Universal Modular Dynamics: The Law of Renormalization

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Abstract

We present a formulation of renormalization within the framework of Universal Modular Dynamics (UMD), in which the density operator ρ is taken as the fundamental object encoding physical structure. In this approach, geometry, locality, and dynamics are not assumed a priori, but emerge from properties of ρ and its associated modular generator $K = -\log \rho$.

We demonstrate that renormalization can be consistently interpreted as a flow in the space of quantum states, parameterized by an internal ordering parameter λ. Within this formulation, the critical scale of the renormalization group (RG) flow is not determined by a single spectral characteristic, such as a gap, but by the full statistical structure of the modular spectrum.

Using explicit numerical constructions, we show that the RG critical point $\lambda^*(\rho)$ is a stable functional of the distribution of modular energies, leading to a law of the form

$$\lambda^*(\rho) = \mathcal{F}\big(\mathrm{Spec}(-\log \rho)\big),$$

where $\mathcal{F}$ depends on statistical properties of the spectrum, including its mean, variance, and quantile structure.

This result establishes a distributional law of renormalization, in which critical behavior is governed by global spectral features rather than isolated eigenvalues. As a consequence, renormalization becomes intrinsically state-dependent, and the notion of scale is replaced by a spectral-statistical structure defined directly at the level of the density operator.

The proposed framework provides a unified perspective on renormalization, quantum structure, and emergent geometry, and suggests a shift from scale-based to distribution-based descriptions of physical laws.

Contents

1 Introduction

The renormalization group (RG) is one of the central structures in modern theoretical physics. It provides a systematic framework for understanding how physical descriptions change with scale, and underlies a wide range of phenomena, from critical behavior in statistical systems to quantum field theory and quantum gravity. In its conventional formulation, renormalization is intrinsically associated with scale transformations, coarse-graining procedures, and the flow of effective parameters.

Despite its success, the standard RG paradigm leaves open a fundamental question: what determines the emergence and location of critical scales in a given physical system? In most approaches, these scales are inferred from model-specific structures, such as coupling constants, spectral gaps, or symmetry-breaking mechanisms. As a result, the notion of a critical scale remains context-dependent and lacks a universal, state-level formulation.

In this work, we propose a different perspective based on the framework of Universal Modular Dynamics (UMD), in which the density operator ρ is taken as the primary object encoding physical structure. Within this approach, geometry, locality, and dynamics are not fundamental inputs, but emergent properties derived from the informational content of ρ and its associated modular generator

$$K = -\log \rho. \tag{1}$$

The central idea of UMD is that physical evolution can be formulated directly in the space of states, without reference to a background spacetime. In this setting, renormalization is reinterpreted as a flow in the space of density operators, parameterized by an internal ordering parameter λ. This shift replaces the notion of scale transformation by a spectral deformation of the state itself.

The key result of the present work is that the critical scale of this flow is not determined by a single spectral quantity, but by the full statistical structure of the spectrum of the modular generator. More precisely, we demonstrate that the RG critical point $\lambda^*(\rho)$ is a stable functional of the distribution of modular energies, leading to a distributional law of renormalization.

This result can be summarized schematically as

$$\lambda^*(\rho) = \mathcal{F}\big(\mathrm{Spec}(-\log \rho)\big), \tag{2}$$

where $\mathcal{F}$ depends on statistical characteristics of the spectrum, such as its mean, variance, and quantile structure.

The emergence of this law has two important consequences. First, it implies that renormalization is inherently state-dependent, rather than controlled by external scale parameters. Second, it replaces the traditional gap-based intuition with a distributional description, in which the global shape of the spectrum determines the behavior of the flow.

An additional conceptual aspect of the present work is that the identification of this structure did not arise from a direct linear extension of the initial formalism. Instead, it emerged through a reorganization of partially developed elements of the theory, followed by systematic numerical verification and formal reconstruction. While the final result satisfies all requirements of consistency and reproducibility, its origin reflects a nontrivial interplay between structured derivation and exploratory insight.

The remainder of the paper is organized as follows. In Section 2 we outline the pre-quantum and informational foundations of the UMD framework. Section 3 introduces the modular formulation of dynamics and its CPTP-consistent structure. Section 4 discusses the emergence of locality and geometry as phase properties of the state space. Section 5 develops the spectral and distributional characterization of modular structure. In Section 6, renormalization is formulated as a spectral flow. Section 7 presents the main result, the law of critical scales. Section 8 analyzes the role of fluctuations in the formation of theoretical structures. We conclude with a discussion of implications and open directions.

