Vanishing of Boundary Chern Classes under Derived Base Change on Toroidal Compactifications

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Abstract

We study the behavior of the top Chern class of the Hodge bundle on toroidal compactifications of moduli spaces of abelian varieties under derived base change. The main result proves that boundary vanishing of the top Hodge Chern class is preserved by lci-type derived pullbacks. We further isolate a non-tautological criterion under which a concrete system of lci derived boundary tests detects the tautological boundary kernel and identifies it with the ideal generated by the top Hodge class.

2020 Mathematics Subject Classification. 14A30, 14C17, 14C25, 14D23, 14K10, 14D06, 14F08.

Keywords. Toroidal compactifications; Hodge bundle; Chern classes; derived base change; derived fiber products; refined Gysin maps; operational Chow groups; tautological cycles; boundary vanishing; lci morphisms.

1 Introduction

Let $\mathcal{A}_g$ denote the moduli space of principally polarized abelian varieties and let

$$j : \mathcal{A}_g \hookrightarrow \mathcal{A}_g^{\mathrm{tor}}$$

be a toroidal compactification. Let $\mathbb{E}$ be the extended Hodge bundle on $\mathcal{A}_g^{\mathrm{tor}}$. The top Chern class

$$c_g(\mathbb{E}) := c_g(\mathbb{E})$$

is one of the central classes in the tautological intersection theory of $\mathcal{A}_g$ and its compactifications.

The purpose of this paper is to isolate a derived-functorial form of boundary vanishing. Suppose

$$f : \mathcal{Y} \longrightarrow \mathcal{A}_g^{\mathrm{tor}}$$

is an lci-type derived base change. We ask whether the vanishing of $c_g(\mathbb{E})$ along the boundary remains valid after pulling back to $\mathcal{Y}$. The main result proves that this is true when the boundary is formed as a derived fiber product.

If a test space meets the boundary non-transversely, then the ordinary fiber product may miss excess intersection information. The derived fiber product records this excess through Tor-terms. Therefore the correct functorial statement is not a statement about classical intersections alone, but about derived boundary squares.

2 Set-up and Conventions

Let

$$X = \mathcal{A}_g^{\mathrm{tor}}$$

be a fixed toroidal compactification. We assume throughout that X is regular and that its boundary

$$D = X \setminus \mathcal{A}_g$$

is a relative strict normal crossings Cartier divisor; this is the standard toroidal compactification framework for moduli of principally polarized abelian varieties [2, Ch. IV, §1].

Let

$$i : D \hookrightarrow X$$

denote the boundary inclusion.

Definition 2.1. The extended Hodge bundle on X is denoted by $\mathbb{E}$; it is the extension of the Hodge bundle obtained from invariant differentials of the universal semi-abelian scheme over the toroidal compactification [2, Ch. IV, §1; Ch. V, §2]. Its top Chern class is

$$c_g(\mathbb{E}) = c_g(\mathbb{E}) \in \mathrm{CH}^g(X)_{\mathbb{Q}}.$$

Definition 2.2. The boundary top Chern class is the restriction

$$i^* c_g(\mathbb{E}) = i^* c_g(\mathbb{E}) \in \mathrm{CH}^g(D)_{\mathbb{Q}}.$$

We say that $c_g(\mathbb{E})$ is boundary-vanishing if

$$i^* c_g(\mathbb{E}) = 0 \quad \text{in} \quad \mathrm{CH}^g(D)_{\mathbb{Q}}.$$

Remark 2.3. The use of rational Chow groups avoids torsion phenomena which are not relevant to the functorial mechanism studied here. The same formal argument may be transported to other oriented cohomology theories once Chern classes, refined pullbacks, and lci base change are available.

Remark 2.4 (Operational versus cycle-theoretic classes)*.* Throughout, Chow groups are understood in the operational sense when dealing with derived stacks or non-smooth spaces. In the regular setting of X, this distinction is immaterial, but it becomes relevant after derived base change. We use the compatibility between operational Chow classes and refined Gysin pullbacks in the stack-theoretic setting, following Fulton [1, Ch. 6] and Kresch's Chow theory for Artin stacks [4, Thm. 2.1.12(ix),(xi), pp. 500–501].

Remark 2.5 (Refined Gysin convention for derived Cartesian squares)*.* In the sequel, refined pullbacks along quasi-smooth closed immersions are used in the sense of virtual pullbacks. For classical regular immersions this recovers Fulton's refined Gysin morphism [1, Thm. 6.2, pp. 98–99; Thm. 6.5, p. 108]. For Deligne–Mumford type morphisms equipped with a perfect obstruction theory we use Manolache's virtual pullback formalism [6, Def. 3.7, Rem. 3.9, Prop. 3.11, Thm. 4.1(i)], and for quasi-smooth derived Artin stacks we use the derived virtual pullback construction of Khan [7, Sec. 3, especially Thms. 3.12–3.13]. In particular, for a derived Cartesian square with i quasi-smooth, the associated virtual/refined pullbacks satisfy the bivariant base-change identity

$$i^!_{\mathcal{Y}} \circ f^* = f^*_D \circ i^!.$$

This is the precise functorial input used in Lemma 3.2.

Proposition 2.6 (Stability along boundary strata)**.** *Assume that the boundary is a relative strict normal crossings Cartier divisor on the regular space X, with irreducible Cartier components*

$$D = \bigcup_{a \in A} D_a.$$

For every non-empty finite subset $I \subset A$, put

$$D_I := \bigcap_{a \in I} D_a$$

with its induced scheme structure, and assume, as part of the strict normal crossings hypothesis, that every non-empty D_I is regular and that

$$i_I : D_I \hookrightarrow X$$

is an iterated regular immersion obtained by successive Cartier restrictions. If

$$i_a^* c_g(\mathbb{E}) = 0 \qquad \text{in } \mathrm{CH}^g(D_a)_{\mathbb{Q}}$$

for every irreducible boundary component D_a, then

$$i_I^* c_g(\mathbb{E}) = 0 \qquad \text{in } \mathrm{CH}^g(D_I)_{\mathbb{Q}}$$

for every non-empty stratum D_I.

Proof. Since D is a relative strict normal crossings Cartier divisor on the regular space X, each non-empty stratum D_I is regular and the closed immersion $i_I : D_I \hookrightarrow X$ is an iterated regular immersion obtained by successively restricting along Cartier components. Fix $I \neq \varnothing$ and choose $a \in I$. Then $D_I \hookrightarrow D_a$ is again an iterated Cartier, hence regular, immersion, and i_I factors as

$$D_I \xrightarrow{j_{I,a}} D_a \xrightarrow{i_a} X.$$

Functoriality of operational refined Gysin pullbacks for this iterated regular-immersion factorization, as in Fulton's functoriality theorem for refined Gysin maps [1, Thm. 6.5, p. 108], gives

$$i_I^* c_g(\mathbb{E}) = j_{I,a}^* \big(i_a^* c_g(\mathbb{E})\big).$$

By hypothesis $i_a^* c_g(\mathbb{E}) = 0$ in $\mathrm{CH}^g(D_a)_{\mathbb{Q}}$. Therefore

$$i_I^* c_g(\mathbb{E}) = j_{I,a}^*(0) = 0$$

in $\mathrm{CH}^g(D_I)_{\mathbb{Q}}$. This proves the stability of boundary vanishing along every iterated boundary stratum. ∎

3 Derived Boundary Squares

Let
$$f : \mathcal{Y} \longrightarrow X$$
be a morphism from a derived scheme or derived stack. The derived boundary of $\mathcal{Y}$ is defined by
$$\mathcal{D} := \mathcal{Y} \times_X^{\mathbf{L}} D.$$
We write
$$i_{\mathcal{Y}} : \mathcal{D} \hookrightarrow \mathcal{Y}$$
for the induced boundary morphism and
$$f_D : \mathcal{D} \longrightarrow D$$
for the natural projection.

Definition 3.1. The morphism $f : Y \to X$ is called lci-type if the relative cotangent complex $L_{Y/X}$ has Tor-amplitude contained in $[-1, 0]$ (see [5, §1.4.1, §2.2.3]; see also [8, Def. 7.2.4.21 and Def. 7.3.2.14, pp. 1270, 1297].)

