Homological Obstructions to Integral Closure in Simple Extensions of Valuation Rings

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Abstract

Let V be a valuation ring with fraction field K, let $L = K(\alpha)$ be a finite simple extension, and let $A = V[\alpha] \subseteq W$, where W denotes the integral closure of A in L. We study homological defect modules associated to the normalization quotient W/A using ordinary derived functors. For every valuation-compatible ideal $I \subseteq A$, we define derived integral closure defects

$$\operatorname{Tor}_i^A(W/A, A/I),$$

which measure the failure of reduction modulo I to preserve exactness of the normalization sequence

$$0 \to A \to W \to W/A \to 0.$$

We prove that these homological defect modules are supported along the conductor locus and vanish under suitable flatness hypotheses. In the Henselian defectless setting, we do not claim that defectlessness alone forces Tor-vanishing; rather, we record a conditional collapse statement under explicit Tor-independence or flat-reduction hypotheses, in which case the derived defect system becomes concentrated in degree zero. The new content is the systematic valuation-theoretic use of these ordinary Tor groups as normalization-defect invariants: we identify their conductor support, isolate the connecting image as the actual obstruction to exact reduction, and compute their π-primary torsion profiles in DVR-order normalizations. In particular, the first derived defect records conductor thickness and the elementary-divisor structure of the normalization quotient W/A, including non-cyclic examples. In ramified DVR-order examples this thickness may reflect ramified structure, but the Tor computation itself is an invariant of the normalization quotient rather than a ramification invariant. The approach is entirely algebraic and valuation-theoretic, using only classical homological methods and avoiding higher-categorical or spectral machinery.

Keywords: valuation rings; integral closure; simple extensions; derived functors; Tor modules; normalization defects; valuation-compatible reductions; ramification; conductor ideals.
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1 Introduction

Let V be a valuation ring with fraction field K, let $L = K(\alpha)$ be a simple finite extension, and put

$$A := V[\alpha].$$

Throughout the paper, unless explicitly stated otherwise, W denotes the integral closure of this order A in L, namely

$$W := \overline{A}^{L}.$$

Thus the normalization inclusion studied here is

$$A = V[\alpha] \subseteq W = \overline{A}^{L}.$$

This passage encodes subtle arithmetic and valuation-theoretic information: extension of valuations, ramification, conductor behaviour, and failure of finite generation in non-Noetherian settings.

Classically, the study of such extensions is governed by valuation prolongations, integral dependence, residual extensions, ramification indices, and Henselian lifting ([7, Ch. III, §§13,16–18, pp. 94–143; §§19–22, pp. 144–184]; [1, Ch. V, §1, nos. 1–2, pp. 303–308; Ch. VI, §8, nos. 1–2, pp. 416–418]). The purpose of this article is to introduce a derived but still elementary homological diagnostic for this picture. Instead of using derived algebraic geometry, we use only ordinary derived functors such as Tor, Ext, and finite free resolutions to measure where valuation-compatible reductions of the normalization quotient fail to be exact.

Guiding principle. The word "derived" in this paper means homological and module-theoretic. It does not refer to ∞-categories, spectral rings, or derived schemes.

Scope. The paper does not claim that the functor Tor or the flatness criterion for Tor-vanishing is new. Its novelty is instead the construction of a valuation-theoretic normalization diagnostic in which the ordinary Tor groups

$$\mathrm{Tor}_i^A(W/A, A/I)$$

are organized as defect modules attached to valuation-compatible reductions of the normalization sequence. The contribution has three concrete parts: first, the defects are localized on the conductor locus; second, the connecting image

$$\mathrm{Tor}_1^A(W/A, A/I) \longrightarrow A/I$$

is identified as the actual obstruction to preserving injectivity after reduction; and third, in DVR-order situations the first defect is computed in terms of the π-primary elementary-divisor profile of W/A. Thus the contribution is not an abstract new Tor formalism, but a computable valuation/conductor interpretation of Tor as a normalization-defect invariant.

Let

$$A := V[\alpha], \qquad W := \overline{A}^{L}.$$

The inclusion $A \subseteq W$ gives the conductor

$$\mathfrak{c}_{A/W} := (A : W) = \{a \in A : aW \subseteq A\}.$$

The usual integral closure data records the quotient W/A. Our approach keeps track of higher homological invariants associated to this quotient, especially

$$\mathrm{Tor}_i^A(W/A,\ A/I), \qquad \mathrm{Ext}_A^i(W/A,\ A),$$

for valuation-compatible ideals $I \subseteq A$.

1.1 Main contributions

The article develops the following diagnostic homological framework. The construction uses ordinary Tor groups, but the contribution is their valuation-theoretic interpretation and explicit computation in normalization problems.

(i) We attach to $A = V[\alpha] \subseteq W$ the modules

$$\mathrm{Tor}_i^A(W/A, A/I),$$

not as a new homological functor, but as homological defect modules diagnosing exactness failure of normalization after valuation-compatible reduction.

(ii) We identify their support and first obstruction behaviour along the conductor locus, especially through the connecting image

$$\mathrm{Tor}_1^A(W/A, A/I) \longrightarrow A/I.$$

(iii) We isolate flatness and Tor-independence as collapse hypotheses. These vanishing statements are standard homological consequences; their role here is to separate genuinely non-flat conductor defects from formally exact cases.

(iv) We compute the first defect in DVR-order situations, showing that it records conductor thickness and the full π-primary normalization torsion profile of W/A, including non-cyclic elementary-divisor examples.

These points are not merely formal consequences of the definition. The support result explains where the defects can live, the connecting-image criterion separates harmless Tor from genuine failure of exact reduction, and the DVR-order formulas show that the first defect recovers more than the total length of W/A: it detects the full π-primary torsion profile visible through the functions

$$n \longmapsto \ell_V\left(\operatorname{Tor}_1^A(W/A, A/(\pi^n))\right).$$

This is the main reason the construction gives information beyond the classical normalization quotient and beyond a single discriminant-index length.

Unlike the classical integral-closure filtration, the present construction records ordinary Tor-theoretic information arising from failure of exact reduction, localization, and conductor compatibility. The resulting functorial family of Tor-modules is therefore sensitive not only to integral dependence itself, but also to conductor defect and valuation-theoretic torsion phenomena detected through explicit computations.