2 Pre-Quantum Foundations

2.1 Informational Ontology

The starting point of Universal Modular Dynamics (UMD) is the assumption that physical structure is fundamentally informational. Rather than taking particles, fields, or spacetime as primitive, we begin with the notion of distinguishability between configurations.

Let $\mathcal{S}$ denote a set of admissible configurations. We introduce a distinguishability functional

$$\Delta : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_{\geq 0}, \tag{3}$$

satisfying

$$\Delta(x, x) = 0, \qquad \Delta(x, y) = \Delta(y, x). \tag{4}$$

This structure encodes the minimal information required to compare configurations, without assuming any underlying geometry or dynamics.

2.2 Emergence of Probabilistic Structure

From the distinguishability structure $(\mathcal{S}, \Delta)$, one constructs an informational measure that quantifies uncertainty over configurations. The requirement of consistency under composition and coarse-graining leads to a probabilistic description.

The canonical representation of such a probabilistic structure is given by a density operator ρ acting on a Hilbert space $\mathcal{H}$:

$$\rho \geq 0, \qquad \mathrm{Tr}(\rho) = 1. \tag{5}$$

This step should not be interpreted as an independent postulate of quantum mechanics, but rather as a natural encoding of informational uncertainty under the constraints of linearity and basis invariance.

2.3 Modular Generator

Given a full-rank density operator ρ, we define the modular generator

$$K = -\log \rho. \tag{6}$$

The operator K plays a central role in UMD. It provides a canonical representation of the informational content of the state and determines its intrinsic structure.

Unlike Hamiltonians in conventional formulations, K is not introduced as an external generator of dynamics. Instead, it is derived directly from the state and therefore encodes internal, state-dependent structure.

2.4 Relative Modularity and Non-Trivial Dynamics

A key observation is that the commutator

$$[-\log \rho, \rho] = 0 \tag{7}$$

is identically zero, implying that the modular generator alone does not produce non-trivial evolution.

To obtain a non-degenerate dynamical structure, one introduces a reference state σ and defines the relative modular generator

$$K_{\rho|\sigma} = -\log \rho + \log \sigma. \tag{8}$$

In general,

$$[K_{\rho|\sigma}, \rho] \neq 0, \tag{9}$$

which allows for non-trivial state-dependent evolution.

2.5 State-Based Dynamics

Within the UMD framework, dynamics is formulated directly in the space of states. A minimal form of evolution is given by

$$\frac{d\rho}{d\lambda} = -i[K_{\rho|\sigma}, \rho] + \mathcal{D}[\rho], \tag{10}$$

where λ is an internal ordering parameter and $\mathcal{D}$ is a completely positive trace-preserving (CPTP) dissipator.

The parameter λ does not correspond to physical time, but rather to an ordering of transformations in the space of states. This interpretation aligns naturally with the concept of renormalization as a flow.

2.6 From State Space to Physical Structure

In this formulation, all physical structures emerge from properties of ρ and its modular generator:

- Geometry arises from relations of distinguishability between subsystems.
- Locality emerges as a phase property of the state space.
- Dynamics is encoded in modular and relative modular flows.
- Renormalization corresponds to trajectories in the space of density operators.

Thus, UMD replaces the traditional hierarchy

$$\text{geometry} \to \text{fields} \to \text{states} \tag{11}$$

by the inverted structure

$$\rho \to K \to \text{geometry and dynamics.} \tag{12}$$

This inversion is the defining feature of the framework and underlies all subsequent constructions in this work.

3 Modular Dynamics

3.1 State-Based Formulation of Dynamics

In the UMD framework, dynamics is formulated directly in the space of quantum states. The density operator ρ is treated as the primary object, and its evolution is described without reference to an external spacetime or Hamiltonian structure.

Given a full-rank density operator ρ, we define the modular generator

$$K = -\log \rho. \tag{13}$$

This operator encodes the intrinsic informational structure of the state. However, as noted in Section 2, the commutator

$$[K, \rho] = 0 \tag{14}$$

vanishes identically, implying that K alone cannot generate non-trivial dynamics.

3.2 Relative Modular Generator

To obtain a non-degenerate evolution, we introduce a reference state σ and define the relative modular generator

$$K_{\rho|\sigma} = -\log \rho + \log \sigma. \tag{15}$$

In general,

$$[K_{\rho|\sigma}, \rho] \neq 0, \tag{16}$$

which allows for non-trivial state-dependent evolution.

The role of σ is not to introduce new physical structure, but to define a reference frame within the space of states. In phase-stable regimes, σ is naturally identified with a maximum-entropy state subject to macroscopic constraints.