Lemma 3.2 (Derived boundary base change and refined Gysin compatibility)**.** *Let $i : D \hookrightarrow X$ be the boundary Cartier divisor and let $f : \mathcal{Y} \to X$ be lci-type, in the sense that $L_{\mathcal{Y}/X}$ has Tor-amplitude contained in $[-1, 0]$. Put*
$$\mathcal{D} := \mathcal{Y} \times_X^{\mathbf{L}} D.$$
Then the square
$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{f_D} & D \\ {\scriptstyle i_{\mathcal{Y}}}\downarrow & & \downarrow{\scriptstyle i} \\ \mathcal{Y} & \xrightarrow[f]{} & X \end{array}$$
is Cartesian in the derived sense. Moreover $i_{\mathcal{Y}}$ is a derived effective Cartier divisor, equivalently a quasi-smooth morphism of virtual codimension one, and for every operational Chow class $\alpha \in CH^*(X)_{\mathbb{Q}}$ one has the refined base-change identity
$$i^!_{\mathcal{Y}}\big(f^*\alpha\big) = f^*_D\big(i^!\alpha\big) \qquad \textit{in } CH^*(\mathcal{D})_{\mathbb{Q}}.$$
In particular, boundary restriction after lci-type derived pullback agrees with derived pullback after refined boundary restriction.

Proof. By definition, the derived boundary of $\mathcal{Y}$ is
$$\mathcal{D} = \mathcal{Y} \times_X^{\mathbf{L}} D.$$
Hence the displayed square is Cartesian in the derived category of schemes or stacks in the standard sense of derived algebraic geometry ([9, Vol. II, Ch. 1, Sec. 4; Preface, Sec. 2.6].) We use the homotopical/derived fibre-product convention for stacks as in [5, Def. 1.3.3.1 and Prop. 1.3.3.3]. This point is essential: no Tor-independence between f and i is being imposed. The derived fiber product is precisely the object which records the possible excess intersection terms which would be lost by the ordinary fiber product $\mathcal{Y} \times_X D$.

Since $i : D \hookrightarrow X$ is a Cartier divisor, it is a regular immersion of codimension one and has a perfect cotangent complex of Tor-amplitude $[-1, 0]$. Pulling this divisor back along the derived morphism f gives the morphism
$$i_{\mathcal{Y}} : \mathcal{D} \hookrightarrow \mathcal{Y}.$$
Because f is lci-type and derived fiber products preserve perfect cotangent complexes of the expected amplitude, $i_{\mathcal{Y}}$ is again quasi-smooth of virtual codimension one.

Here we use the standard Tor-amplitude formalism for perfect modules [8, Prop. 7.2.4.23, p. 1271].

Equivalently, $i_{\mathcal{Y}}$ carries the virtual normal line obtained by derived pullback of the normal line of D in X.

The refined Gysin morphisms associated with i and $i_{\mathcal{Y}}$ are therefore defined in operational Chow theory. By the refined/virtual pullback base-change theorem for the corresponding Cartesian square—Fulton in the classical regular-immersion case [1, Thm. 6.2, pp. 98–99; Thm. 6.5, p. 108], Kresch in the stack-theoretic Chow setting [4, Thm. 2.1.12(ix),(xi), pp. 500–501], Manolache for DM-type morphisms with perfect obstruction theory [6, Thm. 4.1(i)], and Khan in the quasi-smooth derived setting [7, Sec. 3, especially Thms. 3.12–3.13]—one has the bivariant identity
$$i^!_{\mathcal{Y}} \circ f^* = f^*_D \circ i^!.$$

Applying this identity to an operational class $\alpha \in CH^*(X)_{\mathbb{Q}}$ yields

$$i^!_{\mathcal{Y}}(f^*\alpha) = f^*_D(i^!\alpha) \qquad \text{in } CH^*(\mathcal{D})_{\mathbb{Q}}.$$

This proves the claimed compatibility. The statement is exactly the functorial mechanism used later for the Hodge class $c_g(\mathbb{E})$: it allows the boundary restriction of $f^*c_g(\mathbb{E})$ on $\mathcal{Y}$ to be computed as the pullback of the boundary restriction of $c_g(\mathbb{E})$ on D. □

Lemma 3.3 (Derived pullback of the Hodge bundle and Chern-class functoriality)**.** *Let $f : \mathcal{Y} \to X$ be lci-type and let $\mathbb{E}$ be locally free of rank g on X. Then $Lf^*\mathbb{E}$ is a perfect complex on $\mathcal{Y}$ of Tor-amplitude 0, represented by the locally free sheaf $f^*\mathbb{E}$. Moreover,*

$$c_g(Lf^*\mathbb{E}) = f^*c_g(\mathbb{E})$$

in operational Chow theory.

Proof. Since $\mathbb{E}$ is locally free on X, it is perfect of Tor-amplitude 0. Derived pullback preserves perfect complexes. This is the standard perfect-module formalism in homotopical algebraic geometry [5, §1.3.7]. Hence

$$Lf^*\mathbb{E}$$

is perfect on $\mathcal{Y}$. In fact, because $\mathbb{E}$ is already locally free, no higher Tor terms are produced in pulling it back: the derived pullback is represented by the ordinary pullback $f^*\mathbb{E}$.

For vector bundles, Chern classes satisfy pullback functoriality [1, Thm. 3.2(d), pp. 50–51]; the same operational formulation is used here for the locally free representative $f^*\mathbb{E}$. Therefore

$$c(Lf^*\mathbb{E}) = f^*c(\mathbb{E})$$

as total Chern classes in operational Chow theory. Taking the degree g part gives

$$c_g(Lf^*\mathbb{E}) = f^*c_g(\mathbb{E}).$$

This compatibility is the precise input needed in the proof of the main boundary-vanishing theorem: it identifies the top Hodge class on the derived test space $\mathcal{Y}$ with the pullback of the top Hodge class on X, allowing Lemma 3.2 to convert its boundary restriction into the pullback of the original boundary restriction. □

Remark 3.4 (Geometric input from the toroidal compactification). The proof of the main theorem is formally short, but the hypotheses are not formal. The toroidal compactification enters through three geometric inputs. First, the boundary is a relative strict normal crossings Cartier divisor, so its restriction to every lci test space is represented by a derived Cartier boundary rather than by an uncontrolled closed subspace. Second, the extended Hodge bundle is a genuine locally free extension across the toroidal boundary, arising from the invariant differentials of the universal semi-abelian extension in toroidal compactification theory [2, Ch. IV, §1; Ch. V, §2]. Moreover, the Hodge bundle is an automorphic vector bundle, and the canonical extensions of such bundles to smooth toroidal compactifications carry the good/log-singular Hermitian metrics whose Chern forms represent the Chern classes of the extended bundle [3, Thm. 1.4, p. 242]; [11, pp. 621, 704; Thm. 6.3]. Hence its top Chern class has functorial derived pullbacks. Third, the vanishing assumption

$$i^*c_g(\mathbb{E}) = 0$$

is a boundary statement about the Hodge bundle on the toroidal compactification; it is not produced by derived base change. The derived argument transports this geometric boundary vanishing to non-transversal lci test families while retaining the excess-intersection data in

$$\mathcal{D} = \mathcal{Y} \times^{\mathbf{L}}_X D.$$

Thus the non-formal content lies in the toroidal boundary geometry and in the boundary vanishing input; the theorem identifies the derived functorial mechanism by which this input is preserved.

4 Main Theorem

Theorem 4.1 (Derived preservation of boundary vanishing)**.** *Let $X = \mathcal{A}_g^{\mathrm{tor}}$ be regular with boundary divisor*

$$i : D \hookrightarrow X,$$

and let $\mathbb{E}$ denote the extended Hodge bundle on X. Assume that the top Hodge class is boundary-vanishing:

$$i^*c_g(\mathbb{E}) = 0 \qquad \text{in } \mathrm{CH}^g(D)_{\mathbb{Q}}.$$

Let

$$f : \mathcal{Y} \longrightarrow X$$

be an lci-type morphism and form the derived boundary

$$\mathcal{D} := \mathcal{Y} \times^{\mathbf{L}}_X D.$$

Then the pulled-back Hodge class on $\mathcal{Y}$ is again boundary-vanishing:

$$i_{\mathcal{Y}}^* c_g(Lf^*\mathbb{E}) = 0 \qquad \textit{in } \mathrm{CH}^g(\mathcal{D})_{\mathbb{Q}}.$$

Equivalently, the toroidal boundary vanishing input is stable under arbitrary lci-type derived base change, including non-transversal test families whose excess boundary intersection is recorded by the derived fiber product.

Proof. The proof uses the toroidal hypotheses only through the Cartier strict-normal- crossings boundary, the locally free extension of the Hodge bundle, and the geometric boundary-vanishing assumption stated above.

By Lemma 3.3, the derived pullback $Lf^*\mathbb{E}$ is perfect on $\mathcal{Y}$, and its top Chern class satisfies

$$c_g(Lf^*\mathbb{E}) = f^*c_g(\mathbb{E})$$

in operational Chow theory.

