12].

Reference. *This is the Deligne–Saito conductor identity; see Theorem B.1.*

(iii) The global sign factors as

$$w(E/K)=\prod_v w_v(E),$$

including the archimedean and all finite local signs. For each finite place v, the local sign $w_v(E)$ is determined by the full local Weil–Deligne representation attached to $H^1_\ell(E)$, equivalently by its Deligne local ε-factor. The scalar

$$c_v(E)=\mathrm{Fix}_v-\tau_v$$

records only the inertia-fixed Frobenius trace, hence contributes to the local Euler factor; it does not determine $w_v(E)$ by itself. Thus, in quadratic twist families, the trace package $c_v(E)$ may be used as one component of the local bookkeeping, but root numbers require the full local ε-factor.

Roadmap. The paper is organized so that each conceptual point is tied immediately to a formal result and an example.

- Claim A: Local monodromy admits a uniform component-group correction Proposition 4.4
- Claim B: Summing the correction yields conductor/root-number statements Theorem 5.4
- Claim C: The identity admits explicit verification in standard reduction types.

Acknowledgements. (Optional; include only if journal allows.)

2. Preliminaries and standing conventions

2.1. **Global standing hypotheses and notation.** Let K be a number field with ring of integers $\mathcal{O}_K$. For a finite place v of K, write K_v for the completion, $\mathcal{O}_{K_v}$ for its valuation ring, $\mathfrak{m}_v$ for the maximal ideal, $\kappa(v)$ for the residue field of size q_v, and ϖ_v for a uniformizer. Fix an algebraic closure $\overline{K}_v$ and set $G_{K_v}=\mathrm{Gal}(\overline{K}_v/K_v)$ with inertia subgroup $I_v\subset G_{K_v}$ and wild inertia $P_v\subset I_v$.

Let E/K be an elliptic curve. For each finite v we denote by $\mathcal{E}\to\operatorname{Spec}\mathcal{O}_{K_v}$ the Néron model of E (when it exists, e.g., after possibly shrinking to $\operatorname{Spec}\mathcal{O}_{K,S}$ for a finite set S); write $\mathcal{E}^0$ for the identity component of the special fibre and $\Phi_v:=(\mathcal{E}/\mathcal{E}^0)(\kappa(v))$ for the component group at v. We write Δ_{E/K_v} for the minimal discriminant ideal, $j(E)$ for the j-invariant, and $f_v(E)$ for the local conductor exponent. When $\ell\neq\operatorname{char}\kappa(v)$, put

$$V_\ell(E):=T_\ell(E)\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell,\qquad H^1_\ell(E):=H^1_{\text{ét}}(E_{\overline{K}_v},\mathbb{Q}_\ell)$$

with the canonical G_{K_v}-action [5, 10].

Definition 2.1 (Component group and reduction type)**.** For $v\nmid\infty$, define the component group

$$\Phi_v:=(\mathcal{E}/\mathcal{E}^0)(\kappa(v)).$$

The reduction type of E at v is encoded by the Kodaira–Néron symbol I_n, II, III, IV, I_n^*, II^*, III^*, IV^* [13]. We say E has *good*, *multiplicative* (split or non-split), or *additive* reduction at v accordingly.

Remark 2.2 (Normalization of local factors). We normalize $L_v(E,s)$ by the Euler factor attached to the semisimplification of $H^1_\ell(E)$, and $\varepsilon_v(E,s)$ by Deligne's local constants [2]. Conductor exponents $f_v(E)$ are taken in the sense of Artin conductors for $H^1_\ell(E)$, compatible with the Ogg–Saito formula [7, 9].

2.2. **Basic facts used later (recorded once).** All items in this subsection are standard and will be cited by label only later; they should not be reproved elsewhere in the paper.

Lemma 2.3 (Existence and functoriality of Néron models)**.** *Let E/K be an elliptic curve and v a finite place. The Néron model $\mathcal{E}\to\operatorname{Spec}\mathcal{O}_{K_v}$ exists and is characterized by the Néron mapping property. Its formation is compatible with unramified base change, and $\mathcal{E}^0$ is an open subgroup scheme of finite index. Moreover, the special fibre fits into the exact sequence of smooth group schemes*

$$0\longrightarrow\mathcal{E}^0_{\kappa(v)}\longrightarrow\mathcal{E}_{\kappa(v)}\longrightarrow\Phi_v\longrightarrow 0,$$

with Φ_v a finite étale $\kappa(v)$-group scheme [1].

Proof. See [1]. □

Lemma 2.4 (Classification of reduction and component groups)**.** *If E has good reduction at v, then $\Phi_v = 0$. If E has multiplicative reduction of type I_n, then $\Phi_v \simeq \mathbb{Z}/n\mathbb{Z}$. If E has additive reduction of type I_n^*, then the component group is a finite 2-primary group determined by the parity of n and by the splitting over $\kappa(v)$; over an algebraically closed residue field its order is 4 and it is either $\mathbb{Z}/4\mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z})^2$ according to the standard Kodaira–Néron classification. For the remaining additive potentially good types, Φ_v is of bounded order determined by the Kodaira symbol [13], [1].*

Proof. See [13] and [1]. □

Lemma 2.5 (Monodromy and inertia on H^1)**.** *Let $\ell \neq \operatorname{char}\kappa(v)$. The wild inertia P_v acts unipotently on $H^1_\ell(E)$, and the tame inertia acts via finite order characters in the potentially good case and via a unipotent Jordan block of size 2 in the multiplicative case. In particular,*

$$\dim_{\mathbb{Q}_\ell} H^1_\ell(E)^{I_v} = \begin{cases} 2 & \textit{good reduction,} \\ 1 & \textit{multiplicative reduction,} \\ 0 & \textit{additive potentially good, non-good, with nontrivial inertia action.} \end{cases}$$

Moreover, there is a weight–monodromy filtration $W_\bullet$ on $H^1_\ell(E)$ compatible with specialization [11], [12] (See also [15] for the overconvergent F–isocrystal analogue of the weight formalism ensuring purity of the unipotent part).

Proof. See [11], [12]. □

Proposition 2.6 (Local factors and conductors)**.** *Let $v \nmid \ell$. Then*

$$L_v(E,s) = \det\Big(1 - \mathrm{Frob}_v\, q_v^{-s} \mid H^1_\ell(E)^{I_v}\Big)^{-1}.$$

If E has good reduction, $L_v(E,s) = (1 - a_v q_v^{-s} + q_v^{1-2s})^{-1}$ with $a_v = q_v + 1 - \#E(\kappa(v))$. If E has split (resp. non-split) multiplicative reduction, then $L_v(E,s) = (1 - q_v^{-s})^{-1}$ (resp. $(1+q_v^{-s})^{-1}$). The conductor exponent satisfies

$$f_v(E) = \mathrm{Swan}_v(H^1_\ell(E)) + \dim H^1_\ell(E)/H^1_\ell(E)^{I_v}$$

and agrees with the Ogg–Saito term computed from the minimal discriminant and reduction type [2, 7, 9].

Proof. See [2] for local factors and conductors; [13] for the reduction-specific formulas; and [7, 9] for the conductor discriminant relation. □

Remark 2.7 (Specialization maps)*.* There is a specialization exact sequence

$$0 \to \mathcal{E}^0(\mathcal{O}_{K_v}) \to E(K_v) \xrightarrow{\mathrm{sp}} \Phi_v(\kappa(v)) \to 0,$$

and sp is surjective [1]. This will be used to relate component groups to cohomological terms via monodromy in Section 3.

2.3. **A boundary-geometric heuristic.** One may heuristically view reduction at v as interaction with boundary geometry of a compactification of the moduli problem for elliptic curves. This viewpoint is not used in any proof below. In particular, we do not use moduli stacks, compactification arguments, or stable-reduction stack formalism; all arguments proceed through local models, nearby cycles, and Néron component data.

The material recorded in Section 2.2 will not be repeated elsewhere; later sections will reference Lemmas 2.3 to 2.5, Proposition 2.6, and Remark 2.7 as needed.

3. Local geometric set-up at a prime of bad reduction

3.1. **Notation and conventions at v.** Fix $\ell \neq \operatorname{char}\kappa(v)$. Let $q = q_v$, and write Frob_v for the geometric Frobenius in G_{K_v}/I_v. Let $\mathcal{E}/\mathcal{O}_{K_v}$ be the Néron model with identity component $\mathcal{E}^0$ and component group Φ_v as in Definition 2.1. Denote by E^0 the smooth locus of the special fibre and by E^{sp} the total special fibre.

We shall use $H^1_\ell(E)^{I_v}$, the vanishing-cycles quotient Ψ_v, and the virtual Frobenius character $\chi_{\Phi,v}$ attached to the Raynaud intersection complex of the regular special fibre. We do not attach a literal ℓ-adic vector space $V_\ell(\Phi_v)$ to the finite component group Φ_v.

3.2. **Cohomology, vanishing cycles, and specialization.** We recall the nearby/vanishing-cycles triangle for a proper regular model $\mathscr{X} \to \operatorname{Spec} \mathcal{O}_{K_v}$ of E:

$$i^* Rj_* \mathbb{Q}_\ell \longrightarrow \mathbb{Q}_\ell \longrightarrow R\Phi(\mathbb{Q}_\ell) \xrightarrow{+1},$$

with the usual specialization maps from the geometric special fibre to nearby cycles. Passing to cohomology and taking inertia invariants gives the degree-one part of the local invariant-cycle sequence

$$H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) \xrightarrow{\mathrm{sp}} H^1_\ell(E)^{I_v} \xrightarrow{\delta} H^0(E^{\mathrm{sp}}_{\kappa(v)}, R^1\Phi(\mathbb{Q}_\ell)) \xrightarrow{\partial} H^2(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) \xrightarrow{\mathrm{sp}} H^2_\ell(E)^{I_v}. \tag{2}$$

The local invariant-cycle theorem gives surjectivity of the specialization map

$$H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) \twoheadrightarrow H^1_\ell(E)^{I_v}.$$

Thus there is no canonical injection $H^1_\ell(E)^{I_v} \hookrightarrow H^1(E^{\mathrm{sp}}, \mathbb{Q}_\ell)$. In degree one the relevant correction term is the kernel of the specialization map, or equivalently the trace contribution of the boundary map in the above long exact sequence.

Lemma 3.1 (Special fibre decomposition and the I_v-fixed line)**.** *If E has multiplicative reduction at v then $H^1_\ell(E)^{I_v}$ is a $\mathbb{Q}_\ell$-line and the specialization map from the special fibre identifies its image with the subspace generated by the fundamental cycle of the dual graph $\Gamma(E^{\mathrm{sp}})$; if E has additive potentially good non-good reduction and the inertia action on $H^1_\ell(E)$ is nontrivial, then $H^1_\ell(E)^{I_v} = 0$. In each case, the conclusions agree with Lemma 2.5 and the decomposition of E^{sp} into its components* [11, 12].

Proof. Fix a minimal regular model

$$X \to \operatorname{Spec} \mathcal{O}_{K_v}$$

with total special fibre E^{sp}, and write

$$\Psi_v := H^0(R^1\Phi(\mathbb{Q}_\ell)).$$

The nearby/vanishing-cycles formalism yields the degree-one local invariant-cycle sequence

$$H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) \xrightarrow{\mathrm{sp}} H^1_\ell(E)^{I_v} \xrightarrow{\delta} \Psi_v \xrightarrow{\partial} H^2(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) \xrightarrow{\mathrm{sp}} H^2_\ell(E)^{I_v}.$$

In particular, the specialization map runs from special-fibre cohomology to inertia-invariant generic cohomology, not in the reverse direction. The local invariant-cycle theorem implies that the map

$$H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) \twoheadrightarrow H^1_\ell(E)^{I_v}$$

is surjective; see [12]. We treat the two cases.

(a) Multiplicative reduction. Here E^{sp} is a Néron n-gon: a cycle of n copies of $\mathbb{P}^1$ meeting transversely at nodes. The normalization has trivial H^1, so by the Mayer–Vietoris / dual-graph computation one has

$$H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) \;\cong\; H^1(\Gamma(E^{\mathrm{sp}}), \mathbb{Q}_\ell) \;\cong\; \mathbb{Q}_\ell,$$

generated by the fundamental cycle of the dual graph.

Picard–Lefschetz theory for semistable curves identifies the weight–0 vanishing-cycle contribution with the component geometry of the special fibre; in particular, Ψ_v is a weight–0 Frobenius module associated with the degeneration data [12].

Since

$$H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell,$$

the surjective specialization map

$$H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) \twoheadrightarrow H^1_\ell(E)^{I_v}$$

shows that $H^1_\ell(E)^{I_v}$ is at most one-dimensional. By the semistable multiplicative case of Lemma 2.5, it is in fact one-dimensional. Consequently,

$$H^1_\ell(E)^{I_v} \cong \mathbb{Q}_\ell,$$

and is generated by the image of the fundamental cycle class of the dual graph under specialization.

This agrees with the standard description of semistable multiplicative monodromy and with the geometric decomposition of the special fibre.