3.3 CPTP-Consistent Evolution

The minimal form of evolution consistent with complete positivity and trace preservation is given by a GKSL-type equation

$$\frac{d\rho}{d\lambda} = -i[K_{\rho|\sigma}, \rho] + \mathcal{D}[\rho], \tag{17}$$

where $\mathcal{D}$ is a completely positive trace-preserving (CPTP) dissipator of the form

$$\mathcal{D}[\rho] = \sum_{\alpha} \gamma_\alpha(\lambda) \left(L_\alpha \rho L_\alpha^\dagger - \frac{1}{2}\{L_\alpha^\dagger L_\alpha, \rho\} \right), \qquad \gamma_\alpha \geq 0. \tag{18}$$

This structure ensures that the evolution remains within the space of physical states.

3.4 Interpretation of the Flow Parameter

The parameter λ plays the role of an internal ordering parameter. It does not correspond to physical time, but rather to an ordering of transformations in the space of states.

In particular, λ naturally aligns with the concept of renormalization, where evolution corresponds to a systematic transformation of the state.

Thus, we interpret the equation

$$\frac{d\rho}{d\lambda} = -i[K_{\rho|\sigma}, \rho] + \mathcal{D}[\rho] \tag{19}$$

as defining a renormalization flow in the space of density operators.

3.5 Spectral Representation of the Flow

Let $\{k_i\}$ denote the eigenvalues of the modular generator K. The corresponding eigenvalues of ρ are given by

$$\lambda_i = e^{-k_i}. \tag{20}$$

Under the flow, the spectral structure of ρ evolves nonlinearly. This evolution can be interpreted as a redistribution of modular energies.

Importantly, the flow does not act on individual eigenvalues independently, but on the statistical structure of the spectrum as a whole.

3.6 Observables and Modular Response

To probe the dynamics, we introduce observables O such that

$$[K_{\rho|\sigma}, O] \neq 0. \tag{21}$$

We define the modular response functional

$$F(\lambda) = \|[K(\lambda), O]\|_F, \tag{22}$$

where $\|\cdot\|_F$ denotes the Frobenius norm.

This functional provides a quantitative measure of how the spectral structure of the state interacts with observables under the flow.

3.7 Toward Renormalization

The function $F(\lambda)$ generally exhibits non-monotonic behavior as the spectrum of K is reshaped. This behavior reflects a competition between spectral compression and operator sensitivity.

As will be shown in subsequent sections, the structure of $F(\lambda)$ leads to the emergence of a well-defined critical scale λ^*, which depends on the statistical properties of the spectrum of K.

This observation forms the basis for the formulation of renormalization as a spectral-statistical flow.

4 Geometry and Locality

4.1 From States to Spatial Structure

In the UMD framework, geometry is not assumed as a fundamental background structure. Instead, it is reconstructed from properties of the quantum state ρ.

The key idea is that spatial relations between subsystems can be encoded in their mutual distinguishability. This leads naturally to an information-theoretic notion of distance.

4.2 Mutual Information and Effective Distance

Let the system be decomposed into subsystems labeled by indices i, j. For a given state ρ, we define the mutual information

$$I(i:j) = S(\rho_i) + S(\rho_j) - S(\rho_{ij}), \tag{23}$$

where ρ_i, ρ_j, and ρ_{ij} are reduced density operators, and $S(\rho) = -\mathrm{Tr}(\rho \log \rho)$ is the von Neumann entropy.

We define an effective distance between subsystems as

$$d(i,j) = \frac{1}{\big(I(i:j) + \varepsilon\big)^{\nu}}, \tag{24}$$

where $\varepsilon > 0$ is a regularization parameter and $\nu > 0$ controls scaling.

This definition implies that strongly correlated subsystems are close, while weakly correlated ones are far apart.

4.3 Emergent Geometry

The collection of distances $\{d(i,j)\}$ defines a weighted graph structure over subsystems. In suitable regimes, this graph admits a geometric interpretation.

In particular, when the mutual information decays smoothly with subsystem separation, the effective distance can be embedded into a low-dimensional manifold. In this sense, spatial geometry emerges as a coarse-grained description of correlation structure.

Thus, geometry is not fundamental, but arises as a phase of the state space.

4.4 Optimal Factorization and Local Structure

To formalize locality, we introduce a partition $P = \{X_1, X_2, \dots\}$ of the system into subsystems.

We define the functional

$$J_\eta(P;\rho) = \Phi(P;\rho) + \eta\,\Omega(P), \tag{25}$$

where Φ measures total correlations between parts and Ω penalizes complexity of the partition.

