Homological Ramification Filtrations in Residually Transcendental Valued Function Fields

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Abstract

Classical valuation theory studies residually transcendental extensions through value-group growth, residue-field extensions, and prolongation behavior. The purpose of this paper is more modest: we introduce a homological framework for measuring layerwise nonflatness and torsion phenomena inside chosen valuation filtrations of residually transcendental valued function fields.

Given a valued field (K, v) and a simple transcendental extension $K(x)$ endowed with a prolongation w of v, we construct valuation filtration modules associated to the layers of the valuation. Applying derived functor methods to these filtration layers, we define derived residue obstruction modules

$$\mathrm{HRD}_i(n) := \mathrm{Tor}_i^{\mathcal{O}_v}(F_n, k_v),$$

which detect failure of homological exactness in valuation descent, persistent derived obstructions arising from nonflat valuation layers, and residual transcendence torsion phenomena.

We establish Tor-vanishing criteria under explicit filtration-flat hypotheses, prove persistence results under explicit torsion hypotheses in valuation-filtration layers, and attach annihilator ideals to the first derived residue obstruction layers. The examples are divided into residually transcendental model cases and boundary torsion computations over discrete valuation rings; the latter are included only to illustrate the homological obstruction mechanism and are not presented as the main geometric source of residual transcendence.

The theory proposed here is therefore best viewed as a Tor-theoretic filtration theory over valuation rings, motivated by residually transcendental valued function fields, rather than as a new numerical ramification theory.

Keywords. Valuation theory; valued function fields; residually transcendental extensions; valuation filtrations; Tor functors; derived residue obstruction modules; valuation rings; nonflatness; torsion; annihilator ideals.

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1 Introduction

Valuation theory traditionally studies extensions of valued fields through value groups, residue fields, and prolongation structures ([15, Thms. 2.4–2.5, pp. 3475–3476]; [3, Thms. 1–2, p. 387]).

In residually transcendental extensions, these invariants do not by themselves describe all module-theoretic phenomena arising from a chosen valuation filtration.

The central philosophy of this paper is that the graded pieces of a valuation filtration may be studied homologically over the base valuation ring. More precisely, we introduce derived residue obstruction layers measuring:

- failure of flat residue descent for graded valuation pieces,
- persistence of Tor obstructions in valuation-filtration layers,
- annihilator ideals attached to first derived residue obstruction modules,
- layerwise nonflatness and torsion phenomena.

Thus the paper develops a Tor-theoretic obstruction framework attached to chosen valuation filtrations. It is not claimed that the resulting modules are classical ramification groups, nor that they recover classical conductor or valuation-defect invariants.

Conceptual Novelty

This paper introduces a derived-homological framework for valuation filtrations in residually transcendental extensions. The main novelty is the construction of *derived residue layers* associated to valuation filtrations, together with derived obstruction modules detecting nonflatness and torsion in the chosen valuation-filtration layers.

Scope of the examples. The main framework of the paper is formulated for residually transcendental valued function fields. Some later examples deliberately specialize to discrete valuation rings and exhibit explicit $\mathcal{O}_v/(\pi)$-torsion in filtration layers. These examples should be read as mechanism examples for the derived obstruction $\mathrm{Tor}_1^{\mathcal{O}_v}(-, k_v)$, not as evidence that the whole theory is generated by DVR torsion alone.

2 Preliminaries on Valued Function Fields

2.1 Valued Fields

Let (K, v) be a valued field with valuation ring $\mathcal{O}_v$, maximal ideal $\mathfrak{m}_v$, value group Γ_v, and residue field k_v ([11, Ch. 2, §2.1, p. 28].; [12, §6, pp. 40–45]).

We denote by

$$v : K^\times \to \Gamma_v$$

the associated valuation map.

For the standard background on discrete valuation rings, valuation rings, uniformizers, and residue fields, see [14, Ch. I, §1, Prop. 1, pp. 5–6].

2.2 Residually Transcendental Extensions

Let

$$K(x)$$

be a simple transcendental extension of K, and let w be a prolongation of v to $K(x)$ ([8, Thm. 2.1, p. 94]).

Definition 2.1. The extension $(K(x), w)/(K, v)$ is called *residually transcendental* if the residue field extension

$$k_w/k_v$$

is transcendental.

This terminology follows the classical residue-field viewpoint on places and valuations [16, Ch. VI, §2, pp. 4–5]; for the corresponding description of extensions of valuations to $K(x)$, including the residue-field and value-group alternatives, see [2, Thm. 1.1 and Cor. 1.2, pp. 381–382]; [4, Cor. 1.2 and Thm. 1.3, p. 421; Thm. 1.4, pp. 421–423]. For explicit constructions of extensions of valuations to simple transcendental extensions with prescribed residue-field behavior, see [1, Introduction, p. 147; Thm. 4, p. 155].

2.3 Classical Valuation Filtrations

Fix once and for all a positive element $\gamma \in \Gamma_v$, and view it in Γ_w through the ordered-group embedding induced by the prolongation $w|_K = v$. All filtration layers below are taken with respect to this chosen integer-indexed valuation step. For $n \geq 0$, set

$$A_n := \{f \in K(x) : w(f) \geq n\gamma\}, \qquad F_n := A_n/A_{n+1}.$$

This is the integer-step analogue of the usual valuation ideals and associated graded construction for a valuation ([15, Sec. 1.1, p. 3441]; [12, §6, Thm. 6.3 and Cor. 6.5, pp. 40–42]).

Lemma 2.2 (Module structure of valuation filtration layers)**.** *For every $n \geq 0$, the subset $A_n \subseteq K(x)$ is an $\mathcal{O}_v$-submodule of $K(x)$, and $A_{n+1} \subseteq A_n$ is an $\mathcal{O}_v$-submodule. Consequently*

$$F_n = A_n/A_{n+1}$$

is a well-defined $\mathcal{O}_v$-module.

Proof. Let $f, g \in A_n$. Since w is a valuation,

$$w(f + g) \geq \min\{w(f), w(g)\} \geq n\gamma,$$

whenever $f + g \neq 0$, while $0 \in A_n$. Hence A_n is closed under addition.

Now let $a \in \mathcal{O}_v$. Since w extends v, one has $w(a) = v(a) \geq 0$. Therefore, for $f \in A_n$,

$$w(af) = w(a) + w(f) = v(a) + w(f) \geq n\gamma.$$

Thus $af \in A_n$, so A_n is an $\mathcal{O}_v$-submodule of $K(x)$. The same argument applies to A_{n+1}, and the inclusion $A_{n+1} \subseteq A_n$ follows from $(n + 1)\gamma \geq n\gamma$. Therefore the quotient $F_n = A_n/A_{n+1}$ is a well-defined $\mathcal{O}_v$-module. □

Thus the theory below is attached to the chosen discrete cofinal step γ, and is not asserted to be independent of γ for arbitrary value groups.

These form the chosen integer-indexed valuation filtration layers associated to the fixed step γ.

Remark 2.3. The modules F_n encode the graded structure associated to the valuation filtration induced by w. The principal aim of this paper is to derive these layers homologically.

3 Derived Residue Filtrations

3.1 Derived Residue Obstruction Modules

By Theorem 2.2, each F_n is an $\mathcal{O}_v$-module, so the groups $\operatorname{Tor}_i^{\mathcal{O}_v}(F_n, k_v)$ are well-defined.

Definition 3.1. For each integer $n \geq 0$ and $i \geq 0$, define the *derived residue obstruction module* by

$$\mathrm{HRD}_i(n) := \operatorname{Tor}_i^{\mathcal{O}_v}(F_n, k_v).$$

Remark 3.2. The modules $\mathrm{HRD}_i(n)$ are invariants of the chosen $\mathcal{O}_v$-module filtration $\{F_n\}_{n\geq 0}$ and of the chosen step γ. They are not asserted to be intrinsic invariants of the valued field extension unless additional invariance hypotheses on the filtration and the step are imposed.

Remark 3.3. The modules $\mathrm{HRD}_i(n)$ measure the failure of exactness of the ordinary residue-base-change functor

$$- \otimes_{\mathcal{O}_v} k_v$$

applied to the specific valuation-filtration layer F_n. Thus they are Tor-theoretic obstructions attached to the chosen $\mathcal{O}_v$-module F_n, not by themselves classical valuation-theoretic ramification, conductor, or defect invariants.

3.2 Spectral Sequence of Valuation Filtration Layers

With the fixed element $\gamma \in \Gamma_v \subseteq \Gamma_w$ chosen above, let

$$A_n := \{f \in K(x) : w(f) \geq n\gamma\}$$

so that $F_n = A_n/A_{n+1}$. The family $\{A_n\}_{n\geq 0}$ is a decreasing filtration of the $\mathcal{O}_v$-module A_0, and

$$\mathrm{gr}_w(A_0) := \bigoplus_{n\geq 0} A_n/A_{n+1} = \bigoplus_{n\geq 0} F_n$$

is the associated graded valuation module (Using the standard associated-graded construction [16, Ch. VIII, §1, pp. 248–249]).