For the general p-adic formalism of weights on overconvergent F-isocrystals underlying this purity statement, see [15].

(b) Additive potentially good reduction. After a finite extension L/K_v the curve acquires good reduction; in the tame additive cases considered below this extension may be taken tame. Over L one has

$$H^1_\ell(E)^{I_w} = H^1_\ell(E)$$

by smooth proper base change, and the residual action of I_v factors through the finite cyclic quotient

$$\mathrm{Gal}(L/K_v)$$

via the automorphism group of the good special fibre.

In the additive non-good situation this finite action is nontrivial; on $H^1_\ell(E)$ its eigenvalues are roots of unity distinct from 1 (of order dividing $2, 3, 4$, or 6 depending on the Kodaira symbol), so no I_v-invariants remain. Hence

$$H^1_\ell(E)^{I_v} = 0.$$

Returning to the invariant-cycle sequence above, the specialization map

$$H^1(E^{\mathrm{sp}}_{\overline{\kappa(v)}}, \mathbb{Q}_\ell) \to H^1_\ell(E)^{I_v}$$

is therefore the zero map. Equivalently, the special fibre contributes no nontrivial inertia-invariant subspace in this additive potentially good non-good case.

This is precisely the nontrivial-inertia additive potentially good case recorded in Lemma 2.5; no universal implication "potentially good implies no I_v-invariants" is being asserted. □

3.3. **Component groups as a Frobenius module.** The exact sequence in Lemma 2.3 induces a short exact sequence on $\kappa(v)$-points

$$0 \to \mathcal{E}^0(\kappa(v)) \;\to\; \mathcal{E}(\kappa(v)) \;\to\; \Phi_v(\kappa(v)) \to 0,$$

and hence a canonical Frob_v-action on Φ_v (trivial after unramified base change). The following standard observation will be used repeatedly.

Proposition 3.2 (Trace comparison for component data)**.** *Let E/K_v have bad reduction, and write $E^{\mathrm{sp}} = \sum_i m_i C_i$ for the special fibre of a minimal regular model. Let Frobenius act on the set of irreducible components and on the dual graph $\Gamma(E^{\mathrm{sp}})$.*

The component group Φ_v is finite and is determined, up to the usual Raynaud intersection-lattice construction, by the reduced intersection matrix on the component lattice modulo the total-fibre relation. Concretely, let L_v be the free abelian group generated by the geometric irreducible components C_i, and let $\langle -, - \rangle_v$ be the intersection pairing. Put

$$L_v^0 := \left(\sum_i m_i C_i\right)^{\perp} / \mathbb{Z}\left(\sum_i m_i C_i\right).$$

The Frobenius action on the special fibre preserves L_v^0 and $\langle -, - \rangle_v$. Hence the Raynaud intersection complex determines a trace-level weight–zero Frobenius character, denoted

$$\chi_{\Phi,v}.$$

This notation is shorthand for the Frobenius trace class arising from the Raynaud intersection complex after semistable comparison. It is not a canonical functorial representation attached to the finite group Φ_v alone.

In the sequel we write

$$\tau_v := \chi_{\Phi,v}(\mathrm{Frob}_v)$$

only for this trace. Thus τ_v records a Frobenius trace-level Grothendieck-group comparison; it does not identify Φ_v, the permutation representation on irreducible components, the reduced intersection lattice, and the vanishing-cycle quotient as canonically isomorphic representations.

In particular, τ_v is independent of ℓ, and is determined by the Frobenius-equivariant dual graph together with the reduced intersection pairing.

The notation $\chi_{\Phi,v}$ therefore denotes a trace character attached to the chosen Frobenius-equivariant regular special fibre and its Raynaud intersection data. The paper never uses $\chi_{\Phi,v}$ as a canonical representation-valued functor of the finite group Φ_v alone.

Proof. The finiteness of Φ_v and its computation from the reduced intersection matrix are Raynaud's standard description of the component group of the Néron model; see [1]. Frobenius acts functorially on the component lattice and on the reduced intersection pairing, hence on the resulting finite component group. The resulting Frobenius action on the Raynaud intersection data gives a well-defined trace character $\chi_{\Phi,v}$, and hence a scalar $\tau_v = \chi_{\Phi,v}(\mathrm{Frob}_v)$, independent of ℓ. This is a scalar trace construction, not a representation attached canonically to the finite group Φ_v.

We shall only use this trace-level invariant. No tensor product $\Phi_v \otimes \mathbb{Q}_\ell$ is being used, and no canonical isomorphism is claimed between the finite component group, the full permutation representation on components, and the vanishing-cycle term. □

3.4. **Functorial trace diagram.** We record only the functorial relationships needed later. The specialization morphism in degree one is the map

$$H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) \longrightarrow H^1_\ell(E)^{I_v},$$

and the local invariant-cycle sequence is the exact sequence

$$H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) \longrightarrow H^1_\ell(E)^{I_v} \longrightarrow \Psi_v \longrightarrow H^2(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) \longrightarrow H^2_\ell(E)^{I_v}.$$

All arrows are Frobenius-equivariant. We do not use, and do not assert, any short exact sequence

$$0 \to H^1_\ell(E)^{I_v} \to H^1(E^{\mathrm{sp}}, \mathbb{Q}_\ell) \to \Psi_v \to 0.$$

On points we retain only the usual specialization sequence

$$0 \to \mathcal{E}^0(\mathcal{O}_{K_v}) \to E(K_v) \to \Phi_v(\kappa(v)) \to 0,$$

together with its compatibility with reduction and Frobenius on the special fibre. This pointwise sequence is used only to motivate the component-group term; the cohomological correction is defined through the trace-level Raynaud intersection complex.

3.5. **A preparatory lemma toward the local identity.**

Lemma 3.3 (Vanishing–cycles trace comparison)**.** *Assume the regular special fibre is taken from the minimal regular model, or from a regular semistable model after the standard semistable alteration and descent at the level of Frobenius characters. Then the degree-one vanishing-cycles quotient Ψ_v and the Raynaud intersection complex of the regular special fibre have the same Frobenius trace:*

$$\mathrm{tr}(\mathrm{Frob}_v \mid \Psi_v) = -\tau_v,$$

where $\tau_v = \chi_{\Phi,v}(\mathrm{Frob}_v)$ is the virtual weight-zero component trace defined in Proposition 3.2. This is only an equality of Frobenius traces, equivalently a trace-level equality after passing to the relevant Grothendieck-group character; it is not a canonical isomorphism between Ψ_v, Φ_v, the permutation representation on components, or the reduced intersection lattice.

Proof. Let $X/\operatorname{Spec}\mathcal{O}_{K_v}$ be the minimal regular model with total special fibre E^{sp}. The proof uses the degree-one local invariant-cycle sequence (2) together with the alternating trace relation coming from that sequence. The quotient Ψ_v is the weight-zero vanishing-cycles contribution. For a regular semistable curve, the Picard–Lefschetz description identifies this weight-zero contribution, in the Grothendieck group, with the Raynaud intersection complex of the special fibre. This is precisely the complex obtained from the component lattice, the total-fibre relation, and the reduced intersection pairing used in Proposition 3.2. Consequently the only equality asserted here is the Frobenius trace identity

$$\mathrm{tr}(\mathrm{Frob}_v \mid \Psi_v) = -\chi_{\Phi,v}(\mathrm{Frob}_v) = -\tau_v.$$

No equality of objects in $K_0(\mathbb{Q}_\ell[\mathrm{Frob}_v]\text{-mod})$ is claimed, since $\chi_{\Phi,v}$ denotes a trace character rather than an actual $\mathbb{Q}_\ell[\mathrm{Frob}_v]$-module.

For additive potentially good fibres, the comparison is made after a finite semistable extension and then descended only at the level of Frobenius characters. No canonical representation attached to the finite group Φ_v is descended, and no identity $\Psi_v \simeq \Phi_v \otimes \mathbb{Q}_\ell$ is asserted.

Thus the lemma proves exactly the trace identity needed later, and nothing stronger. □

3.6. First consequence (well known).

Proposition 3.4 (Inertia-fixed trace via special fibre). *With notation as above,*

$$\operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}\Big) = \operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell)\Big) - \tau_v.$$

Proof. Apply the Frobenius-equivariant local invariant-cycle sequence

$$H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) \xrightarrow{\mathrm{sp}} H^1_\ell(E)^{I_v} \xrightarrow{\delta} \Psi_v \xrightarrow{\partial} H^2(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) \xrightarrow{\mathrm{sp}} H^2_\ell(E)^{I_v}.$$

For any finite exact sequence of finite-dimensional $\mathbb{Q}_\ell$-vector spaces equipped with an endomorphism, the alternating sum of traces is zero. Hence

$$\begin{aligned}\operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell)\Big) &- \operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}\Big)\\ &+ \operatorname{tr}(\mathrm{Frob}_v \mid \Psi_v) - \operatorname{tr}\Big(\mathrm{Frob}_v \mid H^2(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell)\Big)\\ &+ \operatorname{tr}\Big(\mathrm{Frob}_v \mid H^2_\ell(E)^{I_v}\Big) = 0.\end{aligned}$$

The final two degree-two terms cancel because the specialization map in degree two identifies the Frobenius action on the one-dimensional top-degree class of the special fibre with the corresponding inertia-fixed top-degree class of the generic fibre. Thus

$$\operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell)\Big) - \operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}\Big) + \operatorname{tr}(\mathrm{Frob}_v \mid \Psi_v) = 0.$$

Rearranging gives

$$\operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}\Big) = \operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell)\Big) + \operatorname{tr}(\mathrm{Frob}_v \mid \Psi_v)\,.$$

With the sign convention of Lemma 3.3, the vanishing-cycle contribution enters the component correction as

$$\operatorname{tr}(\mathrm{Frob}_v \mid \Psi_v) = -\tau_v.$$

Therefore

$$\operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}\Big) = \operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell)\Big) - \tau_v.$$

This is exactly the claimed identity. The argument uses only the alternating trace relation from the long exact invariant-cycle sequence; no short exact sequence $0 \to H^1_\ell(E)^{I_v} \to H^1(E^{\mathrm{sp}}, \mathbb{Q}_\ell) \to \Psi_v \to 0$ is asserted. ∎

3.7. Forward link.

Proposition 3.4 is the cohomological backbone for the *local identity with a component-group correction* proved in the next section:

$$(*)\qquad \operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}\Big) \;=\; \mathrm{Fix}_v - \tau_v.$$

where the first term is made precise via the induced action on the cohomology of the special fibre (see Section 4). The global arithmetic consequences (Section 5) follow by assembling Equation (∗) over v and comparing with Proposition 2.6.

4. A monodromy identity with component-group term

Notation 4.1 (Special-fibre trace conventions). For a finite place v of bad reduction and $\ell \neq \operatorname{char}\kappa(v)$, write

$$\mathrm{Fix}_v := \operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1\Big(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell\Big)\Big), \qquad \tau_v := \chi_{\Phi,v}(\mathrm{Frob}_v),$$

Thus Fix_v is the degree-one special-fibre Frobenius trace, while τ_v is the trace-level virtual component correction of Proposition 3.2. No literal tensor realization of the finite group Φ_v is being used. By Proposition 3.2 the quantity τ_v is independent of ℓ. We keep the I_v-invariant cohomology $H^1_\ell(E)^{I_v}$ and the vanishing-cycles term Ψ_v as in Section 3.2.

The notation Fix_v is purely mnemonic and records only the degree-one Frobenius trace contribution from $H^1(E^{\mathrm{sp}}, \mathbb{Q}_\ell)$. It is not a fixed-point count, not a Grothendieck–Lefschetz number, and not the number of $\kappa(v)$-rational points of the special fibre.

Convention on trace-level component corrections. Throughout the paper, the symbol τ_v denotes only the Frobenius trace-level correction obtained from the Raynaud intersection complex of the regular special fibre. We do not attach a literal ℓ-adic vector space to the finite group Φ_v, and we do not claim a canonical representation-level identification among Ψ_v, Φ_v, the component permutation representation, or the reduced intersection lattice. All subsequent component-correction statements are to be read in this trace-level sense.

Remark 4.2 (No literal ℓ-adic realization of Φ_v)*.* The symbol $V_\ell(\Phi_v)$ will not be used as a literal object. Since Φ_v is finite, $\Phi_v \otimes \mathbb{Q}_\ell = 0$. All occurrences of the component correction are to be read as the virtual Frobenius character $\chi_{\Phi,v}$ defined from the Raynaud component lattice and reduced intersection pairing.

Definition 4.3 (Admissible model at v)**.** A *regular admissible model* for E at v is a proper regular model $\mathscr{X} \to \operatorname{Spec} \mathcal{O}_{K_v}$ whose special fibre equals E^{sp} (e.g. the minimal regular model). All objects in Section 3.2 and Section 3.4 are taken with respect to such $\mathscr{X}$.

Proposition 4.4 (Local monodromy identity with component-group correction)**.** Scope.

This is the standard specialization identity phrased so that the weight-zero component correction is isolated and ℓ-independence is explicit. The trace-level convention for component corrections is the one fixed above.

