Inverse spectral problem for the Hill operator on the graph with a loop

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Abstract

In this paper, we investigate a generalization of the classical a PT-symmetric Hill operator to lasso graph. The definition of the PT-symmetric Hill operator on lasso graph is given and derived its spectral properties. We solved the inverse problem, proved the uniqueness theorem and provided a constructive procedure for the solution of the inverse problem.

Key words: Lasso graphs, Inverse spectral problems, PT-symmetric Hill Operator, Reflection coefficient

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1 Introduction

The main purpose of the present work is to solve the inverse problem for the PT-symmetric Hill operator on the lasso graph. By lasso graph, half-line attached to a loop is to be understood.

Since the quantum graphs are the corresponding generalization of Feynman diagrams that provide for this theory an equally procedure for calculation, our aim here will be to investigate one more solvable model of this type.

Let there be given the non-compact graph G where an edge is attached to a loop. The non-compact part of the graph is a ray $\gamma_0 = \{x| \, 0 < x < \infty\}$, compact part is the loop $\gamma_1 = \{z| \, 0 < z < 2\pi\}$ whose length we shall, to be definite, take equal to 2π and with $\gamma_2 = \{\{x = 0\} = \{z = 0\} = \{z = 2\pi\}\}$ corresponding to the attachment point. We investigate the spectral problem describing the one-dimensional scattering of a quantum particle on G.

$$\begin{array}{l} -Y'' + \{q(X) - \lambda^2\}Y = 0, \qquad X \in G\backslash\{\gamma_2\} \\ Y(x = 0) = Y(z = 0) = Y(z = 2\pi), \\ Y'(x = 0 + 0) + Y'(z = 0 + 0) - Y'(z = 2\pi - 0) = 0 \end{array} \tag{1}$$

In (1) differentiation with respect to the variable X is understood as differentiation with respect to x, when $X \in \gamma_0$, and as differentiation with respect to z, when $X \in \gamma_1$. Differentiation is not defined at the vertices.

We assume that the potential

$$q(X) = \begin{cases} q_1(x) = \sum\limits_{n=1}^{\infty} q_{1n} e^{inx}, & X \in \gamma_0 \\ q_2(z) = \sum\limits_{n=1}^{\infty} q_{2n} e^{inz}, & X \in \gamma_1 \end{cases} \tag{2}$$

is defined as complex valued function on the G with $\sum_{n=1}^{\infty} |q_{kn}| < \infty, k = 1, 2;$ and λ is a spectral parameter.

Then the resulting Hill operator will be

$$Y''(X) + q(X)Y(X), X \in G.$$

More precisely, on the Hilbert space $L_2(G)$ with norm

$$\|f\|_{L_2(G)} = \{\|f\|^2_{L_2(\gamma_0)} + \|f\|^2_{L_2(\gamma_1)}\}^{1/2}$$

we introduce the operator L with domain

$$D(L) = \left\{ Y(X), X \in G| \ ; \begin{array}{c} Y(X) \in H^2(\gamma_0) \cup H^2(\gamma_1) \\ Y(x=0) = Y(z=0) = Y(z=2\pi) \\ Y'(x=0+0) + Y'(z=0+0) - Y'(z=2\pi-0) = 0 \end{array} \right\}$$

where $H^k(k = 1, 2, ..)$ are the usual Sobolev spaces.

The potentials considered in the paper have the form

$$q(x) = \sum_{n=1}^{\infty} q_n e^{inx}$$

where, in particular, for the numbers $q_n = \overline{q_n}$ the potential will be PT-symmetric, i.e. $q(x) = \overline{q(-x)}$.

Spectral analysis of operator with the potential of the type (2) firstly has been studied by M.G.Gasymov [2], where he proved the existence of a solution $f(x, \lambda)$ of the equation

$$-y''(x) + q(x)y(x) = \lambda^2 y(x)$$

in $L_2(-\infty, +\infty)$ of the form

$$f(x, \lambda) = e^{i\lambda x} \left(1 + \sum_{n=1}^{\infty} \frac{1}{n + 2\lambda} \sum_{\alpha=n}^{\infty} V_{n\alpha} e^{i\alpha x} \right)$$

where the series

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\alpha=n+1}^{\infty} \alpha(\alpha - n)|V_{n\alpha}|; \quad \sum_{n=1}^{\infty} n|V_{nn}|$$

converge.