Proposition 3.4 (Associated graded valuation module)**.** *Let*

$$A_n = \{f \in K(x) : w(f) \geq n\gamma\} \qquad (n \geq 0),$$

where $\gamma \in \Gamma_v$ is the fixed positive base step, viewed in Γ_w via $w|_K = v$, and set

$$F_n = A_n/A_{n+1}.$$

The associated graded object

$$\mathrm{gr}_F(K(x)) := \bigoplus_{n\geq 0} F_n = \bigoplus_{n\geq 0} A_n/A_{n+1}$$

is a graded module over

$$\mathrm{gr}_v(\mathcal{O}_v) := \bigoplus_{n\geq 0} \{a \in \mathcal{O}_v : v(a) \geq n\gamma\}/\{a \in \mathcal{O}_v : v(a) \geq (n+1)\gamma\}.$$

Moreover, the derived residue layers

$$\mathrm{HRD}_i(n) = \mathrm{Tor}_i^{\mathcal{O}_v}(F_n, k_v)$$

are attached degreewise to the graded pieces of $\mathrm{gr}_F(K(x))$*. In particular, nonvanishing of* $\mathrm{HRD}_i(n)$ *for some* $i \geq 1$ *detects non-exactness of residue base change for the specific graded piece* F_n*. Under the explicit finiteness and torsion-free hypotheses used below, vanishing of these higher Tor layers is equivalent to the flatness condition on* F_n *used in the paper.*

(This associated graded viewpoint follows the standard valuation-graded formalism [15, Sec. 1.1, p. 3441]).

Proof. For $r, n \geq 0$, put

$$I_r := \{a \in \mathcal{O}_v : v(a) \geq r\gamma\}.$$

Since w extends v, one has

$$w(af) = v(a) + w(f)$$

whenever $a \in K^\times$ and $f \in K(x)^\times$. Hence

$$a \in I_r, \quad f \in A_n \quad \Longrightarrow \quad af \in A_{r+n}.$$

Moreover, if $a' \equiv a \pmod{I_{r+1}}$ and $f' \equiv f \pmod{A_{n+1}}$, then

$$a'f' - af = (a' - a)f + a'(f' - f).$$

The first term lies in A_{r+n+1}, since $a' - a \in I_{r+1}$ and $f \in A_n$. The second term lies in A_{r+n+1}, since $a' \in I_r$ and $f' - f \in A_{n+1}$. Therefore multiplication induces a well-defined bilinear map

$$\frac{I_r}{I_{r+1}} \otimes_{\mathbb{Z}} \frac{A_n}{A_{n+1}} \longrightarrow \frac{A_{r+n}}{A_{r+n+1}}.$$

No k_v-vector-space structure on the graded pieces is asserted here for an arbitrary value group or arbitrary positive step γ. The graded-module structure is meant over the displayed associated graded ring $\mathrm{gr}_v(\mathcal{O}_v)$.

Associativity of multiplication in $K(x)$ gives compatibility with degree addition. Consequently

$$\mathrm{gr}_F(K(x)) = \bigoplus_{n \geq 0} A_n/A_{n+1}$$

is a graded module over

$$\mathrm{gr}_v(\mathcal{O}_v) = \bigoplus_{r \geq 0} I_r/I_{r+1}.$$

It remains to explain the homological part in a way compatible with the later filtration sequence. By definition,

$$\mathrm{HRD}_i(n) = \mathrm{Tor}_i^{\mathcal{O}_v}(F_n, k_v) = H_i(F_n \otimes^{\mathbf{L}}_{\mathcal{O}_v} k_v).$$

Thus the n-th HRD-layer is precisely the derived residue obstruction of the n-th graded valuation piece. In particular, $\mathrm{HRD}_i(n)$ is not an invariant of the whole filtration at once, but a degreewise obstruction attached to the summand

$$F_n = A_n/A_{n+1}$$

of $\mathrm{gr}_F(K(x))$.

If F_n is flat over $\mathcal{O}_v$, then

$$\mathrm{Tor}_i^{\mathcal{O}_v}(F_n, k_v) = 0 \qquad (i \geq 1).$$

Hence nonvanishing of some $\mathrm{HRD}_i(n)$, $i \geq 1$, gives an explicit Tor obstruction to exact residue base change for the graded piece F_n. This is a module-theoretic obstruction over $\mathcal{O}_v$. It becomes valuation-theoretically meaningful only through the chosen valuation filtration producing the module F_n. Conversely, under the finitely generated torsion-free hypotheses over the valuation ring used later in the paper, the flatness criterion forces the vanishing of these higher layers. □

Lemma 3.5 (Derived long exact sequence for adjacent valuation steps)**.** *For every $n \geq 0$, the short exact sequence of $\mathcal{O}_v$-modules*

$$0 \longrightarrow A_{n+1} \longrightarrow A_n \longrightarrow F_n \longrightarrow 0$$

induces a natural long exact Tor sequence

$$\cdots \longrightarrow \mathrm{Tor}_1^{\mathcal{O}_v}(F_n, k_v) \longrightarrow A_{n+1} \otimes_{\mathcal{O}_v} k_v \longrightarrow A_n \otimes_{\mathcal{O}_v} k_v \longrightarrow F_n \otimes_{\mathcal{O}_v} k_v \longrightarrow 0.$$

Equivalently,

$$\cdots \longrightarrow \mathrm{HRD}_1(n) \longrightarrow A_{n+1} \otimes_{\mathcal{O}_v} k_v \longrightarrow A_n \otimes_{\mathcal{O}_v} k_v \longrightarrow F_n \otimes_{\mathcal{O}_v} k_v \longrightarrow 0.$$

Proof. By definition,

$$F_n = A_n / A_{n+1}.$$

Hence there is a canonical short exact sequence of $\mathcal{O}_v$-modules

$$0 \longrightarrow A_{n+1} \longrightarrow A_n \longrightarrow F_n \longrightarrow 0,$$

where the second map is the natural quotient morphism.

Apply the right-exact functor

$$- \otimes_{\mathcal{O}_v} k_v.$$

Since tensor product over $\mathcal{O}_v$ is only right exact in general, exactness need not be preserved at the left-hand side after passage to the residue field. The obstruction is measured by the left derived functors of tensor product, namely

$$\mathrm{Tor}_i^{\mathcal{O}_v}(-, k_v).$$

The standard long exact Tor sequence associated to a short exact sequence therefore yields ([13, Appendix B, Tor functors, property (4), p. 279])

$$\cdots \longrightarrow \mathrm{Tor}_1^{\mathcal{O}_v}(A_n, k_v) \longrightarrow \mathrm{Tor}_1^{\mathcal{O}_v}(F_n, k_v) \xrightarrow{\delta_n} A_{n+1} \otimes_{\mathcal{O}_v} k_v$$

$$\longrightarrow A_n \otimes_{\mathcal{O}_v} k_v \longrightarrow F_n \otimes_{\mathcal{O}_v} k_v \longrightarrow 0.$$

In particular, the connecting morphism

$$\delta_n : \mathrm{Tor}_1^{\mathcal{O}_v}(F_n, k_v) \longrightarrow A_{n+1} \otimes_{\mathcal{O}_v} k_v$$

measures the precise failure of exact residue descent between the adjacent valuation layers $A_{n+1} \subseteq A_n$.

Using the notation

$$\mathrm{HRD}_i(n) := \mathrm{Tor}_i^{\mathcal{O}_v}(F_n, k_v),$$

the sequence becomes

$$\cdots \longrightarrow \mathrm{Tor}_1^{\mathcal{O}_v}(A_n, k_v) \longrightarrow \mathrm{HRD}_1(n) \xrightarrow{\delta_n} A_{n+1} \otimes_{\mathcal{O}_v} k_v$$

$$\longrightarrow A_n \otimes_{\mathcal{O}_v} k_v \longrightarrow F_n \otimes_{\mathcal{O}_v} k_v \longrightarrow 0.$$

Consequently, if

$$\mathrm{HRD}_1(n) = 0,$$

then the induced morphism

$$A_{n+1} \otimes_{\mathcal{O}_v} k_v \longrightarrow A_n \otimes_{\mathcal{O}_v} k_v$$

has no first derived obstruction coming from the quotient layer F_n. Conversely, nonvanishing of $\mathrm{HRD}_1(n)$ gives a possible first derived obstruction. It records an actual failure visible in the adjacent residue sequence only through the image of the connecting morphism δ_n; thus the statement does not identify $\mathrm{HRD}_1(n)$ itself with failure of exact residue descent unless the preceding Tor contribution and the connecting map are controlled.

Thus the derived residue layers provide the recursive derived mechanism controlling passage between consecutive valuation filtration layers. ☐

Remark 3.6. Theorem 3.5 is the first recursive derived mechanism in the paper. It shows that the valuation filtration is controlled not only by the individual graded pieces F_n, but also by the connecting morphisms arising after residue-field descent. The module $\mathrm{HRD}_1(n)$ therefore gives a possible first derived obstruction to exact compatibility between adjacent valuation layers after passage to k_v. More precisely, an actual failure visible in the adjacent residue sequence is detected only through the image of the connecting morphism δ_n, together with the preceding Tor term. This adjacent-step obstruction is the local input for the conditional spectral-sequence formalism recorded later in Theorem 3.7, whenever the required compatible filtered flat resolution is available.