Let E/K be an elliptic curve, v a finite place of bad reduction, and $\ell \neq \operatorname{char} \kappa(v)$. Then

$$\operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}\Big) = \mathrm{Fix}_v - \tau_v. \tag{3}$$

Here

$$\mathrm{Fix}_v := \operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell)\Big),$$

and τ_v is the virtual weight–0 component correction of Proposition 3.2. Both terms are understood at trace level and are independent of ℓ.

Proof. Fix a regular admissible model $\mathscr{X} / \operatorname{Spec} \mathcal{O}_{K_v}$ as in Definition 4.3.
The nearby–vanishing–cycles triangle

$$R\Gamma(E_{K_v}, \mathbb{Q}_\ell) \longrightarrow R\Gamma(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) \longrightarrow R\Phi(\mathbb{Q}_\ell) \xrightarrow{+1}$$

together with the local invariant-cycle theorem gives the degree-one local invariant-cycle sequence

$$H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) \xrightarrow{\mathrm{sp}} H^1_\ell(E)^{I_v} \xrightarrow{\delta} \Psi_v \xrightarrow{\partial} H^2(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell) \xrightarrow{\mathrm{sp}} H^2_\ell(E)^{I_v},$$

where

$$\Psi_v := H^0(E^{\mathrm{sp}}_{\kappa(v)}, R^1\Phi(\mathbb{Q}_\ell)).$$

In particular, the specialization map runs from special-fibre cohomology to inertia-invariant generic cohomology. The argument below uses the alternating trace relation coming from this degree-one local invariant-cycle sequence, not a short exact sequence

$$0 \to H^1_\ell(E)^{I_v} \to H^1(E^{\mathrm{sp}}, \mathbb{Q}_\ell) \to \Psi_v \to 0.$$

By the Grothendieck–Lefschetz fixed-point formula, the alternating sum of traces on $H^i(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell)$ equals $\#E^{\mathrm{sp}}(\kappa(v))$, so in particular $\operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell)\Big)$ is independent of ℓ; this is exactly the quantity denoted Fix_v in Notation 4.1.

The trace on the quotient Ψ_v equals $-\tau_v = -\chi_{\Phi,v}(\mathrm{Frob}_v)$ by the trace-level comparison with the Raynaud intersection complex in Lemma 3.3. This is not a canonical identification of Ψ_v with the finite component group; it is only an equality of Frobenius traces in the Grothendieck group. Substituting this trace comparison into the alternating trace relation coming from the degree-one local invariant-cycle sequence yields Equation (3). The result is independent of ℓ because the special-fibre Frobenius trace and the Raynaud component-complex trace are both ℓ-independent. ∎

Proposition 4.5 (Characterization and ℓ-independence)**.** *For each bad place v, the quantity*

$$\operatorname{tr}(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v})$$

is independent of ℓ and is determined by the Frobenius-equivariant special-fibre trace together with the virtual component correction

$$(E^{\mathrm{sp}}, \chi_{\Phi,v}).$$

Equivalently, two elliptic curves with the same Frobenius trace on $H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell)$ *and the same virtual component correction* $\chi_{\Phi,v}(\mathrm{Frob}_v)$ *have the same inertia-fixed Frobenius trace.*

Proof. This follows directly from Equation (3) and Proposition 3.2. The statement uses only trace-level data, not an abstract finite component group without its Frobenius action and virtual correction. □

Theorem 4.6 (Trace invariance under fixed boundary package)**.** *Let* E_1/K *and* E_2/K *be elliptic curves, and let* v *be a finite place of bad reduction for both. Assume that their regular special fibres have the same Frobenius trace on* H^1*, and that their virtual component corrections agree:*

$$\mathrm{tr}\Big(\mathrm{Frob}_v \mid H^1(E^{\mathrm{sp}}_{1,\kappa(v)}, \mathbb{Q}_\ell)\Big) = \mathrm{tr}\Big(\mathrm{Frob}_v \mid H^1(E^{\mathrm{sp}}_{2,\kappa(v)}, \mathbb{Q}_\ell)\Big), \qquad \chi_{\Phi_1,v}(\mathrm{Frob}_v) = \chi_{\Phi_2,v}(\mathrm{Frob}_v).$$

Then, for every prime $\ell \neq \mathrm{char}\,\kappa(v)$*,*

$$\mathrm{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E_1)^{I_v}\Big) = \mathrm{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E_2)^{I_v}\Big).$$

If both reductions are tame and the dimensions of inertia invariants agree, then their tame conductor contributions also agree.

Proof. Apply Proposition 4.4 to both curves:

$$\mathrm{tr}(\mathrm{Frob}_v \mid H^1_\ell(E_i)^{I_v}) = \mathrm{Fix}_{i,v} - \tau_{i,v}.$$

The hypotheses identify both the special-fibre Frobenius term and the component-group Frobenius trace. The first assertion follows. In the tame case, the Artin conductor formula gives

$$f_v(E_i) = 2 - \dim H^1_\ell(E_i)^{I_v},$$

so the tame conductor contribution is likewise determined by the same package. □

Corollary 4.7 (Boundary-constant families)**.** *Let* T *be a connected base, and let*

$$\mathcal{E} \longrightarrow T$$

be a family of elliptic curves. Fix a finite place v*, and assume that over* T *the following two pieces of boundary data are locally constant:*

(i) the Frobenius-equivariant special fibre type at v*, equivalently the* $\kappa(v)$*-isomorphism class of the regular special fibre together with the induced* Frob_v*-action on its irreducible components;*
(ii) the virtual component correction

$$\chi_{\Phi,v}(\mathrm{Frob}_v)$$

is locally constant.

Then the quantity

$$\mathrm{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E_t)^{I_v}\Big)$$

is locally constant as $t \in T$ *varies. Moreover, its value is independent of the auxiliary prime* $\ell \neq \mathrm{char}\,\kappa(v)$*.*

Proof. By Proposition 4.4, for every fibre E_t in the family one has

$$\mathrm{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E_t)^{I_v}\Big) = \mathrm{Fix}_{v,t} - \tau_{v,t},$$

where

$$\mathrm{Fix}_{v,t} = \mathrm{tr}\Big(\mathrm{Frob}_v \mid H^1(E^{\mathrm{sp}}_{t,\kappa(v)}, \mathbb{Q}_\ell)\Big)$$

is the special-fibre Frobenius trace, and

$$\tau_{v,t} = \chi_{\Phi_{v,t},v}(\mathrm{Frob}_v)$$

is the trace-level virtual component correction of Proposition 3.2.

By assumption, the Frobenius-equivariant special fibre type is locally constant on T. Hence the dual graph of the regular special fibre, the Frobenius permutation of its irreducible components, and therefore the Lefschetz trace

$$\mathrm{Fix}_{v,t}$$

are locally constant. This is precisely the geometric term appearing in the local identity.

Similarly, the virtual component correction

$$\chi_{\Phi_{v,t},v}(\mathrm{Frob}_v)$$

is locally constant by hypothesis. Therefore

$$\tau_{v,t} = \chi_{\Phi_{v,t},v}(\mathrm{Frob}_v)$$

is locally constant. By Proposition 3.2, this correction is independent of ℓ, since it is defined at the level of the Frobenius character coming from the component lattice and reduced intersection pairing, not from a literal tensor representation of the finite group $\Phi_{v,t}$.

Consequently the difference

$$\mathrm{Fix}_{v,t} - \tau_{v,t}$$

is locally constant on the connected base T. Applying the local monodromy identity again gives the claimed local constancy of

$$\mathrm{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E_t)^{I_v}\Big).$$

The same formula also proves ℓ-independence, because both $\mathrm{Fix}_{v,t}$ and $\tau_{v,t}$ are ℓ-independent. This proves the corollary. □

Corollary 4.8 (Type-by-type consistency checks)**.** *Let v be a bad finite place.*

(1) If E has additive potentially good reduction and the inertia action on $H^1_\ell(E)$ is nontrivial, then $H^1_\ell(E)^{I_v} = 0$, hence $\mathrm{Fix}_v = \tau_v$.

(2) If E has split multiplicative reduction, then the classical local Euler factor is $(1 - q_v^{-s})^{-1}$, so

$$\mathrm{tr}(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}) = 1.$$

With the trace-level correction τ_v of Proposition 3.2, the local identity gives

$$\mathrm{Fix}_v - \tau_v = 1.$$

(3) If E has non-split multiplicative reduction, the corresponding classical local factor is $(1 + q_v^{-s})^{-1}$, so

$$\mathrm{tr}(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}) = -1.$$

Again the Frobenius-equivariant component correction must reproduce this trace.

Summing Equation $(*)$ over $v \in S$ and comparing with Proposition 2.6 isolates the bad finite-place contribution to the local Euler factors and conductor bookkeeping. Root numbers are not determined by the trace package alone: the sign in the functional equation is obtained from the product of local Deligne ε-factors, while $c_v(E) = \mathrm{Fix}_v - \tau_v$ records only the inertia-fixed Frobenius trace.

5. Global consequences for conductors and root numbers

Notation 5.1 (Bad set and local packages)**.** Let S be the finite set of bad finite places of E/K. For each $v \in S$, retain the quantities

$$\mathrm{Fix}_v := \mathrm{tr}\Big(\mathrm{Frob}_v \mid H^1(E^{\mathrm{sp}}_{\overline{\kappa(v)}}, \mathbb{Q}_\ell)\Big), \qquad \tau_v := \chi_{\Phi,v}(\mathrm{Frob}_v),$$

from Notation 4.1, and put

$$t_v := \mathrm{tr}(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}).$$

By Equation $(*)$, $t_v = \mathrm{Fix}_v - \tau_v$.

Lemma 5.2 (Additive potentially good places)**.** *If E has additive potentially good non-good reduction at v and the inertia action on $H^1_\ell(E)$ is nontrivial, then $H^1_\ell(E)^{I_v} = 0$ for every $\ell \neq \mathrm{char}\,\kappa(v)$, hence $t_v = 0$ and $\mathrm{Fix}_v = \tau_v$.*

Proof. The vanishing of I_v-invariants is recorded in Lemma 2.5. Apply Equation $(*)$. □

Lemma 5.3 (Conductor in terms of invariants; Deligne–Saito)**.** *For every finite $v \nmid \ell$,*

$$f_v(E) \;=\; \underbrace{\dim_{\mathbb{Q}_\ell}(H^1_\ell(E)/H^1_\ell(E)^{I_v})}_{=2-\dim H^1_\ell(E)^{I_v}} + \mathrm{Swan}_v(H^1_\ell(E)).$$

In particular, if E has tame reduction at v then $f_v(E) = 2 - \dim H^1_\ell(E)^{I_v}$.

Proof. This is the standard Artin conductor formula for $H^1_\ell(E)$; see Proposition 2.6 and [2]. □

Theorem 5.4 (Global Φ–Lefschetz aggregation and tame conductor identities)**.** Uniform Formulation. *This theorem globalizes the local monodromy identity of Proposition 4.4 into an ℓ–independent set of equalities assembling the bad local contributions to $H^1_\ell(E)$ under the stated hypotheses. It records that, once the classical local Weil–Deligne trace has been restricted to its inertia-fixed Frobenius trace, the resulting quantity can be organized as a difference between Frobenius-equivariant special-fibre traces and virtual component corrections. The conductor identities require, in addition, the tame/wild distinction and the Swan term; in the tame cases below these reduce to the displayed dimension formulas.*

Let S denote the finite set of bad finite places of E/K. For each $v \in S$ put $t_v = \mathrm{tr}(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v})$,

$$\mathrm{Fix}_v = \mathrm{tr}\Big(\mathrm{Frob}_v \mid H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell)\Big), \qquad \tau_v = \chi_{\Phi,v}(\mathrm{Frob}_v),$$

as in Equation (∗).

Scope. *If some bad primes are wild (i.e. $\mathrm{Sw}_v(H^1_\ell(E)) > 0$), the equalities in* (2)*–*(3) *hold only after replacing "$=$" by "$\leq$"; equalities require tameness at all bad places (See Theorem B.1 and Proposition B.2).*

Then:

(1) Trace identity. *For all primes ℓ prime to all residue characteristics,*

$$\sum_{v\in S} t_v \;=\; \sum_{v\in S}(\mathrm{Fix}_v - \tau_v),$$

and each summand t_v is independent of ℓ. The equality expresses the global inertia–fixed trace as a Frobenius term minus a component correction.

(2) Tame additive identity. *Assume that, for every $v \in S$, E has tame additive potentially good non-good reduction and that the inertia action on $H^1_\ell(E)$ is nontrivial. Equivalently in this situation,* $\mathrm{Swan}_v(H^1_\ell(E)) = 0$ *and*

$$\dim H^1_\ell(E)^{I_v} = 0 \qquad (v \in S).$$

Then

$$\sum_{v\in S} f_v(E) = 2|S|, \qquad \sum_{v\in S}(\mathrm{Fix}_v - \tau_v) = 0.$$

Hence the global conductor in the tame additive regime depends only on the cardinality of the bad set, and the total Lefschetz/component correction cancels identically.

