

Filtered Homological Newton Stratifications and Factorization Stability over Valuation Rings

Kundnani Rahul Thakurdas^{1,*}, Vinoth Marimuthu²,
Dr. Khursheed Alam^{3,*}, Dr. Shri Kant Ojha⁴

^{1,3,4}Department of Mathematics & Data Science, Sharda University, Greater Noida, India

²University of Colorado Boulder, Boulder, CO 80309, United States

*Corresponding Author: khursheed.alam@sharda.ac.in

Abstract

We study the behaviour of Newton polygons attached to monic polynomial families over valuation rings under specialization. Using valuation-theoretic Newton polygon stratifications together with filtered coefficient modules and admissible filtered free resolutions, we introduce a derived Newton defect measuring the failure of exactness after passage to associated graded Newton data. We prove that, under Henselianity, residual coprimality, torsion-control, and bounded filtered-resolution hypotheses, constancy of the derived Newton defect along a specialization implies stability of lifted slope-block factorization type. The resulting framework provides a filtered-homological refinement of classical Newton polygon methods and Henselian factorization theory for deformation families over valuation rings.

Keywords: valuation rings, Newton polygons, Henselian rings, factorization, filtered complexes, derived functors, integral extensions.

MSC 2020: 13F30, 13A18, 13D02, 12J10, 14B05.

1 Introduction

Newton polygons provide a classical method for studying factorization and ramification over valued fields and valuation rings. In the presence of Henselian hypotheses, slopes and residual polynomials often control factorization behaviour. However, in deformation families, Newton polygon data may jump, and the precise homological obstruction to stability is not always visible at the level of the polynomial alone.

The purpose of this paper is to introduce a valuation-theoretic derived refinement of Newton polygon stratification. The word “derived” is used here in the commutative-algebraic sense: filtered modules, resolutions, and Tor–Ext obstructions. No higher-categorical or spectral algebraic geometry machinery is used.

Main contribution

We attach to a family of monic polynomials over a valuation ring a chosen admissible filtered Newton complex whose associated graded pieces record Newton polygon data. The homology of the associated graded augmented complex defines a relative derived Newton defect, depending on the chosen admissible filtered resolution unless strict filtered comparison data are specified. Our main theorem shows that factorization stability follows under suitable Henselian assumptions when the chosen admissible filtered Newton complexes are strictly compared at the associated graded level; the resulting constancy of the relative defect is therefore used together with compatible graded Newton-complex data, not as a bare homology isomorphism.

Theorem 1.1 (Main theorem, informal). *Let V be a Henselian valuation ring and let $F(T) \in V[S][T]$ be a monic polynomial family. Restrict to V -valued specializations $\sigma : V[S] \rightarrow V$, so that the specialized polynomials $F_\sigma(T)$ lie in $V[T]$. Fix admissible filtered Newton complexes for the specialized coefficient modules, or assume strict filtered comparison data between the chosen complexes. If the associated Newton polygon stratification has constant slope data and the corresponding relative derived Newton defect vanishes, or remains constant, along such V -valued specializations, then the lifted slope-block factorization type is stable over $V[T]$.*

2 Preliminaries on Valuation Rings and Newton Polygons

Definition 2.1 (Valuation ring). *Let K be a field. A subring $V \subset K$ is a valuation ring if for every $x \in K^\times$, either $x \in V$ or $x^{-1} \in V$ ([5, Ch. VI, §1, no. 2]; [1, Ch. VI, §2, Thm. 1, p. 5]; see also [9, §6, p. 40; §10, p. 71]).*

Definition 2.2 (Newton polygon). *Let V be a valuation ring with valuation ν , and let*

$$f(T) = a_0 + a_1T + \cdots + a_nT^n \in V[T], \quad a_n \neq 0.$$

The Newton polygon $\text{NP}(f)$ is the lower convex hull of the points

$$(i, \nu(a_i)), \quad 0 \leq i \leq n$$

([8, Ch. 5, §5.1, pp. 133–134]).

Remark 2.3. *Throughout the paper, valuation-theoretic conventions are chosen so that slopes of Newton polygons correspond to valuations of roots after passing to suitable extensions ([8, Ch. 5, §5.1, pp. 134–138, especially (C) and (F)]).*

3 Newton Polygon Stratifications

Definition 3.1 (Newton polygon stratum). *Let V be a valuation ring with fraction field K , valuation ν , and closed point \mathfrak{m}_V . Let A be a V -algebra and let*

$$F(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_0 \in A[T].$$

For $x \in \text{Spec } A$ lying over the closed point of $\text{Spec } V$, choose a valuation ν_x on $\kappa(x)$ extending the valuation induced by V . For a fixed polygon Δ , the Newton polygon stratum associated to Δ is the locus of such valued points (x, ν_x) for which

$$\text{NP}_{\nu_x}(F_x) = \Delta.$$

Thus the stratum is a valuation-theoretic stratum of valued specializations, not an intrinsic Zariski subset of $\text{Spec } A$ without the chosen extension of valuation.

In later statements where the specialized polynomial is required to lie in $V[T]$, we restrict from arbitrary valued residue-field points (x, ν_x) to V -valued specializations, i.e. V -algebra maps $\sigma : A \rightarrow V$. For such a specialization we write $F_\sigma(T) := (\sigma \otimes \text{id})(F) \in V[T]$.

Lemma 3.2 (Valuation-wall description of polygon jumps). *Let V be a valuation ring with valuation ν , and let A be a finitely generated V -algebra. Let*

$$F(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_0 \in A[T]$$

be monic. On the space of valued specializations (x, ν_x) , where $x \in \text{Spec } A$ lies over the closed point of $\text{Spec } V$ and ν_x is a chosen extension of the valuation induced from V to $\kappa(x)$, the Newton polygon of F_x is determined by finitely many valuation comparisons among the specialized coefficients.

In particular, Newton polygon data can change only when one of these comparisons changes status. We use the informal term “valuation wall” to denote loci where one of these defining valuation comparisons changes status. Thus polygon jumps occur along such valuation walls determined by coefficient-valuation relations.

Proof. Write

$$F(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_0.$$

Let (x, ν_x) be a valued specialization, with $x \in \text{Spec } A$ lying over the closed point of $\text{Spec } V$, and with ν_x an extension of the valuation induced from V to the residue field $\kappa(x)$. The specialized polynomial is

$$F_x(T) = T^n + a_{n-1}(x)T^{n-1} + \cdots + a_0(x) \in \kappa(x)[T].$$

Its Newton polygon is the lower convex hull of the finite set of points

$$(i, \nu_x(a_i(x))) \quad 0 \leq i \leq n,$$

where the leading coefficient has valuation 0, and where a coefficient specializing to zero is assigned valuation $+\infty$, equivalently omitted from the lower hull. Hence the polygon is determined by finitely many comparisons among the values

$$\nu_x(a_0(x)), \dots, \nu_x(a_{n-1}(x)), 0.$$

We now make these comparisons explicit. Fix indices $0 \leq i < j \leq n$. The segment joining the coefficient points

$$(i, \nu_x(a_i(x))) \quad \text{and} \quad (j, \nu_x(a_j(x)))$$

can occur as a lower face only if every intermediate coefficient point lies on or above the affine line through these two endpoints. For each $i < \ell < j$, this condition is precisely

$$(j - i)\nu_x(a_\ell(x)) \geq (j - \ell)\nu_x(a_i(x)) + (\ell - i)\nu_x(a_j(x)).$$

Equality records that the intermediate point lies on the same face, while strict inequality records that it lies strictly above the face. Therefore, for every candidate lower convex hull, the corresponding polygon type is determined by finitely many valuation inequalities and equalities among the specialized coefficients.

Since the polynomial has only the finitely many coefficients $a_0, \dots, a_{n-1}, 1$, only finitely many systems of such comparisons can occur. Consequently, only finitely many Newton polygon configurations are possible for the family F . Passing from one configuration to another can occur only when at least one defining comparison changes status: an inequality becomes an equality, an equality becomes strict, or a coefficient specializes to zero and therefore acquires valuation $+\infty$.

These transition conditions are valuation-theoretic conditions on the specialized coefficients. They should be understood as valuation walls in the space of valued specializations, not as intrinsic Zariski-closed subsets of $\text{Spec } A$ without additional hypotheses. Thus the Newton polygon is stable as long as the finite system of valuation comparisons remains unchanged, and any jump of polygon data is forced by crossing one of these coefficient-valuation walls. \square

4 Filtered Complexes Attached to Newton Data

Definition 4.1 (Newton coefficient module and Newton filtration). *Let*

$$F(T) = \sum_{i=0}^n a_i T^i \in V[T],$$

and let

$$E := \bigoplus_{i=0}^n V e_i$$

be the free V -module with distinguished basis e_0, \dots, e_n . We use the convention

$$\nu(0) := +\infty.$$

The Newton coefficient module of F is the labelled filtered submodule

$$M_F := \bigoplus_{\substack{0 \leq i \leq n \\ a_i \neq 0}} V a_i e_i \subseteq E,$$

together with the zero-labelled set

$$Z(F) := \{i \mid a_i = 0\}.$$

For each real number λ , define the decreasing Newton filtration by

$$\mathcal{F}^\lambda M_F := \bigoplus_{\substack{0 \leq i \leq n \\ a_i \neq 0, \nu(a_i) \geq \lambda}} V a_i e_i.$$

The indices in $Z(F)$ are not generators of M_F ; they are recorded separately as coefficient labels of valuation $+\infty$, equivalently as omitted points in the lower Newton hull. Thus M_F , together with its filtration and the zero-labelled set $Z(F)$, is the filtered coefficient datum attached to the Newton data of F .