The optimal partition is given by

$$P^*(\rho) = \arg\min_P J_\eta(P;\rho). \tag{26}$$

This partition identifies the effective local structure of the system.

4.5 Locality as a Phase Property

We define the locality functional

$$T_{P^*}(\rho) = D\big(\rho \,\|\, \bigotimes_{X\in P^*} \rho_X\big), \tag{27}$$

where $D(\cdot\|\cdot)$ is the quantum relative entropy.

When $T_{P^*}(\rho)$ is small, the state is approximately factorized, and a local description is valid. When it is large, the system exhibits nonlocal correlations.

Thus, locality is not a fundamental property, but a phase-dependent feature of the state.

4.6 Phase Structure of Geometry

Different regimes of ρ correspond to distinct geometric phases:

- **Geometric phase:** stable partition P^* and smooth correlation decay;
- **Critical phase:** competing partitions and large fluctuations in P^*;
- **Non-geometric phase:** absence of stable factorization.

Transitions between these regimes correspond to structural changes in the state space.

4.7 Connection to Modular Dynamics

The emergence of geometry is tightly linked to the modular generator $K = -\log\rho$. Changes in the spectrum of K affect correlation structure and therefore the effective geometry.

Under modular evolution, the system may move between different geometric phases. In particular, the onset of critical behavior is associated with instability of the optimal partition.

This provides a direct connection between modular dynamics and emergent geometry.

4.8 Summary

In the UMD framework, geometry and locality arise as emergent, state-dependent structures derived from the informational content of ρ.

This perspective replaces the traditional assumption of a fixed background spacetime by a dynamical, correlation-based notion of geometry, setting the stage for a spectral formulation of renormalization.

5 Spectral Modes and Distributional Structure

5.1 Spectral Decomposition of the State

Let ρ be a full-rank density operator with spectral decomposition

$$\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|, \tag{28}$$

where $\lambda_i > 0$ and $\sum_i \lambda_i = 1$.

The modular generator is given by

$$K = -\log \rho = \sum_i k_i |\psi_i\rangle\langle\psi_i|, \tag{29}$$

with

$$k_i = -\log \lambda_i. \tag{30}$$

The set $\{k_i\}$ defines the modular spectrum, which encodes the informational structure of the state.

5.2 From Eigenvalues to Distribution

Rather than treating the eigenvalues $\{k_i\}$ individually, we interpret them as samples of a distribution.

We introduce the empirical spectral distribution

$$p(k) = \frac{1}{N} \sum_i \delta(k - k_i), \tag{31}$$

where N is the dimension of the Hilbert space.

This distribution captures the global structure of modular energies.

5.3 Statistical Descriptors

The spectral distribution can be characterized by standard statistical quantities:

$$\mu_K = \frac{1}{N} \sum_i k_i, \tag{32}$$

$$\sigma_K^2 = \frac{1}{N} \sum_i (k_i - \mu_K)^2. \tag{33}$$

In addition, quantile-based descriptors provide information about tail structure. For example, we define q_α such that

$$\int_{-\infty}^{q_\alpha} p(k)\, dk = \alpha. \tag{34}$$

In particular, the difference

$$\Delta_K = q_{90} - q_{10} \tag{35}$$

captures the width of the distribution beyond its central region.

5.4 Spectral Modes and Structure

The modular spectrum is not merely a set of numbers, but exhibits structured patterns.

In particular, clusters of eigenvalues can be interpreted as spectral modes, corresponding to coherent sectors of the state. These modes may be stable under evolution or may interact under modular dynamics.

The presence of such structure indicates that the state cannot be characterized by a single scale, but requires a distributional description.

5.5 Distributional Nature of Physical Structure

A central observation of this work is that physical behavior is governed not by isolated spectral features, but by the distribution as a whole.

In particular:

- Mean values set the global energy scale;
- Variance controls fluctuations;
- Tail structure encodes rare but influential modes;
- Clustering reflects internal organization of the state.

This suggests that physical laws should be formulated in terms of spectral distributions rather than individual eigenvalues.

5.6 Evolution of the Spectrum

Under modular dynamics, the spectrum $\{k_i\}$ evolves in a nonlinear manner.

This evolution can be interpreted as a redistribution of modular energies, leading to changes in the shape of the distribution $p(k)$.

Importantly, the evolution does not preserve simple spectral characteristics such as gaps, but reshapes the distribution globally.

5.7 Observables and Spectral Sensitivity

The response functional introduced in Section 3,

$$F(\lambda) = \|[K(\lambda), O]\|_F, \tag{36}$$

depends explicitly on differences between eigenvalues:

$$F^2(\lambda) = \sum_{i,j} (k_i(\lambda) - k_j(\lambda))^2 |O_{ij}|^2. \tag{37}$$

This expression shows that observables are sensitive to the distribution of spectral differences, rather than individual eigenvalues.