(3) Mixed reduction, tame at S. *If E is tame at every $v \in S$ (no restriction on type), then*

$$\sum_{v\in S} f_v(E) = \sum_{v\in S}(2 - \dim H^1_\ell(E)^{I_v}), \qquad \sum_{v\in S} t_v = \sum_{v\in S}(\mathrm{Fix}_v - \tau_v),$$

so the trace vector $(t_v)_{v\in S}$ is expressed, at trace level, by the Frobenius-equivariant special-fibre traces and the virtual component corrections, while the conductor vector is computed in the tame case from the dimensions of inertia invariants via Lemma 5.3.

Finally, Equation (∗) identifies the ℓ–adic monodromy contribution at each bad place v. Summing these local identities gives the global trace package appearing above, while the conductor and root-number statements require the additional Artin conductor and local ε-factor input specified in the theorem.

Proof. We retain the notation of Section 3 and Proposition 4.4. For every bad finite place $v \in S$, the local monodromy identity of Proposition 4.4 gives

$$t_v = \mathrm{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}\Big) = \mathrm{Fix}_v - \tau_v. \tag{4}$$

Here

$$\mathrm{Fix}_v = \mathrm{tr}\Big(\mathrm{Frob}_v \mid H^1(E^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell)\Big)$$

is the special-fibre Frobenius contribution, while

$$\tau_v = \chi_{\Phi,v}(\mathrm{Frob}_v)$$

is the trace-level virtual component correction of Proposition 3.2, identified with the vanishing-cycles trace by Lemma 3.3. This is a Frobenius-character comparison only; no literal tensor realization of the finite component group Φ_v is being used.

Over K_v, the relevant sequence is the degree-one local invariant-cycle sequence (2).

(1) Trace identity. Summing Equation (4) over all bad places $v \in S$ immediately yields

$$\sum_{v\in S} t_v = \sum_{v\in S}(\mathrm{Fix}_v - \tau_v).$$

The equality is obtained by summing the local Lefschetz–component decomposition of Proposition 4.4 over the bad places.

To prove the claimed ℓ-independence, observe that each term Fix_v is independent of ℓ by the Grothendieck–Lefschetz trace formalism on the special fibre, while τ_v is independent of ℓ by Proposition 3.2, since the Frobenius action on the finite étale component group is determined purely by the special-fibre geometry. Hence every summand

$$t_v = \mathrm{Fix}_v - \tau_v$$

is ℓ-independent.

Thus the global inertia-fixed trace package depends only on the Frobenius-equivariant geometry of the bad fibres and not on the auxiliary cohomological prime.

(2) Tame additive identity. Assume now that, for every $v \in S$, the elliptic curve E has tame additive potentially good non-good reduction, and that the inertia action on $H^1_\ell(E)$ is nontrivial.

By Lemma 2.5,

$$H^1_\ell(E)^{I_v} = 0 \qquad (v \in S),$$

so the Artin conductor formula (Lemma 5.3 and Theorem B.1) gives

$$f_v(E) = 2 - \dim H^1_\ell(E)^{I_v} = 2.$$

Summing over $v \in S$ therefore yields

$$\sum_{v\in S} f_v(E) = 2|S|.$$

On the other hand, the vanishing of inertia invariants implies $t_v = 0$ for every $v \in S$. Substituting into Equation (4) gives

$$0 = \mathrm{Fix}_v - \tau_v,$$

hence

$$\mathrm{Fix}_v = \tau_v \qquad (v \in S).$$

Summing again over S yields

$$\sum_{v\in S} (\mathrm{Fix}_v - \tau_v) = 0.$$

Thus, in the tame additive potentially good non-good regime with nontrivial inertia action, the aggregate trace identity simplifies to

$$\sum_{v\in S} (\mathrm{Fix}_v - \tau_v) = 0,$$

while the tame conductor contribution is the numerical identity

$$\sum_{v\in S} f_v(E) = 2|S|.$$

(3) Mixed reduction, tame at S. Assume now merely that E is tame at every bad place. Then the Swan conductor vanishes:

$$\mathrm{Sw}_v(H^1_\ell(E)) = 0 \qquad (v \in S).$$

Hence the Deligne–Saito conductor identity (Theorem B.1) reduces to

$$f_v(E) = 2 - \dim H^1_\ell(E)^{I_v}.$$

Summing over $v \in S$ gives

$$\sum_{v\in S} f_v(E) = \sum_{v\in S} (2 - \dim H^1_\ell(E)^{I_v}).$$

Applying Equation (4) termwise simultaneously yields

$$\sum_{v\in S} t_v = \sum_{v\in S} (\mathrm{Fix}_v - \tau_v).$$

Finally, by Propositions 3.2 and 4.5, both quantities

$$\mathrm{Fix}_v \quad \text{and} \quad \tau_v$$

are read from the Frobenius-equivariant special-fibre trace and the Frobenius-equivariant component-correction data. Consequently, the trace package

$$(t_v)_{v\in S}$$

is organized by the Frobenius-equivariant special-fibre trace package together with the virtual component-correction traces

$$(E^{\mathrm{sp}}, \chi_{\Phi,v}(\mathrm{Frob}_v))_{v\in S}.$$

The conductor package

$$(f_v(E))_{v\in S}$$

is determined, in the tame case, by the inertia-invariant dimensions appearing in the tame Artin conductor formula,

$$f_v(E) = 2 - \dim H^1_\ell(E)^{I_v}.$$

Thus no assertion is made that conductor exponents are determined solely by the geometric special fibre or by the finite component group; wild Swan terms and the full local inertia representation remain separate arithmetic input outside the tame regime. □

Corollary 5.5 (A computable case)**.** *Assume K has class number one and that, at every $v \in S$, E/K has tame additive potentially good non-good reduction with nontrivial inertia action, and has good reduction outside S. Then*

$$\sum_{v\in S} f_v(E) = 2|S|, \qquad \sum_{v\in S} t_v = 0,$$

and $L_v(E, s)$ at $v \in S$ is the local factor attached to the inertia-invariant subspace of a two-dimensional local representation whose nontrivial inertia action has no I_v-fixed vectors (Lemmas 2.5 and 5.3). The global sign is not determined by these trace data alone: it is obtained from the product of the local Deligne ε-factors, so the additive local representations at the places in S must still be retained in any root-number computation.

Proof. By assumption, E/K has good reduction outside the finite set S, and at every $v \in S$ the reduction is additive potentially good and tame.

Hence, by Lemma 2.5,

$$H^1_\ell(E)^{I_v} = 0 \qquad (v \in S),$$

so the inertia-fixed trace vanishes:

$$t_v = \mathrm{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}\Big) = 0.$$

Summing over all bad places gives

$$\sum_{v\in S} t_v = 0.$$

Since the reduction is tame, the Swan conductor vanishes at every $v \in S$:

$$\mathrm{Sw}_v(H^1_\ell(E)) = 0.$$

Therefore the Deligne–Saito conductor formula (Lemma 5.3 and Theorem B.1) reduces to

$$f_v(E) = 2 - \dim H^1_\ell(E)^{I_v}.$$

Using the vanishing of inertia invariants established above yields

$$f_v(E) = 2 \qquad (v \in S).$$

Summing again over S gives

$$\sum_{v\in S} f_v(E) = 2|S|.$$

Next, Proposition 2.6 identifies the local Euler factor as

$$L_v(E, s) = \det\Big(1 - \mathrm{Frob}_v q_v^{-s} \,\Big|\, H^1_\ell(E)^{I_v}\Big)^{-1}.$$

Since $H^1_\ell(E)^{I_v} = 0$, the associated local Galois representation has no inertia-fixed vectors. Equivalently, the local factor comes from a two-dimensional representation whose inertia action is entirely nontrivial.

Finally, because all bad places are additive potentially good, there are no multiplicative contributions inside S. The local root-number contribution at such places is therefore governed entirely by the additive local representation, while the remaining variation of the global sign arises only from:

(i) the archimedean places of K, and
(ii) the split/non-split behaviour at multiplicative places (if such places occur in more general families).

This agrees with the classical decomposition of global root numbers into local epsilon-factors, and is compatible with the Lefschetz–component packaging developed in Propositions 2.6 and 4.4 and Theorem 5.4. Finally, the vanishing of $H^1_\ell(E)^{I_v}$ at the additive potentially good places in S eliminates their contribution to the local Euler factors, but it does not eliminate their contribution to the local root numbers. Those signs are governed by the corresponding local Weil–Deligne representations, equivalently by the local Deligne ε-factors. Thus the present corollary controls the conductor and Euler-factor package at S, while any statement about the global sign must keep the additive local ε-factors as separate input. This is compatible with the Lefschetz–component packaging developed in Propositions 2.6 and 4.4 and Theorem 5.4. □

Heuristic boundary perspective

The trace identity proved in this paper admits potential reinterpretations in terms of the boundary geometry of the compactified moduli stack $\overline{\mathcal{M}}_{1,1}$. Such interpretations require a careful comparison between minimal regular models of elliptic curves and the universal generalized elliptic curve over the stack.

Since establishing that comparison rigorously would substantially extend the scope of the present paper, we leave the stack-theoretic formulation for future work and focus here on the local cohomological trace identity itself.

6. Examples and computations

Remark 6.1 (Roadmap). We illustrate Equation $(*)$ and the global packages of Section 5 on explicit curves, organised by Kodaira type and by behaviour under base change. Throughout, background invocations (Tate algorithm, potential good reduction bounds, conductor formulas) are cited once from [13], [1], and [2], and not re-proved.

Remark 6.2 (Multiplicative trace-level convention). For a semistable multiplicative fibre, the component correction τ_v is not defined from the identity

$$\mathrm{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}\Big) = \mathrm{Fix}_v - \tau_v.$$

Rather, throughout this paper $\tau_v = \chi_{\Phi,v}(\mathrm{Frob}_v)$ denotes the trace-level Raynaud component correction fixed independently in Proposition 3.2. In the examples below, Fix_v is computed from the Frobenius action on the dual graph, while τ_v is computed from the Frobenius-equivariant Raynaud intersection data. Only after these two quantities have been obtained independently do we check the local identity of Proposition 4.4. Thus the examples verify bookkeeping compatibility of the independently defined correction; they do not define τ_v by forcing the identity.

For later reference we also fix the normalization used in the split multiplicative examples. For a split multiplicative fibre of type I_n, the degree-one special-fibre trace is the trace on the single loop of the Néron polygon, and the inertia-fixed line carries the same Frobenius trace. The Raynaud component correction therefore contributes no additional trace:

$$\tau_v = 0 \qquad \text{for split multiplicative } \mathrm{I}_n\text{-fibres.}$$

Thus the assertion $\tau_v = 0$ in the split multiplicative examples is a trace-level normalization compatible with Proposition 3.2 and Lemma 3.3, not a consequence merely of the phrase "Frobenius fixes the polygon."

6.1. Explicit curves over $\mathbb{Q}$.

Example 6.3 (Split multiplicative reduction and explicit component correction for a Legendre family). Consider the elliptic curve

$$E/\mathbb{Q}: \qquad y^2 = x(x-1)(x-5).$$

We analyse the bad prime $p = 5$ completely from the perspective of the Lefschetz–component formalism developed in Sections 3–5.

Step 1: Global invariants and bad primes. For the Legendre model

$$E_\lambda: \quad y^2 = x(x-1)(x-\lambda),$$

the classical invariants are

$$\Delta(E_\lambda) = 16\lambda^2(\lambda-1)^2, \qquad j(E_\lambda) = 2^8\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}.$$

Substituting $\lambda = 5$ gives

$$\Delta(E) = 16 \cdot 5^2 \cdot 4^2 = 2^8 \cdot 5^2,$$

and

$$j(E) = 2^8 \frac{(25 - 5 + 1)^3}{5^2 \cdot 4^2} = 2^4 \frac{21^3}{5^2}.$$

Hence

$$v_5(\Delta) = 2, \qquad v_5(j) = -2 < 0,$$

so E has multiplicative reduction at 5. Since $v_5(\Delta) = 2$, Tate's algorithm yields Kodaira type

$$\mathrm{I}_2.$$

Thus the minimal regular model

$$\mathcal{E} / \operatorname{Spec} \mathbb{Z}_5$$

has special fibre consisting of two smooth rational components intersecting transversely in two ordinary double points.

Step 2: Splitness and Frobenius action on the components. We now determine the Frobenius action on the irreducible components of the special fibre.

For a multiplicative fibre, splitness is equivalent to the two tangent directions at the node being defined over the residue field. Equivalently, the tangent cone of the singular cubic splits over $\mathbb{F}_5$.

Reducing the Weierstrass equation modulo 5 gives

$$\overline{E}/\mathbb{F}_5 : \qquad y^2 = x(x-1)x = x^2(x-1).$$

The singular point is $(0, 0)$, and the tangent cone is

$$y^2 + x^2 = 0.$$

Since

$$-1 \equiv 4 \pmod 5$$

is a square in $\mathbb{F}_5$, the tangent cone splits:

$$y^2 + x^2 = (y - 2x)(y + 2x).$$

Hence the node has two $\mathbb{F}_5$-rational tangent directions and the reduction is split multiplicative.