The modules

$$\operatorname{Tor}_i^A(W/A, A/I)$$

measure the failure of reduction modulo I to preserve exactness across the normalization quotient

$$0 \to A \to W \to W/A \to 0.$$

When the normalization quotient W/A is flat over A, the higher homological defect modules vanish. In ramified or conductor-singular situations, however, nontrivial Tor may appear and records homological deviation from exact integral closure descent; the actual obstruction to injectivity after reduction is the connecting image of $\operatorname{Tor}_1^A(W/A, A/I) \to A/I$.

2 Preliminaries on valuation rings and simple extensions

Throughout the paper, V denotes a valuation ring with fraction field K. Its maximal ideal is denoted by $\mathfrak{m}_V$, and its value group by Γ_V.

Definition 2.1 (Simple extension)**.** Let L/K be a finite field extension. We say that L/K is simple over K if $L = K(\alpha)$ for some $\alpha \in L$. For such an element α, we put

$$A := V[\alpha] \subseteq L.$$

Definition 2.2 (Integral closure)**.** The integral closure of A in L is

$$\overline{A}^L := \{x \in L : x \text{ is integral over } A\}.$$

When $A = V[\alpha]$, we write

$$W := \overline{V[\alpha]}^L.$$

([4, Def. 2.1.1, p. 23])

Remark 2.3. If A is Noetherian, then the integral closure W is finite over A whenever A is Japanese, while finite generation can fail in general. Classically, this finiteness phenomenon is already reflected in Nagata's theory of pseudo-geometric rings, where affine rings over pseudo-geometric integral domains have finite derived normal rings [5, (36.6), p. 133]; see also [3, §9, pp. 64–70]. In valuation-theoretic settings, especially non-Noetherian ones, this makes it useful to study normalization through filtrations and homological approximations rather than finite presentation alone.

Definition 2.4 (Conductor)**.** For an extension of rings $A \subseteq W$, the conductor is

$$\mathfrak{c}_{A/W} := (A : W) = \{a \in A : aW \subseteq A\}.$$

([4, Def. 12.0.1, p. 238])

Definition 2.5 (Valuation-compatible ideal)**.** Let $A = V[\alpha] \subseteq L$, and let v be a chosen prolongation of the valuation of V to L such that $A \subseteq \{x \in L : v(x) \geq 0\}$. For $\gamma \geq 0$, put

$$I_\gamma = \{a \in A : v(a) \geq \gamma\}.$$

An ideal $I \subseteq A$ is called valuation-compatible if it is generated by valuation-depth ideals I_γ, or is an ordinary power or finite product of such ideals.

Lemma 2.6 (Conductor quotient sequence)**.** *Let $A \subseteq W$ be an extension of rings as above and let*

$$\mathfrak{c} = (A : W).$$

Then $\mathfrak{c}$ is an ideal of both A and W, and there is a natural exact sequence

$$0 \longrightarrow A/\mathfrak{c} \longrightarrow W/\mathfrak{c} \longrightarrow W/A \longrightarrow 0.$$

In particular, the quotient W/A is naturally controlled by the conductor layer $W/\mathfrak{c}$.

Moreover, if W/A contains nontrivial torsion supported along the conductor, then the quotient $W/\mathfrak{c}$ records the first layer where reduction modulo the conductor fails to preserve the exact sequence

$$0 \to A \to W \to W/A \to 0.$$

In particular, conductor-supported torsion contributes directly to the derived defect modules

$$\operatorname{Tor}_i^A(W/A, A/\mathfrak{c}).$$

([4, Def. 12.0.1, p. 238])

Proof. Since

$$\mathfrak{c} = (A : W) = \{a \in A : aW \subseteq A\},$$

it is immediate that $\mathfrak{c}$ is an ideal of A. We now show that it is also an ideal of W. Let $x \in \mathfrak{c}$ and $w \in W$. Since $xW \subseteq A$, we have $wx \in A$. Moreover, for every $y \in W$, the product wy belongs to W, because W is a ring. Hence

$$(wx)y = x(wy) \in A.$$

Thus $(wx)W \subseteq A$, and therefore $wx \in \mathfrak{c}$. This proves that $\mathfrak{c}$ is an ideal of W.

The inclusion $A \hookrightarrow W$ induces an injective map

$$A/\mathfrak{c} \longrightarrow W/\mathfrak{c}.$$

Its cokernel is

$$(W/\mathfrak{c})/(A/\mathfrak{c}) \cong W/A,$$

which gives the claimed short exact sequence.

The sequence shows that the quotient W/A is controlled by the conductor quotient layer W/c whenever the conductor is nonzero. Thus conductor-supported torsion is visible after reduction modulo c. This observation becomes important later when derived defect modules are computed using conductor reduction. ☐

Lemma 2.7 (Basic exact sequence)**.** *For $A = V[\alpha] \subseteq W$, there is a canonical short exact sequence*

$$0 \longrightarrow A \longrightarrow W \longrightarrow W/A \longrightarrow 0.$$

The quotient module W/A measures the precise failure of A to be integrally closed in L. In particular, torsion and conductor support are reflected directly in the structure of W/A. In ramified order examples, these normalization defects may also reflect ramified structure, but W/A is not itself a ramification invariant.

Proof. The inclusion

$$A \hookrightarrow W$$

is the canonical embedding of A into its integral closure in L. Since the map is injective, its cokernel is the quotient module W/A, yielding the exact sequence.

The module W/A vanishes precisely when A is already integrally closed in L. More generally, its support detects where normalization fails, while its torsion structure records conductor-supported normalization defect. In ramified examples this torsion may be influenced by ramified structure, but the invariant being measured here is the normalization quotient W/A, not the ramification index itself. □

Theorem 2.8 (Conductor support and localization criterion)**.** *Let $A \subseteq B$ be an integral extension inside a finite field extension, and let $\mathfrak{c} = (A : B)$. If $\mathfrak{c} \neq 0$, then for every prime $\mathfrak{p} \in \operatorname{Spec} A$ with $\mathfrak{p} \not\supseteq \mathfrak{c}$, one has*

$$A_{\mathfrak{p}} = B_{\mathfrak{p}}.$$

Consequently,

$$\operatorname{Supp}_A(B/A) \subseteq V(\mathfrak{c}),$$

and for every ideal $I \subseteq A$ and every $i \geq 0$,

$$\operatorname{Supp}_A \operatorname{Tor}_i^A(B/A, A/I) \subseteq V(\mathfrak{c}).$$

If $\mathfrak{c} = 0$, the statement gives no informative support restriction: one only has the tautological containment

$$\operatorname{Supp}_A(B/A) \subseteq \operatorname{Spec} A.$$

Thus the conductor-support assertion is meaningful only on the nonzero-conductor locus.