5.8 Toward a Distributional Law

The preceding analysis leads to a fundamental conclusion: the behavior of the system under modular flow is governed by statistical properties of the spectral distribution.

In particular, the emergence of critical behavior cannot be attributed to a single spectral parameter, but must be understood in terms of the distribution as a whole.

This observation provides the conceptual basis for the formulation of renormalization as a distributional law, which will be developed in the following sections.

6 Renormalization as Spectral Flow

6.1 Renormalization Without External Scale

In conventional formulations, renormalization is defined as a transformation of a system under changes of scale. This requires an external notion of length or energy scale.

Within the UMD framework, no such external structure is assumed. Instead, renormalization is formulated intrinsically as a flow in the space of states.

Given a density operator ρ, we consider a one-parameter family of states $\rho(\lambda)$ generated by modular dynamics. The parameter λ serves as an internal ordering parameter rather than a physical scale.

6.2 Spectral Flow of Modular Energies

Let $\{k_i(\lambda)\}$ denote the eigenvalues of the modular generator

$$K(\lambda) = -\log \rho(\lambda). \tag{38}$$

The evolution of $\rho(\lambda)$ induces a nonlinear flow of the spectrum:

$$\lambda \longrightarrow \{k_i(\lambda)\}. \tag{39}$$

Equivalently, the empirical spectral distribution evolves as

$$p(k;\lambda) = \frac{1}{N}\sum_i \delta(k - k_i(\lambda)). \tag{40}$$

Thus, renormalization is naturally interpreted as a flow of distributions

$$p(k;\lambda_1) \longrightarrow p(k;\lambda_2). \tag{41}$$

6.3 Definition of the RG Functional

To probe the flow, we introduce the response functional

$$F(\lambda) = \|[K(\lambda), O]\|_F, \tag{42}$$

where O is an observable not commuting with K.

The RG flow is then characterized by the beta function

$$\beta_F(\lambda) = \frac{dF}{d\lambda}. \tag{43}$$

This definition does not rely on coupling constants or external parameters, but is defined directly at the level of the state.

6.4 Interpretation of the Flow

The function $F(\lambda)$ captures how the spectral structure of the state interacts with observables.

Its evolution reflects two competing effects:

- **Spectral compression:** redistribution of eigenvalues tends to reduce spectral differences;
- **Operator sensitivity:** coupling between spectral sectors enhances response.

The interplay of these effects leads to non-monotonic behavior of $F(\lambda)$.

6.5 Critical Scale

We define the critical scale λ^* as the point where the flow changes direction:

$$\beta_F(\lambda^*) = 0. \tag{44}$$

At this point, the competing effects of spectral compression and operator sensitivity are balanced.

This definition provides an intrinsic notion of criticality that does not rely on external scales.

6.6 Renormalization as Distributional Evolution

The key structural observation is that $F(\lambda)$ depends on spectral differences

$$F^2(\lambda) = \sum_{i,j} (k_i(\lambda) - k_j(\lambda))^2 |O_{ij}|^2. \tag{45}$$

Therefore, the RG flow is governed not by individual eigenvalues, but by the statistical structure of spectral differences.

This implies that renormalization is fundamentally a distributional process:

$$\lambda \longrightarrow p(k;\lambda), \tag{46}$$

and all observable consequences of the flow are determined by properties of this evolving distribution.

6.7 Relation to Conventional RG

In traditional RG, flow equations are written in terms of coupling constants and beta functions defined with respect to an external scale.

In contrast, the UMD formulation replaces:

$$\text{scale} \to \lambda, \qquad \text{couplings} \to \text{spectral distribution}. \tag{47}$$

Thus, the RG flow becomes a trajectory in the space of spectral distributions.

6.8 Toward a Law of Renormalization

The identification of renormalization as a spectral flow leads to a natural question: what determines the location of the critical scale λ^*?

Since both $F(\lambda)$ and its derivative depend on the spectral distribution $p(k;\lambda)$, it follows that λ^* must be determined by the structure of this distribution.

This observation sets the stage for the formulation of a law of renormalization, which will be presented in the next section.

7 Critical Scale Law

7.1 Definition of the Critical Scale

Within the UMD framework, the renormalization flow is characterized by the response functional

$$F(\lambda) = \|[K(\lambda), O]\|_F, \tag{48}$$

and its associated beta function

$$\beta_F(\lambda) = \frac{dF}{d\lambda}. \tag{49}$$

We define the critical scale λ^* as the point where the flow changes direction:

$$\beta_F(\lambda^*) = 0. \tag{50}$$

This definition provides an intrinsic notion of criticality formulated entirely at the level of the state.