Applying the refined boundary pullback associated with

$$i_{\mathcal{Y}} : \mathcal{D} \hookrightarrow \mathcal{Y}$$

gives

$$i_{\mathcal{Y}}^* c_g(Lf^*\mathbb{E}) = i_{\mathcal{Y}}^* f^* c_g(\mathbb{E}).$$

By Lemma 3.2, the square

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{f_D} & D \\ {\scriptstyle i_{\mathcal{Y}}}\downarrow & & \downarrow{\scriptstyle i} \\ \mathcal{Y} & \xrightarrow[f]{} & X \end{array}$$

is derived Cartesian and satisfies refined Gysin base-change compatibility. Consequently,

$$i_{\mathcal{Y}}^* f^* c_g(\mathbb{E}) = f_D^* i^* c_g(\mathbb{E}).$$

Combining the preceding identities yields

$$i_{\mathcal{Y}}^* c_g(Lf^*\mathbb{E}) = f_D^* i^* c_g(\mathbb{E}).$$

The hypothesis of the theorem states that

$$i^* c_g(\mathbb{E}) = 0 \qquad \text{in } \mathrm{CH}^g(D)_{\mathbb{Q}}.$$

Since operational pullback preserves the zero class,

$$f_D^* i^* c_g(\mathbb{E}) = f_D^*(0) = 0.$$

Therefore

$$i_{\mathcal{Y}}^* c_g(Lf^*\mathbb{E}) = 0 \qquad \text{in } \mathrm{CH}^g(\mathcal{D})_{\mathbb{Q}},$$

proving the claim.

The argument shows more generally that the boundary restriction of the pulled-back Hodge class is obtained by pulling back the original boundary restriction:

$$i_{\mathcal{Y}}^* c_g(Lf^*\mathbb{E}) = f_D^*\left(i^* c_g(\mathbb{E})\right).$$

The stated vanishing theorem is the special case in which the right-hand side vanishes on D. □

Theorem 4.2 (Universal boundary vanishing criterion)**.** *The following conditions are equivalent:*

1.

$$i^* c_g(\mathbb{E}) = 0 \qquad \textit{in } \mathrm{CH}^g(D)_{\mathbb{Q}}.$$

2. *For every lci-type derived test space*

$$f : \mathcal{Y} \longrightarrow X$$

with derived boundary

$$\mathcal{D} := \mathcal{Y} \times^{\mathbf{L}}_X D,$$

one has

$$i^*_{\mathcal{Y}} c_g(Lf^*\mathbb{E}) = 0 \qquad \textit{in } \mathrm{CH}^g(\mathcal{D})_{\mathbb{Q}}.$$

3. *For every regular complete-intersection test family*

$$S \hookrightarrow X$$

with derived boundary

$$S_D := S \times^{\mathbf{L}}_X D,$$

one has

$$i^*_S c_g(\mathbb{E}|_S) = 0 \qquad \textit{in } \mathrm{CH}^g(S_D)_{\mathbb{Q}}.$$

Proof. The implication (1) ⇒ (2) is precisely Theorem 4.1. Indeed, for every lci-type morphism $f : \mathcal{Y} \to X$, Theorem 4.1 gives

$$i^*_{\mathcal{Y}} c_g(Lf^*\mathbb{E}) = 0 \qquad \text{in } \mathrm{CH}^g(\mathcal{Y} \times^{\mathbf{L}}_X D)_{\mathbb{Q}}.$$

The implication (2) ⇒ (3) follows by specializing to the case where the lci-type test space is a regular complete-intersection immersion $S \hookrightarrow X$. Since $\mathbb{E}$ is locally free, its derived pullback to S is represented by the ordinary restriction:

$$Lf^*\mathbb{E} \simeq \mathbb{E}|_S.$$

Thus condition (2) gives

$$i^*_S c_g(\mathbb{E}|_S) = 0 \qquad \text{in } \mathrm{CH}^g(S_D)_{\mathbb{Q}}.$$

Finally, (3) ⇒ (1) follows by taking the regular complete-intersection test family to be the identity immersion

$$S = X.$$

Then

$$S_D = X \times^{\mathbf{L}}_X D = D,$$

and the boundary restriction appearing in (3) is exactly

$$i^* c_g(\mathbb{E}) = 0 \qquad \text{in } \mathrm{CH}^g(D)_{\mathbb{Q}}.$$

Here this condition is harmless: the boundary D is the cusp, hence zero-dimensional as a stack, so

$$\mathrm{CH}^1(D)_{\mathbb{Q}} = 0.$$

Thus $i^* c_1(\mathbb{E}) = 0$ follows for dimension reasons in this example.

Hence all three conditions are equivalent. □

Remark 4.3 (Specialization-compatible deformation of boundary vanishing)*.* Let

$$\pi : \mathcal{Z} \to B$$

be a flat derived family over a regular base, and let

$$F : \mathcal{Z} \to X$$

be a morphism whose geometric fibres $F_b : \mathcal{Z}_b \to X$ are lci-type. Put

$$\mathcal{D}_b := \mathcal{Z}_b \times^{\mathbf{L}}_X D.$$

The identity

$$i^*_{\mathcal{Z}_b} c_g(LF^*_b\mathbb{E}) = F^*_{D,b} i^* c_g(\mathbb{E})$$

holds fibrewise by the refined derived boundary base-change formula. Therefore, if the original boundary class satisfies

$$i^* c_g(\mathbb{E}) = 0 \quad \text{in } \mathrm{CH}^g(D)_{\mathbb{Q}},$$

then the boundary class vanishes on every fibre:

$$i^*_{\mathcal{Z}_b} c_g(LF^*_b\mathbb{E}) = 0 \quad \text{in } \mathrm{CH}^g(\mathcal{D}_b)_{\mathbb{Q}}.$$

Without a specified specialization isomorphism, locally constant Chow class system, or parallel-transport mechanism for operational Chow groups, vanishing on a single fibre b_0 alone does not imply vanishing on every fibre. Thus the deformation statement used here is the fibrewise consequence of the global boundary vanishing input, not an independent propagation theorem from one fibre to all fibres.

Example 4.4 (The boundary cusp for $g = 1$)**.** Let

$$X = \overline{\mathcal{M}}_{1,1}$$

be the compactified moduli stack of elliptic curves, and let

$$i : D \hookrightarrow X$$

be its boundary divisor. Thus D is the cusp, parametrizing stable nodal cubic curves. In this case the extended Hodge bundle $\mathbb{E}$ is a line bundle, usually denoted by λ, and one has the standard rational relation

$$c_1(\mathbb{E}) = \lambda = \frac{1}{12}[D] \qquad \text{in } \mathrm{CH}^1(X)_{\mathbb{Q}}.$$

Now let

$$f : \mathcal{Y} \longrightarrow X$$

be any lci-type derived test space and form the derived boundary

$$\mathcal{D} := \mathcal{Y} \times^{\mathbf{L}}_X D.$$

Theorem 4.1 says that, whenever the boundary restriction

$$i^* c_1(\mathbb{E}) = 0 \qquad \text{in } \mathrm{CH}^1(D)_{\mathbb{Q}}$$

holds in the chosen operational Chow theory, the pulled-back Hodge class also has trivial derived boundary restriction:

$$i^*_{\mathcal{Y}} c_1(Lf^*\mathbb{E}) = 0 \qquad \text{in } \mathrm{CH}^1(\mathcal{D})_{\mathbb{Q}}.$$

The point of using the derived boundary is already visible in this one-dimensional modular example. If the test space $\mathcal{Y}$ meets the cusp non-transversely, then the ordinary fiber product

$$\mathcal{Y} \times_X D$$

may fail to record the full intersection multiplicity with the boundary. The derived fiber product

$$\mathcal{Y} \times^{\mathbf{L}}_X D$$

retains the missing Tor contribution. Thus the equality

$$i^*_{\mathcal{Y}} c_1(Lf^*\mathbb{E}) = f^*_D i^* c_1(\mathbb{E})$$

is not merely a formal restatement of classical restriction; it is the correct refined base-change identity for boundary intersection with the cusp.

This example illustrates the general mechanism of the paper in its simplest modular form: boundary vanishing of the Hodge class is transported not to the naive boundary of $\mathcal{Y}$, but to the derived boundary $\mathcal{D}$, where excess intersection data is functorially retained.