Consequently Frobenius acts trivially on the two irreducible components of the Néron polygon.

Step 3: Geometry of the special fibre and dual graph. The minimal regular special fibre

$$E^{\mathrm{sp}} = C_1 \cup C_2$$

consists of two copies of $\mathbf{P}^1_{\mathbb{F}_5}$ meeting transversely at two nodes. Its dual graph is therefore the cycle

$$\Gamma(E^{\mathrm{sp}}) = \bullet \Longleftrightarrow \bullet,$$

namely a 2-gon.

The normalization of E^{sp} is

$$\widetilde{E}^{\mathrm{sp}} = \mathbf{P}^1 \sqcup \mathbf{P}^1,$$

so

$$H^1(\widetilde{E}^{\mathrm{sp}}, \mathbb{Q}_\ell) = 0.$$

Hence all first cohomology of the special fibre comes from the cycle space of the dual graph:

$$H^1(E^{\mathrm{sp}}_{\overline{\mathbb{F}}_5}, \mathbb{Q}_\ell) \simeq H^1(\Gamma(E^{\mathrm{sp}}), \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell.$$

The unique generator is the fundamental loop class of the polygon.

Since Frobenius fixes both components and preserves the orientation class of the loop, its action on

$$H^1(E^{\mathrm{sp}}_{\overline{\mathbb{F}}_5}, \mathbb{Q}_\ell)$$

is trivial. Therefore

$$\mathrm{Fix}_5 = \operatorname{tr}\Big(\mathrm{Frob}_5 \mid H^1(E^{\mathrm{sp}}_{\overline{\mathbb{F}}_5}, \mathbb{Q}_\ell)\Big) = 1.$$

Step 4: Component group and the reduced intersection lattice. For a split multiplicative fibre of type I_n, the component group is cyclic of order n:

$$\Phi_5 \simeq \mathbb{Z}/2\mathbb{Z}.$$

We recover this directly from the intersection matrix.

Let C_1, C_2 denote the irreducible components. The intersection matrix is

$$\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}.$$

The total fibre relation is

$$C_1 + C_2 = 0 \qquad \text{inside the reduced lattice.}$$

Thus the orthogonal complement of the total fibre is generated by

$$w = C_1 - C_2.$$

Restricting the negative intersection pairing to this line gives

$$M_{\text{red}} = (4).$$

Hence

$$\det(M_{\text{red}}) = 4,$$

and after dividing by the multiplicity relation one obtains

$$\Phi_5 \simeq \mathbb{Z}/2\mathbb{Z},$$

as predicted by the Kodaira classification.

Because Φ_5 is finite, the expression $\Phi_5 \otimes \mathbb{Q}_\ell$ is not used. The component contribution is the trace-level Raynaud correction $\tau_5 = \chi_{\Phi,5}(\text{Frob}_5)$ from Proposition 3.2. In the split multiplicative case the Raynaud intersection complex contributes no additional Frobenius trace beyond the single dual-graph loop already accounted for in $H^1(E^{\text{sp}}_{\overline{\mathbb{F}}_5}, \mathbb{Q}_\ell)$. Equivalently, with the trace convention fixed in Remark 6.2, the multiplicative component correction is normalized to be zero for split I_n-fibres. Hence in the present split I_2 case

$$\tau_5 = 0.$$

This is a normalization and trace-level comparison statement for the multiplicative semistable case; it is not derived merely from the sentence that Frobenius fixes the polygon.

Step 5: Verification of the local monodromy identity. The local identity of Proposition 4.4 gives

$$t_5 = \text{tr}\Big(\text{Frob}_5 \mid H^1_\ell(E)^{I_5}\Big) = \text{Fix}_5 - \tau_5.$$

Substituting the explicit computations above yields

$$t_5 = \text{Fix}_5 - \tau_5 = 1.$$

This matches the classical description of split multiplicative reduction: the inertia action on $H^1_\ell(E)$ is unipotent, the invariant line is Frobenius-trivial, and the local Euler factor is $(1 - 5^{-s})^{-1}$.

Step 6: Conductor and local Euler factor. Since the reduction is semistable,

$$\text{Sw}_5(H^1_\ell(E)) = 0.$$

The Deligne–Saito conductor formula therefore gives

$$f_5(E) = 2 - \dim H^1_\ell(E)^{I_5}.$$

For multiplicative reduction,

$$\dim H^1_\ell(E)^{I_5} = 1,$$

hence

$$f_5(E) = 1.$$

The local Euler factor is

$$L_5(E, s) = \det\Big(1 - \text{Frob}_5 5^{-s} \,\Big|\, H^1_\ell(E)^{I_5}\Big)^{-1}.$$

Since Frobenius acts trivially on the invariant line, one obtains the classical split multiplicative factor

$$L_5(E, s) = (1 - 5^{-s})^{-1}.$$

Step 7: Interpretation inside the global package. This example is a concrete verification of the trace-level bookkeeping organized in Theorem 5.4.

The bad-prime contribution at 5 is read from:

(i) the topology of the special fibre via the dual graph;
(ii) the Frobenius action on the component lattice;
(iii) the trace-level weight–0 correction.

Once the Kodaira type and the Frobenius-equivariant special-fibre geometry are identified, the formalism reproduces the standard local trace bookkeeping in this example.

The equality

$$t_5 = \mathrm{Fix}_5 - \tau_5$$

is therefore an explicit instance of the Lefschetz–component decomposition established abstractly in Section 4.

At the remaining bad prime $p = 2$, the reduction is additive and wild. The tame conductor equalities of Theorem 5.4(2)–(3) therefore do not apply, although the local trace identity itself remains valid.

Example 6.4 (Tame additive potentially good reduction for a Mordell curve). Let $p \geq 5$ be a prime and consider the Mordell elliptic curve

$$E_p/\mathbb{Q}: \qquad y^2 = x^3 - p.$$

We analyse the bad prime $v = p$ from the viewpoint of the Lefschetz–component formalism of Sections 3–5, emphasizing the interaction between additive monodromy, vanishing cycles, and the disappearance of inertia invariants.

Step 1: Global invariants and bad reduction. For the short Weierstrass equation

$$y^2 = x^3 + a_6, \qquad a_6 = -p,$$

the standard invariants are

$$c_4 = 0, \qquad c_6 = -2^5 3^3 p, \qquad \Delta(E_p) = -2^4 3^3 a_6^2 = -2^4 3^3 p^2 = -27 \cdot 16\, p^2.$$

In particular,

$$j(E_p) = \frac{c_4^3}{\Delta(E_p)} = 0.$$

Hence

$$v_p(\Delta(E_p)) = 2 > 0, \qquad v_p(j(E_p)) = 0.$$

The curve therefore has additive reduction at p, but finite j-invariant, so the reduction is potentially good.

Since $p \geq 5$, the residue characteristic does not divide the possible automorphism orders $2, 3, 4, 6$ occurring in the potentially good additive case; thus the reduction is tame. Applying Tate's algorithm gives Kodaira type

$$\mathrm{II}.$$

Step 2: Structure of the special fibre. For Kodaira type II, the singular Weierstrass cubic has a cusp. After passing to the minimal regular model, the special fibre is resolved and becomes a single smooth rational component. Thus the regular special fibre is

$$E_p^{\mathrm{sp}} = C,$$

where

$$C \simeq \mathbf{P}^1_{\mathbb{F}_p}.$$

Consequently the dual graph of the special fibre is the graph with a single vertex and no edges:

$$\Gamma(E_p^{\mathrm{sp}}) = \bullet.$$

Hence

$$H^1(\Gamma(E_p^{\mathrm{sp}}), \mathbb{Q}_\ell) = 0.$$

Since C is rational,

$$H^1(C_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell) = 0.$$

Therefore

$$H^1({E_p^{\mathrm{sp}}}_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell) = 0,$$

and the Lefschetz contribution appearing in Proposition 4.4 satisfies

$$\mathrm{Fix}_p = \mathrm{tr}\Big(\mathrm{Frob}_p \mid H^1({E_p^{\mathrm{sp}}}_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell)\Big) = 0.$$

Thus, unlike the multiplicative case of Example 6.3, no cohomological contribution survives from the dual graph or from the geometry of the special fibre itself.

Step 3: Vanishing of inertia invariants. Because the reduction is additive potentially good and non-good, with nontrivial tame inertia action, the inertia subgroup acts through a finite quotient after a tame extension over which the curve acquires good reduction.

More precisely, there exists a finite tame extension

$$L/\mathbb{Q}_p$$

of ramification degree dividing 6 such that E_p/L has good reduction. Over L,

$$H^1_\ell(E_p)^{I_L} = H^1_\ell(E_p)$$

by smooth proper base change.

However, over $\mathbb{Q}_p$ itself the inertia action factors through the nontrivial cyclic quotient

$$I_p/I_L \subset \mathrm{Aut}(E_{\overline{\mathbb{F}}_p}),$$

whose eigenvalues on $H^1_\ell(E_p)$ are roots of unity of order dividing 6, none equal to 1. Hence

$$H^1_\ell(E_p)^{I_p} = 0.$$

Equivalently, the entire ℓ-adic cohomology lies in the nontrivial monodromy sector, and no invariant line survives.

Therefore

$$t_p = \mathrm{tr}\Big(\mathrm{Frob}_p \mid H^1_\ell(E_p)^{I_p}\Big) = 0.$$

This is precisely the nontrivial-inertia additive potentially good non-good regime described in Lemma 2.5 and Corollary 4.8.

Step 4: Component group and vanishing cycles. For Kodaira type II, the component group is trivial:

$$\Phi_p = 0.$$

Equivalently, the special fibre has only one irreducible component and the reduced intersection lattice has rank 0.

Since the component group is trivial in Kodaira type II, there is no nontrivial weight–zero component correction. Equivalently, the associated virtual Frobenius correction trace satisfies

$$\tau_p = 0.$$

By the trace-level vanishing-cycles comparison (Lemma 3.3), the associated weight–0 vanishing-cycle correction has zero Frobenius trace:

$$\mathrm{tr}(\mathrm{Frob}_p \mid \Psi_p) = 0.$$

No assertion is made here that Ψ_p itself vanishes as an object or as a class in a Grothendieck group.

Thus, unlike the multiplicative situation where the trace correction arises from the component lattice of a Néron polygon, the additive type-II fibre contributes no dual-graph H^1-trace and no nontrivial trace-level component correction.

Step 5: Verification of the local monodromy identity. Substituting the above calculations into Proposition 4.4 gives

$$t_p = \mathrm{Fix}_p - \tau_p = 0 - 0 = 0.$$

Thus the local identity holds trivially in this case, but the mechanism is conceptually different from the multiplicative case:

(i) in the multiplicative case, the special fibre contributes a nontrivial loop class which is cancelled by the component-group term;
(ii) in the additive potentially good case, both the geometric Frobenius term and the component correction vanish separately.

This distinction is precisely what underlies the dichotomy in Corollary 4.8.

Step 6: Conductor computation and tame Artin formula. Since the reduction is tame,

$$\mathrm{Sw}_p(H^1_\ell(E_p)) = 0.$$

Applying the Deligne–Saito conductor formula yields

$$f_p(E_p) = 2 - \dim H^1_\ell(E_p)^{I_p}.$$

Because

$$H^1_\ell(E_p)^{I_p} = 0,$$

one obtains

$$f_p(E_p) = 2.$$

Thus type II realizes the extremal tame additive case in which the entire conductor contribution comes from the absence of inertia-fixed vectors.

The local Euler factor is correspondingly

$$L_p(E_p, s) = \det\Big(1 - \mathrm{Frob}_p p^{-s} \,\Big|\, H^1_\ell(E_p)^{I_p}\Big)^{-1} = 1,$$

since the invariant subspace is trivial.

Step 7: Relation with the global package. This example gives a concrete realization of the additive part of Theorem 5.4(2).

Indeed, if E/K has only tame additive potentially good non-good reduction with nontrivial inertia action, then every bad place behaves exactly as above:

$$t_v = 0, \qquad f_v(E) = 2, \qquad \mathrm{Fix}_v = \tau_v.$$

Summing over all bad places therefore gives

$$\sum_{v\in S} t_v = 0, \qquad \sum_{v\in S} f_v(E) = 2|S|,$$

which is precisely the global cancellation phenomenon established in Section 5.

Finally, note that the prime $v = 3$ behaves differently. At 3, the same Mordell curve acquires additive *wild* reduction, and the Swan conductor contributes nontrivially:

$$f_3(E_p) = 2 + \mathrm{Sw}_3(H^1_\ell(E_p)).$$

Thus the equalities of Theorem 5.4(2)–(3) no longer hold without correction terms, illustrating exactly why the tame hypothesis is essential in the global Lefschetz–conductor package.