Theorem 3.7 (Conditional spectral sequence associated to a filtered flat resolution)**.** *Let*

$$A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$$

be the valuation filtration associated to the chosen step $\gamma \in \Gamma_v$, and let

$$P_\bullet \longrightarrow A_0$$

be a flat resolution over $\mathcal{O}_v$ equipped with a decreasing filtration $F^p P_\bullet$ such that the induced filtration on homology satisfies

$$H_0(F^p P_\bullet) = A_p,$$

and the associated graded complex

$$\mathrm{gr}_p(P_\bullet) := F^p P_\bullet / F^{p+1} P_\bullet$$

is a flat resolution of

$$F_p = A_p / A_{p+1}.$$

No general existence of such a compatible filtered flat resolution is asserted here; the statement is conditional on the existence of the displayed filtered flat data.

Assume moreover that the filtration is exhaustive, separated, and bounded below. Then, with the standard homological convention

$$E^0_{p,q} = \mathrm{gr}_p(P_{p+q} \otimes_{\mathcal{O}_v} k_v), \qquad d^r : E^r_{p,q} \to E^r_{p-r,q+r-1},$$

there exists a convergent spectral sequence

$$E^1_{p,q} \cong \mathrm{Tor}^{\mathcal{O}_v}_{p+q}(F_p, k_v) \Longrightarrow \mathrm{Tor}^{\mathcal{O}_v}_{p+q}(A_0, k_v).$$

Proof. Tensoring the filtered complex $P_\bullet$ with the residue field k_v gives a filtered complex

$$P_\bullet \otimes_{\mathcal{O}_v} k_v$$

with induced filtration

$$F^p(P_\bullet \otimes_{\mathcal{O}_v} k_v) = F^p P_\bullet \otimes_{\mathcal{O}_v} k_v.$$

Since the terms of $P_\bullet$ are flat over $\mathcal{O}_v$, passage to associated graded objects commutes with tensor product:

$$\mathrm{gr}_p(P_\bullet \otimes_{\mathcal{O}_v} k_v) \cong \mathrm{gr}_p(P_\bullet) \otimes_{\mathcal{O}_v} k_v.$$

The spectral sequence of the decreasing filtered complex $P_\bullet \otimes_{\mathcal{O}_v} k_v$ has

$$E^0_{p,q} = \mathrm{gr}_p(P_{p+q} \otimes_{\mathcal{O}_v} k_v), \qquad d^r : E^r_{p,q} \to E^r_{p-r,q+r-1},$$

and hence

$$E^1_{p,q} = H_{p+q}\Big(\mathrm{gr}_p(P_\bullet \otimes_{\mathcal{O}_v} k_v)\Big).$$

Using the preceding identification, this becomes

$$E^1_{p,q} \cong H_{p+q}\Big(\mathrm{gr}_p(P_\bullet) \otimes_{\mathcal{O}_v} k_v\Big).$$

By hypothesis, $\mathrm{gr}_p(P_\bullet)$ is a flat resolution of $F_p = A_p/A_{p+1}$. Therefore the last homology group computes

$$\mathrm{Tor}^{\mathcal{O}_v}_{p+q}(F_p, k_v).$$

Thus

$$E^1_{p,q} \cong \mathrm{Tor}^{\mathcal{O}_v}_{p+q}(F_p, k_v).$$

Since $P_\bullet \to A_0$ is a flat resolution, the homology of the total tensor complex is

$$H_{p+q}(P_\bullet \otimes_{\mathcal{O}_v} k_v) \cong \mathrm{Tor}^{\mathcal{O}_v}_{p+q}(A_0, k_v).$$

The exhaustiveness, separatedness, and bounded-below hypotheses give the standard convergence of the filtered-complex spectral sequence to this total homology. Hence

$$E^1_{p,q} \cong \mathrm{Tor}^{\mathcal{O}_v}_{p+q}(F_p, k_v) \Longrightarrow \mathrm{Tor}^{\mathcal{O}_v}_{p+q}(A_0, k_v).$$

□

Remark 3.8. Theorem 3.7 shows that once the valuation filtration is represented by a filtered flat resolution, the derived residue obstruction modules organize into the E^1-page of a filtered-complex spectral sequence. The individual layers

$$\mathrm{HRD}_i(p) = \mathrm{Tor}_i^{\mathcal{O}_v}(F_p, k_v)$$

therefore appear degreewise in a global homological mechanism converging to the derived residue obstruction of A_0.

This formulation is deliberately conditional. We do not prove that every valuation filtration admits a compatible filtered flat resolution whose associated graded complexes resolve the layers F_p. The theorem should therefore be read only as the standard filtered-complex spectral sequence attached to such data when it is available. The later vanishing and stabilization results do not depend on this existence question; they use only the degreewise modules $\mathrm{HRD}_i(n)$ and the stated flatness criteria.

3.3 Philosophical Interpretation

The invariants $\mathrm{HRD}_i(n)$ detect:

- failure of flat residue descent for graded valuation pieces,
- torsion inside the chosen filtration layers,
- annihilator ideals attached to first derived residue obstruction modules,
- nonflat valuation growth,
- persistent layerwise obstructions not detected by value-group or residue-field data alone.

4 Structural Properties of Derived Filtrations

4.1 Functoriality

Definition 4.1 (Preservation of the chosen filtration step)**.** Let $(K, v) \hookrightarrow (L, u)$ be an extension of valued fields, and let w and u be the chosen prolongations on $K(x)$ and $L(y)$, respectively. We say that the induced prolongation preserves the chosen valuation filtration step if the fixed element $\gamma \in \Gamma_v$ maps to the corresponding chosen step in Γ_u, and if the induced embedding

$$\iota : K(x) \hookrightarrow L(y)$$

satisfies

$$w(f) \geq n\gamma \quad \Longrightarrow \quad u(\iota(f)) \geq n\gamma$$

for every $f \in K(x)$ and every $n \geq 0$. Equivalently,

$$\iota(A_n(K)) \subseteq A_n(L) \qquad \text{for all } n \geq 0.$$

Proposition 4.2 (Functoriality after residue-field base change)**.** *Let*

$$(K, v) \hookrightarrow (L, u)$$

be an extension of valued fields, and suppose that the induced prolongation preserves the chosen valuation filtration step. Then, for every $n \geq 0$*, there is a natural morphism of derived residue complexes*

$$(F_n(K) \otimes^{\mathbf{L}}_{\mathcal{O}_v} k_v) \otimes^{\mathbf{L}}_{k_v} k_u \longrightarrow F_n(L) \otimes^{\mathbf{L}}_{\mathcal{O}_u} k_u.$$

Equivalently, since k_u is a field extension of k_v, there are natural maps

$$\mathrm{HRD}_i^K(n) \otimes_{k_v} k_u \longrightarrow \mathrm{HRD}_i^L(n) \qquad (i \geq 0),$$

where

$$\mathrm{HRD}_i^K(n) := \mathrm{Tor}_i^{\mathcal{O}_v}(F_n(K), k_v), \qquad \mathrm{HRD}_i^L(n) := \mathrm{Tor}_i^{\mathcal{O}_u}(F_n(L), k_u).$$

Proof. Let

$$\iota : K(x) \hookrightarrow L(y)$$

be the induced embedding, and assume that the prolongations are compatible with the chosen filtration step, so that

$$w(f) \geq n\gamma \quad \Longrightarrow \quad u(\iota(f)) \geq n\gamma$$

for every $f \in K(x)$ and every $n \geq 0$. Hence

$$\iota(A_n(K)) \subseteq A_n(L), \qquad \iota(A_{n+1}(K)) \subseteq A_{n+1}(L).$$

Therefore ι induces an $\mathcal{O}_v$-linear morphism

$$F_n(K) = A_n(K)/A_{n+1}(K) \longrightarrow F_n(L) = A_n(L)/A_{n+1}(L),$$

where $F_n(L)$ is viewed by restriction of scalars along $\mathcal{O}_v \to \mathcal{O}_u$.

Since $F_n(L)$ is naturally an $\mathcal{O}_u$-module, the correct functorial comparison is obtained after extension of scalars. The preceding map induces an $\mathcal{O}_u$-linear morphism

$$F_n(K) \otimes^{\mathbf{L}}_{\mathcal{O}_v} \mathcal{O}_u \longrightarrow F_n(L).$$

Applying the derived residue functor

$$- \otimes^{\mathbf{L}}_{\mathcal{O}_u} k_u$$

gives

$$(F_n(K) \otimes^{\mathbf{L}}_{\mathcal{O}_v} \mathcal{O}_u) \otimes^{\mathbf{L}}_{\mathcal{O}_u} k_u \longrightarrow F_n(L) \otimes^{\mathbf{L}}_{\mathcal{O}_u} k_u.$$

By associativity of derived tensor product, the source identifies with

$$F_n(K) \otimes^{\mathbf{L}}_{\mathcal{O}_v} k_u.$$

Since the residue embedding $k_v \hookrightarrow k_u$ is induced by $\mathcal{O}_v \to \mathcal{O}_u$, this is equivalently

$$(F_n(K) \otimes^{\mathbf{L}}_{\mathcal{O}_v} k_v) \otimes^{\mathbf{L}}_{k_v} k_u.$$

Thus we obtain a natural morphism

$$(F_n(K) \otimes^{\mathbf{L}}_{\mathcal{O}_v} k_v) \otimes^{\mathbf{L}}_{k_v} k_u \longrightarrow F_n(L) \otimes^{\mathbf{L}}_{\mathcal{O}_u} k_u.$$

Taking homology yields

$$\mathrm{HRD}_i^K(n) \otimes_{k_v} k_u \longrightarrow \mathrm{HRD}_i^L(n),$$

because k_u is flat over the field k_v. Naturality follows from functoriality of extension of scalars, derived tensor product, and homology. □

Remark 4.3. Theorem 4.2 should be read only as a functoriality statement producing a natural comparison morphism after residue-field base change. It does not assert that this morphism is injective, surjective, an isomorphism, or dimension-preserving.