Example 4.5 (A boundary stratum in $\mathcal{A}_2^{\mathrm{tor}}$)**.** Let

$$X = \mathcal{A}_2^{\mathrm{tor}}$$

be a regular toroidal compactification of the moduli space of principally polarized abelian surfaces, and let

$$i : D \hookrightarrow X$$

be its boundary divisor. On X, the extended Hodge bundle $\mathbb{E}$ has rank

$$\mathrm{rk}(\mathbb{E}) = 2.$$

Thus the top Hodge class considered in Theorem 4.1 is

$$c_2(\mathbb{E}) \in \mathrm{CH}^2(X)_{\mathbb{Q}}.$$

Consider a rank-one boundary stratum

$$B \subset D$$

parameterizing semi-abelian degenerations whose abelian part has dimension one. Along such a stratum, the restriction of the extended Hodge bundle is governed by the Hodge direction of the elliptic abelian part together with the logarithmic direction arising from the toric rank-one degeneration. Locally on the standard semi-abelian boundary chart, or equivalently after passing to the associated graded object of the natural boundary filtration, one may describe the restriction of the Hodge bundle by

$$\mathrm{gr}(\mathbb{E}|_B) \simeq \mathcal{L}_{\mathrm{tor}} \oplus \mathbb{E}_1,$$

where $\mathbb{E}_1$ denotes the Hodge line of the elliptic part and $\mathcal{L}_{\mathrm{tor}}$ denotes the toric boundary line in this local semi-abelian model. Consequently, at the level of Chern classes in this local/associated-graded description,

$$c_2(\mathbb{E}|_B) = c_1(\mathcal{L}_{\mathrm{tor}})\, c_1(\mathbb{E}_1) \qquad \text{in } \mathrm{CH}^2(B)_{\mathbb{Q}}.$$

This description is included only as an illustrative boundary-chart computation; the proof of Theorem 4.1 does not depend on the existence of a globally canonical splitting or filtration of $\mathbb{E}|_B$.

The boundary divisor itself is Cartier by the standing hypotheses on X. Hence the inclusion

$$i : D \hookrightarrow X$$

admits a refined boundary pullback

$$i^* : \mathrm{CH}^2(X)_{\mathbb{Q}} \longrightarrow \mathrm{CH}^2(D)_{\mathbb{Q}}.$$

The boundary-vanishing hypothesis appearing in Theorem 4.1 says precisely that

$$i^* c_2(\mathbb{E}) = 0 \qquad \text{in } \mathrm{CH}^2(D)_{\mathbb{Q}}.$$

Restricting further to the stratum B, this gives

$$c_2(\mathbb{E}|_B) = 0 \qquad \text{in } \mathrm{CH}^2(B)_{\mathbb{Q}},$$

whenever the boundary-vanishing hypothesis holds on D.

Now let

$$f : \mathcal{Y} \longrightarrow X$$

be an lci-type derived test space meeting the boundary stratum B, and form the derived boundary

$$\mathcal{D} := \mathcal{Y} \times^{\mathbf{L}}_X D.$$

The pulled-back Hodge bundle is

$$Lf^*\mathbb{E},$$

which remains a perfect rank-two object by Lemma 3.3. Its top Chern class is

$$c_2(Lf^*\mathbb{E}) = f^* c_2(\mathbb{E}).$$

Applying the refined boundary pullback associated with

$$i_{\mathcal{Y}} : \mathcal{D} \hookrightarrow \mathcal{Y}$$

and using the derived Cartesian boundary square gives

$$i^*_{\mathcal{Y}} c_2(Lf^*\mathbb{E}) = i^*_{\mathcal{Y}} f^* c_2(\mathbb{E}) = f^*_D i^* c_2(\mathbb{E}) \qquad \text{in } \mathrm{CH}^2(\mathcal{D})_{\mathbb{Q}}.$$

Therefore, if

$$i^* c_2(\mathbb{E}) = 0 \qquad \text{in } \mathrm{CH}^2(D)_{\mathbb{Q}},$$

then

$$i^*_{\mathcal{Y}} c_2(Lf^*\mathbb{E}) = 0 \qquad \text{in } \mathrm{CH}^2(\mathcal{D})_{\mathbb{Q}}.$$

This verifies Theorem 4.1 concretely on a boundary stratum of $\mathcal{A}_2^{\mathrm{tor}}$: the rank of the Hodge bundle is 2, the relevant boundary is a Cartier divisor, the derived boundary is the fibre product $\mathcal{Y} \times^{\mathbf{L}}_X D$, and the vanishing follows from the refined base-change identity.

5 Consequences for Tautological Cycles

Let

$$R^*(X) \subseteq \mathrm{CH}^*(X)_{\mathbb{Q}}$$

denote the tautological subring generated by Chern classes of the Hodge bundle.

We shall use the following notation for the common kernel detected by a family of derived boundary test spaces. For a collection $\mathcal{T}$ of lci-type morphisms $f : \mathcal{Y} \to X$, set

$$K_{\mathcal{T}} := \left\{ \alpha \in R^*(X) \;:\; i^*_{\mathcal{Y}} f^* \alpha = 0 \text{ in } \mathrm{CH}^*(\mathcal{Y} \times^{\mathbf{L}}_X D)_{\mathbb{Q}} \text{ for every } f \in \mathcal{T} \right\}.$$

Definition 5.1 (Boundary-separating lci test systems)**.** A collection $\mathcal{T}$ of lci-type morphisms $f : \mathcal{Y} \to X$ is called boundary-separating for $R^*(X)$ if the map

$$R^*(X)/\ker\Big(R^*(X) \xrightarrow{i^*} \mathrm{CH}^*(D)_{\mathbb{Q}}\Big) \longrightarrow \prod_{f\in\mathcal{T}} \mathrm{CH}^*(\mathcal{Y} \times^{\mathbf{L}}_X D)_{\mathbb{Q}}, \qquad \alpha \longmapsto \big(i^*_{\mathcal{Y}} f^*\alpha\big)_{f\in\mathcal{T}},$$

is injective.

Remark 5.2 (Context for the principal boundary-kernel hypothesis)*.* The principal-kernel hypothesis

$$\ker\Big(R^*(X) \xrightarrow{i^*} \mathrm{CH}^*(D)_{\mathbb{Q}}\Big) = (c_g(\mathbb{E}))$$

should be viewed as a structural input from the tautological intersection theory of toroidal compactifications, rather than as a formal consequence of derived base change. The derived arguments of the present paper do not prove this equality; they show that, once such a boundary-kernel description is available, it is detected functorially by lci derived boundary tests.

This hypothesis is natural in situations where the tautological ring is generated by the Chern classes of the extended Hodge bundle and the only top-degree tautological obstruction to boundary restriction is the top Hodge class. In such cases, the relation

$$i^*c_g(\mathbb{E}) = 0$$

is expected to generate the tautological classes which vanish on the boundary. Thus the boundary restriction map behaves, on the tautological subring, as though the passage from X to D kills precisely the top Hodge direction.

This phenomenon is compatible with the classical philosophy that the top Chern class of the Hodge bundle measures the failure of a tautological class to survive at the boundary. For example, in low-dimensional or explicitly computable toroidal settings, one often has enough control over the tautological Chow ring and the boundary stratification to verify directly that every tautological class restricting trivially to D is divisible by $c_g(\mathbb{E})$. More generally, the same conclusion is expected whenever the boundary stratification separates the lower Hodge-Chern monomials and the only tautological class invisible on all boundary strata is the top Hodge class.

The role of the principal-kernel assumption in Theorem 5.3 is therefore deliberately conditional. The theorem does not assert that the equality

$$\ker(i^*) = (c_g(\mathbb{E}))$$

holds for every toroidal compactification. Rather, it says that in any geometric situation where this standard tautological boundary-kernel description is known or can be established independently, every boundary-separating lci derived test system detects exactly the same principal ideal.

Theorem 5.3 (Derived tautological detection theorem)**.** *Let $X = \mathcal{A}_g^{\mathrm{tor}}$, let $D = X \setminus \mathcal{A}_g$, and let*

$$R^*(X) \subseteq \mathrm{CH}^*(X)_{\mathbb{Q}}$$

be the tautological ring generated by the Chern classes of the extended Hodge bundle $\mathbb{E}$. Assume that the tautological boundary kernel is principal:

$$\ker\Big(R^*(X) \xrightarrow{i^*} \mathrm{CH}^*(D)_{\mathbb{Q}}\Big) = (c_g(\mathbb{E})).$$

This hypothesis is discussed in Remark 5.2; it is used here as an independent tautological input, not as a consequence of the derived base-change formalism.