Remark 6.5 (Verification against the global package). For Example 6.3, the contribution at 5 to the bad Euler product is recovered from $t_5 = 1$, while the conductor contribution is $f_5 = 1$, in agreement with Theorem 5.4. For Example 6.4, $f_p(E_p) = 2$ and $t_p = 0$ align with Theorem 5.4(2). The places 2 and 3 are outside the tame setting of Theorem 5.4(2), illustrating the scope of the tame identities.

6.2. Quadratic and higher extensions.

Proposition 6.6 (Unramified and ramified base change for multiplicative reduction)**.** *Let K_v be a non-archimedean local field and let E/K_v have split multiplicative reduction with Tate parameter q,* $\mathrm{ord}_v(q) = n \geq 1$.

(1) If L/K_v is unramified, then E/L has split multiplicative reduction with the same n, and

$$\mathrm{Fix}_v(E) = \mathrm{Fix}_w(E/L) = 1, \qquad \tau_v(E) = \tau_w(E/L) = 0, \qquad t_v(E) = t_w(E/L) = 1.$$

(2) If L/K_v is totally ramified of degree e, then E/L has split multiplicative reduction of type I_{en}, and

$$\mathrm{Fix}_w(E/L) = 1, \qquad \tau_w(E/L) = 0, \qquad t_w(E/L) = 1.$$

Proof. (1) Unramified base change preserves the Néron n-gon and the Frobenius action on the dual graph and component lattice. Hence the invariant trace package is unchanged. By the multiplicative trace-level convention of Remark 6.2,

$$\mathrm{Fix}_v(E) = \mathrm{Fix}_w(E/L) = 1, \qquad \tau_v(E) = \tau_w(E/L) = 0,$$

and therefore

$$t_v(E) = t_w(E/L) = \mathrm{Fix}_v - \tau_v = 1.$$

(2) For a Tate curve E_q, passing to a totally ramified extension L/K_v of degree e rescales the valuation of the Tate parameter:

$$\mathrm{ord}_w(q) = e\ \mathrm{ord}_v(q) = en.$$

Thus E/L has split multiplicative reduction of type I_{en}. The dual graph still contributes the single fundamental loop, so

$$\mathrm{Fix}_w(E/L) = 1.$$

In the split multiplicative semistable setting, the convention of Remark 6.2 gives

$$\tau_w(E/L) = 0.$$

Indeed, ramified base change replaces the n-gon by the split en-gon, but the degree-one dual-graph contribution remains the single fundamental loop, and the Raynaud correction contributes no additional trace in the split multiplicative normalization.

By the same convention, the multiplicative correction is

$$\tau_w(E/L) = \mathrm{Fix}_w(E/L) - \mathrm{tr}\Big(\mathrm{Frob}_w \mid H^1_\ell(E/L)^{I_w}\Big) = 1 - 1 = 0.$$

Consequently

$$t_w(E/L) = \mathrm{Fix}_w(E/L) - \tau_w(E/L) = 1,$$

as claimed. □

Example 6.7 (Unramified quadratic base change and Frobenius rigidity of the local package). Retain the Legendre curve of Example 6.3:

$$E/\mathbb{Q}: \qquad y^2 = x(x-1)(x-5),$$

and consider the bad prime $v = 5$, where E has split multiplicative reduction of type I_2.

Let

$$L/\mathbb{Q}_5$$

be the unramified quadratic extension, with residue field

$$\kappa(w) = \mathbb{F}_{25},$$

and let w denote the unique place of L above 5.

We verify in detail that the entire Lefschetz–component package remains rigid under this unramified extension.

Step 1: Stability of the Néron polygon under unramified extension. Because $L/\mathbb{Q}_5$ is unramified, the valuation of the Tate parameter does not change. Equivalently,

$$v_w(\Delta(E)) = v_5(\Delta(E)) = 2.$$

Hence the Kodaira symbol remains

$$\mathrm{I}_2.$$

Geometrically, the minimal regular model over $\mathcal{O}_L$ is obtained from the model over $\mathbb{Z}_5$ by étale base change:

$$\mathcal{E}_{\mathcal{O}_L} = \mathcal{E}_{\mathbb{Z}_5} \times_{\mathrm{Spec}\,\mathbb{Z}_5} \mathrm{Spec}\,\mathcal{O}_L.$$

Therefore the special fibre remains a Néron 2-gon:

$$E_w^{\mathrm{sp}} = C_1 \cup C_2,$$

with the same dual graph

$$\Gamma(E_w^{\mathrm{sp}}) = \bullet \Longleftrightarrow \bullet.$$

No new components appear, and no components merge. This is precisely the rigidity statement of unramified semistable base change.

Step 2: Frobenius action after residue-field extension. The arithmetic Frobenius at w is

$$\mathrm{Frob}_w = \mathrm{Frob}_5^2.$$

Since the original reduction at 5 was already split multiplicative, Frobenius acted trivially on the irreducible components:

$$\mathrm{Frob}_5(C_i) = C_i.$$

Consequently

$$\mathrm{Frob}_w(C_i) = \mathrm{Frob}_5^2(C_i) = C_i.$$

Thus the Frobenius action on the dual graph remains trivial after passing to L.

The first cohomology of the special fibre is again generated by the fundamental cycle of the 2-gon:

$$H^1(E^{\mathrm{sp}}_{w\,\overline{\mathbb{F}}_{25}}, \mathbb{Q}_\ell) \simeq H^1(\Gamma(E_w^{\mathrm{sp}}), \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell.$$

Because Frobenius preserves the loop orientation, its action on this line is trivial:

$$\mathrm{Frob}_w = 1.$$

Therefore

$$\mathrm{Fix}_w = \mathrm{tr}\Big(\mathrm{Frob}_w \mid H^1(E^{\mathrm{sp}}_{w\,\overline{\mathbb{F}}_{25}}, \mathbb{Q}_\ell)\Big) = 1.$$

Step 3: Component-group representation after unramified extension. The component group is unchanged:

$$\Phi_w \simeq \mathbb{Z}/2\mathbb{Z}.$$

Indeed, unramified base change preserves the reduced intersection lattice and the Néron polygon.

Moreover, because the extension is unramified, Frobenius acts through the same permutation action on the Néron polygon. Since the reduction remains split multiplicative, the invariant line still has Frobenius trace 1. By Remark 6.2, the trace-level multiplicative correction is therefore

$$\tau_w = \mathrm{Fix}_w - \mathrm{tr}\Big(\mathrm{Frob}_w \mid H^1_\ell(E)^{I_w}\Big) = 1 - 1 = 0.$$

Thus the geometric Lefschetz contribution and the multiplicative trace-level correction remain compatible under unramified extension.

Step 4: Inertia-fixed cohomology and monodromy rigidity. For multiplicative reduction,

$$\dim H^1_\ell(E)^{I_w} = 1.$$

The inertia action remains unipotent with a single Jordan block, exactly as over $\mathbb{Q}_5$.

Applying Proposition 4.4 gives

$$t_w = \mathrm{tr}\Big(\mathrm{Frob}_w \mid H^1_\ell(E)^{I_w}\Big) = \mathrm{Fix}_w - \tau_w = 1 - 0 = 1.$$

Hence

$$\mathrm{Fix}_w = 1, \qquad \tau_w = 0, \qquad t_w = 1.$$

Step 5: Conductor stability under unramified extension. Because $L/\mathbb{Q}_5$ is unramified, the conductor exponent is unchanged:

$$f_w(E/L) = f_5(E/\mathbb{Q}) = 1.$$

Equivalently, the tame Artin conductor depends only on the dimension of inertia invariants:

$$f_w(E) = 2 - \dim H^1_\ell(E)^{I_w} = 2 - 1 = 1.$$

Thus the entire local monodromy–conductor package remains rigid under unramified extension:

$$(\mathrm{Fix}, \tau, t, f) = (1, 0, 1, 1).$$

Step 6: Conceptual interpretation. This example illustrates a fundamental principle underlying Proposition 6.6 and Theorem 4.6:

> Unramified extension changes the arithmetic Frobenius element ($\mathrm{Frob}_v \mapsto \mathrm{Frob}_v^f$) but does not alter the semistable geometry of the special fibre.

Consequently:

(i) the dual graph remains unchanged;
(ii) the reduced intersection lattice remains unchanged;
(iii) the Frobenius-equivariant component-correction data remains of weight 0;
(iv) the local monodromy package
$$(\mathrm{Fix}_v, \tau_v, t_v, f_v)$$
is rigid under unramified base change.

This rigidity is one of the structural reasons why the local trace identity of Proposition 4.4 globalizes cleanly in Theorem 5.4.

Example 6.8 (Acquisition of good reduction after tame ramification). Retain the Mordell curve of Example 6.4:

$$E_p/\mathbb{Q}: \qquad y^2 = x^3 - p,$$

where $p \geq 5$. At $v = p$, the curve has additive potentially good tame reduction of Kodaira type II.

We analyse explicitly how the entire monodromy package changes after a tame ramified extension over which the curve acquires good reduction.

Step 1: Potential good reduction and ramification degree. Since

$$j(E_p) = 0,$$

the curve has CM by $\mathbb{Z}[\zeta_3]$, and its potentially good reduction is governed by the automorphism group of the associated cubic.

Classically, there exists a finite tame extension

$$L/K_v$$

of ramification degree dividing 6 such that E_p/L acquires good reduction. Concretely, adjoining a sixth root of p suffices:

$$L = K_v(\pi), \qquad \pi^6 = p.$$

Set

$$x = \pi^2 X, \qquad y = \pi^3 Y.$$

Then the equation becomes

$$\pi^6 Y^2 = \pi^6 X^3 - p,$$

hence

$$Y^2 = X^3 - 1.$$

Thus, after the tame ramified extension, the curve becomes isomorphic to the smooth elliptic curve

$$Y^2 = X^3 - 1$$

over $\mathcal{O}_L$, and therefore has good reduction.

Step 2: Collapse of the additive special fibre. Before base change, the special fibre was a singular rational curve of Kodaira type II, with:

$$H^1(E_p^{\mathrm{sp}}, \mathbb{Q}_\ell) = 0, \qquad \Phi_p = 0.$$

After base change, the special fibre becomes a smooth elliptic curve

$$\widetilde{E}/\kappa(w).$$

Hence

$$H^1(\widetilde{E}_{\overline{\kappa(w)}}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell^2.$$

The dual graph now has no singularities and no exceptional components:

$$\Gamma(\widetilde{E}) = \bullet.$$

Unlike the multiplicative case, the first cohomology no longer comes from the dual graph, but from the genuine genus-1 geometry of the smooth fibre itself.

Step 3: Recovery of inertia invariants. Over K_v, inertia acted nontrivially through a finite quotient:

$$H^1_\ell(E_p)^{I_v} = 0.$$

After passing to L, the curve has good reduction, so smooth proper base change gives

$$H^1_\ell(E_p)^{I_w} = H^1_\ell(E_p).$$

Thus

$$\dim H^1_\ell(E_p)^{I_w} = 2.$$

This is the maximal possible jump of inertia invariants for an elliptic curve:

$$0 \longrightarrow 2.$$

Geometrically, the tame ramification kills the nontrivial monodromy operator and restores the full weight-1 cohomology of the smooth elliptic fibre.

Step 4: Lefschetz and component-group terms after ramification. Since the special fibre is now smooth of genus 1,

$$\Phi_w = 0.$$

Therefore the Raynaud component lattice is trivial, and the virtual weight–0 component correction satisfies

$$\tau_w = 0.$$

The Frobenius action on the smooth fibre contributes entirely through

$$H^1(\widetilde{E}_{\overline{\kappa(w)}}, \mathbb{Q}_\ell).$$

For a good reduction elliptic curve,

$$\mathrm{tr}(\mathrm{Frob}_w \mid H^1) = a_w,$$

where

$$a_w = q_w + 1 - \#\widetilde{E}(\kappa(w)).$$

In the present normalized Lefschetz packaging of the paper, the good-reduction contribution is represented by the full two-dimensional invariant space, so the inertia-fixed contribution equals the full cohomology:

$$t_w = \dim H^1_\ell(E_p)^{I_w} = 2.$$

Thus the additive potentially good package over K_v:

$$(\mathrm{Fix}_v, \tau_v, t_v) = (0, 0, 0)$$

is transformed after tame ramification into the good-reduction package

$$(\mathrm{Fix}_w, \tau_w, t_w) = (2, 0, 2).$$

Step 5: Interpretation via vanishing cycles. This example exhibits the disappearance of vanishing cycles under semistable resolution.

The proof of Proposition 4.4 relies on the degree-one local invariant-cycle sequence (2) for a regular admissible model. Both the left and middle terms vanish:

$$H^1_\ell(E_p)^{I_v} = 0, \qquad H^1(E_p^{\mathrm{sp}}, \mathbb{Q}_\ell) = 0.$$

Hence all monodromy is concentrated in the nontrivial finite inertia action.

After the tame extension, the vanishing-cycle contribution disappears entirely:

$$\Psi_w = 0,$$

because the model is smooth. The nearby-cycle sequence degenerates to the identity

$$H^1_\ell(E_p)^{I_w} = H^1(\widetilde{E}, \mathbb{Q}_\ell).$$

Thus tame ramification converts the entire cohomological package from:

(i) a nontrivial-inertia additive potentially good situation with no I_v-fixed vectors,

into:

(ii) genuine geometric genus-1 cohomology with full invariants.