He also discussed the corresponding inverse spectral problem of finding the potential $q(x)$ for given so-called "normalizing" numbers V_{nn} where the key role played relation

$$\lim_{\lambda \to \frac{n}{2}} (n - 2\lambda) f(x, -\lambda) = V_{nn} f\left(x, \frac{n}{2}\right) \tag{3}$$

As a final remark relating to the potential of the type (2), we mention some works K.Shin [5], R.Carlson[6,7],Guillemin and V., Uribe A[8], L.Pastur and V. Tkachenko[9] and [10,11,12,13,14]. More information about the potentials can be found in [14].

Let us to mention some results closely related to ours.

Without a claim of completeness of investigation of inverse problems on graphs with loop here are listed the works of Akhyamov A.M, Trooshin I.Y [1], Gomilko A.M. and Pivovarchik V.N[3], Exner P. [15], Yang, Chuan-Fu [16], Berkolaiko G.[17], Kurasov P [18], Mochizuki K. and Trooshin I.Yu. [19] .

Moreover, the potential on graphs with loop (including the potential on loop edge) can be constructed by reflection coefficients and two spectra. In order to solve the inverse problem effective algorithm is given.

Let us review briefly the contents of the paper. The Hamiltonian of the model that is introduced in Section 1. Next, its spectral properties are derived in Section 2. In Section 3 we give a formulation of the inverse problem, prove the uniqueness theorem and provide a constructive procedure for the solution of the inverse problem.

2 General solution

In the present paper, the inverse spectral problem for the operator L is investigated . Here we will study the fundamental solutions of the main equation

$$-Y'' + \{q(X) - \lambda^2\}Y = 0 \tag{4}$$

in γ_0 and γ_1 respectively.

Theorem 1: Let $q(X)$ be in the form (2) and $\sum\limits_{n=1}^{\infty} |q_{nk}| < \infty;\ k = 1,2$ converge. Then the equation (4) has on γ_0 linearly independent solutions of the form

$$f(x, \pm\lambda) = e^{i\lambda x}(1 + \sum_{n=1}^{\infty} \frac{1}{n \pm 2\lambda} \sum_{\alpha=n}^{\infty} V_{n\alpha}^{\gamma_0} e^{i\alpha x}), \tag{5}$$

where the numbers $V_{n\alpha}^{\gamma_0}$ are determined by the following recurrent relations

$$\begin{cases} \alpha(\alpha - n)V_{n\alpha}^{\gamma_0} + \sum\limits_{s=n}^{\alpha-1} q_{1\alpha-s}V_{ns}^{\gamma_0} = 0, & 1 \le n < \alpha \\ \alpha \sum\limits_{n=1}^{\alpha} V_{n\alpha}^{\gamma_0} + q_{1\alpha} = 0; \end{cases} \tag{6}$$

the series

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\alpha=n+1}^{\infty} \alpha\,(\alpha - n)\,|V_{n\alpha}^{\gamma_0}|; \quad \sum_{n=1}^{\infty} n\,|V_{nn}^{\gamma_0}|$$

are converged and fulfilled the relation

$$\lim_{\lambda \to \mp \frac{n}{2}} (n \pm 2\lambda) f(x, \pm\lambda) = V_{nn}^{\gamma_0} f(x, \mp \frac{n}{2}) \tag{7}$$

or that the same

$$V_{m\alpha+m}^{\gamma_0} = V_{mm}^{\gamma_0} \sum_{n=1}^{\alpha} \frac{V_{nn}^{\gamma_0}}{n+m}, \quad \alpha = 1, 2, ...; \tag{8}$$

The proof of the theorem is similar to that of [2] and therefore we do not cite it here.

The next stage is to investigation of the equation (4) on loop γ_1.

Let $\varphi(z, \lambda), \theta(z, \lambda)$ be linear independent solutions of equation (4) on the loop γ_1 , satisfying the initial conditions

$$\begin{array}{l} \varphi(0, \lambda) = \theta'(0, \lambda) = 1 \\ \varphi'(0, \lambda) = \theta(0, \lambda) = 0 \end{array}.$$

Note that their Wronskian is $W[\varphi(z, \lambda), \theta(z, \lambda)] = 1$.