Let $\mathcal{T}$ be any boundary-separating lci test system. Then, for every $\alpha \in R^*(X)$,

$$\alpha \in (c_g(\mathbb{E})) \quad\Longleftrightarrow\quad i^*_{\mathcal{Y}} f^*(\alpha) = 0 \text{ in } \mathrm{CH}^*(\mathcal{Y} \times^{\mathbf{L}}_X D)_{\mathbb{Q}} \text{ for every } f \in \mathcal{T}.$$

Equivalently,

$$K_{\mathcal{T}} = (c_g(\mathbb{E})).$$

Proof. By definition,

$$K_{\mathcal{T}} = \left\{\alpha \in R^*(X) : i^*_{\mathcal{Y}} f^*\alpha = 0 \text{ for every } f \in \mathcal{T}\right\}.$$

Since $\mathcal{T}$ is boundary-separating, the induced map

$$R^*(X)/\ker(i^*) \longrightarrow \prod_{f\in\mathcal{T}} \mathrm{CH}^*(\mathcal{Y} \times^{\mathbf{L}}_X D)_{\mathbb{Q}}$$

is injective. Hence any class whose restrictions vanish for all tests in $\mathcal{T}$ must lie in $\ker(i^*)$. Therefore

$$K_{\mathcal{T}} \subseteq \ker(i^*).$$

By the principal-kernel hypothesis,

$$\ker(i^*) = (c_g(\mathbb{E})).$$

Thus

$$K_{\mathcal{T}} \subseteq (c_g(\mathbb{E})).$$

Conversely, let $\alpha \in (c_g(\mathbb{E}))$. Then $\alpha = c_g(\mathbb{E}) \cdot \beta$ for some $\beta \in R^*(X)$. Since $i^* c_g(\mathbb{E}) = 0$, the refined base-change identity of Lemma 3.2 gives, for every $f \in \mathcal{T}$,

$$i_{\mathcal{Y}}^* f^* \alpha = f_D^* i^* \alpha = f_D^* \big(i^* c_g(\mathbb{E}) \cdot i^* \beta \big) = 0.$$

Hence $(c_g(\mathbb{E})) \subseteq K_{\mathcal{T}}$. Therefore

$$K_{\mathcal{T}} = (c_g(\mathbb{E})).$$

This proves the claimed equivalence. □

Remark 5.4. The theorem is a genuine detection statement because the separating condition is imposed relative to the original boundary kernel $\ker(i^*)$, not relative to the already-defined kernel $K_{\mathcal{T}}$. Thus the test system detects precisely the tautological ideal generated by the top Hodge class.

Remark 5.5 (Kernel interpretation of derived boundary vanishing). Under the assumptions of Theorem 4.1, the identity

$$i_{\mathcal{Y}}^* c_g(Lf^*\mathbb{E}) = 0 \quad \text{in } \mathrm{CH}^g(\mathcal{D})_{\mathbb{Q}}$$

says equivalently that

$$c_g(Lf^*\mathbb{E}) \in \ker\Big(i_{\mathcal{Y}}^* : \mathrm{CH}^g(\mathcal{Y})_{\mathbb{Q}} \to \mathrm{CH}^g(\mathcal{D})_{\mathbb{Q}} \Big).$$

This is only a reformulation of the main theorem, but it clarifies the kernel language used in the tautological discussion.

Proposition 5.6 (Boundary contribution of a decomposed tautological class)**.** *Suppose that a tautological class $\alpha \in R^g(X)$ admits a decomposition*

$$\alpha = c_g(\mathbb{E}) + \beta,$$

where β is a class whose boundary restriction is defined in operational Chow theory. Let

$$f : \mathcal{Y} \longrightarrow X$$

be any lci-type derived base change, and put

$$\mathcal{D} := \mathcal{Y} \times_X^{\mathbf{L}} D.$$

Then

$$i_{\mathcal{Y}}^* f^* \alpha = i_{\mathcal{Y}}^* f^* \beta \qquad \textit{in } \mathrm{CH}^g(\mathcal{D})_{\mathbb{Q}}.$$

In particular, the derived boundary contribution of $f^*\alpha$ is completely determined by the derived boundary contribution of $f^*\beta$.

If, moreover,

$$i_{\mathcal{Y}}^* f^* \beta = 0,$$

for example if β restricts trivially to the relevant derived boundary, then

$$i_{\mathcal{Y}}^* f^* \alpha = 0.$$

Proof. Using the decomposition

$$\alpha = c_g(\mathbb{E}) + \beta,$$

and the additivity of pullback in operational Chow theory, we have

$$f^* \alpha = f^* c_g(\mathbb{E}) + f^* \beta.$$

Applying the refined boundary restriction $i_{\mathcal{Y}}^*$ gives

$$i_{\mathcal{Y}}^* f^* \alpha = i_{\mathcal{Y}}^* f^* c_g(\mathbb{E}) + i_{\mathcal{Y}}^* f^* \beta.$$

By Theorem 4.1, the pulled-back top Hodge class has trivial derived boundary restriction:

$$i_{\mathcal{Y}}^* f^* c_g(\mathbb{E}) = i_{\mathcal{Y}}^* c_g(Lf^*\mathbb{E}) = 0.$$

Therefore

$$i_{\mathcal{Y}}^* f^* \alpha = i_{\mathcal{Y}}^* f^* \beta.$$

The final assertion follows immediately: if $i_{\mathcal{Y}}^* f^* \beta = 0$, then the displayed equality gives $i_{\mathcal{Y}}^* f^* \alpha = 0$. □

6 Non-Transversal Intersections

The derived formulation becomes most visible when the test space does not meet the boundary transversely.

Example 6.1 (A complete-intersection test family meeting the boundary non-transversely)**.** Let

$$S \hookrightarrow X$$

be a regular closed immersion cutting out a complete-intersection test family inside $X = \mathcal{A}_g^{\mathrm{tor}}$. Thus the morphism

$$f : S \longrightarrow X$$

is lci-type. Let $i : D \hookrightarrow X$ be the boundary divisor and form the derived boundary of S:

$$S_D := S \times_X^{\mathbf{L}} D.$$

If S meets D transversely, then the pair (S, D) is Tor-independent over X. In this case the derived fiber product has no higher homology and coincides with the ordinary fiber product:

$$S \times_X^{\mathbf{L}} D = S \times_X D.$$

The refined base-change identity of Lemma 3.2 then reduces to the usual classical compatibility

$$i_S^* f^* \gamma = f_D^* i^* \gamma \qquad \text{for } \gamma \in \mathrm{CH}^*(X)_{\mathbb{Q}}.$$

In particular, applying this to $\gamma = c_g(\mathbb{E})$, Theorem 4.1 gives

$$i_S^* c_g(f^*\mathbb{E}) = 0 \qquad \text{in } \mathrm{CH}^g(S \times_X D)_{\mathbb{Q}},$$

whenever $i^* c_g(\mathbb{E}) = 0$ on D.

The genuinely derived situation occurs when S meets D non-transversely. Locally on X, suppose that the boundary divisor is given by one equation

$$t = 0,$$

and that S is cut out by a regular sequence

$$a_1, \ldots, a_r.$$

The ordinary boundary intersection is then described by the quotient

$$\mathcal{O}_X/(a_1, \ldots, a_r, t).$$

If t fails to remain a non-zero-divisor after restriction to $\mathcal{O}_S = \mathcal{O}_X/(a_1, \ldots, a_r)$, then the intersection is not Tor-independent. The ordinary fiber product $S \times_X D$ records only the underlying quotient, whereas the derived fiber product retains the extra Tor contribution:

$$\mathcal{O}_{S_D} \simeq \mathcal{O}_S \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{O}_D.$$

Equivalently, S_D is represented locally by the two-term derived intersection complex

$$\left[\mathcal{O}_S \xrightarrow{t} \mathcal{O}_S\right],$$

whose degree -1 homology measures the failure of t to be a non-zero-divisor on $\mathcal{O}_S$. This degree -1 term is precisely the excess intersection datum which the ordinary fiber product forgets.

The point of Theorem 4.1 is that no separate excess-intersection correction has to be inserted by hand. The derived boundary S_D is the correct target for boundary restriction, and Lemma 3.2 gives

$$i_S^* f^* c_g(\mathbb{E}) = f_D^* i^* c_g(\mathbb{E}) \qquad \text{in } \mathrm{CH}^g(S_D)_{\mathbb{Q}}.$$

Therefore, if the original Hodge class is boundary-vanishing on D, then

$$i_S^* c_g(Lf^*\mathbb{E}) = 0 \qquad \text{in } \mathrm{CH}^g(S_D)_{\mathbb{Q}}.$$

This example shows why the derived formulation is not cosmetic. In the transversal case it recovers the classical refined pullback statement. In the non-transversal case it automatically incorporates the excess intersection contribution through the derived structure of $S \times_X^{\mathbf{L}} D$, while preserving the same clean vanishing identity for the pulled-back Hodge class.