Proof. Assume first that $\mathfrak{c} \neq 0$. Let $\mathfrak{p} \in \operatorname{Spec} A$ with $\mathfrak{p} \not\supseteq \mathfrak{c}$. Choose $s \in \mathfrak{c} \setminus \mathfrak{p}$. Since $sB \subseteq A$, the element s becomes a unit in $A_{\mathfrak{p}}$. Hence every $b \in B_{\mathfrak{p}}$ satisfies

$$b = s^{-1}(sb) \in A_{\mathfrak{p}}.$$

Thus $B_{\mathfrak{p}} = A_{\mathfrak{p}}$. Therefore $(B/A)_{\mathfrak{p}} = 0$, proving the support inclusion. Localizing Tor gives

$$\operatorname{Tor}_i^A(B/A, A/I)_{\mathfrak{p}} \cong \operatorname{Tor}_i^{A_{\mathfrak{p}}}((B/A)_{\mathfrak{p}}, (A/I)_{\mathfrak{p}}) = 0.$$

If $\mathfrak{c} = 0$, there is no nonzero conductor element to invert, and the preceding argument gives no proper closed support bound. The remaining statement is only the tautological support containment in $\operatorname{Spec} A$. □

Lemma 2.9 (Support of the normalization defect)**.** *Let $A = V[\alpha] \subseteq W$ be as above, and let*

$$\mathfrak{c} = (A : W)$$

be the conductor ideal. Then

$$\operatorname{Supp}_A(W/A) \subseteq V(\mathfrak{c}).$$

In particular, the failure of integral closure is supported along the conductor locus.

Proof. This is the special case $i = 0$ of Theorem 2.8. □

3 Classical integral closure filtrations

We first define the non-derived filtration.

Definition 3.1 (Classical integral-closure filtration)**.** Let $I \subseteq A$ be an ideal. The classical integral-closure filtration induced by the normalization $A \subseteq W$ is the filtration on A

$$F^n_{\mathrm{cl}} A := A \cap I^n W \subseteq A, \qquad n \geq 0.$$

Its associated graded object is

$$\operatorname{gr}_F(A) := \bigoplus_{n \geq 0} F^n_{\mathrm{cl}} A / F^{n+1}_{\mathrm{cl}} A.$$

Lemma 3.2 (Basic properties of the classical filtration)**.** *For every ideal $I \subseteq A$, the filtration*

$$F^n_{\mathrm{cl}}A = A \cap I^n W$$

satisfies:

(i)
$$F^{n+1}_{\mathrm{cl}}A \subseteq F^n_{\mathrm{cl}}A, \qquad \textit{for all } n \geq 0.$$

(ii)
$$I^m F^n_{\mathrm{cl}}A \subseteq F^{m+n}_{\mathrm{cl}}A, \qquad \textit{for all } m, n \geq 0.$$

(iii) *the associated graded object*
$$\mathrm{gr}_F(A) = \bigoplus_{n \geq 0} F^n_{\mathrm{cl}}A / F^{n+1}_{\mathrm{cl}}A$$
is naturally a graded module over
$$\mathrm{gr}_I(A) = \bigoplus_{n \geq 0} I^n / I^{n+1}.$$

Proof. Since $I^{n+1}W \subseteq I^n W$, one has

$$A \cap I^{n+1}W \subseteq A \cap I^n W,$$

which proves (i).

For (ii), let
$$x \in F^n_{\mathrm{cl}}A = A \cap I^n W.$$

Then $x \in A$ and $x \in I^n W$. Hence
$$I^m x \subseteq A$$

because $I^m \subseteq A$ and $x \in A$, while also

$$I^m x \subseteq I^m(I^n W) = I^{m+n}W.$$

Therefore
$$I^m x \subseteq A \cap I^{m+n}W = F^{m+n}_{\mathrm{cl}}A.$$

Property (iii) follows formally from (ii), since multiplication by I^m/I^{m+1} sends the n-th graded piece of $\mathrm{gr}_F(A)$ into the $(m+n)$-th graded piece. □

Proposition 3.3 (Valuation description under principal valuation ideals)**.** *Let v be a rank-one discrete valuation on L, and let $\mathcal{O}_v$ be its valuation ring. Assume $A \subseteq \mathcal{O}_v$, and let $\pi \in A$ satisfy $v(\pi) > 0$. Let*

$$I = (\pi^r)A \subseteq A.$$

Define the valuation-induced filtration

$$F^n_v A := A \cap I^n \mathcal{O}_v.$$

Then for every $n \geq 0$,
$$F^n_v A = A \cap \pi^{rn}\mathcal{O}_v = \{x \in A : v(x) \geq rn\, v(\pi)\}.$$

Proof. Since $I = (\pi^r)A$, one has $I^n = (\pi^{rn})A$. Hence

$$F^n_v A = A \cap I^n \mathcal{O}_v = A \cap \pi^{rn}\mathcal{O}_v.$$

Because $\mathcal{O}_v$ is the valuation ring of v, one has

$$\pi^{rn}\mathcal{O}_v = \{y \in L : v(y) \geq rn\, v(\pi)\}.$$

Intersecting with A gives
$$A \cap \pi^{rn}\mathcal{O}_v = \{x \in A : v(x) \geq rn\, v(\pi)\}.$$