Remark 4.2 (Standing finiteness convention for derived Newton data). *Throughout Sections 4–6, whenever the derived Newton complex or the derived Newton defect $\mathfrak{D}_{\text{NP}}(F)$ is used, we restrict attention to monic polynomials F for which the Newton coefficient module M_F , with its Newton filtration, admits a bounded filtered free resolution. All subsequent statements involving $\mathfrak{D}_{\text{NP}}(F)$ are made under this standing hypothesis.*

Remark 4.3 (Existence convention). *The paper does not claim existence of admissible derived Newton complexes for arbitrary filtered coefficient modules over arbitrary valuation rings. Throughout, all statements involving derived Newton defects are understood only for those monic polynomials F for which such bounded filtered free resolutions are given or known to exist.*

Definition 4.4 (Admissible derived Newton complex). *An admissible derived Newton complex for F is a bounded augmented complex of filtered free V -modules*

$$C_F^\bullet \longrightarrow M_F$$

satisfying the following four conditions.

- (a) *The augmentation $C_F^0 \rightarrow M_F$ is a filtered surjection and the augmented complex $C_F^\bullet \rightarrow M_F$ is exact after forgetting the filtration.*
- (b) *Each differential $d^q : C_F^q \rightarrow C_F^{q+1}$ and the augmentation are strict filtered morphisms, namely*

$$d^q(C_F^q) \cap \mathcal{F}^\lambda C_F^{q+1} = d^q(\mathcal{F}^\lambda C_F^q)$$

for every relevant filtration level λ , and similarly for the augmentation $C_F^0 \rightarrow M_F$.

(c) The associated graded augmented complex

$$\mathrm{gr} C_F^\bullet \longrightarrow \mathrm{gr} M_F$$

is generated, in degree zero, by the initial forms of the nonzero labelled coefficient generators $a_i e_i$ with $a_i \neq 0$, while the zero-labelled indices $Z(F)$ are carried as omitted $+\infty$ -level coefficient labels; its higher terms present the syzygies among these nonzero initial forms in the associated graded valuation module.

(d) The filtration on every free summand of C_F^q is induced by assigning a Newton level to each chosen basis element; equivalently, the associated graded complex is a graded free presentation of the initial Newton coefficient module $\mathrm{gr} M_F$.

Thus admissibility is not merely a choice of a filtered resolution. It is the requirement that the resolution be strict levelwise and that its associated graded complex present the initial Newton-level coefficient relations. The complex is not asserted to be unique. Two choices are regarded as equivalent only when they are connected by a strict filtered quasi-isomorphism over M_F .

Definition 4.5 (Relative derived Newton defect). For an admissible derived Newton complex $C_F^\bullet \rightarrow M_F$, the relative derived Newton defect of the pair (F, C_F^\bullet) is

$$\mathfrak{D}_{\mathrm{NP}}(F; C_F^\bullet) = H^\bullet(\mathrm{gr} C_F^\bullet \longrightarrow \mathrm{gr} M_F).$$

It measures the failure of this chosen admissible filtered resolution to remain exact after passage to associated graded Newton data.

When the admissible complex is fixed throughout the discussion, we write $\mathfrak{D}_{\mathrm{NP}}(F)$ for brevity. This abbreviation is not meant to assert independence from the chosen filtered resolution.

Proposition 4.6 (Interpretation of the derived Newton defect). Let V be a valuation ring and let $F \in V[T]$ be monic. Assume that the Newton coefficient module M_F admits a bounded filtered free resolution, and let

$$C_F^\bullet \longrightarrow M_F$$

be one such resolution. Let

$$\mathrm{gr} C_F^\bullet \longrightarrow \mathrm{gr} M_F$$

denote the associated augmented graded complex. Then

$$\mathfrak{D}_{\mathrm{NP}}(F) = 0$$

if and only if the augmented associated graded complex

$$\mathrm{gr} C_F^\bullet \longrightarrow \mathrm{gr} M_F$$

is exact. Equivalently, passage to associated graded Newton data preserves exactness of the chosen filtered resolution. In this sense, the filtered resolution is compatible with the Newton filtration at the level of associated graded objects.

In particular, $\mathfrak{D}_{\mathrm{NP}}(F)$ is precisely the obstruction to exactness with respect to the Newton filtration after passage from filtered coefficient data to associated graded Newton data.

Proof. By construction, the Newton coefficient module M_F is equipped with its decreasing Newton filtration, and $C_F^\bullet \rightarrow M_F$ is a bounded filtered free resolution. Thus each differential

$$d^q : C_F^q \longrightarrow C_F^{q+1}$$

is filtration-preserving, and the augmentation

$$C_F^0 \longrightarrow M_F$$

is also a filtered morphism. Passing to associated graded objects therefore gives an augmented complex of graded V -modules

$$\mathrm{gr} C_F^\bullet \longrightarrow \mathrm{gr} M_F.$$

By definition of the derived Newton defect,

$$\mathfrak{D}_{\mathrm{NP}}(F) = H^\bullet(\mathrm{gr} C_F^\bullet \rightarrow \mathrm{gr} M_F),$$

where the homology is taken for the augmented associated graded complex. Consequently

$$\mathfrak{D}_{\mathrm{NP}}(F) = 0$$

if and only if this augmented associated graded complex is exact.

It remains to interpret this exactness as strictness of the Newton filtration. For each filtration level λ , write

$$F^\lambda C_F^q \subseteq C_F^q$$

for the corresponding filtered piece. Since d^q is filtered, one has

$$d^q(F^\lambda C_F^q) \subseteq F^\lambda C_F^{q+1}.$$

The associated graded differential in degree λ is induced by

$$F^\lambda C_F^q / F^{>\lambda} C_F^q \longrightarrow F^\lambda C_F^{q+1} / F^{>\lambda} C_F^{q+1}.$$

Thus a homogeneous cycle in $\mathrm{gr} C_F^q$ is represented by an element $z \in F^\lambda C_F^q$ such that

$$d^q z \in F^{>\lambda} C_F^{q+1}.$$

Such a cycle is a boundary in the augmented associated graded complex exactly when there exists $y \in F^\lambda C_F^{q-1}$ such that

$$z - d^{q-1} y \in F^{>\lambda} C_F^q.$$

Therefore exactness says that every relation which becomes a cycle at initial Newton level λ is already the initial form of a filtered boundary at the same Newton level.

This is precisely the filtration-compatibility condition described above. Indeed, compatibility of the filtered resolution with the Newton filtration means that the equality between cycles and boundaries is compatible with the Newton filtration, not merely after forgetting the filtration. Equivalently, no element becomes a boundary only after increasing its valuation level. If such an element existed, its initial form would define a nonzero homology class in the associated graded complex. Conversely, any nonzero homology class in the associated graded complex is represented by an initial Newton form of a filtered cycle which cannot be lifted to a boundary at the same filtration level; this is exactly a failure of compatibility with the Newton filtration at the associated graded level.

Hence the vanishing of

$$H^\bullet(\mathrm{gr} C_F^\bullet \rightarrow \mathrm{gr} M_F)$$

is equivalent to exactness compatible with the Newton filtration of the filtered resolution. In this sense $\mathfrak{D}_{\mathrm{NP}}(F)$ measures the obstruction created by hidden filtered extension terms: cycles and boundaries may coincide after forgetting the Newton filtration, but fail to coincide at their initial Newton levels. Therefore $\mathfrak{D}_{\mathrm{NP}}(F) = 0$ if and only if passage from filtered coefficient data to associated graded Newton data introduces no new homology. \square

Remark 4.7 (Dependence on the admissible filtered resolution). *The graded module*

$$H^\bullet(\mathrm{gr} C_F^\bullet \rightarrow \mathrm{gr} M_F)$$

is attached to the chosen admissible filtered resolution $C_F^\bullet \rightarrow M_F$. In this paper, no intrinsic independence statement is claimed for arbitrary bounded filtered free resolutions.

If two admissible filtered resolutions are connected by a strict filtered quasi-isomorphism over M_F , then their associated graded augmented complexes are quasi-isomorphic, and hence they define canonically isomorphic graded homology modules. Thus all subsequent uses of $\mathfrak{D}_{\text{NP}}(F)$ are to be read as $\mathfrak{D}_{\text{NP}}(F; C_F^\bullet)$ for the fixed admissible filtered Newton complex under consideration, unless strict filtered comparison data identify the defects attached to different admissible complexes.