7.2 Statement of the Law

[Law of Renormalization in UMD] Let ρ be a full-rank density operator on a finite-dimensional Hilbert space, and let

$$K = -\log \rho \tag{51}$$

be its modular generator.

Then the critical scale $\lambda^*(\rho)$ of the renormalization flow is determined by the spectral distribution of K:

$$\lambda^*(\rho) = \mathcal{F}\big(p(k)\big), \tag{52}$$

where $p(k)$ is the empirical distribution of eigenvalues of K.

In particular, to leading order, $\lambda^*(\rho)$ depends on statistical descriptors of the distribution, including

$$\mu_K = \langle K \rangle, \qquad \sigma_K^2 = \mathrm{Var}(K), \qquad \Delta_K = q_{90} - q_{10}. \tag{53}$$

7.3 Proof Sketch

Step 1: Spectral representation. The modular generator admits a spectral decomposition

$$K = \sum_i k_i |\psi_i\rangle\langle\psi_i|. \tag{54}$$

The response functional can be written as

$$F^2(\lambda) = \sum_{i,j} (k_i(\lambda) - k_j(\lambda))^2 |O_{ij}|^2, \tag{55}$$

which depends explicitly on spectral differences.

Step 2: Dependence on distribution. Since $F(\lambda)$ involves all pairwise differences $(k_i - k_j)$, its behavior is determined by the global structure of the spectrum, rather than individual eigenvalues.

Step 3: Emergence of a critical point. Under modular flow, the spectrum undergoes redistribution. The resulting competition between spectral compression and operator sensitivity produces a non-monotonic $F(\lambda)$.

The condition $\beta_F(\lambda^*) = 0$ therefore depends on the full distribution of spectral values.

Step 4: Reduction to statistical descriptors. To leading order, the behavior of $F(\lambda)$ can be captured by low-order statistical descriptors of the distribution $p(k)$, such as mean, variance, and quantile structure.

This establishes that $\lambda^*(\rho)$ is a functional of the spectral distribution.

□

7.4 Explicit Form and Numerical Evidence

Numerical analysis shows that $\lambda^*(\rho)$ can be approximated by a low-dimensional function of spectral descriptors:

$$\lambda^*(\rho) \approx a\,\mu_K + b\,\sigma_K^2 + c\,\Delta_K + d, \tag{56}$$

where the coefficients a, b, c, d are stable across ensembles.

This demonstrates that the critical scale is governed by the global structure of the modular spectrum.

7.5 Physical Interpretation

The law implies that renormalization is controlled not by isolated spectral features, such as gaps, but by the full distribution of modular energies.

In particular:

- μ_K sets the overall scale;
- σ_K^2 encodes fluctuations;
- Δ_K captures tail effects and rare modes.

Thus, critical behavior is inherently distributional.

7.6 Consequences

The Law of Renormalization in UMD implies:

1. Renormalization is intrinsically state-dependent;
2. Critical scales are not externally imposed, but emerge from spectral structure;
3. The notion of scale is replaced by a distributional description of modular energies.

This result establishes renormalization as a law governing spectral-statistical structure in the space of states.

8 Role of Fluctuations in Theory Formation

8.1 Emergence of Structure Beyond Linear Derivation

The formulation of the law presented in the previous section did not arise as a direct linear extension of the initial formal framework. Instead, it emerged through a reorganization of partially developed structures, followed by systematic numerical validation and formal reconstruction.

This process highlights an important aspect of theoretical development: not all structurally valid results are obtained through step-by-step deduction. In sufficiently complex frameworks, new structures may appear through non-linear transitions in the space of admissible formulations.

8.2 Fluctuations in Conceptual Space

Within the UMD perspective, physical structure originates from fluctuations in informational distinguishability. By analogy, the process of theory formation may involve fluctuations in the space of conceptual configurations.

We define a conceptual configuration as a consistent assignment of mathematical structures to physical interpretation. A fluctuation in this space corresponds to a temporary instability that reorganizes existing elements into a new, coherent structure.

In this sense, the emergence of the distributional formulation of renormalization can be viewed as a fluctuation-driven transition in conceptual space.

8.3 Stabilization Through Consistency

While fluctuations may initiate the formation of new structures, they do not by themselves establish validity. A fluctuation-generated structure becomes part of the theory only if it satisfies:

- internal mathematical consistency,
- compatibility with the existing formal framework,
- reproducibility under independent constructions,
- stability under perturbations of the initial setup.

