An infinitesimal proof of the Riemann hypothesis on nontrivial zeros of the zeta function

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UDC 511

Setting the problem

Let $s = \sigma + it$ be a complex variable, where $\sigma = \operatorname{Re} s, t = \operatorname{Im} s$, and $x \in \mathbb{R}$ be a real variable.

For $\operatorname{Re} s > 0, s \neq 1$, it is known [1] that the Riemann zeta function $\zeta(s)$ can be expressed by the formula

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^{\infty} \frac{\{x\}}{x^{s+1}}dx. \tag{1}$$

Here, $\{x\}$ denotes the fractional part of a number x.

Let us rewrite equality 1 in the form

$$\zeta(s) = s\left(\frac{1}{s-1} - \int_1^{\infty} \frac{\{x\}}{x^{s+1}}dx\right).$$

Thus, to obtain nontrivial zeros of the function $\zeta(s)$, we must solve the following equation:

$$\int_1^{\infty} \frac{\{x\}}{x^{s+1}}dx = \frac{1}{s-1}. \tag{2}$$

This implies two equations:

$$\frac{1}{x^{s+1}} = \frac{1}{x^{\sigma+1}}\left(\cos(t\ln x) - i\sin(t\ln x)\right),$$
$$\frac{1}{s-1} = \frac{\sigma-1}{(\sigma-1)^2+t^2} - i\frac{t}{(\sigma-1)^2+t^2}.$$

Therefore, equation 2 is equivalent to the following system:

$$\begin{cases} \int\limits_1^\infty \frac{\{x\}}{x^{\sigma+1}} \cos(t \ln x) dx = \frac{\sigma - 1}{(\sigma-1)^2 + t^2}, \\ \int\limits_1^\infty \frac{\{x\}}{x^{\sigma+1}} \sin(t \ln x) dx = \frac{t}{(\sigma-1)^2 + t^2}. \end{cases} \tag{3}$$

It is known that nontrivial zeros are symmetric about the real axis, therefore we consider only the case $t > 0$.

We always assume that $0 < \sigma < 1,\ t > 0$.

Let $s_0 = \sigma_0 + it_0$ be a nontrivial zero.

The Riemann hypothesis states that $\sigma_0 = \frac{1}{2}$.

Left and right sides of the equations of system 3

Let us introduce four useful functions as follows:

$$\begin{aligned} u_1(\sigma, t) &= \int\limits_1^\infty \frac{\{x\}}{x^{\sigma+1}} \cos(t \ln x) dx, \\ v_1(\sigma, t) &= \int\limits_1^\infty \frac{\{x\}}{x^{\sigma+1}} \sin(t \ln x) dx, \\ u_2(\sigma, t) &= \frac{\sigma - 1}{(\sigma-1)^2 + t^2}, \\ v_2(\sigma, t) &= \frac{t}{(\sigma-1)^2 + t^2}. \end{aligned}$$

We represent system 3 in the form

$$\begin{cases} u_1(\sigma, t) = u_2(\sigma, t), \\ v_1(\sigma, t) = v_2(\sigma, t). \end{cases} \tag{4}$$

$s = \sigma + it$ is a nontrivial zero if and only if (σ, t) is a solution to system 4.

Lemma 1. *The function $w = v_2(\sigma, t_0)$ increases as a function of one variable $\sigma \in (0; 1)$.*

Proof. It follows from the inequality

$$\frac{\mathrm{d}v_2}{\mathrm{d}\sigma} = -\frac{2(\sigma-1)t_0}{(t_0^2 + (\sigma-1)^2)^2} > 0.$$