Then, any solution of equation (4) on the loop γ_1 can be represented as

$$u(z, \lambda) = A(\lambda)\varphi(z, \lambda) + B(\lambda)\theta(z, \lambda).$$

We will attempt to find $u(z, \lambda)$ via the Green function of the equation (4) on γ_1.

The Green's function on the loop can be constructed by means of the fundamental solutions $\varphi(z, \lambda), \theta(z, \lambda)$.

The boundary conditions for the $G(z, t, \lambda)$ are

$$\begin{array}{l} G(0, t, \lambda) = G(2\pi, t, \lambda) \\ G'(0, t, \lambda) = G'(2\pi, t, \lambda) \\ G'_z(t+0, t, \lambda) - G'_z(t-0, t, \lambda) = -1 \\ \lim\limits_{z \to t+0} G(z, t, \lambda) = \lim\limits_{z \to t-0} G(z, t, \lambda) \end{array} \tag{9}$$

Then by virtue of (9), we have

$$\begin{array}{rl} G(z, t, \lambda) = & \frac{\theta(t,\lambda)+\varphi(t,\lambda)\theta(2\pi,\lambda)-\theta(t,\lambda)\varphi(2\pi,\lambda)}{\varphi(2\pi,\lambda)+\theta'(2\pi,\lambda)-2} \varphi(z, \lambda) + \\ & + \frac{\varphi(t,\lambda)\theta'(2\pi,\lambda)-\theta(t,\lambda)\varphi'(2\pi,\lambda)-\varphi(t,\lambda)}{\varphi(2\pi,\lambda)+\theta'(2\pi,\lambda)-2} \theta(z, \lambda), \quad t \geq z \end{array}$$

$$\begin{array}{rl} G(z, t, \lambda) = & \frac{\varphi(t,\lambda)\theta(2\pi,\lambda)+\theta'(2\pi,\lambda)\varphi(t,\lambda)-\theta(,\lambda)}{\varphi(2\pi,\lambda)+\theta'(2\pi,\lambda)-2} \varphi(z, \lambda) + \\ & + \frac{\varphi(y,\lambda)-\varphi(y,\lambda)\varphi(2\pi,\lambda)-\theta(y,\lambda)\varphi'(2\pi,\lambda)}{\varphi(2\pi,\lambda)+\theta'(2\pi,\lambda)-2} \theta(z, \lambda), \quad t \leq z \end{array}$$

Then easy to see that the function

$$\begin{array}{rl} G(z, 0, \lambda) = G(z, 2\pi, \lambda) = & \frac{\theta(2\pi,\lambda)}{\varphi(2\pi,\lambda)+\theta'(2\pi,\lambda)-2} \varphi(z, \lambda) + \\ & + \frac{1-\varphi(2\pi,\lambda)}{\varphi(2\pi,\lambda)+\theta'(2\pi,\lambda)-2} \theta(z, \lambda). \end{array} \tag{10}$$

is a solution of the equation equation (4) on the loop γ_1 up to constant . So, we can take

$$u(z,\lambda) = \alpha G(z,0,\lambda) \tag{11}$$

where α is constant.

Theorem 2: For any real $\lambda \neq 0$ there exists a solution

$$Y(X,\lambda) = \begin{cases} y(x,\lambda), & X \in \gamma_0 \\ u(z,\lambda), & X \in \gamma_1 \end{cases}$$

of the problem (1-2) which is represented as follows;
On γ_0 we have

$$y(x,\lambda) = f(x,-\lambda) + R_{11}(\lambda) f(x,\lambda) \tag{12}$$

here $R_{11}(\lambda)$ is a reflection coefficient and

$$R_{11}(\lambda) = \frac{\alpha - f(0,-\lambda)}{f(0,\lambda)}. \tag{13}$$

On the loop γ_1 we have the solution of the form (11) or

$$u(z,\lambda) = \alpha\left(\frac{\theta\,(2\pi,\lambda)}{\varphi\,(2\pi,\lambda) + \theta'\,(2\pi,\lambda) - 2}\varphi\,(z,\lambda) + \frac{1 - \varphi\,(2\pi,\lambda)}{\varphi\,(2\pi,\lambda) + \theta'\,(2\pi,\lambda) - 2}\theta(z,\lambda)\right).$$