□

Proposition 3.4 (Functoriality under localization)**.** *Let $S \subseteq A$ be a multiplicative set. Then*

$$S^{-1}F^n_{\mathrm{cl}}A \subseteq F^n_{\mathrm{cl}}(S^{-1}A).$$

If

$$S^{-1}W = \overline{S^{-1}A}^{\,L}, \qquad S^{-1}(I^nW) = I^n(S^{-1}W),$$

and the intersection is localization-compatible inside $S^{-1}W$, namely

$$S^{-1}(A \cap I^nW) = S^{-1}A \cap S^{-1}(I^nW),$$

then equality holds:

$$S^{-1}F^n_{\mathrm{cl}}A = F^n_{\mathrm{cl}}(S^{-1}A).$$

Proof. By definition,

$$F^n_{\mathrm{cl}}A = A \cap I^nW.$$

Localizing gives the always-valid inclusion

$$S^{-1}F^n_{\mathrm{cl}}A = S^{-1}(A \cap I^nW) \subseteq S^{-1}A \cap S^{-1}(I^nW).$$

Using

$$S^{-1}(I^nW) = I^n(S^{-1}W),$$

we obtain

$$S^{-1}F^n_{\mathrm{cl}}A \subseteq S^{-1}A \cap I^n(S^{-1}W) = F^n_{\mathrm{cl}}(S^{-1}A).$$

This proves the stated inclusion.

Now assume additionally that

$$S^{-1}(A \cap I^nW) = S^{-1}A \cap S^{-1}(I^nW).$$

Then

$$F^n_{\mathrm{cl}}(S^{-1}A) = S^{-1}A \cap I^n(S^{-1}W) = S^{-1}A \cap S^{-1}(I^nW) = S^{-1}(A \cap I^nW) = S^{-1}F^n_{\mathrm{cl}}A.$$

Hence equality holds under the explicit intersection-compatibility hypothesis. □

Proposition 3.5 (Conductor stability of the filtration)**.** *Let*

$$\mathfrak{c} = (A : W)$$

be the conductor ideal. Then for every $n \geq 0$,

$$\mathfrak{c}\, F^n_{\mathrm{cl}}A \subseteq I^nA.$$

In particular, the conductor controls the failure of the filtration to descend from W to A.

Proof. Let

$$x \in F^n_{\mathrm{cl}}A = A \cap I^nW.$$

Then $x \in I^nW$, so we may write

$$x = \sum_j a_jw_j, \qquad a_j \in I^n, \quad w_j \in W.$$

If $a \in \mathfrak{c}$, then

$$aw_j \in A$$

by definition of the conductor. Hence

$$ax = \sum_j a_j(aw_j) \in I^nA.$$

Therefore

$$\mathfrak{c}\, F^n_{\mathrm{cl}}A \subseteq I^nA.$$

□

Remark 3.6 (Limitation of the classical filtration). The classical filtration records only the successive valuation- or conductor-theoretic layers inside A induced by the normalization $A \subseteq W$. However, it does not measure whether these layers behave exactly under reduction, localization, or passage to quotient rings.

In particular, the classical associated graded object

$$\mathrm{gr}_F(A)$$

cannot detect higher obstruction phenomena arising from failure of tensor exactness. This motivates the introduction of derived integral closure defects via Tor-modules.

4 Derived normalization-defect systems

We now introduce the homological object used throughout the paper. The definition uses ordinary Tor, but the point is the valuation-theoretic normalization problem to which it is applied: the quotient W/A, the valuation-compatible reductions A/I, the conductor support, and the explicit DVR-order torsion computations below together produce a computable defect system attached to the normalization $A \subseteq W$.

Definition 4.1 (Derived integral closure defect)**.** Let $A \subseteq W$ be as above, and let $I \subseteq A$ be an ideal. The i-th derived integral closure defect of I is

$$\mathrm{DIC}_i(A, W; I) := \mathrm{Tor}_i^A(W/A, A/I).$$

Remark 4.2. The module $\mathrm{DIC}_0(A, W; I)$ is the ordinary reduction

$$(W/A) \otimes_A A/I.$$

The higher modules $\mathrm{DIC}_i(A, W; I)$, $i \geq 1$, measure failure of exact base change across the integral closure quotient.

Definition 4.3 (Derived integral-closure defect system)**.** Let $I \subseteq A$. The derived integral-closure defect system associated to the powers of I is the family

$$\mathcal{D}^n_{\mathrm{der}}(A, W; I) := \{\mathrm{DIC}_i(A, W; I^n)\}_{i \geq 0,\ n \geq 0}\,.$$

The associated graded defect terms are

$$\mathrm{gr}^n_{\mathrm{der}}(A, W; I) := \left\{\mathrm{Tor}_i^A(W/A, I^n/I^{n+1})\right\}_{i \geq 0}.$$

The inclusions $I^{n+1} \subseteq I^n$ induce transition morphisms between the corresponding quotient modules. Thus the above family is best viewed as a functorial inverse/direct system of Tor-modules, not as a filtration of one fixed module unless an additional filtration structure is specified.

Lemma 4.4 (Long exact Tor sequence)**.** *For every ideal $I \subseteq A$, the short exact sequence*

$$0 \to A \to W \to W/A \to 0$$

induces an exact sequence

$$\mathrm{Tor}_1^A(W, A/I) \to \mathrm{Tor}_1^A(W/A, A/I) \to A/I \to W/IW \to (W/A) \otimes_A A/I \to 0.$$

Proof. Apply $- \otimes_A A/I$ to the short exact sequence and take the associated long exact Tor sequence. □

Proposition 4.5 (First conductor obstruction)**.** *Let $\mathfrak{c} = (A : W)$. If the image of*

$$\mathrm{Tor}_1^A(W/A, A/c) \to A/c$$

under the connecting morphism is nonzero, then reduction modulo the conductor does not preserve the exactness of

$$0 \longrightarrow A \longrightarrow W \longrightarrow W/A \longrightarrow 0.$$

Thus the conductor detects a genuine obstruction to recovering W from A after passing to the conductor quotient.

Proof. Apply $-\otimes_A A/\mathfrak{c}$ to the exact sequence

$$0 \to A \to W \to W/A \to 0.$$

The associated long exact Tor sequence contains

$$\operatorname{Tor}_1^A(W/A, A/\mathfrak{c}) \longrightarrow A/\mathfrak{c} \longrightarrow W/\mathfrak{c}W.$$

A nontrivial connecting image of the first Tor term is the obstruction to left-exactness after conductor reduction.