Remark 4.8. *This definition is intentionally homological rather than higher-categorical. It records the failure of Newton filtration data to be exact after passage to associated graded objects. The invariant is used relative to the chosen admissible filtered Newton complex, or under strict filtered comparison data when two such complexes are compared.*

Proposition 4.9 (Slopes decomposition under a split slope refinement). *Let V be a valuation ring and let $F \in V[T]$ be monic. Suppose that the Newton polygon of F decomposes into pairwise distinct slope blocks*

$$\Delta(F) = \Delta_1 \cup \cdots \cup \Delta_r.$$

Assume that the residual polynomials attached to the distinct slope blocks are pairwise coprime. Suppose moreover that the chosen admissible filtered Newton complex $C_F^\bullet \rightarrow M_F$ admits a slope-compatible splitting after passing to associated graded objects:

$$\text{gr } C_F^\bullet \cong \bigoplus_{i=1}^r \text{gr } C_{F, \Delta_i}^\bullet, \quad \text{gr } M_F \cong \bigoplus_{i=1}^r \text{gr } M_{F, \Delta_i},$$

and that the augmentation $\text{gr } C_F^\bullet \rightarrow \text{gr } M_F$ respects these direct sum decompositions. Then

$$\mathfrak{D}_{\text{NP}}(F) \cong \bigoplus_{i=1}^r \mathfrak{D}_{\text{NP}, \Delta_i}(F).$$

Proof. Under the stated slope-compatible splitting, the augmented associated graded Newton complex decomposes as a finite direct sum of augmented complexes:

$$(\text{gr } C_F^\bullet \rightarrow \text{gr } M_F) \cong \bigoplus_{i=1}^r (\text{gr } C_{F, \Delta_i}^\bullet \rightarrow \text{gr } M_{F, \Delta_i}).$$

Taking homology commutes with finite direct sums. Hence

$$H^\bullet(\text{gr } C_F^\bullet \rightarrow \text{gr } M_F) \cong \bigoplus_{i=1}^r H^\bullet(\text{gr } C_{F, \Delta_i}^\bullet \rightarrow \text{gr } M_{F, \Delta_i}).$$

By definition, the left-hand side is $\mathfrak{D}_{\text{NP}}(F)$, and the i -th summand is $\mathfrak{D}_{\text{NP}, \Delta_i}(F)$. Therefore

$$\mathfrak{D}_{\text{NP}}(F) \cong \bigoplus_{i=1}^r \mathfrak{D}_{\text{NP}, \Delta_i}(F).$$

The residual coprimality hypothesis motivates and, in classical Henselian Newton settings, often supplies such slope separation; the formal conclusion used here is the stated split slope refinement. \square

5 Henselian Factorization and Stability

Lemma 5.1 (Henselian lifting of factorization). *Let V be a Henselian valuation ring and let $f \in V[T]$ be monic. If the reduction of f admits a coprime factorization, then this factorization lifts uniquely to $V[T]$.*

Proof of Lemma 5.1. Let \mathfrak{m} be the maximal ideal of V , and write

$$\bar{f} \in (V/\mathfrak{m})[T]$$

for the reduction of f . Suppose that

$$\bar{f} = \bar{g}\bar{h}$$

with $\bar{g}, \bar{h} \in (V/\mathfrak{m})[T]$ monic and

$$(\bar{g}, \bar{h}) = 1 \quad \text{in } (V/\mathfrak{m})[T].$$

Choose arbitrary monic lifts $g_0, h_0 \in V[T]$ of \bar{g}, \bar{h} , respectively. Since the reductions of g_0 and h_0 are coprime, there exist polynomials $\bar{a}, \bar{b} \in (V/\mathfrak{m})[T]$ such that

$$\bar{a}\bar{g} + \bar{b}\bar{h} = 1.$$

Lifting \bar{a}, \bar{b} to $a, b \in V[T]$, we obtain

$$ag_0 + bh_0 \equiv 1 \pmod{\mathfrak{m}}.$$

Thus g_0 and h_0 are comaximal modulo \mathfrak{m} .

The Henselian property of the local ring V , in the form of Hensel's factorization lemma ([10, Thm. 4.1.3(3)–(4), pp. 87–88] ; see also [3, Thm. 44.4, pp. 189–190]), now applies to the approximate factorization

$$f \equiv g_0h_0 \pmod{\mathfrak{m}}.$$

Since f is monic and the two reduced factors are coprime, Hensel's factorization lemma gives monic polynomials $g, h \in V[T]$ ([2, Thm. 8.3, p. 55]) such that

$$f = gh, \quad g \equiv g_0 \pmod{\mathfrak{m}}, \quad h \equiv h_0 \pmod{\mathfrak{m}}.$$

Equivalently,

$$\bar{g} = \bar{g}, \quad \bar{h} = \bar{h},$$

so the prescribed coprime factorization of the reduction lifts to $V[T]$.

It remains to check uniqueness. Suppose that

$$f = g_1h_1 = g_2h_2$$

are two monic factorizations in $V[T]$ lifting the same reduced factorization:

$$\bar{g}_1 = \bar{g}_2 = \bar{g}, \quad \bar{h}_1 = \bar{h}_2 = \bar{h}.$$

Because \bar{g} and \bar{h} are coprime, the Henselian lift of the idempotent decomposition determined by (\bar{g}, \bar{h}) is unique. Hence the two lifted factors must coincide:

$$g_1 = g_2, \quad h_1 = h_2.$$

Thus the coprime factorization of \bar{f} lifts uniquely to a monic factorization of f over $V[T]$. \square

Remark 5.2 (Associated graded valuation algebra). *For a valuation ring V with valuation filtration $\{F^\lambda V\}_\lambda$, we write*

$$\mathrm{gr}_\nu(V) = \bigoplus_{\lambda} F^\lambda V / F^{>\lambda} V$$

for the associated graded valuation algebra. Initial forms and residual polynomials are always understood relative to this associated graded construction. We write $\mathrm{in}_{\mathrm{NP}}(f)$ for the associated graded initial form of f determined by the Newton filtration.

Lemma 5.3 (Newton–Hensel lifting from slopewise residual data). *Let V be a Henselian valuation ring with valuation ν , and let $f \in V[T]$ be monic. Suppose that the Newton polygon of f decomposes into slope blocks*

$$\Delta(f) = \Delta_1 \cup \cdots \cup \Delta_r$$

with horizontal lengths n_i . For each block Δ_i , let ρ_i be the corresponding residual polynomial in the graded valuation algebra after the usual normalization of that face.

Assume that the ρ_i are pairwise coprime and that the associated graded Newton initial form admits a slopewise factorization

$$\mathrm{in}_{\mathrm{NP}}(f) = \rho_1 \cdots \rho_r$$

compatible with the face normalizations and the degrees n_i . Then there exist monic polynomials $f_i \in V[T]$ such that

$$f = f_1 \cdots f_r, \quad \deg f_i = n_i,$$

and the Newton polygon and residual polynomial of f_i are precisely those prescribed by Δ_i and ρ_i .

Moreover, the factorization is unique among monic factorizations with the same normalized residual slope data.

Proof. It is enough to treat two slope blocks; the general case follows by iteration. Let the two blocks have slopes λ_1 and λ_2 , horizontal lengths n_1, n_2 , and residual polynomials ρ_1, ρ_2 , with $(\rho_1, \rho_2) = 1$ in the appropriate graded residual polynomial ring.

Write the two endpoint vertices of the face Δ_i as

$$(a_i, \alpha_i), \quad (b_i, \beta_i), \quad n_i = b_i - a_i,$$

and let its slope be $-\mu_i$, so that

$$\mu_i = \frac{\alpha_i - \beta_i}{b_i - a_i}.$$

Choose a valued algebraic extension (K_i, ν_i) of the fraction field $K = \mathrm{Frac}(V)$ and an element $\pi_i \in K_i$ with $\nu_i(\pi_i) = \mu_i$. Let V_i be the valuation ring of K_i extending V , and replace it by its Henselization V_i^h . The Henselization of a valuation ring is again a valuation ring with the same value group (see [3, §43, Ex. 4, p. 188]). Since V is Henselian, the embedding $V \rightarrow V_i^h$ is local and valuation-preserving on K .

For the face Δ_i , form the normalized polynomial

$$\tilde{f}_i(Y) = \pi_i^{-\alpha_i} T^{-a_i} f(\pi_i Y),$$

retaining the terms whose exponents lie between a_i and b_i . The terms on the face have valuation 0, while all terms strictly above the face have positive valuation. Hence reduction modulo the maximal ideal of V_i^h gives precisely the ordinary residual polynomial $\rho_i(Y)$. Thus the slopewise factorization in $\mathrm{gr}_\nu(V)[T]$ becomes an ordinary coprime factorization modulo the maximal ideal of the Henselian valuation ring V_i^h .