Thus the derived residue obstruction layers are compatible with extensions of valued fields and compatible prolongations only in the sense that the displayed natural maps exist. No base-change theorem or preservation statement for the dimensions of the obstruction modules is claimed here.

5 Vanishing in Stable Residually Transcendental Towers

Proposition 5.1 (Flatness criterion for valuation filtration layers)**.** *Suppose that, for every* $n \geq 0$*, the filtration layer* F_n *is a finitely generated torsion-free module over the valuation ring* $\mathcal{O}_v$*. Then* F_n *is flat over* $\mathcal{O}_v$*. Consequently,*

$$\operatorname{Tor}_i^{\mathcal{O}_v}(F_n, k_v) = 0 \qquad (i \geq 1).$$

Proof. Fix $n \geq 0$. By hypothesis, the valuation filtration layer

$$F_n = A_n / A_{n+1}$$

is finitely generated and torsion-free as an $\mathcal{O}_v$-module.

A fundamental structural property of valuation rings is that every torsion-free module over a valuation ring is flat ([17, Lemma 15.22.10], see also [13, §10, Exercise 10.2, p. 77]). Indeed, valuation rings are Prüfer domains, and finitely generated torsion-free modules over Prüfer domains are flat. Applying this criterion to the module F_n, we conclude that

$$F_n$$

is flat over $\mathcal{O}_v$.

Now let

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be a flat resolution of an $\mathcal{O}_v$-module M. By flatness, the higher Tor groups vanish ([13, Theorem 7.8, p. 51]). Hence, for every $i \geq 1$,

$$\operatorname{Tor}_i^{\mathcal{O}_v}(F_n, k_v) = 0.$$

Using the notation introduced earlier,

$$\mathrm{HRD}_i(n) = \operatorname{Tor}_i^{\mathcal{O}_v}(F_n, k_v),$$

we obtain

$$\mathrm{HRD}_i(n) = 0 \qquad (i \geq 1).$$

Thus torsion-free finite generation of the valuation filtration layers forces homological exactness after residue-field descent. In particular, under the present hypotheses, the higher derived residue obstruction layers disappear completely. □

Remark 5.2. Theorem 5.1 is the fundamental vanishing mechanism for the paper. It identifies the precise algebraic condition under which the derived residue layers become trivial: finite generation together with torsion-freeness of the valuation filtration layers.

Conceptually, the proposition shows that nontrivial higher derived residue obstruction modules arise only from genuinely nonflat phenomena inside the valuation filtration. This explains why the later stability theorems reduce naturally to proving eventual torsion-free behavior of the graded valuation pieces.

Remark 5.3 (Scope of the vanishing mechanism)**.** The vanishing result above is purely module-theoretic: it is a flatness criterion over the valuation ring $\mathcal{O}_v$. In particular, the hypotheses "finitely generated" and "torsion-free" are the actual algebraic inputs used to prove the vanishing of the higher Tor groups. The valuation-theoretic terminology used later describes regimes in which such filtration-flatness may naturally occur, but it is not used as a substitute for the flatness argument itself.

Definition 5.4 (Stable defectless tame filtration-flat regime)**.** Let $(K(x), w)/(K, v)$ be as above, and let

$$A_n = \{f \in K(x) : w(f) \geq n\gamma\}, \qquad F_n = A_n/A_{n+1}.$$

We say that the chosen valuation filtration is in the *stable defectless tame filtration-flat regime* if the following conditions hold. Here the terms "defectless" and "tame" are used in the classical valued-field sense of finite extensions and tame fields ([7, p. 648; Thm. 1.1, p. 648].)

(i) The prolongation w/v is defectless in the sense that the finite algebraic approximations of the valued extension used to control the chosen filtration have trivial valuation defect, in the classical finite-extension sense recalled in [9, p. 401; Thm. 1.1 and Lem. 2.1, p. 402]; see also [10, Thm. 1.3, p. 234]; for the classical valuation-theoretic formulation of defectlessness, unramifiedness and tame ramification, see [12, §22, (22.3)–(22.6), pp. 178–181].

(ii) The residue extension k_w/k_v is *stably transcendental*, meaning that after every finite residue-field base extension k'_v/k_v arising from such finite algebraic approximations, the induced residue extension remains purely transcendental up to finite algebraic extension.

(iii) The associated graded valuation filtration is *tame*, meaning that the graded pieces

$$F_n = A_n/A_{n+1}$$

have no wild residue-characteristic torsion contribution obstructing flat descent over $\mathcal{O}_v$.

(iv) The filtration is *filtration-flat* if, for every $n \geq 0$, the layer F_n is a finitely generated torsion-free $\mathcal{O}_v$-module.

Thus, in this paper, the phrase *defectless filtration-flat regime* always refers to the simultaneous validity of the above defectless, stable-residue, tame-graded, and filtration-flat hypotheses for the chosen step γ.

Proposition 5.5 (Filtration-flat vanishing in the stable regime)**.** *Suppose that the chosen valuation filtration lies in the stable defectless tame regime of Theorem 5.4, and assume in particular that it is filtration-flat, i.e. for every $n \geq 0$,*

$$F_n = A_n/A_{n+1}$$

is a finitely generated torsion-free $\mathcal{O}_v$-module. Then

$$\mathrm{HRD}_i(n) = 0 \qquad (i \geq 1).$$

Proof. Fix $n \geq 0$. The vanishing assertion uses only the filtration-flat part of the stable regime. Namely, by hypothesis the layer

$$F_n = A_n/A_{n+1}$$

is a finitely generated torsion-free $\mathcal{O}_v$-module. Since $\mathcal{O}_v$ is a valuation ring, Theorem 5.1 applies and gives that F_n is flat over $\mathcal{O}_v$. Hence

$$\operatorname{Tor}_i^{\mathcal{O}_v}(F_n, k_v) = 0 \qquad (i \geq 1).$$

By definition,

$$\mathrm{HRD}_i(n) = \operatorname{Tor}_i^{\mathcal{O}_v}(F_n, k_v),$$

so

$$\mathrm{HRD}_i(n) = 0 \qquad (i \geq 1).$$

The defectless, stably transcendental, and tame assumptions are therefore not independent homological inputs in this proof. They specify the valuation-theoretic regime in which the filtration-flat hypothesis is being considered; the actual Tor-vanishing mechanism is the flatness of the $\mathcal{O}_v$-module F_n. ☐

Remark 5.6. Theorem 5.5 should be read as a filtration-flat vanishing statement, not as a theorem deriving flatness from defectlessness, tameness, or residual transcendence alone. Its homological content is that once the graded valuation layer F_n is finitely generated and torsion-free over $\mathcal{O}_v$, all higher derived residue obstruction layers vanish.

Thus the terminology "stable defectless tame" describes the surrounding valuation-theoretic regime, while the decisive algebraic input is filtration-flatness. Nonzero modules $\mathrm{HRD}_i(n)$ therefore detect nonflatness of the chosen graded valuation pieces; they are not, by themselves, classical defect, conductor, or ramification invariants.

Theorem 5.7 (Comparison with the defectless filtration-flat regime)**.** *Assume that the prolongation lies in a defectless filtration-flat regime in the following precise sense:*

- *(i) the prolongation is defectless in the valuation-theoretic sense relevant to the finite algebraic approximations used in the filtration;*
- *(ii) the residue extension k_w/k_v is stably transcendental;*
- *(iii) the associated graded valuation filtration is tame in the sense of Theorem 5.5;*
- *(iv) for every $n \geq 0$, the valuation layer*

  $$F_n = A_n/A_{n+1}$$

  is a finitely generated torsion-free $\mathcal{O}_v$-module.

Then

$$\mathrm{HRD}_i(n) = 0 \qquad (i \geq 1,\ n \geq 0).$$

Conversely, if $\mathrm{HRD}_1(n) \neq 0$ for infinitely many n, then the valuation filtration cannot satisfy all of the above stable defectless tame finite-generation hypotheses. Thus persistent nonvanishing of the first derived residue obstruction layer detects failure of the stated filtration-flatness regime, without being identified with classical valuation defect.

Proof. Assumption (i) places the filtration in the defectless regime required for the comparison; no numerical classical defect invariant is being defined for the transcendental extension itself. Together with assumptions (ii), (iii), and (iv), this is precisely the stable defectless tame finite-generation regime required in Theorem 5.5. Hence Theorem 5.5 applies and gives

$$\mathrm{HRD}_i(n) = 0 \qquad (i \geq 1,\ n \geq 0).$$

Conversely, suppose that

$$\mathrm{HRD}_1(n) \neq 0$$

for infinitely many integers $n \geq 0$. If the valuation filtration satisfied all hypotheses listed above, then the first part of the theorem would imply

$$\mathrm{HRD}_1(n) = 0 \qquad \text{for all } n \geq 0,$$

which contradicts the assumed persistent nonvanishing. Therefore at least one of the stable defectless tame finite-generation hypotheses must fail. ☐

Remark 5.8. Theorem 5.7 should be read as a compatibility statement, not as an identification with classical valuation defect. The implication proved above is that the stated filtration-flatness hypotheses force the higher derived residue obstruction layers to vanish. Conversely, persistent nonvanishing of $\mathrm{HRD}_1(n)$ shows that the chosen filtration cannot satisfy all of those hypotheses.