Proof: For any real $\lambda \neq 0$ the Wronskian of the functions $f(x,\lambda), f(x,-\lambda)$ is

$$W[f(x,\lambda), f(x,-\lambda)] = f(x,\lambda) f'(x,-\lambda) - f'(x,\lambda) f(x,-\lambda) = 2i\lambda$$

this implies that these two functions form a fundamental system of solutions of the equation (4) in γ_0, and thus, if $y(x,\lambda)$ satisfies (4) for any real $\lambda \neq 0$, then we have some constants $C(\lambda)$, $D(\lambda)$

$$y(x,\lambda) = C(\lambda) f(x,\lambda) + D(\lambda) f(x,-\lambda)$$

On the other hand, any solution of equation (4) on the loop γ_1 can be represented as (11).

Then we can seek the solution to the spectral problem on the whole graph in the form

$$Y(X,\lambda) = \begin{cases} f(x,-\lambda) + R_{11}(\lambda) f(x,-\lambda), & X \in \gamma_0 \\ \alpha G(z,0,\lambda), & X \in \gamma_1 \end{cases} \tag{14}$$

Let us find $R_{11}\lambda)$ on such way that the solution in the form (10) would be satisfy boundary conditions in (1)

Then from the the boundary conditions in (1) we have

$$f\,(0,-\lambda)+R_{11}(\lambda)f\,(0,\lambda)=\alpha G(0,0,\lambda)=\alpha G(2\pi,0,\lambda)$$
$$f'\,(0,-\lambda)+R_{11}(\lambda)f'\,(0,\lambda)+\alpha[G'_z(0+0,0,\lambda)-G'_z(0-0,0,\lambda])=0.$$

Taking into account the boundary conditions (9) for the Green function on the loop, we obtain

$$\begin{array}{l} f\,(0,-\lambda)+R_{11}(\lambda)f\,(0,\lambda)=\alpha G(0,0,\lambda) \\ f'\,(0,-\lambda)+R_{11}(\lambda)f'\,(0,\lambda)=\alpha, \end{array} \tag{15}$$

thus

$$f\,(0,-\lambda)+R_{11}(\lambda)f\,(0,\lambda)=[f'\,(0,-\lambda)+R_{11}(\lambda)f'\,(0,\lambda)]G(0,0,\lambda).$$

So, we have the following relations that will be used in future

$$G(0,0,\lambda)=\frac{f(0,-\lambda)+R_{11}(\lambda)f(0,\lambda)}{f'(0,-\lambda)+R_{11}(\lambda)f'(0,\lambda)} \tag{16}$$

and

$$R_{11}\,(\lambda)=\frac{\alpha-f'(0,-\lambda)}{f'(0,\lambda)}=-\frac{f(0,-\lambda)-G(0,0,\lambda)f'(0,-\lambda)}{f(0,\lambda)-G(0,0,\lambda)f'(0,\lambda)} \tag{17}$$

The theorem is proved.

3 The Inverse Spectral Problem On Lasso Graph

If the graph has at least one loop, then the potential on the loop cannot be reconstructed using local methods: calculation of the potential requires consideration of the whole loop at once.

The main idea of the solution of the inverse problem for the considered system is its reduction of two independent problems of reconstruction of the potential $q(X)=[q_1(x),q_2(z)]$, to recover $q_1(x)$ on the edge γ_0 and to recover $q_2(z)$ on the edge γ_1. Since the coefficients $R_{11}(\lambda)$ can be found by using matching conditions

$$y\,(0)=u\,(0)=u\,(2\pi)$$

and

$$y'\,(0+0)+u'\,(0+0)-u'\,(2\pi-0)=0$$

at the central vertex, it is natural to formulate inverse problem - recovering of the potential $q(X)$ at non-compact graph G by reflection coefficient, the set of eigenvalues of Dirichlet problems

$$\begin{array}{l} -u''(z,\lambda)+q_2(z)u(z,\lambda)=\lambda^2u(z,\lambda), \quad z\in[0,2\pi] \\ u(0,\lambda)=u(2\pi,\lambda)=0 \end{array} \tag{18}$$

and the spectrum of Neumann boundary value problem

$$
\begin{aligned}
&-u''(z,\lambda)+q_2(z)u(z,\lambda)=\lambda^2u(z,\lambda),\;\; z\in[0,2\pi]\\
&u'(0,\lambda)=u'(2\pi,\lambda)=0
\end{aligned}
\tag{19}
$$

Inverse problem: Given the spectral data: The spectrum of the Dirichlet problem (18), the spectrum of Neumann boundary value problem (19) and reflection coefficient $R_{11}(\lambda)$, construct the potential $q(X)$.