Applying Lemma 5.1 over V_i^h gives a unique monic lift of this normalized residual factorization; compare the valuation-theoretic lifting formalism and root-extension invariance for lifting

polynomials [6, Def. 1.B, p. 2650; Thm. 1.1, p. 2650]; [4, Prop. 2.3, pp. 111–113; Thms. 1.1–1.2, p. 109], as well as the corresponding prolongation criterion controlling the number of extensions, ramification indices, and residual degrees [7, Thm. 1.1, pp. 324–325]. The lifted factor has coefficients whose valuations lie on or above the prescribed face, with initial form equal to ρ_i .

Undoing the substitution $T = \pi_i Y$ and the endpoint normalization gives a factor over $K_i[T]$ whose Newton polygon is the prescribed face Δ_i . Because the construction is characterized uniquely by the monic factor with the prescribed normalized residual data, it is invariant under automorphisms of the valued extension K_i/K preserving K . Consequently the coefficients are fixed by descent and lie in $K[T]$. Since the original polynomial f is monic with coefficients in the valuation ring V , and the descended factor has coefficients integral over V , the integrally closedness of valuation rings gives coefficients in V (see [1, Ch. VI, §2, p. 6] and see also [2, Thm. 10.3, p. 73]). Hence the factor belongs to $V[T]$.

Repeating this construction for the pairwise coprime residual factors ρ_1, \dots, ρ_r gives

$$f = f_1 \cdots f_r$$

with $\deg f_i = n_i$, with Newton polygon supported on Δ_i , and with residual polynomial ρ_i . Uniqueness follows because, after the same normalization over V_i^h , any two such monic factors give the same ordinary Hensel lift; hence they coincide after descent to $V[T]$. \square

Proposition 5.4 (Slope separation criterion). *Let V be a Henselian valuation ring and let $f \in V[T]$ be monic. Suppose that the Newton polygon of f decomposes into distinct slope blocks and that the corresponding residual polynomials are pairwise coprime.*

Assume moreover that the associated graded initial Newton data admit a compatible coprime slopewise factorization in the graded valuation algebra. Then f admits a corresponding factorization in $V[T]$.

Proof. Let

$$\Delta(f) = \Delta_1 \cup \cdots \cup \Delta_r$$

be the decomposition of the Newton polygon into distinct slope blocks, and let ρ_i denote the residual polynomial attached to Δ_i after the usual normalization of that Newton face. By hypothesis the residual polynomials

$$\rho_1, \dots, \rho_r$$

are pairwise coprime, and the associated graded Newton initial form admits a compatible slopewise factorization

$$\text{in}_{\text{NP}}(f) = \rho_1 \cdots \rho_r.$$

This is not an ordinary factorization modulo the maximal ideal of V . Rather, it is a factorization in the associated graded valuation algebra. Therefore one cannot apply Lemma 5.1 directly at this stage. The necessary passage from graded Newton data to an ordinary Hensel problem is precisely the content of Lemma 5.3: each slope block is first normalized by the valuation dictated by its face, the residual factorization is then read modulo the maximal ideal in that normalized problem, and ordinary Hensel lifting is applied only after this reduction.

Applying Lemma 5.3 gives monic polynomials $f_1, \dots, f_r \in V[T]$ such that

$$f = f_1 f_2 \cdots f_r,$$

where $\deg f_i$ is the horizontal length of Δ_i , the Newton polygon of f_i is supported on Δ_i , and the residual polynomial of f_i is ρ_i . Hence f admits the asserted factorization over $V[T]$ compatible with the slope decomposition. \square

6 Derived Factorization Stability

Definition 6.1 (Torsion-controlled associated graded coefficient modules). *Let V be a valuation ring with valuation filtration and let*

$$R := \text{gr}_\nu(V)$$

be its associated graded valuation algebra. Let M be a Newton coefficient module with Newton filtration, and write

$$\text{gr } M = \bigoplus_{\lambda} \text{gr}^\lambda M.$$

We say that $\text{gr } M$ is torsion-controlled if, for every nonzero homogeneous element $r \in R$, multiplication by r is injective on each graded Newton piece $\text{gr}^\lambda M$ occurring in the chosen associated graded Newton complex.

For two filtered Newton coefficient modules M_0, M_1 equipped with a strict graded comparison

$$\text{gr } M_0 \xrightarrow{\sim} \text{gr } M_1,$$

we say that the pair is torsion-controlled if both $\text{gr } M_0$ and $\text{gr } M_1$ are torsion-controlled in the preceding sense and the comparison identifies the corresponding graded Newton pieces, slope-block summands, and residual factor submodules. Equivalently, no nonzero homogeneous class in any relevant graded Newton piece becomes killed by a nonzero homogeneous element of $\text{gr}_\nu(V)$ during the comparison.

Theorem 6.2 (Derived Newton stability theorem). *Let V be a Henselian valuation ring and let*

$$F(T) \in V[S][T]$$

be a monic polynomial family. Let

$$\sigma_0, \sigma_1 : V[S] \longrightarrow V$$

be two V -valued specializations, and write

$$F_{\sigma_j}(T) := (\sigma_j \otimes \text{id})(F) \in V[T], \quad j = 0, 1.$$

Assume that the corresponding valued specializations lie in the same Newton polygon stratum. Assume:

- (i) the slope decomposition of $\text{NP}(F_{\sigma_0})$ and $\text{NP}(F_{\sigma_1})$ is identical;*
- (ii) the residual factor polynomials remain pairwise coprime;*
- (iii) admissible filtered Newton complexes in the precise sense of Definition 4.4,*

$$C_{F_{\sigma_0}}^\bullet \rightarrow M_{F_{\sigma_0}}, \quad C_{F_{\sigma_1}}^\bullet \rightarrow M_{F_{\sigma_1}},$$

are fixed together with a strict filtered comparison quasi-isomorphism

$$\Phi : C_{F_{\sigma_0}}^\bullet \longrightarrow C_{F_{\sigma_1}}^\bullet$$

compatible with the Newton coefficient modules and inducing an isomorphism of associated graded augmented Newton complexes

$$\text{gr } \Phi : (\text{gr } C_{F_{\sigma_0}}^\bullet \rightarrow \text{gr } M_{F_{\sigma_0}}) \xrightarrow{\sim} (\text{gr } C_{F_{\sigma_1}}^\bullet \rightarrow \text{gr } M_{F_{\sigma_1}}).$$

Moreover, this comparison respects the slope-block decompositions and the residual factor data attached to the common Newton polygon;

(iv) under this strict graded comparison, the induced map on relative derived Newton defects is the homology map of the above isomorphism of associated graded augmented Newton complexes:

$$\mathfrak{D}_{\text{NP}}(F_{\sigma_0}; C_{F_{\sigma_0}}^\bullet) \cong \mathfrak{D}_{\text{NP}}(F_{\sigma_1}; C_{F_{\sigma_1}}^\bullet).$$

Thus the constancy of \mathfrak{D}_{NP} is not used as a bare isomorphism of graded homology modules, but as a consequence of compatible identification of the associated graded Newton complexes themselves;

(v) the associated graded coefficient modules $\text{gr } M_{F_{\sigma_0}}$ and $\text{gr } M_{F_{\sigma_1}}$, together with their strict graded comparison, are torsion-controlled in the sense of Definition 6.1.

Then F_{σ_0} and F_{σ_1} admit compatible lifted factorizations over $V[T]$ with identical slope-block degree data.

Proof. Let

$$F(T) \in V[S][T]$$

be the given monic family, and let

$$\sigma_0, \sigma_1 : V[S] \rightarrow V$$

be the two V -valued specializations. By definition,

$$F_{\sigma_j}(T) := (\sigma_j \otimes \text{id})(F) \in V[T], \quad j = 0, 1.$$

Thus the specialized polynomials really are polynomials over the same valuation ring V , so the later Henselian lifting and factorization arguments take place in $V[T]$, not merely in residue-field polynomial rings attached to residue-field specializations.

Because the two valued specializations lie in the same Newton polygon stratum, Definition 3.1 implies that

$$\text{NP}(F_{\sigma_0}) = \text{NP}(F_{\sigma_1}).$$

In particular, the valuation inequalities defining the lower convex hull remain unchanged under specialization. Hence the two specializations possess identical slope decompositions:

$$\Delta(F_{\sigma_0}) = \Delta_1 \cup \cdots \cup \Delta_r = \Delta(F_{\sigma_1}).$$

The horizontal lengths and slopes of the blocks therefore coincide for the two fibers.

For each slope block Δ_i , let

$$\rho_{i, \sigma_0}, \quad \rho_{i, \sigma_1}$$

be the corresponding residual factor polynomials. By assumption (ii), these residual factors remain pairwise coprime throughout the specialization. Consequently the associated graded initial forms of F_{σ_0} and F_{σ_1} admit compatible coprime decompositions:

$$\text{in}_{\text{NP}}(F_{\sigma_j}) = \rho_{1, \sigma_j} \cdots \rho_{r, \sigma_j}, \quad j = 0, 1.$$

Since the slope data are identical, the degrees and valuation ranges of the corresponding residual factors agree.