Thus $\mathrm{HRD}_i(n)$ detects failure of flatness inside the graded valuation pieces. It may be invisible to the ordinary value-group and residue-field data, but it is not claimed to be equal to a classical defect, conductor, or ramification invariant.

5.1 Euler Characteristic of derived residue layers

Definition 5.9 (Euler characteristic of derived residue obstructions)**.** Assume that, for a fixed $n \geq 0$, the k_v-vector spaces $\mathrm{HRD}_i(n)$ are finite-dimensional and vanish for all sufficiently large i. Define the Euler characteristic of the n-th derived residue obstruction layer by

$$\chi_n := \sum_i (-1)^i \dim_{k_v} \mathrm{HRD}_i(n).$$

Proposition 5.10 (Quasi-isomorphism invariance of the Euler characteristic)**.** *Assume that, for a fixed $n \geq 0$, the derived residue complexes computing*

$$F_n \otimes^{\mathbf{L}}_{\mathcal{O}_v} k_v$$

before and after a stable prolongation are quasi-isomorphic. Then the Euler characteristic χ_n, whenever defined, is preserved.

In particular, under the stronger filtration-flat vanishing hypothesis of Theorem 5.5, this invariant reduces to

$$\chi_n = \dim_{k_v} \mathrm{HRD}_0(n),$$

so it contains no higher-derived contribution in that regime.

Proof. Fix $n \geq 0$. By definition, the n-th derived residue obstruction layers are the homology groups of the derived residue complex:

$$\mathrm{HRD}_i(n) = \mathrm{Tor}_i^{\mathcal{O}_v}(F_n, k_v) = H_i\Big(F_n \otimes^{\mathbf{L}}_{\mathcal{O}_v} k_v\Big).$$

By hypothesis, the derived residue complexes associated to the degree-n filtration layer before and after stable prolongation are quasi-isomorphic in the derived category $D(k_v)$. Hence they represent the same derived object up to isomorphism:

$$F_n \otimes^{\mathbf{L}}_{\mathcal{O}_v} k_v \simeq F'_n \otimes^{\mathbf{L}}_{\mathcal{O}_v} k_v.$$

Quasi-isomorphic complexes have canonically isomorphic homology groups. Therefore, for every $i \geq 0$,

$$H_i\Big(F_n \otimes^{\mathbf{L}}_{\mathcal{O}_v} k_v\Big) \cong H_i\Big(F'_n \otimes^{\mathbf{L}}_{\mathcal{O}_v} k_v\Big).$$

Equivalently,

$$\mathrm{HRD}_i(n) \cong \mathrm{HRD}'_i(n).$$

By the finite-dimensionality hypothesis in Theorem 5.9, the Euler characteristic

$$\chi_n = \sum_i (-1)^i \dim_{k_v} \mathrm{HRD}_i(n)$$

is well defined. Since quasi-isomorphic complexes have identical homology dimensions in every degree, the alternating sum of these dimensions is preserved. Consequently,

$$\chi_n = \chi'_n.$$

Thus the Euler characteristic of the derived residue layer depends only on the quasi-isomorphism class of the derived residue complex associated to the valuation filtration layer. In particular, stable defectless prolongation preserving the derived residue structure leaves the invariant unchanged. ☐

Remark 5.11. Theorem 5.10 is mainly a bookkeeping statement: Euler characteristics are invariant under quasi-isomorphism because quasi-isomorphic complexes have the same homology. Its usefulness in the present framework is that it records what remains of the derived residue complex under stable comparison.

However, in the filtration-flat regime of Theorem 5.5, all higher groups $\mathrm{HRD}_i(n)$ for $i \geq 1$ vanish, and therefore

$$\chi_n = \dim_{k_v} \mathrm{HRD}_0(n).$$

Thus the Euler characteristic carries genuinely higher-derived information only outside the filtration-flat vanishing regime, for example in the nonflat or torsion examples considered later.

Theorem 5.12 (Asymptotic stabilization of first derived residue obstructions)**.** *Assume that there exists an integer $N \geq 0$ such that, for every $n \geq N$, the valuation layer*

$$F_n = A_n/A_{n+1}$$

is a finitely generated torsion-free $\mathcal{O}_v$-module. Then

$$\mathrm{HRD}_1(n) = 0 \qquad \textit{for all } n \geq N.$$

Proof. By hypothesis, for every $n \geq N$, the layer

$$F_n = A_n/A_{n+1}$$

is finitely generated and torsion-free as an $\mathcal{O}_v$-module. Hence Theorem 5.1 applies to each such

F_n, giving

$$\operatorname{Tor}_1^{\mathcal{O}_v}(F_n, k_v) = 0 \qquad \text{for all } n \geq N.$$

Since

$$\mathrm{HRD}_1(n) = \operatorname{Tor}_1^{\mathcal{O}_v}(F_n, k_v),$$

we obtain

$$\mathrm{HRD}_1(n) = 0 \qquad \text{for all } n \geq N.$$

□

Remark 5.13. Theorem 5.12 isolates the precise algebraic input required for eventual vanishing of the first derived residue obstruction layer: asymptotic finite generation and torsion-freeness of the valuation filtration layers.

Under these hypotheses, the filtration eventually enters a derived-flat regime in which the first obstruction layer

$$\mathrm{HRD}_1(n)$$

disappears permanently for all sufficiently large n.

Conceptually, the theorem places the present derived residue obstruction framework within the broader asymptotic philosophy of graded modules, Rees-type filtrations, and stabilization phenomena in commutative algebra. The valuation filtration behaves asymptotically like a stabilized graded system whose higher layers no longer generate new first-order derived obstructions.

The theorem also clarifies the role of the persistence results in Section 6: any infinite family of nonvanishing modules

$$\mathrm{HRD}_1(n)$$

necessarily reflects failure of the asymptotic flatness regime appearing in the theorem.

6 Persistence of Wild Residual Obstructions

Theorem 6.1 (Persistence under explicit torsion direct summands in valuation layers)**.** *Assume that there exists an infinite set $I \subseteq \mathbb{N}$ such that, for every $n \in I$, the valuation layer*

$$F_n = A_n/A_{n+1}$$

contains a nonzero cyclic direct summand M_n with

$$\operatorname{Tor}_1^{\mathcal{O}_v}(M_n, k_v) \neq 0.$$

In particular, this holds if $\mathcal{O}_v$ is a discrete valuation ring with uniformizer π and F_n contains a direct summand isomorphic to $\mathcal{O}_v/(\pi)$ for infinitely many n. Then

$$\mathrm{HRD}_1(n) \neq 0$$

for infinitely many integers $n \geq 0$.

Proof. Let $n \in I$. By hypothesis, F_n contains a nonzero cyclic direct summand M_n with $\operatorname{Tor}_1^{\mathcal{O}_v}(M_n, k_v) \neq 0$. Hence

$$F_n \cong M_n \oplus G_n$$

for some $\mathcal{O}_v$-module G_n. Since Tor is additive in the first variable over direct sums, one has

$$\operatorname{Tor}_1^{\mathcal{O}_v}(F_n, k_v) \cong \operatorname{Tor}_1^{\mathcal{O}_v}(M_n, k_v) \oplus \operatorname{Tor}_1^{\mathcal{O}_v}(G_n, k_v).$$

(Here we use the standard structure of a discrete valuation ring with maximal ideal generated by a uniformizer π; see [14, Ch. I, §1, pp. 5–6].) we use the standard description of a discrete valuation ring as a local Dedekind domain with principal maximal ideal [18, Ch. V, §6, Theorem 15 and the paragraph following it, pp. 277–278]. The standard free resolution

$$0 \longrightarrow \mathcal{O}_v \xrightarrow{\pi} \mathcal{O}_v \longrightarrow \mathcal{O}_v/(\pi) \longrightarrow 0$$

after tensoring with $k_v = \mathcal{O}_v/(\pi)$ becomes

$$0 \longrightarrow k_v \xrightarrow{0} k_v \longrightarrow 0.$$

Therefore

$$\operatorname{Tor}_1^{\mathcal{O}_v}(\mathcal{O}_v/(\pi), k_v) \cong k_v \neq 0.$$

Consequently

$$\mathrm{HRD}_1(n) = \operatorname{Tor}_1^{\mathcal{O}_v}(F_n, k_v) \neq 0$$

for every $n \in I$. Since I is infinite, the asserted infinite nonvanishing follows. □

Remark 6.2. Theorem 6.1 gives a sufficient mechanism for persistent nonvanishing of the first derived residue obstruction layer. It should not be read as a converse to the asymptotic stabilization theorem: eventual vanishing of $\mathrm{HRD}_1(n)$ alone does not, without additional finiteness and tameness hypotheses, force valuation-theoretic stability or exclude conductor oscillation. Rather, the theorem isolates an explicit torsion mechanism inside the valuation layers which guarantees persistent derived obstruction.