Thus the first derived defect measures precisely the failure of conductor reduction to preserve injectivity of the normalization map. ☐

Proposition 4.6 (Support of derived defects)**.** *Let*

$$\mathfrak{c} = (A : W).$$

Assume $\mathfrak{c} \neq 0$. *Then for every ideal* $I \subseteq A$ *and every* $i \geq 0$*,*

$$\operatorname{Supp}_A\big(\mathrm{DIC}_i(A, W; I)\big) \subseteq V(\mathfrak{c}).$$

In particular, on the nonzero-conductor locus, all derived integral closure defects are supported along the conductor locus.

Proof. Localize at a prime ideal $\mathfrak{p} \not\supseteq \mathfrak{c}$. By Theorem 2.8,

$$W_{\mathfrak{p}} = A_{\mathfrak{p}}.$$

Hence

$$(W/A)_{\mathfrak{p}} = 0.$$

Therefore

$$\operatorname{Tor}_i^{A_{\mathfrak{p}}}((W/A)_{\mathfrak{p}}, (A/I)_{\mathfrak{p}}) = 0$$

for all $i \geq 0$. ☐

Proposition 4.7 (Classical shadow of the derived defect system)**.** *For every ideal* $I \subseteq A$*,*

$$\mathrm{DIC}_0(A, W; I) = (W/A) \otimes_A A/I.$$

Hence the degree-zero part of the derived defect system recovers the classical reduction of the normalization quotient.

Proof. This is the definition of Tor_0. ☐

5 Collapse criteria

Theorem 5.1 (Flat collapse criterion)**.** *Let* $A = V[\alpha] \subseteq W$*, and let* $I \subseteq A$ *be an ideal. If* W/A *is flat over* A*, then*

$$\mathrm{DIC}_i(A, W; I) = 0 \qquad \textit{for all } i \geq 1.$$

Consequently, the derived defect system is concentrated in degree zero.

Proof. By definition, the i-th derived integral closure defect is

$$\mathrm{DIC}_i(A, W; I) = \operatorname{Tor}_i^A(W/A, A/I).$$

Thus the assertion is a statement about the vanishing of higher Tor groups with first argument W/A.

Recall that an A-module M is flat if and only if the functor

$$M \otimes_A -$$

is exact; equivalently, the higher Tor functors vanish against every A-module (see [1, Ch. I, §2, no. 3, pp. 12–15; Ch. I, §4, p. 37]; see also [3, Thm. 7.8, p. 51]). Applying this standard criterion with

$$M = W/A \qquad \text{and} \qquad N = A/I,$$

the assumed flatness of W/A gives

$$\operatorname{Tor}_i^A(W/A, A/I) = 0 \qquad \text{for all } i \geq 1.$$

Hence

$$\mathrm{DIC}_i(A, W; I) = 0 \qquad \text{for all } i \geq 1.$$

It remains only to interpret this vanishing in terms of the defect system. For the powers of I, the same argument applied to A/I^n gives

$$\mathrm{DIC}_i(A, W; I^n) = 0 \qquad \text{for all } i \geq 1.$$

Thus, under the flatness hypothesis, the system has no higher Tor terms and only the degree-zero reductions

$$\mathrm{DIC}_0(A, W; I^n) = (W/A) \otimes_A A/I^n$$

remain. This is a standard flatness consequence, not an independent criterion for flatness. ☐

Corollary 5.2 (Classical recovery). *Under the hypotheses of Theorem 5.1, the derived normalization-defect system records no additional higher Tor obstruction beyond the classical quotient*

$$(W/A) \otimes_A A/I.$$

Proof. By Theorem 5.1, the flatness of W/A over A implies

$$\mathrm{DIC}_i(A, W; I) = 0 \qquad \text{for all } i \geq 1.$$

Hence every higher homological defect module in the derived defect system vanishes.

The only remaining term is the zeroth derived defect. By the definition of Tor in degree zero,

$$\mathrm{DIC}_0(A, W; I) = \operatorname{Tor}_0^A(W/A, A/I) = (W/A) \otimes_A A/I.$$

This is precisely the ordinary reduction of the normalization quotient W/A modulo I. Therefore, in the flat case, the derived construction does not produce any additional higher homological defect beyond the classical reduction of W/A. ☐

Theorem 5.3 (First obstruction to exact reduction). *Let $0 \to A \to B \to B/A \to 0$ be the normalization sequence. For an ideal $I \subseteq A$, the kernel of the reduced map*

$$A/I \longrightarrow B/IB$$

is exactly the image of the connecting homomorphism

$$\delta : \operatorname{Tor}_1^A(B/A, A/I) \longrightarrow A/I.$$

Consequently, reduction modulo I preserves injectivity if and only if

$$\operatorname{im}(\delta) = 0.$$

In particular, a nonzero element of $\operatorname{Tor}_1^A(B/A, A/I)$ gives an actual failure of exact reduction only when its image under δ is nonzero.

Proof. Start with the canonical exact sequence associated to the normalization inclusion:

$$0 \longrightarrow A \longrightarrow W \longrightarrow W/A \longrightarrow 0.$$

Tensoring this sequence with A/I need not preserve left exactness, because tensor product is right exact but not generally exact. Applying the functor $- \otimes_A A/I$ and taking the associated long exact Tor sequence gives

$$\operatorname{Tor}_1^A(W, A/I) \longrightarrow \operatorname{Tor}_1^A(W/A, A/I) \xrightarrow{\delta} A \otimes_A A/I \longrightarrow W \otimes_A A/I \longrightarrow (W/A) \otimes_A A/I \longrightarrow 0.$$

Using the natural identifications

$$A \otimes_A A/I \cong A/I, \qquad W \otimes_A A/I \cong W/IW,$$

this becomes

$$\operatorname{Tor}_1^A(W, A/I) \longrightarrow \operatorname{Tor}_1^A(W/A, A/I) \xrightarrow{\delta} A/I \longrightarrow W/IW \longrightarrow (W/A) \otimes_A A/I \longrightarrow 0.$$

The map

$$A/I \longrightarrow W/IW$$

is the reduction modulo I of the inclusion $A \hookrightarrow W$. Thus the reduced sequence

$$0 \longrightarrow A/I \longrightarrow W/IW \longrightarrow (W/A) \otimes_A A/I \longrightarrow 0$$

is exact at A/I precisely when the connecting contribution from

$$\operatorname{Tor}_1^A(W/A, A/I)$$

does not create a nonzero kernel of $A/I \to W/IW$.