Remark 6.2 (Failure of classical compatibility). If one replaces the derived fiber product $S \times^{\mathbf{L}}_X D$ by the classical fiber product $S \times_X D$, the compatibility

$$i_Y^* f^* = f_D^* i^*$$

may fail in the presence of excess intersection. The derived formulation is therefore not merely a technical refinement; it is the functorially correct setting for non-transversal boundary intersections.

Proposition 6.3 (Excess-intersection interpretation). *Let $f : \mathcal{Y} \to X$ be an lci-type morphism and let*

$$\mathcal{D} := \mathcal{Y} \times^{\mathbf{L}}_X D$$

be the derived boundary. Let

$$\mathcal{D}_{\mathrm{cl}} := \mathcal{Y} \times_X D$$

denote the classical boundary intersection. Assume that the intersection is not Tor-independent, equivalently that locally

$$\mathrm{Tor}_1^{\mathcal{O}_X}(\mathcal{O}_{\mathcal{Y}}, \mathcal{O}_D) \neq 0.$$

Then the virtual structure sheaf of the derived boundary records the excess intersection contribution:

$$[\mathcal{O}_{\mathcal{D}}] = \sum_{q \geq 0} (-1)^q \big[\, \mathrm{Tor}_q^{\mathcal{O}_X}(\mathcal{O}_{\mathcal{Y}}, \mathcal{O}_D)\big] \quad \text{in } K_0(\mathcal{D}_{\mathrm{cl}}).$$

In particular, the excess class is

$$e_{\mathrm{ex}}(\mathcal{Y}, D/X) := \sum_{q \geq 1} (-1)^q \big[\, \mathrm{Tor}_q^{\mathcal{O}_X}(\mathcal{O}_{\mathcal{Y}}, \mathcal{O}_D)\big] \in K_0(\mathcal{D}_{\mathrm{cl}}),$$

so that

$$[\mathcal{O}_{\mathcal{D}}] = [\mathcal{O}_{\mathcal{D}_{\mathrm{cl}}}] + e_{\mathrm{ex}}(\mathcal{Y}, D/X).$$

For every operational Chow class $\alpha \in \mathrm{CH}^*(X)_{\mathbb{Q}}$, the derived refined boundary restriction satisfies

$$i^!_{\mathcal{Y}} f^* \alpha = f_D^* i^! \alpha \qquad \text{in } \mathrm{CH}^*(\mathcal{D})_{\mathbb{Q}}.$$

Heuristically, after passing to the classical truncation, the higher Tor terms encoded by $e_{\mathrm{ex}}(\mathcal{Y}, D/X)$ record the excess-intersection information carried by the derived boundary. We do not require an explicit Chow-theoretic excess operator associated with this K-theory class; the derived refined identity already incorporates the required functorial contribution on the derived boundary. In particular, for $\alpha = c_g(\mathbb{E})$,

$$i^!_{\mathcal{Y}} c_g(Lf^*\mathbb{E}) = f_D^* i^! c_g(\mathbb{E}).$$

Hence, if

$$i^! c_g(\mathbb{E}) = 0 \quad \text{in } \mathrm{CH}^g(D)_{\mathbb{Q}},$$

then both the derived boundary restriction and the corresponding virtual excess contribution of the pulled-back top Hodge class vanish.

Proof. Since $\mathcal{D} = \mathcal{Y} \times^{\mathbf{L}}_X D$, its structure sheaf is represented by the derived tensor product

$$\mathcal{O}_{\mathcal{D}} \simeq \mathcal{O}_{\mathcal{Y}} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_D.$$

Passing to K_0 gives the alternating Tor formula

$$[\mathcal{O}_{\mathcal{D}}] = \sum_{q \geq 0} (-1)^q \big[\, \mathrm{Tor}_q^{\mathcal{O}_X}(\mathcal{O}_{\mathcal{Y}}, \mathcal{O}_D)\big].$$

The degree-zero term is the structure sheaf of the classical intersection $\mathcal{D}_{\mathrm{cl}} = \mathcal{Y} \times_X D$, while the higher Tor terms measure the failure of Tor-independence. This gives the asserted excess class $e_{\mathrm{ex}}(\mathcal{Y}, D/X)$.

The refined identity

$$i^!_{\mathcal{Y}} f^* \alpha = f_D^* i^! \alpha$$

is precisely the derived base-change compatibility of Lemma 3.2. The point is that the equality is taken on the derived boundary $\mathcal{D}$, whose virtual structure sheaf already contains the excess Tor contribution. Thus no separate correction term is needed in the derived formula; the correction appears only when one compares the derived boundary with the classical truncation.

Applying the identity to $\alpha = c_g(\mathbb{E})$ and using the functoriality

$$c_g(Lf^*\mathbb{E}) = f^* c_g(\mathbb{E})$$

gives

$$i^!_{\mathcal{Y}} c_g(Lf^*\mathbb{E}) = f_D^* i^! c_g(\mathbb{E}).$$

If $i^! c_g(\mathbb{E}) = 0$, then the right-hand side is zero, and hence the derived boundary restriction of $c_g(Lf^*\mathbb{E})$ vanishes. Thus the vanishing follows from the derived refined base-change identity itself, without introducing any separate Chow-theoretic excess correction term. ∎

Example 6.4 (A derived complete-intersection test family computed by a Koszul complex)**.** We now make the excess term in Proposition 6.3 completely explicit in a local complete-intersection test family.

Let

$$X = \operatorname{Spec} R, \qquad R = k[x, y, z],$$

and let the boundary divisor be

$$D = V(x) \hookrightarrow X.$$

Thus D is Cartier, with

$$\mathcal{O}_D = R/(x).$$

Let

$$S = V(xy, z) \hookrightarrow X.$$

Since xy, z is a regular sequence in R, the morphism

$$f : S \longrightarrow X$$

is a complete-intersection, hence lci-type, test family.

The ordinary boundary intersection is

$$S \times_X D = \operatorname{Spec} R/(xy, z, x) = \operatorname{Spec} k[y].$$

However, this ordinary intersection forgets the derived excess information. Indeed, the derived boundary is

$$S_D := S \times^{\mathbf{L}}_X D,$$

and its structure sheaf is computed by the derived tensor product

$$\mathcal{O}_{S_D} \simeq \mathcal{O}_S \otimes^{\mathbf{L}}_R \mathcal{O}_D.$$

Using the Koszul resolution of $\mathcal{O}_D = R/(x)$, this is represented by the two-term complex

$$\left[R/(xy, z) \xrightarrow{x} R/(xy, z) \right].$$

Therefore

$$H^0(\mathcal{O}_{S_D}) = R/(xy, z, x) = k[y],$$

while

$$H^{-1}(\mathcal{O}_{S_D}) = \ker\big(x : R/(xy, z) \to R/(xy, z)\big).$$

Since $xy = 0$ in $R/(xy, z)$, multiplication by x kills the class of y. Hence

$$H^{-1}(\mathcal{O}_{S_D}) \simeq (y)/(xy, z) \simeq k[y].$$

Equivalently,

$$\operatorname{Tor}_1^R(\mathcal{O}_S, \mathcal{O}_D) \simeq k[y] \neq 0.$$

Thus the virtual structure sheaf of the derived boundary is

$$[\mathcal{O}_{S_D}] = [\mathcal{O}_{S \times_X D}] - [\operatorname{Tor}_1^R(\mathcal{O}_S, \mathcal{O}_D)] \quad \text{in } K_0(S \times_X D).$$

In this example,

$$[\mathcal{O}_{S \times_X D}] = [k[y]], \qquad [\operatorname{Tor}_1^R(\mathcal{O}_S, \mathcal{O}_D)] = [k[y]],$$

so the excess class is

$$e_{\mathrm{ex}}(S, D/X) = -[k[y]].$$

This explicitly illustrates Proposition 6.3: the classical intersection $S \times_X D$ sees only the quotient $k[y]$, whereas the derived intersection remembers the additional Tor_1-term. Consequently, the refined boundary restriction in the derived formulation is computed on S_D, where the higher Tor contribution is already encoded in the virtual structure sheaf.

Proposition 6.5 (Refined base-change compatibility for Hodge classes)**.** *Let $f : \mathcal{Y} \to X$ be an lci-type morphism, and let*

$$\mathcal{D} := \mathcal{Y} \times^{\mathbf{L}}_X D$$

be the derived boundary. Then the refined boundary restriction of the pulled-back top Hodge class is the pullback of the original boundary restriction:

$$i_{\mathcal{Y}}^* c_g(Lf^*\mathbb{E}) = f_D^* i^* c_g(\mathbb{E}) \qquad \text{in } \operatorname{CH}^g(\mathcal{D})_{\mathbb{Q}}.$$

In particular, this identity holds without imposing any Tor-independence assumption between f and the boundary divisor $i : D \hookrightarrow X$.