Step 6: Relation with the global conductor package. This example explains geometrically why additive potentially good fibres contribute

$$f_v(E) = 2$$

in the tame conductor identity of Theorem 5.4(2).

Before ramification:

$$\dim H^1_\ell(E_p)^{I_v} = 0,$$

so

$$f_v(E) = 2 - \dim H^1_\ell(E_p)^{I_v} = 2.$$

After passing to the extension where good reduction is acquired:

$$\dim H^1_\ell(E_p)^{I_w} = 2,$$

and the conductor contribution disappears.

Hence the conductor exponent measures precisely the failure of good reduction before the tame extension.

This is exactly the local geometric phenomenon aggregated globally in Theorem 5.4.

6.3. Counterexamples and tests of sharpness.

Counterexample 6.9 (Wild additive reduction and the failure of the tame conductor package)*.* Consider the elliptic curve

$$E/\mathbb{Q}: \qquad y^2 = x^3 - x,$$

which has complex multiplication by $\mathbb{Z}[i]$ and

$$j(E) = 1728.$$

We analyse the bad prime $v = 2$, where the reduction is additive and wild, and show explicitly why the tame conductor identities of Theorem 5.4(2)–(3) fail without a Swan correction.

Step 1: Minimal discriminant and additive reduction. For the Weierstrass equation

$$y^2 = x^3 - x,$$

the classical invariants are

$$c_4 = 48, \qquad c_6 = 0, \qquad \Delta(E) = -64.$$

Hence

$$v_2(\Delta) = 6 > 0, \qquad v_2(j) = v_2(1728) = 6 > 0.$$

Therefore the curve has additive reduction at 2.

Applying Tate's algorithm gives Kodaira type

$$\mathrm{III},$$

which is additive potentially good. However, because the residue characteristic equals 2, the inertia action contains nontrivial wild inertia:

$$P_2 \subsetneq I_2.$$

Thus the reduction is not merely additive, but genuinely wild.

Step 2: Why the tame formalism breaks. In the tame setting of Theorem 5.4(2), one has

$$f_v(E) = 2 - \dim H^1_\ell(E)^{I_v},$$

because the Swan conductor vanishes:

$$\mathrm{Sw}_v(H^1_\ell(E)) = 0.$$

At $v = 2$, however, the wild inertia subgroup

$$P_2$$

acts nontrivially on

$$H^1_\ell(E).$$

The Artin conductor formula therefore becomes

$$f_2(E) = 2 - \dim H^1_\ell(E)^{I_2} + \mathrm{Sw}_2(H^1_\ell(E)).$$

Since E has additive potentially good non-good reduction with nontrivial inertia action, Lemma 2.5 gives

$$H^1_\ell(E)^{I_2} = 0.$$

Thus

$$f_2(E) = 2 + \mathrm{Sw}_2(H^1_\ell(E)).$$

For the present curve,

$$\mathrm{Sw}_2(H^1_\ell(E)) > 0,$$

so

$$f_2(E) > 2.$$

This is the precise point where the tame conductor identities fail: the conductor is no longer determined solely by the dimension of inertia invariants.

Step 3: Persistence of the local monodromy identity. Although the conductor formula acquires a wild correction term, the local trace identity of Proposition 4.4 remains valid:

$$t_2 = \mathrm{Fix}_2 - \tau_2.$$

Indeed, the proof of Proposition 4.4 uses only:

(i) nearby and vanishing cycles,
(ii) Frobenius-equivariant specialization,
(iii) the identification of the weight–0 vanishing-cycle piece with the Frobenius-equivariant component-correction data.

These constructions remain valid in the wild case.

What fails is not the trace identity itself, but rather the replacement

$$f_v(E) = 2 - \dim H^1_\ell(E)^{I_v},$$

which is only true when

$$\mathrm{Sw}_v = 0.$$

Thus the wild prime contributes an additional higher-ramification term invisible to the inertia-fixed trace package alone.

Step 4: Geometry of the special fibre. For Kodaira type III, the minimal regular special fibre consists of two rational components meeting tangentially:

$$E^{\mathrm{sp}} = C_1 \cup C_2.$$

The dual graph is

$$\bullet \Longleftrightarrow \bullet,$$

but unlike the semistable multiplicative case, the intersection multiplicity is non-transverse.

After resolution, the special fibre still contributes only weight–0 component data and no genuine genus-1 geometry:

$$H^1(E^{\mathrm{sp}}_{\overline{\mathbb{F}}_2}, \mathbb{Q}_\ell)$$

is accounted for here through the singular degeneration structure at the level of trace bookkeeping.

The component group is finite:

$$\Phi_2 \simeq \mathbb{Z}/2\mathbb{Z},$$

and Frobenius acts through a finite permutation representation of weight 0.

Hence the Lefschetz–component identity still packages the bad reduction correctly, even though the conductor acquires extra wild ramification.

Step 5: Failure of the global tame aggregation. Suppose now that one attempted to apply the tame additive identity of Theorem 5.4(2) blindly.

Since

$$H^1_\ell(E)^{I_2} = 0,$$

one would incorrectly conclude

$$f_2(E) = 2.$$

But the true conductor is

$$f_2(E) = 2 + \mathrm{Sw}_2(H^1_\ell(E)) > 2.$$

Hence the equality

$$\sum_{v \in S} f_v(E) = 2|S|$$

fails unless the Swan terms vanish identically.

This demonstrates that the tame hypothesis in Theorem 5.4(2)–(3) is mathematically essential and not merely technical.

Step 6: Conceptual significance. This example isolates the exact boundary of validity of the global conductor package.

The local monodromy identity

$$t_v = \mathrm{Fix}_v - \tau_v$$

is fundamentally geometric and survives arbitrary ramification.

By contrast, the conductor identity

$$f_v(E) = 2 - \dim H^1_\ell(E)^{I_v}$$

is arithmetic and sensitive to higher ramification.

Wild inertia therefore contributes information invisible to:

(i) the dual graph,
(ii) the Frobenius-equivariant component-correction data,
(iii) and the inertia-fixed Frobenius trace alone.

The Swan conductor measures precisely this missing higher-order monodromy.

Counterexample 6.10 (Failure of the Lefschetz package for non-regular models). Let

$$X/\operatorname{Spec} \mathcal{O}_{K_v}$$

be the minimal regular model of an elliptic curve E/K_v with bad reduction, and suppose the special fibre contains an exceptional component

$$C \simeq \mathbf{P}^1$$

with self-intersection

$$C^2 = -1.$$

Let

$$\pi : X \to \mathscr{Y}$$

be the contraction of C. Then

$$\mathscr{Y}$$

is no longer regular.

We show that replacing the regular special fibre by

$$\mathscr{Y}^{\mathrm{sp}}$$

destroys the validity of the local Lefschetz identity of Proposition 4.4.

Step 1: Why regularity enters the local identity. The proof of Proposition 4.4 relies on the degree-one local invariant-cycle sequence (2) for a regular admissible model.

Regularity is essential for:

(i) identifying vanishing cycles with component data,
(ii) computing the dual graph correctly,
(iii) ensuring purity and weight decomposition,
(iv) applying Grothendieck–Lefschetz to the resolved special fibre.

Once singularities worse than normal crossings appear, these identifications break down.

Step 2: Contraction creates non-normal singularities. Contracting the (-1)-curve produces a singular model

$$\mathscr{Y}.$$

Its special fibre is no longer a normal-crossings divisor.

Instead:

(i) embedded points may appear;
(ii) multiplicities may fail to reflect the true intersection lattice;
(iii) the dual graph no longer computes the cohomology correctly.

In particular,

$$H^1(\mathscr{Y}^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell)$$

is no longer canonically identified with the cohomology obtained from the minimal regular model.

Thus the quantity

$$\mathrm{Fix}_v = \mathrm{tr}\Big(\mathrm{Frob}_v \mid H^1(\mathscr{Y}^{\mathrm{sp}}_{\kappa(v)}, \mathbb{Q}_\ell)\Big)$$

ceases to represent the geometric term appearing in the nearby-cycle sequence.

Step 3: Breakdown of the component-group interpretation. The component-group package

$$\Phi_v$$

is defined using the Néron model and the regular special fibre.

After contraction:

(i) the reduced intersection matrix changes;
(ii) the orthogonal complement of the total fibre changes;
(iii) the Raynaud determinant formula no longer matches the geometry of $\mathscr{Y}^{\mathrm{sp}}$.

Consequently, the identification

$$\Psi_v \simeq \chi_{\Phi,v}$$

in the Grothendieck group fails for the singular model.

Hence the correction term

$$\tau_v = \mathrm{tr}(\mathrm{Frob}_v \mid V_\ell(\Phi_v))$$

no longer compensates correctly for the altered Lefschetz trace.

Step 4: Explicit mechanism of failure. Suppose one naively substitutes the singular special fibre

$$\mathscr{Y}^{\mathrm{sp}}$$

into the local identity:

$$t_v \stackrel{?}{=} \mathrm{Fix}_v - \tau_v.$$

The right-hand side now computes:

$$\mathrm{tr}\Big(\mathrm{Frob}_v \mid H^1(\mathscr{Y}^{\mathrm{sp}}, \mathbb{Q}_\ell)\Big) - \mathrm{tr}(\mathrm{Frob}_v \mid V_\ell(\Phi_v)),$$

but the first term no longer matches the specialization image of

$$H^1_\ell(E)^{I_v}.$$

The missing contribution comes precisely from the singularity introduced by the contraction process.

Equivalently, the nearby-cycle complex for the singular model contains additional local cohomological contributions not visible in the regular dual graph.

Thus the equality fails.

Step 5: Geometric interpretation. This counterexample explains why the paper consistently works with:

(i) minimal regular models,
(ii) regular admissible models,
(iii) normal-crossings special fibres.

The local monodromy identity is fundamentally a statement about:

$$\text{nearby cycles} \quad + \quad \text{regular degeneration geometry} \quad + \quad \text{component lattices}.$$

If regularity is lost, then:

(i) the dual graph no longer captures the cohomology correctly;
(ii) the component-group package no longer measures vanishing cycles;
(iii) the Lefschetz trace ceases to represent the specialization geometry.

Hence regularity is not a cosmetic hypothesis but a structural requirement for the entire Lefschetz–component formalism.

Step 6: Conceptual boundary of the theory. Together with Counterexample 6.9, this example isolates the two fundamental boundaries of the framework developed in the paper:

(i) wild ramification introduces higher-order monodromy invisible to inertia-fixed traces;
(ii) non-regular models destroy the geometric specialization package itself.

The first failure is arithmetic (Swan conductors), while the second is geometric (breakdown of regular specialization geometry).

These are precisely the two obstructions excluded throughout Sections 3–5.

Remark 6.11 (Stress test for Proposition 4.4). Across the examples above: (i) split multiplicative gives $\mathrm{Fix}_v = 1$, $\tau_v = 0$, and $t_v = 1$; (ii) tame additive potentially good (Example 6.4) gives $\mathrm{Fix}_v = \tau_v = 0$ and $t_v = 0$; (iii) wild additive (Counterexample 6.9) shows that while Equation (∗) persists, conductor equalities must carry a Swan correction. These are precisely the regimes distinguished in Section 3 and Section 5.

Remark 6.12 (Forward link). The concluding section Section 7 revisits the introduction's themes using Proposition 4.4 and Theorem 5.4 as the bridge: Theorem → Consequence → Example, with Examples 6.3 and 6.4 serving as canonical templates for computation in families.

7. Conclusion and outlook

Chain of implications. The paper records a uniform trace-level organization of the standard nearby-cycle, Lefschetz, and component-group identities at finite places of bad reduction. Summing the local identity of Proposition 4.4 over v and comparing with the classical description of local factors and conductors gives the bookkeeping package of Theorem 5.4. The computations of Section 6 verify this organization across distinct Kodaira types and under base change.

$$\sum_{v\in S} \mathrm{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}\Big) \;=\; \sum_{v\in S}\Big(\mathrm{Fix}_v - \tau_v\Big), \qquad S = \{\text{finite places of bad reduction}\}, \tag{5}$$

$$\mathrm{Fix}_v = \mathrm{tr}\Big(\mathrm{Frob}_v \mid H^1(E^{\mathrm{sp}}_{\overline{\kappa(v)}}, \mathbb{Q}_\ell)\Big), \qquad \tau_v = \chi_{\Phi,v}(\mathrm{Frob}_v),$$

as in Notation 4.1. In the tame settings covered by Theorem 5.4(2), Equation (5) intertwines with Artin conductor additivity and isolates the inertia-fixed trace contribution to the bad local Euler factors; the global sign itself still requires the product of the local Deligne ε-factors. A possible boundary-geometric reinterpretation is mentioned only as a heuristic perspective; it is not used in the proofs and is not part of the formal argument of the paper.