7 Localization and Annihilator Ideals

7.1 Localization Compatibility

Proposition 7.1 (Localization compatibility)**.** *Let $S \subset \mathcal{O}_v$ be a multiplicative subset. Since localization is exact ([13, Theorem 4.5, p. 26]), the short exact sequence*

$$0 \to A_{n+1} \to A_n \to F_n \to 0$$

localizes to a short exact sequence

$$0 \to S^{-1}A_{n+1} \to S^{-1}A_n \to S^{-1}F_n \to 0.$$

Hence there is a canonical identification

$$S^{-1}F_n \cong \frac{S^{-1}A_n}{S^{-1}A_{n+1}}.$$

Then, for every $i \geq 0$, there is a natural isomorphism

$$S^{-1}\,\mathrm{HRD}_i(n) \cong \operatorname{Tor}_i^{S^{-1}\mathcal{O}_v}(S^{-1}F_n, S^{-1}k_v).$$

Moreover, if $S \cap \mathfrak{m}_v = \varnothing$, then $S^{-1}k_v \cong k_v$. If $S \cap \mathfrak{m}_v \neq \varnothing$, then $S^{-1}k_v = 0$, and both sides of the displayed isomorphism are zero.

Proof. Fix $n \geq 0$. By definition,

$$\mathrm{HRD}_i(n) = \mathrm{Tor}_i^{\mathcal{O}_v}(F_n, k_v).$$

Let $P_\bullet \to F_n$ be a flat resolution over $\mathcal{O}_v$. Localizing gives a flat resolution

$$S^{-1}P_\bullet \to S^{-1}F_n$$

over $S^{-1}\mathcal{O}_v$. Since localization is exact and commutes with tensor products, one has

$$S^{-1}(P_\bullet \otimes_{\mathcal{O}_v} k_v) \cong (S^{-1}P_\bullet) \otimes_{S^{-1}\mathcal{O}_v} S^{-1}k_v.$$

Taking homology gives

$$S^{-1}\,\mathrm{Tor}_i^{\mathcal{O}_v}(F_n, k_v) \cong \mathrm{Tor}_i^{S^{-1}\mathcal{O}_v}(S^{-1}F_n, S^{-1}k_v),$$

which is the asserted isomorphism.

It remains to record the residue-field dichotomy. Since $k_v = \mathcal{O}_v/\mathfrak{m}_v$, if $S \cap \mathfrak{m}_v = \varnothing$, every element of S maps to a nonzero element of the field k_v, hence becomes invertible and $S^{-1}k_v \cong k_v$. If $S \cap \mathfrak{m}_v \neq \varnothing$, some element of S acts as zero on k_v; after localization this forces $S^{-1}k_v = 0$. Since each $\mathrm{HRD}_i(n)$ is naturally a k_v-vector space, the same localization also gives $S^{-1}\,\mathrm{HRD}_i(n) = 0$. Thus in this case both sides of the isomorphism are zero. ☐

Remark 7.2. Theorem 7.1 shows that the derived residue obstruction layers behave coherently under localization of the valuation ring. Consequently, the invariants are not merely global filtration objects attached to a single valued field extension; they also admit a genuinely local interpretation compatible with the scheme-theoretic operations used throughout valuation theory and algebraic geometry.

Conceptually, the proposition identifies the modules

$$\mathrm{HRD}_i(n)$$

as local derived obstruction invariants. Passage to

$$S^{-1}\mathcal{O}_v$$

admits a localization behavior compatible with the valuation filtration; in particular, the derived obstruction theory localizes functorially, including the degenerate case in which $S^{-1}k_v = 0$.

This compatibility is essential for the later derived-category framework developed in Section 9, where valuation filtration objects are studied through local triangulated and derived operations.

7.2 Annihilator Layers

Definition 7.3. Define the annihilator ideal

$$\mathfrak{a}_n := \mathrm{Ann}_{\mathcal{O}_v}(\mathrm{HRD}_1(n)).$$

These ideals are not asserted to be classical valuation conductors; they are annihilator ideals attached to the first Tor obstruction layer.

Proposition 7.4 (Annihilator detection by derived residue layers)**.** *Let*

$$\mathfrak{a}_n := \mathrm{Ann}_{\mathcal{O}_v}(\mathrm{HRD}_1(n)).$$

Then the family $\{\mathfrak{a}_n\}_{n\geq 0}$ *detects stabilization of the first derived residue obstruction annihilators. More precisely, if there exists* $N \geq 0$ *such that*

$$\mathrm{HRD}_1(n) = 0 \qquad \textit{for all } n \geq N,$$

then

$$\mathfrak{a}_n = \mathcal{O}_v \qquad \textit{for all } n \geq N.$$

Conversely, if $\mathfrak{a}_n \neq \mathcal{O}_v$ *for infinitely many* n*, then the first derived residue obstruction does not stabilize to zero.*

Proof. By definition,

$$\mathfrak{a}_n = \mathrm{Ann}_{\mathcal{O}_v}(\mathrm{HRD}_1(n)).$$

If $\mathrm{HRD}_1(n) = 0$, then every element of $\mathcal{O}_v$ annihilates $\mathrm{HRD}_1(n)$. Hence

$$\mathfrak{a}_n = \mathrm{Ann}_{\mathcal{O}_v}(0) = \mathcal{O}_v.$$

Therefore, if there exists $N \geq 0$ such that

$$\mathrm{HRD}_1(n) = 0 \qquad (n \geq N),$$

then

$$\mathfrak{a}_n = \mathcal{O}_v \qquad (n \geq N).$$

Conversely, suppose that

$$\mathfrak{a}_n \neq \mathcal{O}_v$$

for infinitely many n. If $\mathrm{HRD}_1(n)$ vanished for all sufficiently large n, then the first part would imply

$$\mathfrak{a}_n = \mathcal{O}_v$$

for all sufficiently large n, contradicting the assumed infinite nontriviality of the annihilator ideals. Hence $\mathrm{HRD}_1(n)$ cannot stabilize to zero.

Thus the ideals $\mathfrak{a}_n$ record precisely whether the first derived residue obstruction has disappeared in the tail of the valuation filtration. □

Remark 7.5. The ideals $\mathfrak{a}_n$ should be viewed as annihilator shadows of the derived residue layers. When

$$\mathfrak{a}_n = \mathcal{O}_v,$$

the corresponding first obstruction layer has vanished. Persistent properness of $\mathfrak{a}_n$ therefore records persistent derived residue obstruction in the filtration.

Thus the annihilator filtration translates the derived information contained in

$$\mathrm{HRD}_1(n)$$

into ideal-theoretic data inside the valuation ring. No claim is made that these ideals are classical valuation conductors.

8 Examples and Explicit Computations

The examples in this section have two different roles. The Gauss, rank-two, and conic examples illustrate explicit filtration-layer computations, often through visible $\mathcal{O}_v/(\pi)$-torsion over a discrete valuation ring. These examples serve to make the derived obstruction mechanism concrete. They should not be interpreted as claiming that residual transcendence itself is exhausted by DVR torsion phenomena. The immediate-transcendental examples are boundary examples outside the main residually transcendental hypothesis.

8.1 Gauss Valuations

Example 8.1 (Gauss valuation as a torsion-mechanism model)**.** Let (K, v) be discretely valued with valuation ring $\mathcal{O}_v$, uniformizer π, and residue field $k_v = \mathcal{O}_v/(\pi)$ [14, Ch. I, §1, pp. 5–6]. In this example we restrict the valuation filtration to the polynomial subalgebra $K[x] \subset K(x)$. Let w be the Gauss valuation on $K[x]$,

$$w\left(\sum_i a_i x^i\right) = \min_i v(a_i).$$

(This is the classical Gauss-type extension used for polynomial subalgebras; compare [1, p. 147; Thm. 2(i), p. 151], where the residue field of the corresponding valuation is identified as $k_0(\bar{x})$.) Choose the filtration step $\gamma = v(\pi)$. Then

$$A_n = \{f \in K[x] : w(f) \geq n\gamma\} = \pi^n \mathcal{O}_v[x], \qquad A_{n+1} = \pi^{n+1}\mathcal{O}_v[x].$$

Hence

$$F_n = A_n/A_{n+1} \cong \pi^n \mathcal{O}_v[x]/\pi^{n+1}\mathcal{O}_v[x] \cong k_v[x]$$

as k_v-vector spaces, and as an $\mathcal{O}_v$-module F_n is killed by π. Using the free resolution

$$0 \longrightarrow \mathcal{O}_v \xrightarrow{\pi} \mathcal{O}_v \longrightarrow k_v \longrightarrow 0,$$

and tensoring with F_n, multiplication by π becomes zero. Therefore

$$\mathrm{Tor}_1^{\mathcal{O}_v}(F_n, k_v) \cong F_n \cong k_v[x], \qquad \mathrm{Tor}_i^{\mathcal{O}_v}(F_n, k_v) = 0 \quad (i \geq 2).$$

Thus

$$\mathrm{HRD}_1(n) \cong k_v[x], \qquad \mathrm{HRD}_i(n) = 0 \quad (i \geq 2).$$

This computation shows that even the classical Gauss filtration may have a nonzero first derived residue layer when the graded piece is viewed as an $\mathcal{O}_v$-module.