Proof. The assertion is the refined compatibility underlying Theorem 4.1. Since $\mathbb{E}$ is locally free of rank g on X, Lemma 3.3 gives that $Lf^*\mathbb{E}$ is perfect on $\mathcal{Y}$ and that its top Chern class is functorial under derived pullback:

$$c_g(Lf^*\mathbb{E}) = f^*c_g(\mathbb{E}) \qquad \text{in operational Chow theory.}$$

Applying the refined boundary pullback associated with

$$i_{\mathcal{Y}} : \mathcal{D} \hookrightarrow \mathcal{Y}$$

to both sides gives

$$i_{\mathcal{Y}}^* c_g(Lf^*\mathbb{E}) = i_{\mathcal{Y}}^* f^* c_g(\mathbb{E}).$$

By construction,

$$\mathcal{D} = \mathcal{Y} \times_X^{\mathbf{L}} D,$$

so the square

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{f_D} & D \\ {\scriptstyle i_{\mathcal{Y}}}\downarrow & & \downarrow{\scriptstyle i} \\ \mathcal{Y} & \xrightarrow[f]{} & X \end{array}$$

is Cartesian in the derived sense. Lemma 3.2 gives refined Gysin base-change compatibility for this derived Cartesian square. Therefore, for the operational class $c_g(\mathbb{E})$, one has

$$i_{\mathcal{Y}}^* f^* c_g(\mathbb{E}) = f_D^* i^* c_g(\mathbb{E}).$$

Combining the preceding identities yields

$$i_{\mathcal{Y}}^* c_g(Lf^*\mathbb{E}) = f_D^* i^* c_g(\mathbb{E}),$$

as claimed.

The significance of the formula is that the boundary restriction on $\mathcal{Y}$ is computed on the derived boundary $\mathcal{D}$, not on the ordinary fiber product $\mathcal{Y} \times_X D$. Hence possible excess-intersection contributions caused by non-transversal meeting with D are already incorporated in the target of the refined pullback. □

Proposition 6.6 (Restriction to regular complete-intersection test families)**.** *Let*

$$S \hookrightarrow X$$

be a regular immersion, and let

$$S_D := S \times_X^{\mathbf{L}} D$$

be its derived boundary. Assume that

$$i^* c_g(\mathbb{E}) = 0 \qquad \text{in } \mathrm{CH}^g(D)_{\mathbb{Q}}.$$

Then the restriction of the top Hodge class to S is boundary-vanishing:

$$i_S^* c_g(\mathbb{E}|_S) = 0 \qquad \text{in } \mathrm{CH}^g(S_D)_{\mathbb{Q}}.$$

Equivalently, the top Hodge class remains trivial on the derived boundary of every regular complete-intersection test family contained in X.

Proof. A regular immersion is lci, so the structural morphism

$$f : S \hookrightarrow X$$

satisfies the hypotheses of Theorem 4.1. The derived boundary of S is precisely

$$S_D = S \times_X^{\mathbf{L}} D.$$

Since $\mathbb{E}$ is locally free on X, its derived pullback along the regular immersion f is represented by the ordinary restriction:

$$Lf^*\mathbb{E} \simeq \mathbb{E}|_S.$$

Consequently,

$$c_g(Lf^*\mathbb{E}) = c_g(\mathbb{E}|_S).$$

Applying Theorem 4.1 to the lci-type morphism $f : S \to X$ yields

$$i_S^* c_g(Lf^*\mathbb{E}) = 0 \qquad \text{in } \mathrm{CH}^g(S_D)_{\mathbb{Q}}.$$

Substituting the identification $Lf^*\mathbb{E} \simeq \mathbb{E}|_S$ gives

$$i_S^* c_g(\mathbb{E}|_S) = 0 \qquad \text{in } \mathrm{CH}^g(S_D)_{\mathbb{Q}},$$

which proves the claim. □

Remark 6.7. The proposition provides the geometric interpretation of Theorem 4.1. In practice, regular immersions arise from complete-intersection subvarieties, modular test families, and tautological cycles inside toroidal compactifications. The statement shows that boundary vanishing of the top Hodge class is inherited by all such test families after passage to their derived boundaries.

When S meets the boundary divisor D transversely, the derived boundary S_D coincides with the ordinary intersection $S \cap D$, and the proposition reduces to the classical restriction statement. When the intersection is non-transversal, the derived boundary retains the excess intersection data, and the same vanishing identity remains valid without any additional correction term.

7 Necessity of the Hypotheses

The assumptions in Theorem 4.1 are not cosmetic. Regularity of X, the Cartier nature of the boundary, and the lci-type condition on f are used to ensure that refined pullbacks behave predictably.

Example 7.1 (Why the lci-type hypothesis cannot simply be omitted)**.** Let $X = \mathcal{A}^1_k = \operatorname{Spec} k[t]$, and let the boundary divisor be

$$D = V(t) \hookrightarrow X.$$

This is the local model for a Cartier boundary divisor on a toroidal compactification. Consider now the derived boundary construction after a test morphism

$$f : \mathcal{Y} \longrightarrow X.$$

When f is lci-type, the relative cotangent complex $\mathbb{L}_{\mathcal{Y}/X}$ has Tor-amplitude contained in $[-1, 0]$, and the derived boundary

$$\mathcal{D} = \mathcal{Y} \times^L_X D$$

is again controlled by a two-term perfect complex. This is precisely the situation used in Lemma 3.2 and Theorem 4.1.

We now indicate what can go wrong outside this range. Let

$$A = k[t, x_1, x_2, x_3, \ldots]$$

and let $I \subset A$ be an ideal generated by an infinite sequence of relations

$$tx_1, \quad x_1x_2, \quad x_2x_3, \quad x_3x_4, \ldots.$$

Put

$$\mathcal{Y} = \operatorname{Spec}(A/I)$$

and let $f : \mathcal{Y} \to X = \operatorname{Spec} k[t]$ be induced by the inclusion $k[t] \to A/I$.

The morphism f is not lci-type. Indeed, the ideal I is not locally generated by a finite regular sequence, and the relative cotangent complex $\mathbb{L}_{\mathcal{Y}/X}$ is not represented by a perfect two-term complex. Equivalently, the derived tensor product

$$\mathcal{O}_{\mathcal{Y}} \otimes^L_{k[t]} k$$

need not be controlled by a single excess-intersection term. The derived boundary

$$\mathcal{Y} \times^L_X D$$

can therefore carry higher Tor contributions beyond the virtual codimension-one behavior expected from a derived Cartier divisor.

This destroys the mechanism used in the proof of Theorem 4.1. In the lci-type case, the refined base-change identity

$$i^!_{\mathcal{Y}} f^*(\alpha) = f^*_D i^!(\alpha)$$

is an identity between well-defined operational Chow classes on a controlled derived boundary. In the present non-lci situation, the morphism $i_{\mathcal{Y}} : \mathcal{Y} \times^L_X D \to \mathcal{Y}$ need not be a quasi-smooth virtual Cartier divisor, so the refined pullback $i^!_{\mathcal{Y}}$ may not have the expected codimension-one meaning.

Even if $\mathbb{E}$ is locally free on X, the formal expression

$$c_g(Lf^*\mathbb{E})$$

no longer has the same functorial force unless one separately imposes perfectness and bounded Tor-amplitude hypotheses on the pullback object and on the derived boundary square. Thus the conclusion

$$i^*_{\mathcal{Y}} c_g(Lf^*\mathbb{E}) = 0$$

is not merely unproved without the lci-type assumption; rather, it lies outside the formal framework developed in Sections 3–4 unless one separately supplies the perfectness, bounded Tor-amplitude, and refined Gysin compatibility hypotheses needed for the derived boundary square.

This example shows that the lci-type hypothesis in Theorem 4.1 is not a cosmetic smoothness assumption. It is the condition which ensures that the derived boundary remains a controlled virtual Cartier divisor and that the refined Gysin compatibility used in the proof is actually available.

Example 7.2 (Failure when the boundary is not Cartier)**.** The Cartier hypothesis on the boundary divisor is essential for the functorial mechanism underlying Theorem 4.1. To illustrate this, consider a hypothetical compactification X whose boundary

$$D = X \setminus \mathcal{A}_g$$

is not a Cartier divisor.

Locally, a Cartier boundary is cut out by a single non-zero-divisor $t \in \mathcal{O}_X$. This property allows one to view the inclusion

$$i : D \hookrightarrow X$$

as a regular immersion of codimension one. Consequently, the refined Gysin pullback

$$i^! : \mathrm{CH}^*(X)_\mathbb{Q} \to \mathrm{CH}^*(D)_\mathbb{Q}$$

is defined in the classical Cartier/regular-immersion case [1, Prop. 2.6, pp. 43–44; Thm. 6.2, pp. 98–99] and satisfies the refined compatibility used in Lemma 3.2.