In the present case, the relation

$$\lambda^*(\rho) = \mathcal{F}\big(\mathrm{Spec}(-\log \rho)\big) \tag{57}$$

was subjected to systematic numerical and structural analysis, confirming its robustness.

8.4 Constructive Fluctuation Principle

The above considerations motivate the following principle.

> **Constructive Fluctuation Principle.** In sufficiently rich theoretical frameworks, new physically meaningful structures may emerge through non-linear reorganizations of partially developed elements. Such structures become valid components of the theory only after undergoing full stabilization through consistency, reproducibility, and invariance checks.

8.5 Relation to the UMD Framework

This principle is not external to the theory. It reflects the same structural logic that underlies UMD itself.

In UMD, physical objects arise from fluctuations in informational structure, followed by stabilization into coherent phases. The emergence of the renormalization law follows an analogous pattern at the level of theoretical construction.

Thus, fluctuation is not merely a heuristic element, but a structural component linking the formation of physical reality and its theoretical description.

8.6 Implications

The role of fluctuations in theory formation suggests that the boundary between discovery and construction is inherently dynamic.

In particular:

- Theoretical development may involve transitions between distinct structural regimes;
- Non-linear reorganization can reveal structures not accessible through incremental derivation;
- Validation requires strict post-emergence verification, rather than reliance on the origin of the result.

This perspective reinforces the interpretation of the law of renormalization as a structurally stable outcome, independent of the specific path by which it was identified.

9 Discussion

9.1 Summary of the Main Result

We have shown that, within the framework of Universal Modular Dynamics (UMD), renormalization can be formulated as an intrinsic flow in the space of quantum states. In this formulation, the critical scale $\lambda^*(\rho)$ is not determined by a single spectral parameter, but by the full statistical structure of the modular spectrum.

This leads to a law of the form

$$\lambda^*(\rho) = \mathcal{F}\big(\mathrm{Spec}(-\log\rho)\big), \tag{58}$$

which establishes renormalization as a distributional phenomenon.

9.2 Conceptual Implications

The proposed formulation implies a shift in the conceptual understanding of renormalization.

First, the notion of scale is no longer fundamental. Instead, it is replaced by an internal ordering parameter λ, defined directly in the space of states.

Second, the role of coupling constants is replaced by the spectral distribution of the modular generator. This suggests that physical behavior is governed by informational structure rather than externally imposed parameters.

Third, criticality emerges as a global property of the spectrum, rather than as a consequence of isolated features such as spectral gaps.

9.3 Relation to Conventional Renormalization

In conventional renormalization group (RG) theory, flows are defined with respect to external scale transformations and are typically expressed in terms of coupling constants.

In contrast, the UMD formulation replaces:

$$\text{external scale} \to \lambda, \qquad \text{couplings} \to \mathrm{Spec}(-\log\rho). \tag{59}$$

This redefinition suggests that renormalization can be understood as a structural property of quantum states, rather than a procedure applied to a given model.

9.4 Emergent Geometry and Phase Structure

The distributional nature of the law is consistent with the emergence of geometry and locality discussed in previous sections.

In particular, changes in the spectral distribution correspond to transitions between geometric phases. The critical scale λ^* marks the point at which the structure of correlations becomes unstable, leading to reorganization of the effective geometry.

Thus, renormalization, geometry, and phase structure are unified within a single framework.

9.5 Universality and Robustness

An important feature of the proposed law is its robustness across different classes of states.

Numerical analysis indicates that the dependence of $\lambda^*(\rho)$ on statistical descriptors of the spectrum is stable under variations of the system, suggesting a form of universality.

This supports the interpretation of the result as a structural law rather than a model-specific observation.

9.6 Limitations

The present formulation is restricted to finite-dimensional systems and relies on numerical evidence to support the explicit functional form of $\mathcal{F}$.

Further work is required to:

- extend the framework to infinite-dimensional systems,
- establish rigorous bounds on $\mathcal{F}$,
- analyze the role of different choices of observables O,
- connect the formalism to concrete physical models.

9.7 Outlook

The results presented here suggest several directions for future research.

First, the distributional formulation of renormalization may provide new tools for analyzing strongly correlated systems.

Second, the connection between spectral structure and geometry may lead to new approaches to emergent spacetime.

Third, the intrinsic formulation of RG flow may offer insights into quantum gravity, where external notions of scale are problematic.

More broadly, the identification of renormalization as a law governing spectral-statistical structure suggests that fundamental physical behavior may be encoded directly in the informational content of quantum states.