This does not contradict the vanishing philosophy of Theorem 5.5. In the present Gauss example the graded layer $F_n \cong k_v[x]$, when regarded as an $\mathcal{O}_v$-module, is annihilated by π. Hence F_n is not torsion-free, and therefore not flat, over $\mathcal{O}_v$. Thus the decisive flatness hypothesis in Theorem 5.5 fails. The example should therefore be read as showing that even a classically tame-looking Gauss filtration may carry a nonzero derived residue layer once its graded pieces are viewed over the base valuation ring.

Accordingly, the Gauss computation is not a counterexample to the vanishing theorem; it illustrates the boundary of that theorem by showing what happens when the torsion-free flatness hypothesis is deliberately absent.

8.2 Module-Theoretic Torsion Layers

Example 8.2 (A module-theoretic torsion layer)**.** Let (K, v) be discretely valued with uniformizer π, valuation ring $\mathcal{O}_v$, and residue field $k_v = \mathcal{O}_v/(\pi)$. Suppose that, for some filtration index n, the associated graded valuation layer contains a cyclic torsion direct summand

$$F_n \cong \mathcal{O}_v/(\pi) \oplus G_n$$

for an $\mathcal{O}_v$-module G_n. Then additivity of Tor gives

$$\mathrm{Tor}_1^{\mathcal{O}_v}(F_n, k_v) \cong \mathrm{Tor}_1^{\mathcal{O}_v}(\mathcal{O}_v/(\pi), k_v) \oplus \mathrm{Tor}_1^{\mathcal{O}_v}(G_n, k_v).$$

The standard resolution

$$0 \longrightarrow \mathcal{O}_v \xrightarrow{\pi} \mathcal{O}_v \longrightarrow \mathcal{O}_v/(\pi) \longrightarrow 0$$

yields

$$\mathrm{Tor}_1^{\mathcal{O}_v}(\mathcal{O}_v/(\pi), k_v) \cong k_v.$$

Consequently

$$\mathrm{HRD}_1(n) = \mathrm{Tor}_1^{\mathcal{O}_v}(F_n, k_v) \neq 0.$$

If such torsion direct summands occur for infinitely many n, then the first derived residue obstruction layer is nonzero for infinitely many filtration degrees.

This example is purely a module-theoretic torsion computation over $\mathcal{O}_v$. It does not use, and should not be read as relying on, any rank-two value-group structure.

8.3 Conic Function Fields

Example 8.3 (Conic degeneration producing a torsion subquotient)**.** Let (K, v) be discretely valued with uniformizer π, residue field k_v, and assume $\mathrm{char}(k_v) \neq 2$. By the standard diagonal normal form for conic function fields in characteristic different from 2, [5, Lem. 2.1, p. 472], consider the conic function field

$$L = K(x, y)/(y^2 - x^2 - \pi).$$

Equivalently,

$$(y - x)(y + x) = \pi.$$

Near the special fibre, put $u = y - x$ and $z = y + x$. Then the local relation becomes

$$uz = \pi.$$

Modulo π, this degenerates to

$$uz = 0,$$

so the special fibre has two components meeting along the locus $u = z = 0$. This geometric degeneration suggests the possible appearance of π-torsion in suitable valuation-filtration layers. For the homological conclusion used here, we impose the precise filtration-level hypothesis that, for some n, the layer F_n has a cyclic subquotient isomorphic to $\mathcal{O}_v/(\pi)$. Under this explicit hypothesis,

$$\mathrm{Tor}_1^{\mathcal{O}_v}(\mathcal{O}_v/(\pi), k_v) \cong k_v \neq 0.$$

Thus the conic degeneration gives a valid obstruction example only for those chosen valuations and filtration indices for which this cyclic subquotient condition is verified.

$$\text{No implication } uz = \pi \implies \exists\, \mathcal{O}_v/(\pi)\text{-subquotient of } F_n$$

is being asserted here. The conic degeneration is used only to motivate a natural geometric situation in which nonflat filtration layers may be looked for. The actual homological obstruction in this example is entirely conditional on the explicitly stated cyclic-subquotient hypothesis. Thus Example 8.3 is a mechanism example, not a proof that every such conic degeneration produces a torsion layer.

8.4 Immediate Transcendental Prolongations

The examples in this subsection are included only as boundary examples. They lie outside the residually transcendental hypothesis of Theorem 2.1, since an immediate prolongation has $k_w = k_v$. They are not used as examples of the central residually transcendental theory.

Proposition 8.4 (Torsion obstruction with immediate interpretation)**.** *Let (K, v) be a discretely valued field with valuation ring $\mathcal{O}_v$, uniformizer π, and residue field k_v. Let w be a prolongation of v to $K(x)$. Assume that, for infinitely many integers $n \geq 0$, the valuation layer*

$$F_n = A_n/A_{n+1}$$

contains a nonzero cyclic direct summand annihilated by π. Then

$$\mathrm{HRD}_1(n) \neq 0$$

for infinitely many n.

If, in addition, the prolongation w/v is immediate, so that

$$\Gamma_w = \Gamma_v \qquad \text{and} \qquad k_w = k_v,$$

then this nonvanishing occurs although the classical value-group and residue-field invariants remain unchanged.

Proof. The homological assertion uses only the displayed torsion hypothesis on the valuation layers. Let n be one of the infinitely many integers appearing in the hypothesis. Then F_n contains a nonzero cyclic direct summand annihilated by π. Since $\mathcal{O}_v$ is a discrete valuation ring with uniformizer π, such a summand is isomorphic to

$$\mathcal{O}_v/(\pi).$$

Hence there exists an $\mathcal{O}_v$-module decomposition

$$F_n \cong \mathcal{O}_v/(\pi) \oplus G_n$$

for some $\mathcal{O}_v$-module G_n. Additivity of Tor in the first variable gives

$$\mathrm{Tor}_1^{\mathcal{O}_v}(F_n, k_v) \cong \mathrm{Tor}_1^{\mathcal{O}_v}(\mathcal{O}_v/(\pi), k_v) \oplus \mathrm{Tor}_1^{\mathcal{O}_v}(G_n, k_v).$$

Using the standard resolution

$$0 \longrightarrow \mathcal{O}_v \xrightarrow{\pi} \mathcal{O}_v \longrightarrow \mathcal{O}_v/(\pi) \longrightarrow 0,$$

and tensoring with $k_v = \mathcal{O}_v/(\pi)$, multiplication by π becomes zero. Therefore

$$\mathrm{Tor}_1^{\mathcal{O}_v}(\mathcal{O}_v/(\pi), k_v) \cong k_v \neq 0.$$

Thus

$$\mathrm{HRD}_1(n) = \mathrm{Tor}_1^{\mathcal{O}_v}(F_n, k_v) \neq 0$$

for every n in the infinite set.

If the prolongation is furthermore immediate, then $\Gamma_w = \Gamma_v$ and $k_w = k_v$. Hence the same Tor nonvanishing is invisible to the classical value-group and residue-field invariants. This final immediate-extension statement is an interpretation of the torsion computation, not an additional input in the Tor calculation. ☐

Remark 8.5. The proposition is deliberately formulated so that the mathematical input is the torsion summand in F_n. The immediate-transcendental hypothesis is used only to explain why this derived obstruction is not seen by the classical invariants Γ_w/Γ_v and k_w/k_v. Thus the result should be read as a torsion-layer obstruction with an immediate-extension interpretation, not as a theorem whose Tor nonvanishing is caused by immediacy itself.

Proposition 8.6 (Euler-characteristic instability in immediate extensions)**.** *Assume in addition to the immediate case discussed after Theorem 8.4 that assume that for some n the layer F_n contains a direct summand $(\mathcal{O}_v/(\pi))^{\oplus r_n}$ with $r_n > 0$. Then this torsion summand contributes nontrivially to the derived residue complex and produces*

$$\dim_{k_v} \mathrm{HRD}_1(n) \geq r_n.$$

Consequently, the Euler characteristic

$$\chi_n = \sum_i (-1)^i \dim_{k_v} \mathrm{HRD}_i(n)$$

is sensitive to immediate-extension torsion which is invisible to the classical value-group and residue-field invariants.