Now consider the fixed admissible filtered Newton complexes

$$C_{F_{\sigma_0}}^\bullet \rightarrow M_{F_{\sigma_0}}, \quad C_{F_{\sigma_1}}^\bullet \rightarrow M_{F_{\sigma_1}}.$$

By Definition 4.5,

$$\mathfrak{D}_{\text{NP}}(F_{\sigma_j}; C_{F_{\sigma_j}}^\bullet) = H^\bullet(\text{gr } C_{F_{\sigma_j}}^\bullet \rightarrow \text{gr } M_{F_{\sigma_j}}) \quad (j = 0, 1).$$

Assumptions (iii)–(iv) give more than an abstract isomorphism of the homology modules $\mathfrak{D}_{\text{NP}}(F_{\sigma_0})$ and $\mathfrak{D}_{\text{NP}}(F_{\sigma_1})$. They provide a strict filtered comparison of the chosen admissible Newton complexes whose associated graded map identifies the entire augmented graded Newton complexes:

$$(\text{gr } C_{F_{\sigma_0}}^\bullet \rightarrow \text{gr } M_{F_{\sigma_0}}) \cong (\text{gr } C_{F_{\sigma_1}}^\bullet \rightarrow \text{gr } M_{F_{\sigma_1}}).$$

This identification is compatible with the common slope-block decomposition and with the residual factor polynomials. Hence the cycles, boundaries, extension classes, and gluing maps appearing in the associated graded Newton complexes are transported blockwise from σ_0 to σ_1 , rather than merely having isomorphic total homology.

Therefore no new associated graded Newton-complex obstruction can appear between the two specializations: any obstruction class, residual relation, or filtered gluing datum at σ_1 is the image under the strict graded comparison of the corresponding datum at σ_0 .

By Definition 6.1, the torsion-control hypothesis in assumption (v) means that no nonzero homogeneous class in any relevant graded Newton piece is killed by a nonzero homogeneous element of $\text{gr}_\nu(V)$, and that the strict graded comparison identifies the corresponding slope-block and residual-factor submodules. Hence specialization cannot introduce hidden graded torsion or valuation collapse inside the coefficient pieces while preserving only the combinatorial shape of the Newton polygon. Therefore the associated graded slope decompositions remain compatible with the filtered structures of the specialized coefficient modules.

By Proposition 5.4, the pairwise coprimality of the residual factors together with the common slope decomposition produce compatible graded factorizations of

$$F_{\sigma_0} \quad \text{and} \quad F_{\sigma_1}.$$

Since V is Henselian, Lemma 5.1 lifts these graded factorizations uniquely to factorizations in $V[T]$. The slope blocks, their multiplicities, and their residual data coincide for both specializations, and the strict graded comparison of the associated Newton complexes, together with torsion control, guarantees that no additional splitting, merging, extension, or residual gluing obstruction is introduced during specialization.

Consequently the lifted factorizations of F_{σ_0} and F_{σ_1} have the same number of factors, the same degrees, and the same slope distribution. Therefore the two specialized polynomials admit compatible lifted factorizations over $V[T]$ with the same number of slope-block factors, the same block degrees, and the same slope distribution. In particular, under uniqueness hypotheses for the lifted factorization data, the factorization type is preserved. \square

Theorem 6.3 (Invariance under strict unramified base extension). *Let $V \subset W$ be an unramified extension of Henselian valuation rings in the following strict filtered sense: the value group is unchanged, the residue extension is separable, and, for every filtered module occurring below, the natural maps*

$$F^\lambda N \otimes_V W \longrightarrow F^\lambda(N \otimes_V W)$$

are isomorphisms for all relevant filtration levels λ . Let $F \in V[T]$ be monic. Assume that M_F , with its Newton filtration, admits a bounded filtered free resolution, and that the graded pieces occurring in the associated graded Newton complex are W -flat, or equivalently have vanishing higher Tor^V -terms after tensoring with W . Then

$$\mathfrak{D}_{\text{NP}}(F \otimes_V W) \cong \mathfrak{D}_{\text{NP}}(F) \otimes_V W.$$

Proof. Let

$$C_F^\bullet \longrightarrow M_F$$

be a bounded filtered free resolution of the Newton coefficient module M_F . Since W is a V -algebra, tensoring with W gives a bounded filtered complex

$$C_F^\bullet \otimes_V W \longrightarrow M_F \otimes_V W.$$

Because each term of C_F^\bullet is filtered free over V , the base-changed complex remains filtered free over W .

By the strict filtered unramified hypothesis, the value group is not enlarged and the relevant Newton filtration is preserved after base change. Thus, for every filtration level λ , the canonical comparison map

$$F^\lambda M_F \otimes_V W \longrightarrow F^\lambda(M_F \otimes_V W)$$

is an isomorphism. The same strictness assumption applies to the filtered terms of C_F^\bullet . Consequently passage to associated graded objects commutes with this base change:

$$\mathrm{gr}(C_F^\bullet \otimes_V W) \cong (\mathrm{gr} C_F^\bullet) \otimes_V W, \quad \mathrm{gr}(M_F \otimes_V W) \cong (\mathrm{gr} M_F) \otimes_V W.$$

Under these strict compatibility assumptions, the residual initial forms and slope blocks are transported without introducing new ramification or new graded filtration levels. Therefore no additional ramification or filtered-extension term appears in the associated graded augmented complex.

Now consider the graded homology of the base-changed augmented complex:

$$H^\bullet(\mathrm{gr}(C_F^\bullet \otimes_V W) \longrightarrow \mathrm{gr}(M_F \otimes_V W)).$$

Using the preceding compatibility isomorphisms, this becomes

$$H^\bullet((\mathrm{gr} C_F^\bullet \longrightarrow \mathrm{gr} M_F) \otimes_V W).$$

By the assumed W -flatness, equivalently the vanishing of the relevant higher Tor^V -terms, homology commutes with extension of scalars:

$$H^\bullet((\mathrm{gr} C_F^\bullet \longrightarrow \mathrm{gr} M_F) \otimes_V W) \cong H^\bullet(\mathrm{gr} C_F^\bullet \longrightarrow \mathrm{gr} M_F) \otimes_V W.$$

By Definition 4.5,

$$H^\bullet(\mathrm{gr} C_F^\bullet \longrightarrow \mathrm{gr} M_F) = \mathfrak{D}_{\mathrm{NP}}(F).$$

On the other hand,

$$H^\bullet(\mathrm{gr}(C_F^\bullet \otimes_V W) \longrightarrow \mathrm{gr}(M_F \otimes_V W)) = \mathfrak{D}_{\mathrm{NP}}(F \otimes_V W).$$

Combining the previous identifications yields

$$\mathfrak{D}_{\mathrm{NP}}(F \otimes_V W) \cong \mathfrak{D}_{\mathrm{NP}}(F) \otimes_V W,$$

as required. □

Proposition 6.4 (Functoriality under filtered morphisms). *Let $V \rightarrow W$ be a morphism of valuation rings compatible with the valuation filtrations, and let $F \in V[T]$ be monic. Assume that M_F , with its Newton filtration, admits a bounded filtered free resolution. Assume that the Newton filtration on M_F is transported to the Newton filtration on $M_{F \otimes_V W}$. Then there is a natural morphism of graded modules*

$$\mathfrak{D}_{\mathrm{NP}}(F) \otimes_V W \longrightarrow \mathfrak{D}_{\mathrm{NP}}(F \otimes_V W).$$

If, moreover, tensoring with W is exact on the graded pieces occurring in the Newton filtration and the filtered-quotient comparison maps are isomorphisms for all relevant filtration levels, then this morphism is an isomorphism.

Proof. Let

$$C_F^\bullet \longrightarrow M_F$$

be a bounded filtered free resolution of the Newton coefficient module. Thus each term C_F^q is a filtered free V -module and each differential

$$d^q : C_F^q \longrightarrow C_F^{q+1}$$

preserves the Newton filtration.

Since W is a V -algebra, tensoring degreewise with W produces a bounded complex

$$C_F^\bullet \otimes_V W \longrightarrow M_F \otimes_V W.$$

By hypothesis, the valuation filtration on V is compatible with the valuation filtration on W , and the Newton filtration on M_F is transported to the Newton filtration on

$$M_{F \otimes_V W}.$$

Consequently the tensor product complex inherits a natural filtered structure: for every filtration level λ ,

$$F^\lambda(C_F^q \otimes_V W) := \text{Im}(F^\lambda C_F^q \otimes_V W \longrightarrow C_F^q \otimes_V W).$$

Because the differentials on C_F^\bullet are filtered, the induced differentials on $C_F^\bullet \otimes_V W$ are also filtered. Hence

$$C_F^\bullet \otimes_V W \longrightarrow M_F \otimes_V W$$

is a filtered augmented complex resolving the transported Newton coefficient module.