Lemma 1: All numbers $V_{nn}^{\gamma_0}$ can be determined by specifying the reflection coefficients $R_{11}(\lambda)$ as

$$
\lim_{\lambda\to\frac{n}{2}}(n-2\lambda)\,R_{11}(\lambda)=-V_{nn}^{\gamma_0}
$$

Proof :

Indeed , from the relation (17),

$$
R_{11}\left(\lambda\right)=\frac{\alpha-f'(0,-\lambda)}{f'(0,\lambda)}
$$

then by using (7), we obtain

$$
\begin{aligned}
&\lim_{\lambda\to\frac{n}{2}}(n-2\lambda)R_{11}\left(\lambda\right)=\lim_{\lambda\to\frac{n}{2}}(n-2\lambda)\tfrac{\alpha-f'(0,-\lambda)}{f'(0,\lambda)}=\\
&=\lim_{\lambda\to\frac{n}{2}}\tfrac{(n-2\lambda)\alpha-(n-2\lambda)f'(0,-\lambda)}{f'(0,\lambda)}=-\lim_{\lambda\to\frac{n}{2}}\tfrac{(n-2\lambda)f'(0,-\lambda)}{f'(0,\lambda)}=-V_{nn}^{\gamma_0}
\end{aligned}
$$

.

From the identity (7) we have

$$
V_{m,\alpha+m}^{\gamma_0}=V_{m,m}^{\gamma_0}\sum_{n=1}^{\alpha}\frac{V_{n,\alpha}^{\gamma_0}}{n+m},\quad \alpha=1,2,..
$$

These relations are fundamental equations for defining q_{1n} from $V_{nn}^{\gamma_0}$. In fact, if $V_{nn}^{\gamma_0}$ are known, then (4) give recurrent formulas for defining $V_{n\alpha}^{\gamma_0}$. Thus, for the numbers $V_{nn}^{\gamma_0}$, the function $q_1(x)$ may be reconstructed uniquely and effectively.

Theorem 3: The specification of spectral data uniquely determines the potential $q(X)$.

Proof:

All numbers q_{1n} can be determined from (4) by using the "normalizing" numbers $V_{nn}^{\gamma_0}$ and the potential $q_1(x)$ may be reconstructed as the above showed algorithm uniquely and effectively on the edge γ_0.

Since specifying numbers $V_{nn}^{\gamma_0}$ makes possible to construct a function $f\left(x,\lambda\right)$, then knowing the reflection coefficient $R_{11}(k)$, we can find values of the spectral parameter λ that are roots of the equation

$$
f'\left(0,-\lambda\right)+R_{11}(\lambda)f'\left(0,\lambda\right)=0. \tag{20}
$$

Then from (14) we directly see that for these λ , $\alpha=0$ and from (11) obtain that the solution to the spectral problem on the loop must satisfy the boundary conditions

$$u(0,\lambda) = u(2\pi,\lambda) = 0$$
$$u'(0,\lambda) = u'(2\pi,\lambda) = 0$$

Now we will consider the problem of reconstruction of the potential $q_2(z)$ on the loop γ_1. As initial data, we take the sequences $\{\lambda_n\}$ and $\{\mu_n\}$ where the first of which coincides with eigenvalues of the spectral problem (18) and the second determines the eigenvalue of problem (19). Then it is noticeable that the $\{\lambda_n\}$ and $\{\mu_n\}$ coincide with the zeros of

$$\Phi_1(\lambda) = \theta(2\pi,\lambda)$$
$$\Phi_2(\mu) = \varphi'(2\pi,\mu)$$

and the functions $\theta(2\pi,\lambda)$ and $\varphi'(2\pi,\mu)$ can be recovered by using $\{\lambda_n\}$ - eigenvalues of the spectral problem (18) and $\{\mu_n\}$-the eigenvalue of problem (19) respectively.

Let us introduce the function

$$S(\lambda) = \frac{g'(0,\lambda) + i\lambda g(0,\lambda)}{g'(0,-\lambda) + i\lambda g(0,-\lambda)}$$

which will play an important role in solving the inverse problem on the loop.