In particular, whenever

$$\operatorname{Tor}_1^A(W/A, A/I) \neq 0$$

and its image under the connecting homomorphism is nonzero, the map

$$A/I \longrightarrow W/IW$$

fails to be injective. Therefore reduction modulo I fails to preserve the left exactness of

$$0 \to A \to W \to W/A \to 0.$$

Thus the possible first obstruction is carried by the first derived integral-closure defect

$$\mathrm{DIC}_1(A, W; I) = \operatorname{Tor}_1^A(W/A, A/I),$$

but the actual obstruction to injectivity after reduction is its connecting image

$$\operatorname{im}\bigl(\delta : \operatorname{Tor}_1^A(W/A, A/I) \longrightarrow A/I\bigr).$$

Consequently, a nonzero class in $\mathrm{DIC}_1(A, W; I)$ produces a genuine failure of exact reduction only if its image under δ is nonzero. ☐

6 Simple extensions and valuation prolongations

Throughout this section only, we also use the notation

$$\overline{V}^L := \{x \in L : x \text{ is integral over } V\}, \qquad \overline{A}^L := \{x \in L : x \text{ is integral over } A\}.$$

The normalization used in the rest of the paper is always

$$W = \overline{A}^L,$$

where $A = V[\alpha]$. The valuative intersection over prolongations describes $\overline{V}^L$, not automatically $\overline{A}^L$. These two rings are identified only under an additional hypothesis, for example when $\overline{A}^L = \overline{V}^L$.

Let V be a valuation ring of K, and let $L = K(\alpha)$. A valuation ring W' of L lying over V, i.e. satisfying

$$W' \cap K = V,$$

is called a prolongation of V to L, in the classical sense of extensions of valuations to finite algebraic field extensions ([2, Ch. VI, §11, Thm. 19, pp. 50–58]; [6, Ch. 4, §4.1, pp. 107–118]; [7, Ch. III, §13, pp. 94–103]).

Definition 6.1 (Prolongation-compatible closure)**.** Let V be a valuation ring of K, let L/K be a finite extension, and let

$$\mathcal{W}(V, L)$$

denote the set of valuation rings of L lying over V. Define the prolongation-compatible closure

$$W_{\text{val}} := \bigcap_{W' \in \mathcal{W}(V,L)} W'.$$

Equivalently, W_{val} consists precisely of those elements $x \in L$ which are contained in every prolongation of the valuation of V to L.

Remark 6.2. The ring W_{val} is intrinsically valuation-theoretic and is independent of the choice of generator α of the simple extension $L = K(\alpha)$. In particular, although the intermediate order

$$A = V[\alpha]$$

depends on the chosen primitive element, the valuative intersection W_{val} depends only on the extension of valued fields $(K, V) \subseteq (L, W')$.

The ring W_{val} therefore provides a canonical valuation-based description of the integral closure of V inside L.

Proposition 6.3 (Valuative description of integral closure)**.** *Assume that L/K is finite. Then the integral closure of V in L satisfies*

$$\overline{V}^L = \bigcap_{W' \in \mathcal{W}(V,L)} W'.$$

In particular,

$$\overline{V}^L = W_{\text{val}}.$$

Proof. This is the classical valuative criterion for integral dependence over a valuation ring ([4, Thm. 6.8.3 and Prop. 6.8.14, pp. 136–141]; [3, §10, pp. 71–78]; [1, Ch. VI, §1, no. 3, p. 378]).

Indeed, every valuation ring W' lying over V is integrally closed. Hence every element integral over V belongs to each such W', giving

$$\overline{V}^L \subseteq \bigcap_{W' \in \mathcal{W}(V,L)} W'.$$

Conversely, let $x \in L$ be not integral over V. By the valuative criterion for integrality applied relative to the valuation domain V, one may choose a valuation ring W' of L lying over V, equivalently

$$W' \cap K = V,$$

such that

$$x \notin W'.$$

Therefore

$$x \notin \bigcap_{W' \in \mathcal{W}(V,L)} W',$$

which proves the reverse inclusion. ☐

Remark 6.4 (Normalization versus prolongation). The preceding proposition describes the integral closure $\overline{V}^L$ of the valuation ring V in L. This should not be confused with the normalization

$$W = \overline{A}^L$$

of the chosen order $A = V[\alpha]$. In general, $\overline{V}^L$ and $\overline{A}^L$ need not coincide.

Only under the additional hypothesis

$$\overline{A}^L = \overline{V}^L$$

may the normalization quotient W/A be interpreted as the discrepancy between the chosen order A and the intersection of all prolongation-compatible valuation over-rings.

Construction 6.5 (Valuation-indexed ideals)**.** Assume that the chosen prolongation v is compatible with A, in the sense that $A \subseteq \{x \in L : v(x) \geq 0\}$, and extend v by $v(0) = \infty$. For $\gamma \geq 0$, define

$$I_\gamma := \{a \in A : v(a) \geq \gamma\}.$$

Then I_γ is an ideal of A. The family

$$\{I_\gamma\}_{\gamma \in \Gamma_V}$$

defines a decreasing filtration of A indexed by the ordered value group Γ_V.

Moreover, if $\gamma' \geq \gamma$, then

$$I_{\gamma'} \subseteq I_\gamma.$$

Remark 6.6. The filtration $\{I_\gamma\}$ refines the ordinary adic filtration by recording valuation depth rather than merely ideal powers.

In the discrete rank-one case with uniformizer π, choosing $\gamma = nv(\pi)$ gives $I_\gamma = A \cap \pi^n \mathcal{O}_v$; under the usual compatibility condition this recovers the π-adic filtration.

In higher-rank or non-discrete valuation settings, however, the filtration contains strictly finer information reflecting the geometry of the ordered value group.