□

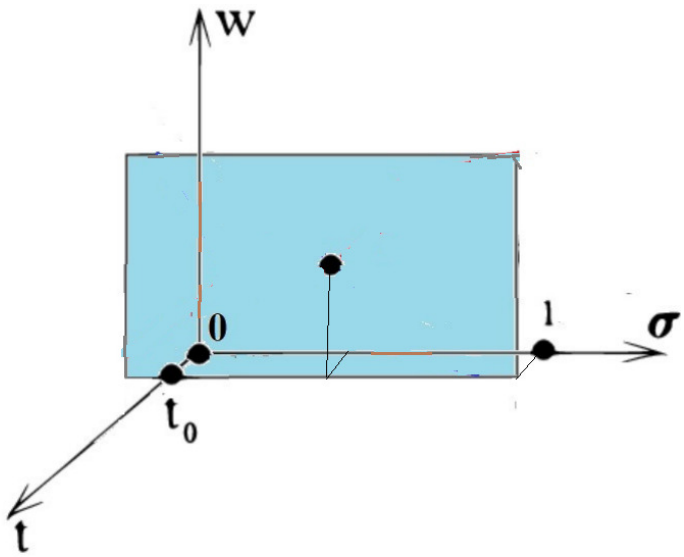

Figure 1: The plane $t = t_0$

The range of the function $w = v_2(\sigma, t_0)$ is $U = \left(\frac{t_0}{1+t_0^2}, \frac{1}{t_0}\right)$.

Obviously, the graph of the function $w = v_2(\sigma, t_0)$ lies in the rectangle $\Pi = \left\{(\sigma, w) \,\middle|\, \sigma \in (0; 1), w \in U\right\}$.

We consider the part of the graph of the function $v_1(\sigma, t_0)$ that lies in this rectangle.

Definition 1. *A rectangle* Π *is critical.*

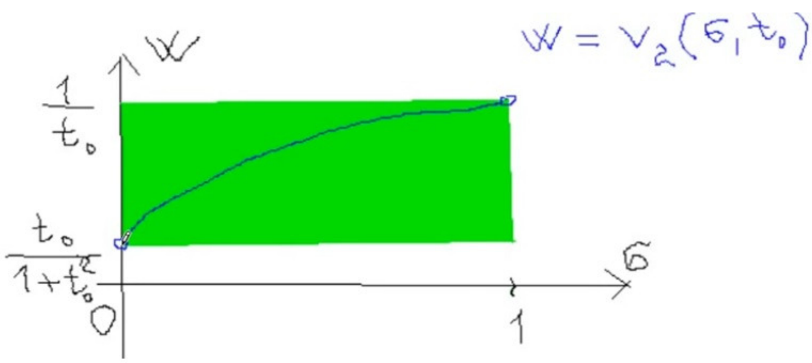

Figure 2: A critical rectangle

Remark 1. *Critical rectangles are very thin, their width equals* $\frac{1}{t_0} - \frac{t_0}{1+t_0^2} = \frac{1}{(1+t_0^2)t_0}$. *Take the nontrivial zero with the least positive imaginary part* $t_0 = 14.134725141...$ *and get the width 0.0003523461812...*

Definition 2. σ *is critical if* $(\sigma, v_1(\sigma, t_0)) \in \Pi$.

Thus the value σ_0 is critical. The graphs of $v_1(\sigma, t_0)$ and $v_2(\sigma, t_0)$ intersect in the point $(\sigma_0, v_1(\sigma_0, t_0)) \in \Pi$.

This implies the inequality

$$v_1(\sigma_0, t_0) = \int\limits_1^{+\infty} \frac{\{x\}}{x^{\sigma_0+1}} \sin(t_0 \ln x) dx = \frac{t_0}{\sigma_0^2 + t_0^2} > 0.$$

Moreover, by definition, we get $v_1(\sigma, t_0) \in \left(\frac{t_0}{1+t_0^2}, \frac{1}{t_0} \right)$ for all critical σ; this implies that $v_1(\sigma, t_0) > 0$.

Let us introduce the function

$$\Psi(\sigma, x) = \frac{\{x\}}{x^{\sigma+1}} \sin(t_0 \ln x).$$

Then we have the equality

$$v_1(\sigma, t_0) = \int\limits_1^{\infty} \Psi(\sigma, x) dx.$$

Lemma 2. *Graphs of the functions $v_1(\sigma, t_0)$ and $v_2(\sigma, t_0)$ intersect only at the point (σ_0, t_0).*

Proof. Let σ' be a positive number such that $\sigma + \sigma'$ is critical.

It is obvious that

$$\Psi(\sigma + \sigma', x) = \frac{1}{x^{\sigma'}} \Psi(\sigma, x).$$

Then we get

$$v_1(\sigma + \sigma', t_0) = \int\limits_1^{\infty} \frac{1}{x^{\sigma'}} \Psi(\sigma, x) dx.$$

Since σ and $\sigma + \sigma'$ are critical, we obtain $v_1(\sigma, t_0) > 0$ and $v_1(\sigma + \sigma', t_0) > 0$. This implies that there exists a X_0 such that for all $X > X_0$ we get the inequalities

$$\int\limits_1^{X} \Psi(\sigma, x) dx > 0 \text{ and } \int\limits_1^{X} \frac{1}{x^{\sigma'}} \Psi(\sigma, x) dx > 0.$$

Let $\Re[a, b]$ be the set of Riemann-integrable functions on an interval $[a, b]$.