Here the function $g(z,\lambda)$ is a solution of the problem

$$-u''(z,\lambda) + q(z)u(z,\lambda) = \lambda^2 u(z,\lambda) \qquad (21)$$

in the space $L_2[0,\infty)$ with the potential $q(z)$

$$q(z) = \begin{cases} q_2(z) & on\ z \in [0,2\pi] \\ 0 & on\quad z > 2\pi \end{cases}$$

with the boundary condition $u'(0) = 0$ and for that fulfilling the condition

$$\lim_{\mathrm{Im}z\to\infty} g(z,\lambda)e^{-i\lambda z} = 1.$$

Then, from [1] follow that , $g(z,\lambda)$ can be represented as

$$g(z,\lambda) = \begin{cases} \tilde{f}(z,\lambda) & on\ z \in [0,2\pi] \\ e^{i\lambda z} & on\quad z > 2\pi \end{cases}$$

where

$$\tilde{f}(z,\lambda) = e^{i\lambda z}\Big(1 + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty} \frac{V_{n\alpha}^{\gamma_1}}{n+2\lambda} e^{i\alpha z}\Big).$$

The numbers $V_{n\alpha}^{\gamma_1}$ are determined by the following recurrent relations

$$\begin{cases} \alpha(\alpha-n)V_{n\alpha}^{\gamma_1} + \sum\limits_{s=n}^{\alpha-1} q_{2\alpha-s}V_{ns}^{\gamma_1} = 0, \qquad 1 \le n < \alpha \\ \alpha \sum\limits_{n=1}^{\alpha} V_{n\alpha}^{\gamma_1} + q_{2\alpha} = 0; \end{cases} \qquad (22)$$

where the series

$$\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\alpha=n+1}^{\infty}\alpha\,(\alpha-n)\,|V_{n\alpha}^{\gamma_1}|;\quad \sum_{n=1}^{\infty}n\,|V_{nn}^{\gamma_1}|$$

are converged and the relation

$$\lim_{\lambda\to\mp\frac{n}{2}}(n\pm 2\lambda)\tilde{f}(z,\pm\lambda)=V_{nn}^{\gamma_1}\tilde{f}(z,\mp\frac{n}{2}) \tag{23}$$

is fulfilled.

Lemma 2: For the solution $g(x,\lambda)$ of the equation (12), the relation

$$g(z,\lambda)=e^{2i\lambda\pi}[\theta'(2\pi,\lambda)-i\lambda\theta(2\pi,\lambda)]\varphi(z,\lambda)+e^{2i\lambda\pi}[i\lambda\varphi(2\pi,\lambda)-\varphi'(2\pi,\lambda)]\theta(z,\lambda)$$

is fulfilled.

From the Lemma 2 we have

$$\begin{array}{l} g(0,\lambda)=e^{2i\lambda\pi}[\theta'(2\pi,\lambda)-i\lambda\theta(2\pi,\lambda)] \\ g'(0,\lambda)=e^{2i\lambda\pi}[i\lambda\varphi(2\pi,\lambda)-\varphi'(2\pi,\lambda)] \end{array}$$

or

$$\begin{array}{l} g'(0,\lambda)+i\lambda g(0,\lambda)=e^{2i\lambda\pi}[i\lambda(\theta'(2\pi,\lambda)+\varphi(2\pi,\lambda))+\lambda^2\theta(2\pi,\lambda)-\varphi'(2\pi,\lambda)]= \\ =e^{2i\lambda\pi}[i\lambda F(\lambda)+\lambda^2\theta(2\pi,\lambda)-\varphi'(2\pi,\lambda)] \end{array}$$

where $F(\lambda)=\theta'(2\pi,\lambda)+\varphi(2\pi,\lambda)$ is a Liapounoff function (Hill discriminant).

Taking into account the formulas (10) and (16) we have

$$G(0,0,\lambda)=\frac{\theta(2\pi,\lambda)}{\varphi(2\pi,\lambda)+\theta'(2\pi,\lambda)-2}=\frac{f(0,-\lambda)+R_{11}\,(\lambda)\,f(0,\lambda)}{f'(0,-\lambda)+R_{11}\,(\lambda)\;\;f'(0,\lambda)}$$

Lemma 3: Zeros of the functions $f(0,-\lambda)+R_{11}\,(\lambda)\,f(0,\lambda)$ and $f'(0,-\lambda)+R_{11}\,(\lambda)\;\;f'(0,\lambda)$ do not coincide.