9.8 Final Perspective

Taken together, these results indicate that renormalization is not merely a computational tool, but a structural principle governing the organization of physical systems.

Within the UMD framework, this principle is expressed as a law relating critical behavior to the distribution of modular energies, providing a unified description of dynamics, geometry, and phase structure.

A Appendix A: Spectral Representation and Response Functional

In this appendix we provide technical details underlying the spectral formulation of the response functional.

Let ρ be a density operator with spectral decomposition

$$\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|, \tag{60}$$

and modular generator

$$K = -\log\rho = \sum_i k_i |\psi_i\rangle\langle\psi_i|. \tag{61}$$

For an observable O, the commutator reads

$$[K, O] = \sum_{i,j} (k_i - k_j) O_{ij} |\psi_i\rangle\langle\psi_j|. \tag{62}$$

Taking the Frobenius norm, we obtain

$$\|[K, O]\|_F^2 = \sum_{i,j} (k_i - k_j)^2 |O_{ij}|^2. \tag{63}$$

This expression shows explicitly that the response functional depends on pairwise spectral differences, rather than individual eigenvalues.

This provides the mathematical basis for the distributional dependence of the renormalization flow.

B Appendix B: Numerical Protocol and Extraction of λ^*

We summarize the numerical procedure used to extract the critical scale λ^*.

Spectral Flow Construction

A family of states $\rho(\lambda)$ is generated through modular evolution:

$$\frac{d\rho}{d\lambda} = -i[K_{\rho|\sigma}, \rho] + \mathcal{D}[\rho]. \tag{64}$$

At each value of λ, the modular generator is computed:

$$K(\lambda) = -\log \rho(\lambda). \tag{65}$$

Response Functional

The response functional is evaluated as

$$F(\lambda) = \|[K(\lambda), O]\|_F. \tag{66}$$

Beta Function and Critical Scale

The beta function is obtained numerically:

$$\beta_F(\lambda) \approx \frac{F(\lambda + \Delta\lambda) - F(\lambda)}{\Delta\lambda}. \tag{67}$$

The critical scale λ^* is identified as the point where

$$\beta_F(\lambda^*) = 0. \tag{68}$$

Stability Checks

To ensure robustness, the following checks are performed:

- variation of initial states ρ,
- variation of observables O,
- smoothing of $F(\lambda)$ to remove numerical noise,
- consistency across different discretizations of λ.

Extraction of Spectral Descriptors

For each state, the spectral distribution of K is characterized by:

- mean μ_K,
- variance σ_K^2,
- quantiles q_{10}, q_{90}.

These quantities are used to construct empirical approximations of the functional $\mathcal{F}$.

This protocol establishes the numerical basis for the law of renormalization formulated in the main text.

C Conclusion

In this work, we have developed a formulation of renormalization within the framework of Universal Modular Dynamics (UMD), in which the density operator ρ serves as the fundamental object.

The central result is the identification of a law governing the emergence of critical scales in renormalization flow. Specifically, we have shown that the critical scale $\lambda^*(\rho)$ is determined by the spectral distribution of the modular generator $K = -\log \rho$, rather than by isolated spectral features or externally imposed parameters.

This leads to a distributional law of the form

$$\lambda^*(\rho) = \mathcal{F}\big(\mathrm{Spec}(-\log \rho)\big), \tag{69}$$

which establishes renormalization as an intrinsic, state-dependent phenomenon.

The implications of this result are threefold.

First, renormalization is no longer understood as a procedure based on external scale transformations, but as a structural property of quantum states. The notion of scale is replaced by an internal ordering parameter, and the flow is defined directly in the space of density operators.

Second, critical behavior is governed by the global statistical structure of the modular spectrum. This replaces traditional gap-based intuition with a distributional description, in which collective spectral properties determine physical behavior.

Third, the framework provides a unified perspective in which dynamics, geometry, and phase structure emerge from a common informational foundation. In this sense, the law of renormalization is not an isolated result, but part of a broader structural picture encoded in the properties of ρ.

An important aspect of the present work is that the identified structure is not tied to a specific model. The dependence of $\lambda^*(\rho)$ on statistical descriptors of the spectrum appears robust across different classes of states, suggesting a degree of universality.

At the same time, the formulation remains open to further refinement. A fully rigorous characterization of the functional $\mathcal{F}$, as well as extensions to more general settings, represent natural directions for future work.

Taken together, these results indicate that renormalization may be understood as a law governing the spectral-statistical organization of quantum states. Within the UMD framework, this law provides a direct link between informational structure and physical behavior, and suggests that fundamental aspects of physical theory may be encoded at the level of state distributions rather than external parameters.

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