We use the following [2, p.357]

Theorem (the second mean-value theorem for the integral[1])**.** *If $f(x), g(x) \in \Re[a, b]$ and $g(x)$ is a monotonic function on [a,b], then there exists a point $\xi \in [a, b]$ such that*

$$\int\limits_a^{b} f(x)g(x)dx = g(a) \int\limits_a^{\xi} f(x)dx + g(b) \int\limits_{\xi}^{b} f(x)dx.$$

[1]It states the equality which is often colled Bonnet's formula

The function $g(x) = \dfrac{1}{x^{\sigma'}}$ is decreasing on $[1, X]$, and the second mean-value theorem for the integral tells that there exists a point $\xi = \xi(X) \in [1, X]$ such that

$$\int\limits_1^X \frac{1}{x^{\sigma'}} \Psi(\sigma, x)dx = \int\limits_1^\xi \Psi(\sigma, x)dx + \frac{1}{X^{\sigma'}} \int\limits_\xi^X \Psi(\sigma, x)dx. \tag{5}$$

If $\int\limits_\xi^X \Psi(\sigma, x)dx \geq 0$, then, as $\dfrac{1}{X^{\sigma'}} < 1$, we get

$$\int\limits_1^X \frac{1}{x^{\sigma'}} \Psi(\sigma, x)dx = \int\limits_1^\xi \Psi(\sigma, x)dx + \frac{1}{X^{\sigma'}} \int\limits_\xi^X \Psi(\sigma, x)dx \leq \int\limits_1^\xi \Psi(\sigma, x)dx + \int\limits_\xi^X \Psi(\sigma, x)dx = \int\limits_1^X \Psi(\sigma, x)dx, \tag{6}$$

so in this case Lemma 2 is true.

Let us go to the case $\int\limits_\xi^X \Psi(\sigma, x)dx < 0$.

In this paper, we assume a construction of the hyperreal number system from real number sequences [3].

Hyperreal number $[a_1, a_2, a_3, \ldots, a_n, \ldots]$ will be denoted briefly as $[a_n]$. Real number r will be identified with the sequence $[r, r, \ldots, r, \ldots] = [r]$.

According to the nonstandard transfer principle, the second mean-value theorem for the integral takes place on nonstandard numbers as well.

Let now $\sigma' = \left[1, \dfrac{1}{2}, \dfrac{1}{3}, \ldots, \dfrac{1}{n}, \ldots\right] = \left[\dfrac{1}{n}\right]$ be a positive infinitely small number, then a nonstadard $\xi \in [1, X]$ corresponds to σ' and real number $[X]$.

If a is a positive real number, then $a^{\sigma'} = [a, \sqrt{a}, \sqrt[3]{a}, \ldots, \sqrt[k]{a}, \ldots] \approx [1]$.

We get $X^{\sigma'} = \left[X^{\sigma'}, X^{\sigma'}, \ldots, X^{\sigma'}, \ldots\right] \approx [1, 1, \ldots, 1, \ldots] = 1$. That means $\dfrac{1}{X^{\sigma'}} \approx [1]$.

Denote

$$\bar{v}_1(\sigma, t) = \int\limits_1^X \frac{\{x\}}{x^{\sigma+1}} \sin(t \ln x)dx.$$

Calculate the derivative

$$\frac{d\bar{v}_1}{d\sigma} = \frac{\dfrac{1}{X^{\sigma'}} - [1]}{\sigma'} \int\limits_\xi^X \Psi(\sigma, x)dx = \frac{\left[\dfrac{1}{X^{\frac{1}{n}}} - 1\right]}{\left[\dfrac{1}{n}\right]} \int\limits_\xi^X \Psi(\sigma, x)dx =$$

$$= \left[\left(\frac{1}{X^{\frac{1}{n}}} - 1\right) n\right] \int\limits_\xi^X \Psi(\sigma, x)dx = -\ln(X) \int\limits_\xi^X \Psi(\sigma, x)dx.$$

This means that along the graph of the function $\bar{v}_1$ the derivative is constant, so this graph is a straight line[2].