Proof: Let us assume contrary. Let the λ^* be a common of root of the both functions. Then

$$f(0,-\lambda^*)+R_{11}\,(\lambda^*)\,f(0,\lambda^*)=0,$$

$$f'(0,-\lambda^*)+R_{11}\,(\lambda^*)\;\;f'(0,\lambda^*)=0,$$

from that we have

$$R_{11}\,(\lambda^*)=-\frac{f(0,-\lambda^*)}{f(0,\lambda^*)}=-\frac{f'(0,-\lambda^*)}{f'(0,\lambda^*)}$$

or

$$f(0,-\lambda^*)f'(0,\lambda^*)-f'(0,-\lambda^*)f(0,\lambda^*)=0$$

which cannot be, since these solutions are linearly independent.

The Lemma is proved.

It turns out that the roots of the equation (20) are eigenvalues of periodic boundary-value problem, that are also the roots of the dispersion relation $F(\lambda) = 2$. Therefore, the Liapounoff function $F(\lambda)$ can be recovered by the roots of the equation (20).

Since $\theta(2\pi, \lambda)$ and $\varphi'(2\pi, \lambda)$ can be recovered by using $\{\lambda_n\}$ - eigenvalues of the spectral problem (18) and $\{\mu_n\}$-the eigenvalue of problem (19) respectively we find out that specifying spectral data: the spectrum of the Dirichlet problem (18), the spectrum of Neumann boundary value problem (19) and reflection coefficient $R_{11}(\lambda)$, the function

$$g'(0,\lambda) + i\lambda g(0,\lambda) = e^{2i\lambda\pi}[i\lambda F(\lambda) + \lambda^2\theta(2\pi,\lambda) - \varphi'(2\pi,\lambda)]$$

can be reconstructed. Thus, specifying the spectral data uniquely determines the function

$$S(\lambda) = \frac{g'(0,\lambda) + i\lambda g(0,\lambda)}{g'(0,-\lambda) + i\lambda g(0,-\lambda)}.$$

Then taking into account (23), we can find

$$\lim_{\lambda \to -\frac{n}{2}} (n + 2\lambda) S(\lambda) = V_{nn}^{\gamma_1}$$

By using the results obtained above, we obtain the following procedure for the solution of the inverse problem recovering the potential $q_2(z)$ uniquely and effectively on the edge γ_1:

1. Taking into account (23), we get

$$V_{m\alpha+m}^{\gamma_1} = V_{mm}^{\gamma_1} \sum_{n=1}^{\alpha} \frac{V_{nn}^{\gamma_1}}{n+m}, \quad \alpha = 1, 2, ...;$$

from which all the numbers $V_{n\alpha}^{\gamma_1}$ are defined.

2. From recurrent formula (22), we can find all numbers q_{2n}.

So, the inverse problem has a unique solution, and the number q_{2n} are defined constructively by spectral data on the edge γ_1.

The theorem is proved.

Using the results obtained above we arrive at the following procedure for the solution of Inverse Problem.

Algoritm:

Let the spectral data : The spectrum of the Dirichlet problem (18), the spectrum of Neumann boundary value problem (19) and reflection coefficient $R_{11}(\lambda)$ be given.

1. Construct the numbers $V_{nn}^{\gamma_0}$ by $R_{11}(\lambda)$ on the γ_0.

2. Find the numbers $V_{n\alpha}^{\gamma_0}$ from (8) and find all numbers $q_{1\alpha}$ to recover potential $q_1(x)$ on γ_0 by (6).

3. Construct the functions $\theta(2\pi, \lambda)$ and $\varphi'(2\pi, \lambda)$ using the spectrum of the Dirichlet problem (18)- λ_n and the spectrum of Neumann boundary value problem (19) - $\{\mu_n\}$ correspondingly.

4. Construct dispersion relation $F(\lambda) = \theta'(2\pi, \lambda) + \varphi(2\pi, \lambda)$ by using roots of (20) and construct the function

$$g'(0, \lambda) + i\lambda g(0, \lambda) = e^{2i\lambda\pi}[i\lambda F(\lambda) + \lambda^2\theta(2\pi, \lambda) - \varphi'(2\pi, \lambda)]$$

5. Use relation

$$\lim_{\lambda \to -\frac{n}{2}} (n + 2\lambda)S(\lambda) = V_{nn}^{\gamma_1}$$

for finding the numbers $V_{nn}^{\gamma_1}$.