Proof. Fix n as in the hypothesis. Since F_n contains $(\mathcal{O}_v/(\pi))^{\oplus r_n}$ as a direct summand, there exists an $\mathcal{O}_v$-module G_n such that

$$F_n \cong (\mathcal{O}_v/(\pi))^{\oplus r_n} \oplus G_n.$$

Applying $\mathrm{Tor}_1^{\mathcal{O}_v}(-, k_v)$ and using additivity of Tor in the first variable gives

$$\mathrm{Tor}_1^{\mathcal{O}_v}(F_n, k_v) \cong \mathrm{Tor}_1^{\mathcal{O}_v}((\mathcal{O}_v/(\pi))^{\oplus r_n}, k_v) \oplus \mathrm{Tor}_1^{\mathcal{O}_v}(G_n, k_v).$$

Again by additivity,

$$\mathrm{Tor}_1^{\mathcal{O}_v}((\mathcal{O}_v/(\pi))^{\oplus r_n}, k_v) \cong \left(\mathrm{Tor}_1^{\mathcal{O}_v}(\mathcal{O}_v/(\pi), k_v)\right)^{\oplus r_n}.$$

Now use the standard free resolution

$$0 \longrightarrow \mathcal{O}_v \xrightarrow{\pi} \mathcal{O}_v \longrightarrow \mathcal{O}_v/(\pi) \longrightarrow 0.$$

Tensoring with $k_v = \mathcal{O}_v/(\pi)$ turns multiplication by π into the zero map, and therefore

$$\operatorname{Tor}_1^{\mathcal{O}_v}(\mathcal{O}_v/(\pi), k_v) \cong k_v.$$

Hence

$$\operatorname{Tor}_1^{\mathcal{O}_v}((\mathcal{O}_v/(\pi))^{\oplus r_n}, k_v) \cong k_v^{\oplus r_n}.$$

Therefore $k_v^{\oplus r_n}$ occurs as a direct summand of $\operatorname{Tor}_1^{\mathcal{O}_v}(F_n, k_v)$. Using the definition

$$\mathrm{HRD}_1(n) = \operatorname{Tor}_1^{\mathcal{O}_v}(F_n, k_v),$$

we obtain

$$\dim_{k_v} \mathrm{HRD}_1(n) \geq r_n.$$

It remains to relate this computation to the Euler characteristic. By Definition 5.9, whenever the derived residue layers are finite-dimensional and vanish in sufficiently high degree, one has

$$\chi_n = \sum_i (-1)^i \dim_{k_v} \mathrm{HRD}_i(n).$$

The r_n independent k_v-directions constructed above occur in degree 1. Consequently they contribute at least

$$-r_n$$

to the alternating sum defining χ_n, up to possible additional contributions from the remaining summand G_n and from degree zero.

Since the prolongation is immediate, the classical invariants satisfy

$$\Gamma_w = \Gamma_v \qquad \text{and} \qquad k_w = k_v.$$

Thus the value group and residue field do not record the torsion summand $(\mathcal{O}_v/(\pi))^{\oplus r_n}$. The Euler characteristic, however, is computed from the derived residue complex

$$F_n \otimes^{\mathbf{L}}_{\mathcal{O}_v} k_v,$$

and therefore detects the nonflat π-torsion inside F_n. This proves that χ_n is sensitive to immediate-extension torsion which is invisible to the classical value-group and residue-field invariants. □

9 A Triangulated Framework for Valuation Filtrations

Definition 9.1 (Derived valuation-filtration category)**.** Let (K, v) be a valued field and let w be a prolongation of v to $K(x)$. Let $\gamma \in \Gamma_v \subseteq \Gamma_w$ be the fixed positive filtration step chosen above, and let

$$A_n = \{f \in K(x) : w(f) \geq n\gamma\}, \qquad F_n = A_n/A_{n+1}.$$

Define

$$\mathcal{D}_{\mathrm{val}}(K(x), w)$$

to be the smallest full triangulated subcategory of $\mathcal{D}(\mathcal{O}_v)$ containing the valuation filtration layers F_n, equivalently the complexes arising from the short exact filtration steps

$$0 \longrightarrow A_{n+1} \longrightarrow A_n \longrightarrow F_n \longrightarrow 0$$

for all $n \geq 0$, and closed under shifts, cones, and finite direct sums.

Remark 9.2 (Valuation dependence of the category)**.** The notation $\mathcal{D}_{\text{val}}(K(x), w)$ does not mean that a new intrinsic valuation category has already been constructed independently of the chosen filtration. In this paper the valuation enters through the prolongation w, the chosen step γ, the filtration

$$A_n = \{f \in K(x) : w(f) \geq n\gamma\},$$

and the associated layers $F_n = A_n/A_{n+1}$. Thus $\mathcal{D}_{\text{val}}(K(x), w)$ is the triangulated envelope of these valuation-produced filtration objects.

Accordingly, the category should be viewed as an organizational derived framework for the modules already constructed from the valuation filtration, not as an independently defined valuation-theoretic invariant. The invariant content presently lies in the derived residue homology

$$H_i(F_n \otimes^{\mathbf{L}}_{\mathcal{O}_v} k_v) = \operatorname{Tor}_i^{\mathcal{O}_v}(F_n, k_v),$$

together with the filtration triangles relating the adjacent valuation layers.

Remark 9.3 (Conditional extension-of-scalars compatibility)**.** Let

$$\phi : (K(x), w) \longrightarrow (L(y), u)$$

be a morphism of valued prolongations preserving the chosen valuation filtration. Assume moreover that, for every $n \geq 0$,

$$F_n \otimes^{\mathbf{L}}_{O_v} O_u \in \langle F'_m : m \geq 0 \rangle_{\text{tri}}.$$

Then the derived extension-of-scalars functor

$$- \otimes^{\mathbf{L}}_{O_v} O_u : D(O_v) \longrightarrow D(O_u)$$

sends the triangulated subcategory generated by the layers F_n into the triangulated subcategory generated by the layers F'_m.

Indeed, $D_{\text{val}}(K(x), w)$ is by definition generated from the objects F_n by shifts, cones, and finite direct sums. Since $- \otimes^{\mathbf{L}}_{O_v} O_u$ is an exact triangulated functor, it preserves these operations. The stated containment of the images of the generators therefore implies the claimed containment for the whole valuation-generated triangulated subcategory. Thus this is an organizational compatibility statement, not an independent invariance theorem.

Proposition 9.4 (Quasi-isomorphic filtrations give identical HRD layers)**.** *Let two valuation filtrations determine objects C_n and C'_n of $\mathcal{D}_{\text{val}}(K(x), w)$. If C_n and C'_n are quasi-isomorphic in $\mathcal{D}(\mathcal{O}_v)$, then*

$$H_i(C_n \otimes^{\mathbf{L}}_{\mathcal{O}_v} k_v) \cong H_i(C'_n \otimes^{\mathbf{L}}_{\mathcal{O}_v} k_v)$$

for all i. In particular, quasi-isomorphic valuation filtrations produce identical derived residue obstruction modules.

Proof. Assume that C_n and C'_n are quasi-isomorphic in $\mathcal{D}(\mathcal{O}_v)$. Equivalently, they represent the same object of the derived category up to isomorphism:

$$C_n \simeq C'_n \qquad \text{in } \mathcal{D}(\mathcal{O}_v).$$

The derived residue functor

$$- \otimes^{\mathbf{L}}_{\mathcal{O}_v} k_v : \mathcal{D}(\mathcal{O}_v) \longrightarrow \mathcal{D}(k_v)$$

is a well-defined derived functor and therefore preserves isomorphisms in the derived category. Applying it to the quasi-isomorphism above gives an isomorphism

$$C_n \otimes^{\mathbf{L}}_{\mathcal{O}_v} k_v \simeq C'_n \otimes^{\mathbf{L}}_{\mathcal{O}_v} k_v \qquad \text{in } \mathcal{D}(k_v).$$

Taking homology in degree i yields

$$H_i(C_n \otimes^{\mathbf{L}}_{\mathcal{O}_v} k_v) \cong H_i(C'_n \otimes^{\mathbf{L}}_{\mathcal{O}_v} k_v)$$

for every i.

In particular, when C_n is represented by the valuation layer

$$F_n = A_n/A_{n+1},$$

the derived residue homology is exactly

$$H_i(F_n \otimes^{\mathbf{L}}_{\mathcal{O}_v} k_v) = \operatorname{Tor}_i^{\mathcal{O}_v}(F_n, k_v) = \mathrm{HRD}_i(n).$$

Thus quasi-isomorphic filtration objects have identical derived residue homology, and hence produce the same derived residue obstruction modules. ☐

Remark 9.5. Thus Theorem 11.1 is no longer left entirely open in its most basic form: the present section constructs a first derived-category framework in which the modules $\mathrm{HRD}_i(n)$ arise as derived residue homology of valuation-filtration objects. The remaining open problem is to determine whether this category is invariant under broader equivalences of valued function fields.

10 Future Directions

Several directions emerge naturally from the present framework:

1. Derived annihilator cohomology.
2. Higher spectral sequences associated to multi-step valuation towers and iterated conductor filtrations.
3. Connections with graded Rees algebras.
4. Derived residue obstruction growth functions.
5. Invariance and comparison theorems for the derived valuation-filtration category $\mathcal{D}_{\mathrm{val}}(K(x), w)$.
6. Applications to nonarchimedean geometry.
7. Stability of derived residue obstruction layers under localization and completion.

11 Open Problems

Question 11.1. Is the triangulated envelope $\mathcal{D}_{\mathrm{val}}(K(x), w)$, constructed from the chosen valuation filtration layers F_n, invariant under natural changes of presentation of the valued function field or of the compatible filtration data? Under what additional hypotheses does it become a genuine invariant of valued function fields rather than only of the chosen valuation filtration?

Question 11.2. Does asymptotic vanishing of

$$\mathrm{HRD}_i(n)$$

characterize defectless residual transcendence?

Question 11.3. Can these invariants be connected to Berkovich skeleta or tropical structures?

12 Conclusion

This paper proposes a homological framework for studying valuation filtrations in residually transcendental valued function fields. The main contribution is the introduction of derived residue obstruction layers capturing layerwise nonflatness and torsion phenomena which may be invisible to value-group and residue-field invariants alone.

The resulting theory opens a possible bridge between valuation theory, derived commutative algebra, and derived residue obstruction theory.

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