If $\int\limits_{\xi}^{X} \Psi(\sigma, x)dx < 0$, the derivative is positive, therefore the straight line $\bar{v}_1$ goes up (in this case, according to the condition, it passes through the point(σ_0, w_0)).

According to Lemma 1, the graph of the function v_2 is increasing; moreover, it is strictly convex upward (it's second derivative is negative). The line $\bar{v}_1$ either is tangent to the curve v_2 at the single point (σ_0, w_0) or intersects it at two points: $A(\sigma_0, w_0)$ and some other point $B(\sigma_1, w_1)$.

Let us assume, for the sake of certainty, that $\sigma_0 < \frac{1}{2}$.

As is known, if a fixed point $\sigma_0 + it_0$ is a nontrivial zero of the zeta function, so is the point $s_1 = 1 - \sigma_0 + it_0$. The graph v_2 is symmetric about the line $\sigma = \frac{1}{2}$, therefore the point B lies on this graph and it is above the point A.

Using these two points, we find the tangent of the angle of inclination $\bar{\varphi}$ of the line $\bar{v}_1$:

$$\mathrm{tg}(\bar{\varphi}) = \frac{v_2(\sigma_1, t_0) - v_2(\sigma_0, t_0)}{\sigma_1 - \sigma_1} = \frac{\dfrac{t_0}{\sigma_0^2 + t_0^2} - \dfrac{t_0}{(1-\sigma_0)^2 + t^2}}{1 - 2\sigma_0}.$$

At the same time $\mathrm{tg}(\bar{\varphi}) = -\ln(X) \int\limits_{\xi}^{X} \Psi(\sigma, x)dx$.

We get the equation

$$-\ln(X) \int\limits_{\xi}^{X} \Psi(\sigma, x)dx = \frac{\dfrac{t_0}{\sigma_0^2 + t_0^2} - \dfrac{t_0}{(1-\sigma_0)^2 + t^2}}{1 - 2\sigma_0}. \tag{7}$$

Let now X be an infinitely large number and let $[a_n]$ be its infinitesimal representation as positive number sequence. Then $\ln(X)$ is infinitely large number, too. We see that $\int\limits_{\xi}^{X} \Psi(\sigma, x)dx$ cannot be finite and not infinitesimal number, otherwise the left side of equality 7 is an infinitely large number as the right-hand side is a real non-zero number. Hence, $\int\limits_{\xi}^{X} \Psi(\sigma, x)dx \approx 0$; let $\varepsilon = [\varepsilon_n]$ be its infinitesimal representation as negative number sequence.

[2]The case $\int\limits_{\xi}^{X} \Psi(\sigma, x)dx \geq 0$ corresponds to the decrease of the function $\bar{v}_1$, which once again confirms the inequality 6.

$$-\ln(X)\int\limits_{\xi}^{X}\Psi(\sigma,x)dx=-\ln[a_n]\cdot\varepsilon=[-\varepsilon\cdot\ln(a_n)]=[\ln(a_n^{-\varepsilon})]\approx[\ln(1)]=[0].$$

This means that the left side of equality7 cannot be equal to the right side of it.

The resulting contradiction proves Lemma 2.

□

The proof of the Riemann hypothesis

Theorem. *Let $s_0 = \sigma_0 + it_0$ be a nontrivial zero of the Riemann zeta function, then $\sigma_0 = \frac{1}{2}$.*

Proof. A nontrivial zero of the zeta function is a solution to equation 2, hence the pair (σ_0, t_0) satisfies system 4, and, in particular, its second equality.

From Lemma 2 it follows that this pair is unique. Suppose $\sigma_0 \neq \frac{1}{2}$.

It is known that nontrivial zeros are symmetric about the line $\mathrm{Re}\, s = \frac{1}{2}$, hence there exists another zero $1-\sigma_0+it_0$ at the same "height" $t = t_0$, therefore the pair $(1-\sigma_0, t_0)$ satisfies the second equality as well. This contradiction establishes the theorem. □

References

[1] Galochkin A.I., Nesterenko Yu.V., Shidlovski A.B.: An Introduction to the Theory of Numbers (In Russian). 2nd ed,

[2] Zorich V. A.: Mathematical Analysis. Springer-Verlag Berlin Heidelberg, 2004.

[3] Goldblatt R. Lectures on the Hyperreals: An Introduction to Nonstandard Analysis. New York etc.: Springer, 1998.