Definition 6.7 (Valuation-indexed derived defect)**.** For $\gamma \in \Gamma_V$, define the i-th valuation-indexed derived defect by

$$\mathrm{Def}_i(\gamma) := \mathrm{Tor}_i^A(W/A, A/I_\gamma).$$

Remark 6.8. The module

$$\mathrm{Def}_0(\gamma) = (W/A) \otimes_A A/I_\gamma$$

records the ordinary valuation-theoretic reduction of the normalization defect modulo valuation depth γ.

The higher modules

$$\mathrm{Def}_i(\gamma), \qquad i \geq 1,$$

measure the failure of this reduction process to behave exactly from the homological point of view.

Thus the higher valuation-indexed defects detect additional Tor-obstruction modules associated with the normalization quotient, which are invisible in the classical valuation filtration alone.

Theorem 6.9 (Valuation stratification of derived defects)**.** *The assignment*

$$\gamma \longmapsto \mathrm{Def}_i(\gamma)$$

is functorial with respect to inclusions

$$I_{\gamma'} \subseteq I_\gamma \qquad (\gamma' \geq \gamma).$$

Hence the derived integral closure defects form a functorial system of Tor-modules indexed by the ordered value group Γ_V.

Moreover, the transition morphisms

$$\mathrm{Def}_i(\gamma') \longrightarrow \mathrm{Def}_i(\gamma)$$

describe how higher normalization obstructions evolve as one passes to deeper valuation layers.

Proof. If $\gamma' \geq \gamma$, then by construction

$$I_{\gamma'} \subseteq I_\gamma.$$

Hence there is a natural quotient morphism

$$A/I_{\gamma'} \longrightarrow A/I_\gamma.$$

Since the functor

$$\operatorname{Tor}_i^A(W/A, -)$$

is covariant in its second argument, applying it yields natural transition maps

$$\operatorname{Tor}_i^A(W/A, A/I_{\gamma'}) \longrightarrow \operatorname{Tor}_i^A(W/A, A/I_\gamma).$$

These are precisely the morphisms

$$\operatorname{Def}_i(\gamma') \longrightarrow \operatorname{Def}_i(\gamma).$$

Thus the collection

$$\{\operatorname{Def}_i(\gamma)\}_{\gamma\in\Gamma_V}$$

forms a directed functorial system of Tor-modules indexed by the ordered value group. ☐

Corollary 6.10 (Conditional survival of defects under transition maps)**.** *Assume that* $\operatorname{Def}_1(\gamma) \neq 0$. *For every* $\gamma' \geq \gamma$*, the transition morphism*

$$\operatorname{Def}_1(\gamma') \longrightarrow \operatorname{Def}_1(\gamma)$$

measures whether the obstruction visible at depth γ *lifts to the deeper valuation layer* γ'*. If this transition map has nonzero image, then the defect persists from* γ' *to* γ*.*

Theorem 6.11 (Asymptotic vanishing under eventual Tor-independence)**.** *Let* $A = V[\alpha] \subseteq W$ *and let* I_γ *be the valuation-indexed ideals of Construction 6.5. Assume that there exists* $\gamma_0 \in \Gamma_V$ *such that, for every* $\gamma \geq \gamma_0$*, the quotient* A/I_γ *is Tor-independent from* W/A*, namely*

$$\operatorname{Tor}_i^A(W/A, A/I_\gamma) = 0 \qquad \textit{for all } i \geq 1.$$

Then the higher valuation-indexed derived defects vanish asymptotically:

$$\operatorname{Def}_i(\gamma) = 0 \qquad \textit{for all } i \geq 1,\ \gamma \geq \gamma_0.$$

Proof. By Definition 6.7, the valuation-indexed derived defect at depth γ is

$$\operatorname{Def}_i(\gamma) := \operatorname{Tor}_i^A(W/A, A/I_\gamma).$$

Here W/A is the normalization defect module associated with the inclusion

$$A = V[\alpha] \subseteq W,$$

and A/I_γ is the valuation-depth reduction coming from Construction 6.5.

The hypothesis says that there exists a valuation depth $\gamma_0 \in \Gamma_V$ such that, for every $\gamma \geq \gamma_0$, the reduction module A/I_γ is Tor-independent from the normalization quotient W/A. In explicit homological terms, this means exactly that

$$\operatorname{Tor}_i^A(W/A, A/I_\gamma) = 0 \qquad \text{for every } i \geq 1$$

whenever $\gamma \geq \gamma_0$.

Substituting the defining identity for $\operatorname{Def}_i(\gamma)$, we therefore obtain

$$\operatorname{Def}_i(\gamma) = \operatorname{Tor}_i^A(W/A, A/I_\gamma) = 0 \qquad \text{for every } i \geq 1,\ \gamma \geq \gamma_0.$$

Thus, beyond the threshold valuation depth γ_0, the reduction of the normalization defect module W/A along the quotients A/I_γ produces no higher Tor obstruction.

Equivalently, the valuation-indexed derived defect system introduced in Definition 6.7 becomes concentrated in degree zero for all sufficiently deep valuation layers. Its remaining degree-zero term is only the ordinary reduction

$$\mathrm{Def}_0(\gamma) = (W/A) \otimes_A A/I_\gamma,$$

while all higher homological defect modules vanish. Hence the higher valuation-indexed derived defects disappear asymptotically, as claimed. □

Remark 6.12. The preceding viewpoint is purely algebraic. The ordered parameter space is the valuation group Γ_V, and the transition maps arise only from valuation-ideal inclusions and functoriality of ordinary Tor. No topological persistence module, spectral sequence, or obstruction tower is being constructed here.

6.1 π-primary normalization thickness and derived defects

Theorem 6.13 (General π-primary formula for finite-length normalization defects)**.** *Let V be a DVR with uniformizer π, and let $A = V[\alpha] \subseteq W$ be as above. Assume that π is a non-zero-divisor on A, and put $M := W/A$. Whenever elementary-divisor decompositions or V-lengths are invoked below, we also assume that M is a finite-length π-primary V-module; equivalently, in the DVR-order applications under consideration, W/A is a finite V-module killed by a power of π. For every $n \geq 1$, there is a natural identification*

$$\mathrm{DIC}_1(A, W; (\pi^n)) = \mathrm{Tor}_1^A(M, A/(\pi^n)) \cong M[\pi^n],$$

where

$$M[\pi^n] := \{m \in M : \pi^n m = 0\}.$$

Here and throughout this subsection, all elementary-divisor decompositions and all lengths are taken over the DVR V, not over A.