Passing to associated graded objects gives augmented graded complexes

$$\text{gr } C_F^\bullet \longrightarrow \text{gr } M_F \quad \text{and} \quad \text{gr}(C_F^\bullet \otimes_V W) \longrightarrow \text{gr}(M_F \otimes_V W).$$

Functoriality of tensor product with respect to filtered quotients produces a canonical morphism of augmented graded complexes

$$(\text{gr } C_F^\bullet \longrightarrow \text{gr } M_F) \otimes_V W \longrightarrow (\text{gr}(C_F^\bullet \otimes_V W) \longrightarrow \text{gr}(M_F \otimes_V W)).$$

Explicitly, in filtration degree λ , the map on the resolution terms is induced from the natural morphisms

$$\frac{F^\lambda C_F^q}{F^{>\lambda} C_F^q} \otimes_V W \longrightarrow \frac{F^\lambda(C_F^q \otimes_V W)}{F^{>\lambda}(C_F^q \otimes_V W)}.$$

The corresponding map on the augmented coefficient term is induced by

$$\frac{F^\lambda M_F}{F^{>\lambda} M_F} \otimes_V W \longrightarrow \frac{F^\lambda(M_F \otimes_V W)}{F^{>\lambda}(M_F \otimes_V W)}.$$

Compatibility of the valuation filtrations guarantees that these maps are well-defined and compatible with the graded differentials and augmentations. Therefore they assemble into a morphism of augmented graded complexes.

Applying homology yields canonical morphisms

$$H^q(\text{gr } C_F^\bullet \longrightarrow \text{gr } M_F) \otimes_V W \longrightarrow H^q(\text{gr}(C_F^\bullet \otimes_V W) \longrightarrow \text{gr}(M_F \otimes_V W))$$

for every degree q . Collecting these morphisms over all degrees gives

$$H^\bullet(\text{gr } C_F^\bullet \longrightarrow \text{gr } M_F) \otimes_V W \longrightarrow H^\bullet(\text{gr}(C_F^\bullet \otimes_V W) \longrightarrow \text{gr}(M_F \otimes_V W)).$$

Now, by Definition 4.5,

$$H^\bullet(\mathrm{gr} C_F^\bullet \longrightarrow \mathrm{gr} M_F) = \mathfrak{D}_{\mathrm{NP}}(F).$$

Moreover,

$$H^\bullet(\mathrm{gr}(C_F^\bullet \otimes_V W) \longrightarrow \mathrm{gr}(M_F \otimes_V W)) = \mathfrak{D}_{\mathrm{NP}}(F \otimes_V W),$$

because the tensor-product complex is a filtered free resolution of the transported Newton coefficient module over W . Hence we obtain the natural graded morphism

$$\mathfrak{D}_{\mathrm{NP}}(F) \otimes_V W \longrightarrow \mathfrak{D}_{\mathrm{NP}}(F \otimes_V W).$$

It remains to prove the final assertion. We now assume not only that tensoring with W is exact on the graded pieces appearing in the Newton filtration, but also that the canonical filtered-quotient comparison maps

$$\frac{F^\lambda C_F^q}{F^{>\lambda} C_F^q} \otimes_V W \longrightarrow \frac{F^\lambda (C_F^q \otimes_V W)}{F^{>\lambda} (C_F^q \otimes_V W)}$$

and

$$\frac{F^\lambda M_F}{F^{>\lambda} M_F} \otimes_V W \longrightarrow \frac{F^\lambda (M_F \otimes_V W)}{F^{>\lambda} (M_F \otimes_V W)}$$

are isomorphisms for all relevant q and λ . Under this strictness hypothesis, the canonical morphism of augmented graded complexes

$$(\mathrm{gr} C_F^\bullet \longrightarrow \mathrm{gr} M_F) \otimes_V W \longrightarrow (\mathrm{gr}(C_F^\bullet \otimes_V W) \longrightarrow \mathrm{gr}(M_F \otimes_V W))$$

is an isomorphism of complexes. Since tensoring with W is exact on the graded pieces, homology commutes with tensor product:

$$H^\bullet((\mathrm{gr} C_F^\bullet \longrightarrow \mathrm{gr} M_F) \otimes_V W) \cong H^\bullet(\mathrm{gr} C_F^\bullet \longrightarrow \mathrm{gr} M_F) \otimes_V W.$$

Using again the definition of the derived Newton defect, this gives

$$\mathfrak{D}_{\mathrm{NP}}(F) \otimes_V W \cong \mathfrak{D}_{\mathrm{NP}}(F \otimes_V W).$$

Therefore the functorial morphism constructed above is an isomorphism under the combined hypotheses of exactness on the graded Newton pieces and strict compatibility of the filtered quotients with base change. \square

7 Examples

Example 7.1 (Quadratic family over a discrete valuation ring). *Let V be a discrete valuation ring with uniformizer π , valuation ν , fraction field K , and residue field $k = V/(\pi)$. Consider the monic quadratic family*

$$F_s(T) = T^2 + a(s)T + b(s) \in V[s][T].$$

For a specialization $s = s_0$, the Newton polygon of F_{s_0} is the lower convex hull of the three points

$$(0, \nu(b(s_0))), \quad (1, \nu(a(s_0))), \quad (2, 0).$$

Set

$$\alpha = \nu(a(s_0)), \quad \beta = \nu(b(s_0)).$$

The middle point $(1, \alpha)$ lies on or below the line segment joining $(0, \beta)$ and $(2, 0)$ precisely when

$$\alpha \leq \frac{\beta}{2}.$$

Thus three boundary regimes must be distinguished.

First, if

$$\alpha > \frac{\beta}{2},$$

then the middle point lies strictly above the lower convex hull. The Newton polygon has one face, namely the segment from $(0, \beta)$ to $(2, 0)$, of slope

$$-\frac{\beta}{2}.$$

In this case both roots, after passage to a suitable valued extension, have the same valuation $\beta/2$. The Newton filtration on the coefficient module

$$M_{F_{s_0}} = Vb(s_0) \oplus Va(s_0) \oplus V$$

has one dominant slope contribution. The associated graded Newton complex therefore has only one slope block, and no slopewise splitting is forced by the polygon alone.

Second, if

$$\alpha = \frac{\beta}{2},$$

then the middle point lies on the same line segment from $(0, \beta)$ to $(2, 0)$. In this case the Newton polygon has one face with an interior lattice point, or equivalently one slope occurring with horizontal length two. It should not be counted as a genuinely two-slope decomposition.

Third, if

$$\alpha < \frac{\beta}{2},$$

then the middle point is a genuine vertex of the lower convex hull. The Newton polygon has two distinct slope blocks:

$$(0, \beta) \longrightarrow (1, \alpha), \quad (1, \alpha) \longrightarrow (2, 0).$$

The corresponding slopes are

$$\alpha - \beta \quad \text{and} \quad -\alpha.$$

Equivalently, the predicted root valuations are

$$\beta - \alpha \quad \text{and} \quad \alpha.$$

If these two distinct slope blocks have coprime residual polynomials, then the associated graded initial form decomposes into two coprime linear slope factors. When V is Henselian, Proposition 5.4 lifts this graded decomposition to an actual factorization in $V[T]$.

This example shows concretely how the comparison between

$$2\nu(a(s)) \quad \text{and} \quad \nu(b(s))$$

controls the slope decomposition of the Newton polygon. The equality case is a boundary case: it gives one face with an interior lattice point, not two distinct slopes. Hence the quadratic family already exhibits the basic stratification phenomenon: inside one stratum the slope data remain constant, while crossing the wall

$$2\nu(a(s)) = \nu(b(s))$$

may change the associated graded Newton complex and therefore the derived Newton defect.

Example 7.2 (Change of associated graded Newton data when the middle coefficient changes filtration level). *Let V be a discrete valuation ring with uniformizer π , and consider*

$$F_s(T) = T^2 + \pi sT + \pi^2 \in V[s][T].$$

For a specialization $s = s_0$, the relevant valuation data are

$$\nu(\pi^2) = 2, \quad \nu(\pi s_0) = 1 + \nu(s_0), \quad \nu(1) = 0.$$

Hence the Newton polygon is determined by

$$(0, 2), \quad (1, 1 + \nu(s_0)), \quad (2, 0).$$

If s_0 is a unit, then $\nu(s_0) = 0$, and the middle point is

$$(1, 1).$$

This point lies exactly on the line joining $(0, 2)$ and $(2, 0)$. The Newton polygon has a single slope repeated twice, namely slope -1 . Equivalently, the lower hull has one face with the interior lattice point $(1, 1)$; this is not a two-slope decomposition. Thus both roots have predicted valuation 1, and the associated graded initial form is governed by

$$T^2 + \overline{s_0}T + 1$$

after dividing by the appropriate power of π .