6. Find all numbers $V_{n\alpha}^{\gamma_1}$ by using

$$V_{m\alpha+m}^{\gamma_1} = V_{mm}^{\gamma_1} \sum_{n=1}^{\alpha} \frac{V_{nn}^{\gamma_1}}{n+m}, \quad \alpha = 1, 2, ...;$$

and use (22) for finding the numbers $q_{2\alpha}$ to recover potential $q_2(x)$ on γ_1.

References

[1] Akhtyamov, AM, Trooshin, IY. Direct and Boundary Inverse Spectral problems for Sturm-Liouville differential operators on noncompact star-shaped graphs. Azerbaijan Journal of Mathematics, V.9, Issue 1,pp:108-124

[2] Gasymov, M. G. Spectral analysis of a class of second-order nonselfadjoint differential operators. (Russian) Funktsional. Anal. i Prilozhen. 14 (1980), no. 1, 14–19, 96.

[3] Gomilko, A.M. and Pivovarchik, V.N. Inverse STURM–LIOUVILLE problem on a figure-eight graph. Ukr Math J (2008) 60: 1360. https://doi.org/10.1007/s11253-009-0145-9

[4] Fegan, H. D. "Special function potentials for the Laplacian." Canad. J. Math 34 (1982): 1183-1194.

[5] Shin, Kwang C. On half-line spectra for a class of non-self-adjoint Hill operators. Math. Nachr. 261/262 (2003), 171–175

[6] Carlson, Robert. A note on analyticity and Floquet isospectrality. Proc. Amer. Math. Soc. 134 (2006), no. 5, 1447–1449 (electronic).

[7] Carlson, Robert. Hill's equation for a homogeneous tree. Electron. J. Differential Equations 1997, No. 23, 30 pp. (electronic).

[8] Guillemin, V., and A. Uribe. "Spectral properties of a certain class of complex potentials." Transactions of the American Mathematical Society 279.2 (1983): 759-771.

[9] Pastur L. A., Tkachenko V. A. Spectral theory of Schrödinger operators with periodic complex-valued potentials , Functional Analysis and its Applications.22 (1988), no 2. 156-158.

[10] Efendiev, R. F. Spectral analysis for one class of second-order indefinite non-self-adjoint differential operator pencil. Appl. Anal. 90 (2011), no. 12, 1837–1849.

[11] Efendiev, R. F. The characterization problem for one class of second order operator pencil with complex periodic coefficients. Mosc. Math. J. 7 (2007), no. 1, 55–65, 166

[12] Efendiev, R. F. Spectral analysis of a class of nonselfadjoint differential operator pencils with a generalized function. (Russian) Teoret. Mat. Fiz. 145 (2005), no. 1, 102–107; translation in Theoret. and Math. Phys. 145 (2005), no. 1, 1457–1461

[13] Efendiev, R. F.; Orudzhev, H. D. Inverse wave spectral problem with discontinuous wave speed. Zh. Mat. Fiz. Anal. Geom. 6 (2010), no. 3, 255–265.

[14] Efendiev, Rakib F., Hamzaga D. Orudzhev, and Zaki FA El-Raheem. "Spectral analysis of wave propagation on branching strings." Boundary Value Problems 2016.1 (2016): 215.

[15] Exner, Pavel. "Magnetoresonances on a lasso graph." Foundations of Physics 27.2 (1997): 171-190.

[16] Yang, Chuan-Fu. "Inverse problems for the differential operator on a graph with cycles." Journal of Mathematical Analysis and Applications 445.2 (2017): 1548-1562.

[17] Berkolaiko, Gregory. "An elementary introduction to quantum graphs." arXiv preprint arXiv:1603.07356 (2016).

[18] Kurasov, Pavel. "Inverse scattering for lasso graph." Journal of Mathematical Physics 54.4 (2013): 042103.

[19] Mochizuki K. and Trooshin I.Yu. "On the scattering on a Loop-shaped Graph" Progress in Mathematics, Vol.301, 227-245, (2012)