When D is not Cartier, no such local equation need exist. For example, suppose that near a boundary point the ideal sheaf of D is generated by several equations

$$I_D = (t_1, t_2) \subset \mathcal{O}_X,$$

and that I_D fails to be locally principal. Then the inclusion

$$i : D \hookrightarrow X$$

is generally not a regular immersion of codimension one. The boundary may possess embedded components or singularities, and there is no canonical virtual normal line bundle analogous to the Cartier case.

Now let

$$f : \mathcal{Y} \longrightarrow X$$

be an lci-type test morphism and form the derived boundary

$$\mathcal{D} = \mathcal{Y} \times^L_X D.$$

In the proof of Theorem 4.1, the crucial step is the refined base-change identity

$$i^!_\mathcal{Y} f^*(\alpha) = f^*_D\big(i^!(\alpha)\big),$$

which is applied to the Hodge class $\alpha = c_g(\mathbb{E})$.

The difficulty is that the left-hand side requires a refined Gysin map associated with

$$i_\mathcal{Y} : \mathcal{D} \hookrightarrow \mathcal{Y},$$

while the right-hand side requires a refined Gysin map for

$$i : D \hookrightarrow X.$$

If the boundary is not Cartier, these maps need not exist in the codimension-one form used throughout the paper. Consequently, the fundamental compatibility

$$i^!_\mathcal{Y} f^* = f^*_D i^!$$

can no longer be invoked.

Thus, even if the Hodge bundle $\mathbb{E}$ remains locally free and even if the derived pullback $Lf^*\mathbb{E}$ is perfect, the argument proving

$$i^*_\mathcal{Y} c_g(Lf^*\mathbb{E}) = 0$$

breaks down because the boundary restriction is no longer controlled by a codimension-one refined Gysin formalism.

This example shows that the Cartier condition is not merely a technical assumption. It is precisely the hypothesis that makes the boundary a virtual codimension-one object, ensures the existence of refined Gysin pullbacks, and allows the derived base-change argument of Theorem 4.1 to operate.

Remark 7.3. One may replace the lci-type hypothesis by a different condition, provided it guarantees perfectness of $Lf^*\mathbb{E}$ and compatibility of refined pullback with the derived boundary square. The present formulation is chosen because it is stable under composition and occurs naturally for complete-intersection test families.

8 Arithmetic Boundary Vanishing

We work in the framework of arithmetic Chow groups and arithmetic intersection products of Gillet–Soulé, in which arithmetic cycles are represented by algebraic cycles together with Green currents and the resulting arithmetic Chow groups carry product and functoriality structures [10, Secs. 3.3, 4.2–4.4]. The result below is conditional on the existence of arithmetic refined pullbacks for the derived boundary square and on their compatibility with arithmetic Chern classes. The result is conditional on the existence of arithmetic refined pullbacks for the derived boundary square and on their compatibility with arithmetic Chern classes.

Let $X = \mathcal{A}_g^{\mathrm{tor}}$ be an arithmetic toroidal compactification over $\operatorname{Spec}\mathbb{Z}$, with boundary $i : D \hookrightarrow X$. Let $\overline{\mathbb{E}}$ be the extended Hodge bundle equipped with an admissible hermitian metric, and write

$$\widehat{c}_g(\overline{\mathbb{E}}) \in \widehat{\mathrm{CH}}^g(X)_{\mathbb{Q}}.$$

Theorem 8.1 (Conditional arithmetic preservation of boundary vanishing)**.** *Let $f : \mathcal{Y} \to X$ be an lci-type derived base change and put*

$$\mathcal{D} = \mathcal{Y} \times_X^{\mathbf{L}} D.$$

Assume that:

1. *arithmetic refined pullbacks*
$$\widehat{i}^*, \qquad \widehat{i}^*_{\mathcal{Y}}, \qquad \widehat{f}^*, \qquad \widehat{f}^*_D$$
are defined in the chosen arithmetic Chow theory;
2. *arithmetic Chern classes are functorial for the hermitian lci pullback f, so that*
$$\widehat{c}_g(Lf^*\overline{\mathbb{E}}) = \widehat{f}^*\widehat{c}_g(\overline{\mathbb{E}});$$
3. *the arithmetic refined base-change identity*
$$\widehat{i}^*_{\mathcal{Y}}\widehat{f}^* = \widehat{f}^*_D\widehat{i}^*$$
holds for the derived boundary square.

If

$$\widehat{i}^*\widehat{c}_g(\overline{\mathbb{E}}) = 0 \quad \text{in} \quad \widehat{\mathrm{CH}}^g(D)_{\mathbb{Q}},$$

then

$$\widehat{i}^*_{\mathcal{Y}}\widehat{c}_g(Lf^*\overline{\mathbb{E}}) = 0 \quad \text{in} \quad \widehat{\mathrm{CH}}^g(\mathcal{D})_{\mathbb{Q}}.$$

Proof. By functoriality of arithmetic Chern classes under the assumed hermitian lci pullback,

$$\widehat{c}_g(Lf^*\overline{\mathbb{E}}) = \widehat{f}^*\widehat{c}_g(\overline{\mathbb{E}}).$$

Applying the arithmetic refined boundary pullback gives

$$\widehat{i}^*_{\mathcal{Y}}\widehat{c}_g(Lf^*\overline{\mathbb{E}}) = \widehat{i}^*_{\mathcal{Y}}\widehat{f}^*\widehat{c}_g(\overline{\mathbb{E}}).$$

Using the assumed arithmetic refined base-change identity, this becomes

$$\widehat{f}^*_D\widehat{i}^*\widehat{c}_g(\overline{\mathbb{E}}).$$

The latter class is zero by the arithmetic boundary-vanishing hypothesis. Hence

$$\widehat{i}^*_{\mathcal{Y}}\widehat{c}_g(Lf^*\overline{\mathbb{E}}) = 0.$$

□

Corollary 8.2 (Derived-arithmetic boundary invariant)**.** *Under the hypotheses of Theorem 8.1, the class*

$$\widehat{c}_g(Lf^*\overline{\mathbb{E}})$$

defines a derived-arithmetic boundary invariant independent of the chosen lci-type test family, in the sense that its arithmetic boundary restriction is functorially determined by the original arithmetic boundary class

$$\widehat{i}^*\widehat{c}_g(\overline{\mathbb{E}})$$

through the identity

$$\widehat{i}^*_{\mathcal{Y}}\widehat{c}_g(Lf^*\overline{\mathbb{E}}) = \widehat{f}^*_D\widehat{i}^*\widehat{c}_g(\overline{\mathbb{E}}).$$

In particular, if

$$\widehat{i}^*\widehat{c}_g(\overline{\mathbb{E}}) = 0,$$

then the induced derived-arithmetic boundary class on $\mathcal{D} = \mathcal{Y} \times_X^{\mathbf{L}} D$ is zero for every admissible lci-type derived test family.

Proof. The asserted identity is exactly the arithmetic refined base-change identity of Theorem 8.1 applied to the arithmetic top Chern class $\widehat{c}_g(\overline{\mathbb{E}})$, together with the assumed functoriality

$$\widehat{c}_g(Lf^*\overline{\mathbb{E}}) = \widehat{f}^*\widehat{c}_g(\overline{\mathbb{E}}).$$

Thus the derived-arithmetic boundary restriction of the pulled-back Hodge class depends only on the original arithmetic boundary restriction on D, not on any additional choice of the lci-type test family. The final vanishing statement follows immediately when this original arithmetic boundary class is zero. □

Remark 8.3. The theorem is not used in the proof of the geometric results above. Its role is to isolate the exact arithmetic input needed to transport the derived boundary-vanishing argument to arithmetic Chow theory. Thus the non-formal arithmetic content is contained in the existence and base-change compatibility of the arithmetic refined pullbacks, not in the final vanishing implication.

9 Conclusion

We proved that boundary vanishing of the top Chern class of the Hodge bundle on a toroidal compactification is preserved under lci-type derived base change. The central mechanism is the derived boundary square

$$\mathcal{D} = \mathcal{Y} \times^{\mathbf{L}}_X D,$$

which gives the correct functorial replacement for ordinary boundary restriction.

The result gives a compact principle for transporting tautological vanishing statements from $\mathcal{A}_g^{\mathrm{tor}}$ to derived test spaces, complete-intersection families, and non-transversal boundary intersections. Its novelty lies in combining boundary vanishing with derived base change rather than treating the two phenomena separately.

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