On the other hand, if $s_0 \in (\pi)$, then $\nu(s_0) > 0$, and the middle point

$$(1, 1 + \nu(s_0))$$

lies strictly above the line from $(0, 2)$ to $(2, 0)$. The Newton polygon again has the segment from $(0, 2)$ to $(2, 0)$, but the middle coefficient no longer contributes to the same associated graded initial form. In the associated graded coefficient module, the class of the middle coefficient has moved to a higher filtration level.

This produces a genuine filtered phenomenon. The ordinary degree of the polynomial remains 2 throughout the family, and the endpoint coefficients still determine the same outer segment. However, the associated graded Newton complex changes because the middle coefficient enters a different filtration degree. Equivalently, the coefficient module

$$M_{F_{s_0}} = V\pi^2 \oplus V(\pi s_0) \oplus V$$

has different graded pieces according as s_0 is a unit or is divisible by π .

Thus the specialization from a unit value of s to a non-unit value of s can preserve the superficial shape of the polynomial while changing the filtered coefficient data. The derived Newton defect is designed precisely to detect this failure of filtered exactness after passing to associated graded objects. Therefore this family illustrates why Newton polygon data alone are not sufficient in deformation problems: one must also control the homological behaviour of the Newton filtration.

Example 7.3 (A fully worked filtered Newton-complex computation). *Let $V = k[[\pi]]$, with its π -adic valuation, and let*

$$F(T) = T^2 + \pi T + \pi^2 \in V[T].$$

The Newton coefficient module is

$$M_F = V\pi^2 e_0 \oplus V\pi e_1 \oplus V e_2.$$

Its Newton levels are 2, 1, 0, respectively. Consider the following filtered free presentation of the initial Newton coefficient data:

$$C_F^1 \xrightarrow{d} C_F^0 \xrightarrow{\epsilon} M_F \longrightarrow 0,$$

where

$$C_F^0 = Vx_0 \oplus Vx_1 \oplus Vx_2, \quad C_F^1 = 0,$$

and

$$\epsilon(x_0) = \pi^2 e_0, \quad \epsilon(x_1) = \pi e_1, \quad \epsilon(x_2) = e_2.$$

Assign Newton levels

$$\ell(x_0) = 2, \quad \ell(x_1) = 1, \quad \ell(x_2) = 0.$$

With these levels, ϵ is a strict filtered isomorphism from C_F^0 onto M_F . Hence the augmented filtered complex is exact after forgetting filtrations and is strict level by level.

Passing to associated graded objects gives

$$\text{gr } C_F^0 = k\bar{x}_0 \oplus k\bar{x}_1 \oplus k\bar{x}_2, \quad \text{gr } M_F = k\overline{\pi^2 e_0} \oplus k\overline{\pi e_1} \oplus k\bar{e}_2.$$

The associated graded augmentation is

$$\text{gr}(\epsilon)(\bar{x}_0) = \overline{\pi^2 e_0}, \quad \text{gr}(\epsilon)(\bar{x}_1) = \overline{\pi e_1}, \quad \text{gr}(\epsilon)(\bar{x}_2) = \bar{e}_2.$$

Therefore $\text{gr}(\epsilon)$ is an isomorphism. Since $C_F^1 = 0$, the associated graded augmented complex is

$$0 \longrightarrow \text{gr } C_F^0 \xrightarrow{\text{gr}(\epsilon)} \text{gr } M_F \longrightarrow 0,$$

and this complex is exact. Consequently

$$\mathfrak{D}_{\text{NP}}(F; C_F^\bullet) = H_\bullet(\text{gr } C_F^\bullet \rightarrow \text{gr } M_F) = 0.$$

This is the simplest admissible case: the Newton filtration is perfectly compatible with the chosen filtered free resolution, and no hidden associated graded homology appears. Thus the derived Newton defect vanishes by a direct computation, not merely by an abstract appeal to the definition.

Corollary 7.4 (Persistence of slope-block factorization type along a Newton stratum). *Let V be a Henselian valuation ring and let $F(T) \in V[S][T]$ be a monic polynomial family. Restrict to V -valued specializations $\sigma : V[S] \rightarrow V$, so that $F_\sigma(T) \in V[T]$. Suppose that these V -valued specializations are parametrized by a Newton polygon stratum. Suppose that:*

- (i) *the slope decomposition remains constant along the stratum;*
- (ii) *the residual factor polynomials attached to the distinct slope blocks remain pairwise coprime;*
- (iii) *admissible filtered Newton complexes in the sense of Definition 4.4 are fixed along the stratum, or are identified by strict filtered quasi-isomorphisms, and the corresponding relative derived Newton defect is constant along the stratum;*
- (iv) *the relevant Newton coefficient modules satisfy the torsion-control hypotheses of Theorem 6.2.*
- (v) *the relevant Newton coefficient modules admit bounded filtered free resolutions and satisfy the torsion-control hypotheses of Theorem 6.2.*

Then the lifted slope-block factorization type is constant along the stratum: the number of slope-block factors and their degrees remain unchanged.

Proof. Let s_0 and s be two specializations in the given Newton polygon stratum. By assumptions (i)–(iv), the hypotheses of Theorem 6.2 apply to the pair $F_{s_0}(T)$ and $F_s(T)$. Hence the two specialized polynomials admit compatible lifted factorizations over $V[T]$ with the same slope-block degree data. Therefore the slope-block factorization type is constant along the stratum.

This conclusion is deliberately weaker than irreducibility persistence. Irreducibility does not follow merely from constancy of slope data, residual coprimality, and vanishing or constancy of the derived Newton defect unless one adds a separate hypothesis excluding any nontrivial lifted slope-block factorization. \square

8 Counterexample(s)

Example 8.1 (Necessity of the coprimality hypothesis). *Let k be a field of characteristic different from 2, let*

$$V = k[[\pi]]$$

with its π -adic valuation, and consider the one-parameter family

$$F_c(T) = (T - \pi)^2 + \pi^3 c = T^2 - 2\pi T + \pi^2(1 + \pi c) \in V[c][T].$$

For every specialization $c = c_0 \in V$, the relevant Newton points are

$$(0, \nu(\pi^2(1 + \pi c_0))), \quad (1, \nu(-2\pi)), \quad (2, 0).$$

Since $2 \in V^\times$ and $1 + \pi c_0 \in V^\times$, these are

$$(0, 2), \quad (1, 1), \quad (2, 0).$$

Thus the Newton polygon is constant throughout the family: it is the single line segment of slope -1 .

The associated slope-one residual polynomial is obtained by writing $T = \pi Y$ and dividing by π^2 :

$$\pi^{-2} F_c(\pi Y) = (Y - 1)^2 + \pi c.$$

Modulo π , this gives

$$\overline{R}(Y) = (Y - 1)^2.$$

The residual factorization is therefore not coprime: the two residual linear factors coincide.

Now compare two specializations. For $c = 0$, one has

$$F_0(T) = (T - \pi)^2,$$

so the polynomial is reducible, indeed a square of a linear factor. For $c = c_0 \in V^\times$, the discriminant is

$$\Delta_c = (-2\pi)^2 - 4\pi^2(1 + \pi c_0) = -4\pi^3 c_0.$$

Since $\nu(\Delta_c) = 3$ is odd, Δ_c is not a square in the fraction field of V . Hence $F_{c_0}(T)$ is irreducible over $V[T]$.

Thus the Newton polygon remains constant, and the slope is unchanged, but the factorization type changes from a repeated linear factor to an irreducible quadratic. The failure is exactly the failure of residual coprimality: Hensel lifting cannot separate two residual factors which have already coalesced in the associated graded residue algebra. Therefore the coprimality condition in Theorem 6.2 is essential and cannot be omitted.

Example 8.2 (A cubic polynomial with separated Newton slopes). *Let V be a discrete valuation ring with uniformizer π , residue field k of characteristic different from 2, and valuation $\nu(\pi) = 1$. Consider*

$$F(T) = T^3 + \pi^2 T + \pi^5 \in V[T].$$

The nonzero coefficient points are

$$(0, 5), \quad (1, 2), \quad (3, 0).$$

The lower convex hull has two segments:

$$(0, 5) \longrightarrow (1, 2), \quad (1, 2) \longrightarrow (3, 0).$$

Their slopes are respectively

$$-3, \quad -1.$$

Thus the Newton polygon predicts one root of valuation 3 and two roots of valuation 1, after passage to a suitable valued extension.