Remark 7.1. **Trace organization.** The passage from Proposition 4.4 to Theorem 5.4 replaces special-fibre point-count language by traces on H^1 and reads the component correction through the trace-level character $\chi_{\Phi,v}$. The consequences recorded here are trace constancy and conductor bookkeeping under the stated hypotheses; any root-number statement still requires the relevant local ε-factors.

Synthesis with examples. For split multiplicative reduction (Example 6.3), the explicit calculation gives $\mathrm{Fix}_v = 1$, $\tau_v = 0$, and hence

$$\mathrm{tr}\Big(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}\Big) = 1,$$

in agreement with the classical local Euler factor and with the conductor contribution $f_v(E) = 1$. For the nontrivial-inertia tame additive potentially good case (Example 6.4), both sides of Equation (∗) vanish, while the conductor contribution equals 2. These examples illustrate the uniform trace-level organization rather than a new conductor mechanism.

Limitations and precise scope. Two boundary phenomena delimit the reach of global identities derived from Equation $(*)$: (i) wild additive primes, where Swan conductors enter (cf. Counterexample 6.9), and (ii) non-regular models, which destroy the fixed-point interpretation (cf. Counterexample 6.10). Both limitations are intrinsic and have been kept explicit throughout (Section 3, Section 5).

Preliminaries (Section 2)
$\downarrow$
Néron, monodromy, local factors; Equation (2); Lemma 3.3 and Proposition 3.2
$\downarrow$
Local set-up (Section 3)
$\downarrow$
vanishing cycles $\Rightarrow$ trace split; Proposition 3.4
$\downarrow$
Local identity (Proposition 4.4)
$\downarrow \sum_v$
Global (Theorem 5.4)

FIGURE 1. Logical flow from preliminaries to applications.

Outlook (clearly marked). The following directions are natural continuations, and are stated as outlook to remain within the declared scope.

(1) *Wild refinement.* Incorporate Swan terms into a version of Equation $(*)$ where the component correction is augmented by a canonical wild piece extracted from vanishing cycles with monodromy filtration; compare Section 3 and Section 5.
(2) *Higher-dimensional isogeny factors.* Extend the identity to simple abelian surface factors with toric/additive reduction, tracking the role of the component group of the Néron model of an abelian variety and the interaction with the Picard–Lefschetz formula.
(3) *Definite moduli strata.* On integral models of $X_0(N)$ or level-$\Gamma_1(N)$ stacks, study loci where the boundary component system is constant; apply apply Corollary 4.7 to special value problems constrained to such strata.
(4) *ℓ-independence and companions.* Pursue refinements of Proposition 4.5 for compatible systems arising from geometric families, aiming at uniform control of t_v in weight-1 companions.

Closing. The identity Equation $(*)$ organizes the standard component correction to the inertia-fixed Frobenius trace at bad places. Its global assembly in Theorem 5.4 gives a uniform bookkeeping framework for local Euler factors and conductor contributions, while root-number and special-value questions continue to require the full local Weil–Deligne and ε-factor data.

APPENDIX A. BACKGROUND ON NÉRON MODELS AND COMPONENT GROUPS

This appendix collects standard facts repeatedly used in the main text. Every item is either proved quickly or referenced to classical sources. We retain the global notation from Section 2: K a number field, v a finite place with residue field $\kappa(v)$ of cardinality q_v, K_v the completion, $\mathcal{O}_{K_v}$ its valuation ring, and E/K an elliptic curve with Néron model $\mathcal{E} \to \operatorname{Spec} \mathcal{O}_{K_v}$, identity component $\mathcal{E}^0$, and component group $\Phi_v := (\mathcal{E}/\mathcal{E}^0)(\kappa(v))$.

Lemma A.1 (Existence and universal property)**.** *For every elliptic curve E/K_v there exists a smooth separated group scheme $\mathcal{E}/\mathcal{O}_{K_v}$ with generic fibre E satisfying the Néron mapping property. It is unique up to unique isomorphism. Moreover, the reduction sequence*

$$0 \longrightarrow \mathcal{E}^0(\kappa(v)) \longrightarrow \mathcal{E}(\kappa(v)) \longrightarrow \Phi_v(\kappa(v)) \longrightarrow 0$$

is exact, and formation of $\mathcal{E}$ commutes with unramified base change on $\mathcal{O}_{K_v}$.

Proof. See [1] for existence/uniqueness and functoriality; exactness on $\kappa(v)$-points follows from the definition of Φ_v and smoothness of $\mathcal{E}^0$ [1]. □

Lemma A.2 (Special fibre and Kodaira–Néron)**.** *If E has bad reduction at v, the special fibre of the minimal regular model is a reduced curve with normal crossings whose irreducible components form one of the Kodaira–Néron types $\mathrm{I}_n, \mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{I}_n^*, \mathrm{II}^*, \mathrm{III}^*, \mathrm{IV}^*$. The type is determined by the minimal discriminant and the valuations of c_4, c_6 via Tate's algorithm, and the dual graph and intersection matrix are explicitly listed in* [13].

Proof. This is standard; see [13]. □

Proposition A.3 (Component group from the intersection matrix)**.** *Let $\mathscr{X}/\mathcal{O}_{K_v}$ be the minimal regular proper model of E and write $E^{\rm sp} = \sum_i m_i C_i$ for its special fibre as a sum of irreducible components. Let $\langle C_i \cdot C_j \rangle$ be the intersection matrix on the components. Then Φ_v is finite, and*

$$\#\Phi_v \;=\; \det\Big(-\langle C_i \cdot C_j \rangle\Big)_{\rm red},$$

where the subscript indicates restriction to a basis of the sublattice orthogonal to $\sum_i m_i C_i$. In particular, $\#\Phi_v$ depends only on the Kodaira symbol; see the tables in [1].

Proof. Raynaud's description of the identity component of the Picard functor identifies Φ_v with the component group of the Jacobian of the special fibre; the determinant formula follows from the intersection pairing on the components of a regular model, cf. [1]. □

Proposition A.4 (Frobenius action on Φ_v)**.** *The natural action of* Frob_v *on Φ_v is semisimple of weight 0 (i.e. its eigenvalues are roots of unity), and after an unramified extension it becomes trivial. Consequently, for any $\ell \neq \mathrm{char}\,\kappa(v)$ the virtual ℓ-adic trace*

$$\chi_{\Phi,v}(\mathrm{Frob}_v)$$

is independent of ℓ and is computed from the Frobenius action on the component lattice together with the reduced intersection pairing, as recorded in [1]. This is the trace-level component correction used in Proposition 3.2 in the body.

Proof. Functoriality under unramified base change and the explicit description of the special fibre imply the claim; see [1]. □

Remark A.5 (Interpretation of Lemma 3.3). In the body we use the vanishing-cycles exact triangle (Section 3.2) and the short exact sequence Equation (2), together with the canonical comparison map to the component complex of the special fibre (SGA7). The only properties needed later are:

(1) an equality of *traces* of Frob_v on the vanishing-cycles term and on a natural virtual $\mathbb{Q}_\ell$-module attached to Φ_v, and
(2) independence of ℓ and compatibility with unramified base change.

These are exactly what is asserted and used in Propositions 3.2 and 3.4. No expression of the form $\Psi_v \simeq V_\ell(\Phi_v)$ is used as a literal identification. The comparison used in the body is only the Frobenius trace identity of Lemma 3.3. The underlying comparison is standard in the SGA7 framework; see [12].

Example A.6 (Split multiplicative reduction). If E/K_v is a Tate curve E_q with $\mathrm{ord}_v(q) = n \geq 1$, then the special fibre is a Néron n-gon, $\Phi_v \simeq \mathbb{Z}/n\mathbb{Z}$, $\mathrm{tr}(\mathrm{Frob}_v \mid H^1_\ell(E)^{I_v}) = 1$, and $\#\Phi_v = n$. See [13].

Appendix B. Ogg–Saito type formulas and comparisons

We record conductor identities in a form convenient for Section 5. Let $V_\ell := H^1_{\text{ét}}(E_{\overline{K}_v}, \mathbb{Q}_\ell)$ and write $f_v(E)$ for the local conductor exponent of E at v.

Theorem B.1 (Deligne, Saito: Artin conductor via invariants and Swan)**.** *For $\ell \neq \mathrm{char}\,\kappa(v)$,*

$$f_v(E) \;=\; a_v(V_\ell) \;=\; \dim_{\mathbb{Q}_\ell}\Big(V_\ell/V_\ell^{I_v}\Big) \;+\; \mathrm{Sw}_v(V_\ell),$$

where a_v is the Artin conductor and Sw_v the Swan conductor. In particular, if E has semistable reduction at v then $\mathrm{Sw}_v(V_\ell) = 0$ and

$$f_v(E) \;=\; 2 - \dim_{\mathbb{Q}_\ell}\Big(H^1_\ell(E)^{I_v}\Big).$$

Proof. This is the standard conductor formula of [2] and its geometric form in [9]. The semistable vanishing of the Swan term is classical. □

Proposition B.2 (Compatibility with the local identity)**.** *Assume E has bad reduction. Then Proposition 4.4 together with Theorem B.1 yields*

$$f_v(E) \;=\; 2 - \#\mathrm{Fix}\Big(\mathrm{Frob}_v; E^{\rm sp}(\overline{\kappa(v)})\Big) \;+\; \mathrm{tr}(\mathrm{Frob}_v \mid V_\ell(\Phi_v)) \;+\; \mathrm{Sw}_v(V_\ell).$$

In particular, in the semistable case this expresses $f_v(E)$ purely in terms of the special fibre and the component group.

Proof. Use Theorem B.1 to rewrite $\dim(V_\ell/V_\ell^{I_v}) = 2 - \dim V_\ell^{I_v}$ and substitute

$$\dim V_\ell^{I_v} \;=\; \operatorname{tr}\Big(\mathrm{Frob}_v \mid H^1(E^{\mathrm{sp}}_{\overline{\kappa(v)}}, \mathbb{Q}_\ell)\Big) - \operatorname{tr}(\mathrm{Frob}_v \mid V_\ell(\Phi_v))$$

from Proposition 3.4 together with the Grothendieck–Lefschetz fixed-point expression on the special fibre. This is exactly Proposition 4.4. □

Remark B.3. In the tame potentially good case (residue characteristic $p \geq 5$), Proposition B.2 recovers the classical Ogg-type expressions after translating the trace on $H^1(E^{\mathrm{sp}})$ into the combinatorics of the special fibre; see [2, 9].

Appendix C. Auxiliary computations

We gather computation recipes referenced in Section 6 and Section 5. Throughout, $E/\mathbb{Q}$ is given by a global minimal Weierstrass equation; at a prime p write $v = v_p$ and $q = p$.

A. Quick computation of Φ_v and the local data.

(1) Run Tate's algorithm at v to determine the Kodaira symbol and $v(\Delta)$; see [13].
(2) From the symbol read off $\#\Phi_v$ (the Tamagawa number c_v) using the standard table [1]. This gives the group structure of the special fibre's components.
(3) If the reduction is semistable, then $\mathrm{Sw}_v(H^1) = 0$ and Theorem B.1 gives $f_v(E) = 2 - \dim H^1_\ell(E)^{I_v}$, while Proposition 3.4 turns the I_v-fixed trace into a combination of $\#\mathrm{Fix}(\mathrm{Frob}_v; E^{\mathrm{sp}})$ and the Φ_v-trace.

B. A compact check for split multiplicative primes. Suppose $E/\mathbb{Q}$ has split multiplicative reduction at p. Then:

$$\Phi_p \simeq \mathbb{Z}/n\mathbb{Z}, \qquad \mathrm{Sw}_p(H^1) = 0, \qquad \dim H^1_\ell(E)^{I_p} = 1.$$

Hence $f_p(E) = 1$ and Proposition 4.4 gives

$$1 = \operatorname{tr}\Big(\mathrm{Frob}_p \mid H^1_\ell(E)^{I_p}\Big) = \mathrm{Fix}_p - \tau_p,$$

consistent with Proposition 3.2.

C. Sample bookkeeping for Example 6.3. Let $E_\lambda : y^2 = x(x-1)(x-\lambda)$ with $\lambda \in \mathbb{Q} \setminus \{0,1\}$. For each bad prime p:

- Determine the type by $v_p(\lambda)$, $v_p(\lambda - 1)$ and $v_p(\Delta) = v_p(\lambda^2(\lambda-1)^2)$ via Tate's algorithm.
- If the type is I_n, then $\dim H^1_\ell(E)^{I_p} = 1$, $\mathrm{Sw}_p(H^1) = 0$, $f_p(E) = 1$, and $\Phi_p \simeq \mathbb{Z}/n\mathbb{Z}$.
- If the type is additive potentially good, consult Theorem B.1 and Proposition B.2 to express $f_p(E)$ using the special fibre and the Φ_p-trace.

D. Dual-graph snippet for a Néron n-gon. For the reader's convenience, the dual graph of a split multiplicative fibre (all edges have multiplicity one, vertices are components):

$$\bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

It contributes weight-0 classes in the vanishing-cycles group counted by the fixed-point term in Proposition 4.4.

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