The first slope block has horizontal length 1. To compute its initial form, set $T = \pi^3 Y$ and divide by π^5 . One obtains

$$\pi^{-5} F(\pi^3 Y) = Y + 1 + \pi^4 Y^3.$$

Modulo π , the residual polynomial of the slope -3 block is

$$R_1(Y) = Y + 1.$$

For the second slope block, set $T = \pi Y$ and divide by the valuation level π^3 corresponding to the segment from $(1, 2)$ to $(3, 0)$. Then

$$\pi^{-3} F(\pi Y) = Y^3 + Y + \pi^2.$$

Modulo π , this gives

$$Y^3 + Y = Y(Y^2 + 1).$$

For the segment from $(1, 2)$ to $(3, 0)$, the residual polynomial is defined by retaining the terms whose coefficient points lie on this segment and then normalizing by the left endpoint $(1, 2)$. Equivalently, after normalization by the left endpoint $(1, 2)$, the face polynomial associated to the segment is

$$Y^3 + Y = Y(Y^2 + 1),$$

and the endpoint factor Y records the left endpoint of the face rather than an additional residual factor. Hence the residual polynomial attached to the length-two slope block is

$$R_2(Y) = Y^2 + 1.$$

If R_1 and R_2 are coprime in $k[Y]$, equivalently if

$$R_2(-1) = 2 \neq 0,$$

then the residual slope factors are pairwise coprime. This is why the assumption $\text{char } k \neq 2$ is convenient here.

Under this coprimality condition, Proposition 5.4 gives a factorization

$$F(T) = G_1(T)G_2(T) \quad \text{in } V[T],$$

where

$$\deg G_1 = 1, \quad \deg G_2 = 2.$$

The Newton polygon of G_1 is supported on the slope -3 block, while the Newton polygon of G_2 is supported on the slope -1 block. In particular, the factor G_1 carries the root valuation 3, and G_2 carries the two root valuations equal to 1.

At the level of the Newton coefficient module,

$$M_F = V\pi^5 \oplus V\pi^2 \oplus V,$$

the two slope blocks define two associated graded pieces: one generated by the endpoint data on $(0, 5) \rightarrow (1, 2)$, and one generated by the data on $(1, 2) \rightarrow (3, 0)$. The residual coprimality condition makes these two pieces independent in the associated graded category. Thus the associated graded Newton complex splits into slopewise complexes, and the derived Newton defect decomposes as

$$\mathfrak{D}_{\text{NP}}(F) \cong \mathfrak{D}_{\text{NP}, \Delta_1}(F) \oplus \mathfrak{D}_{\text{NP}, \Delta_2}(F).$$

This example therefore illustrates the precise mechanism behind the slope separation criterion: separated slopes identify the possible valuation ranges, while residual coprimality is what converts this separation into an actual factorization over $V[T]$.

Example 8.3 (Failure without Henselianity). Let k be a field of characteristic different from 2, and let

$$V = k[t]_{(t)}$$

with its t -adic valuation. Then V is a discrete valuation ring, but it is not Henselian. Consider

$$f(T) = T^2 - (1 + t) \in V[T].$$

The coefficient points are

$$(0, \nu(1 + t)), \quad (2, 0).$$

Since $1 + t \in V^\times$, one has $\nu(1 + t) = 0$. Hence the Newton polygon is the single horizontal segment from $(0, 0)$ to $(2, 0)$. Thus both roots, if they exist, are predicted to have valuation 0.

Modulo t , the polynomial becomes

$$\bar{f}(T) = T^2 - 1 = (T - 1)(T + 1) \in k[T].$$

Because $\text{char } k \neq 2$, the two residual factors $T - 1$ and $T + 1$ are coprime. Therefore the residual factorization is exactly of the type that Hensel's lemma would lift over a Henselian valuation ring.

However, no such lift exists in $V[T]$. Indeed, a factorization into two monic linear factors would have the form

$$T^2 - (1 + t) = (T - u)(T + u)$$

for some $u \in V$, and hence would require

$$u^2 = 1 + t.$$

Equivalently, $1 + t$ would have to be a square in the fraction field $k(t)$. But $1 + t$ is not a square in $k(t)$: its divisor has a simple zero at $t = -1$, hence contains a valuation of odd order. A rational square has even order at every discrete valuation. Therefore

$$\sqrt{1 + t} \notin k(t),$$

and so $f(T)$ is irreducible over $V[T]$.

This example shows that stable Newton polygon data and coprime residual factorization do not by themselves imply factorization stability. The missing ingredient is precisely Henselianity. In the Henselianization of V , the two simple residual roots 1 and -1 lift uniquely; in the non-Henselian ring $k[t]_{(t)}$, the residual roots exist but the actual lifted roots do not. Hence the Henselian hypothesis in Lemma 5.1, Proposition 5.4, and Theorem 6.2 is structurally indispensable.

Example 8.4 (A filtered model pathology when torsion control fails). *Let $V = k[[\pi]]$. The following is a filtered model illustrating the kind of pathology excluded by the torsion-control hypothesis. It is not asserted here to arise from a displayed polynomial family. Consider a filtered subquotient of the form*

$$Q_s = Vx \oplus Vy / (\pi y - sx),$$

with filtration levels chosen so that x and y contribute to two fixed Newton slope blocks. For $s \in V^\times$, the relation $\pi y = sx$ identifies the two generators only after a shift in valuation. For $s = 0$, the relation becomes $\pi y = 0$, so the associated graded coefficient module acquires π -torsion.

The Newton polygon can remain unchanged because the valuations of the visible coefficients defining the lower convex hull are unchanged. The residual slope factors may also remain coprime, since their initial forms still lie in distinct slope components. Nevertheless, the associated graded coefficient module is no longer flat in the Newton filtration: a torsion class appears in one graded piece and produces an additional homology class in the associated graded Newton complex.

Thus the polygonal and residual data alone do not detect the failure. The obstruction appears only after passing to the filtered resolution and then to the associated graded complex. This is exactly the phenomenon excluded by the torsion-control hypothesis in Theorem 6.2. Hence that hypothesis is not merely technical; it prevents hidden valuation collapse inside the graded Newton pieces.

9 Comparison with Classical Newton Polygon Theory

Classical Newton polygon methods control possible valuations of roots and often imply factorization after Henselian lifting. The present construction adds a homological layer: it records whether the filtration induced by the Newton polygon behaves exactly under deformation. Thus the derived Newton defect is not a replacement for Newton polygon theory, but a refinement designed to detect factorization instability in families.

10 Conclusion

We introduced derived Newton polygon stratifications over valuation rings and proved a factorization stability theorem under Henselian, residual coprimality, and torsion-control hypotheses. The central point is that the derived Newton defect is not merely an auxiliary homology group: by Proposition 4.6, it is the obstruction to exactness compatible with the Newton filtration after passing from filtered coefficient data to associated graded Newton data.

Relative to classical Newton polygon theory, the construction preserves the valuation-theoretic core of the subject. Slopes, residual polynomials, and Henselian lifting still provide the factorization mechanism. The new ingredient is the filtered homological layer, which detects whether these Newton data remain exact in deformation families. This is why the framework is naturally compatible with classical work on valuation prolongations, lifting, and factorization over Henselian rings.

From the deformation-theoretic viewpoint, the derived Newton defect measures hidden extension and torsion phenomena that may remain invisible to the polygon alone. The examples above show that residual coprimality, Henselianity, and torsion control each remove a distinct obstruction to factorization stability.

Several directions remain open. One may study ramified extensions of valuation rings, higher-rank valuations, finer defect filtrations, connections with Berkovich or non-archimedean geometry, and algorithmic factorization criteria based on computable associated graded Newton

complexes. These directions remain within commutative algebra and valuation theory and do not require spectral or higher-categorical machinery.

References

- [1] O. Zariski and P. Samuel, *Commutative Algebra, Volume II*, Graduate Texts in Mathematics, vol. 29, Springer-Verlag, New York–Heidelberg–Berlin, 1975; reprint of the 1960 Van Nostrand edition.
- [2] H. Matsumura, *Commutative Ring Theory*, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1986.
- [3] M. Nagata, *Local Rings*, Interscience Tracts in Pure and Applied Mathematics, no. 13, Interscience Publishers, New York, 1962; reprinted with corrections, Robert E. Krieger Publishing Co., Huntington, NY, 1975.
- [4] S. Bhatia and S. K. Khanduja, *On extensions generated by roots of lifting polynomials*, *Mathematika* **49** (2002), 107–118.
- [5] N. Bourbaki, *Commutative Algebra*, Elements of Mathematics, Hermann, Paris, 1972.
- [6] S. K. Khanduja and S. Kumar, *On prolongations of valuations via Newton polygons and liftings of polynomials*, *J. Pure Appl. Algebra* **216** (2012), no. 12, 2648–2656.
- [7] S. K. Khanduja and M. Kumar, *Prolongations of valuations to finite extensions*, *Manuscripta Math.* **131** (2010), no. 3–4, 323–334.
- [8] P. Ribenboim, *The Theory of Classical Valuations*, Springer Monographs in Mathematics, Springer-Verlag, New York, 1999.
- [9] O. Endler, *Valuation Theory*, Universitext, Springer-Verlag, New York–Heidelberg–Berlin, 1972.
- [10] A. J. Engler and A. Prestel, *Valued Fields*, Springer Monographs in Mathematics, Springer-Verlag, Berlin–Heidelberg, 2005.