In particular, if M is a finite-length π-primary V-module and

$$M \cong \bigoplus_{j=1}^{s} V/(\pi^{r_j})$$

as a V-module, then

$$\ell_V\big(\mathrm{DIC}_1(A, W; (\pi^n))\big) = \sum_{j=1}^{s} \min(r_j, n).$$

Thus the first derived normalization defect detects the full π-primary torsion profile of W/A, not only the cyclic case.

Proof. Since π is a non-zero-divisor on A, the quotient $A/(\pi^n)$ has the two-term free resolution

$$0 \longrightarrow A \xrightarrow{\pi^n} A \longrightarrow A/(\pi^n) \longrightarrow 0.$$

Tensoring this resolution with $M = W/A$ gives

$$0 \longrightarrow \mathrm{Tor}_1^A(M, A/(\pi^n)) \longrightarrow M \xrightarrow{\pi^n} M \longrightarrow M \otimes_A A/(\pi^n) \longrightarrow 0.$$

Hence

$$\mathrm{Tor}_1^A(M, A/(\pi^n)) \cong \ker(M \xrightarrow{\pi^n} M) = M[\pi^n].$$

If $M \cong \bigoplus_{j=1}^{s} V/(\pi^{r_j})$, then

$$(V/(\pi^{r_j}))[\pi^n] \cong V/(\pi^{\min(r_j,n)}).$$

Adding lengths over all summands gives

$$\ell_V\big(\mathrm{DIC}_1(A, W; (\pi^n))\big) = \sum_{j=1}^{s} \min(r_j, n).$$

This proves the formula. □

Theorem 6.14 (Cyclic specialization of the π-primary normalization-thickness formula)**.** *Let V be a DVR with uniformizer π, and let $A = V[\alpha] \subseteq W$ be a DVR-order normalization extension. Assume that π is a non-zero-divisor on A. Suppose, in the cyclic finite-length π-primary case, that*

$$W/A \cong V/(\pi^r)$$

as a V-module.

Then, for every $n \geq 1$, the first derived defect

$$\mathrm{DIC}_1(A, W; (\pi^n)) = \mathrm{Tor}_1^A(W/A, A/(\pi^n))$$

is given by the explicit π-primary normalization-thickness formula

$$\mathrm{DIC}_1(A, W; (\pi^n)) \cong V/(\pi^{\min(r,n)})$$

as a V-module. Equivalently,

$$\ell_V\big(\mathrm{DIC}_1(A, W; (\pi^n))\big) = \min(r, n).$$

Thus the computation detects the π-primary torsion thickness of the normalization quotient W/A. In ramified DVR-order examples this thickness may reflect ramified structure, but the formula is a Tor computation for W/A, not a formula for the ramification index.

Proof. This theorem is the cyclic one-summand specialization of Theorem 6.13. Put $M := W/A$. By Theorem 6.13, for every $n \geq 1$,

$$\mathrm{DIC}_1(A, W; (\pi^n)) = \mathrm{Tor}_1^A(M, A/(\pi^n)) \cong M[\pi^n]$$

as a V-module, where

$$M[\pi^n] = \{m \in M : \pi^n m = 0\}.$$

Under the cyclic V-module hypothesis

$$M = W/A \cong V/(\pi^r),$$

one has

$$M[\pi^n] \cong (V/(\pi^r))[\pi^n] \cong V/(\pi^{\min(r,n)}).$$

Therefore

$$\mathrm{DIC}_1(A, W; (\pi^n)) \cong V/(\pi^{\min(r,n)})$$

as a V-module, and hence

$$\ell_V\big(\mathrm{DIC}_1(A, W; (\pi^n))\big) = \min(r, n).$$

This proves that the first derived defect detects the cyclic π-primary normalization thickness of W/A. The computation is a Tor computation for the normalization quotient W/A, not a formula for the ramification index. ☐

Corollary 6.15 (Vanishing converse for cyclic π-primary defect layers)**.** *Keep the hypotheses of Theorem 6.14. Assume that $W/A \cong A/(\pi^r)$ and that π is a non-zero-divisor on A. Then, for $n = 1$,*

$$\mathrm{DIC}_1(A, W; (\pi)) = 0 \quad \Longleftrightarrow \quad r = 0 \quad \Longleftrightarrow \quad W = A.$$

Equivalently, in this cyclic π-primary situation, the first derived defect vanishes modulo (π) if and only if the normalization defect has zero π-primary thickness.

Proof. By Theorem 6.14, for every $n \geq 1$,

$$\mathrm{DIC}_1(A, W; (\pi^n)) \cong A/(\pi^{\min(r,n)}).$$

Taking $n = 1$, we obtain

$$\mathrm{DIC}_1(A, W; (\pi)) \cong A/(\pi^{\min(r,1)}).$$

If $r \geq 1$, then $\min(r, 1) = 1$, so

$$\mathrm{DIC}_1(A, W; (\pi)) \cong A/(\pi) \neq 0.$$

Therefore $\mathrm{DIC}_1(A, W; (\pi)) = 0$ forces $r = 0$.

Conversely, if $r = 0$, then the assumed cyclic defect layer $W/A \cong A/(\pi^r)$ is trivial in the intended thickness sense, namely the normalization quotient has no positive π-primary layer. Hence $W/A = 0$, and therefore $W = A$. In that case all derived defects vanish by Corollary 10.2. This proves the equivalence. ☐

Theorem 6.16 (DVR-order discriminant-index length formula for the normalization defect)**.** *Let V be a DVR with uniformizer π, fraction field K, and residue field k. Let L/K be a finite separable extension, and let W be the maximal V-order in L. Let*

$$A = V[\alpha] \subseteq W$$

be a finite V-order.

This is a strictly DVR-order statement. It is not asserted in the general valuation-ring setting of Sections 2–6, where the relevant normalizations may be non-finite, non-free, or outside the finite Noetherian order framework required for the discriminant-index formula.

Let

$$\mathfrak{P}_1, \ldots, \mathfrak{P}_g$$

be the maximal ideals of W lying over the maximal ideal of V. For each j, write

$$e_j = e(\mathfrak{P}_j/V), \qquad f_j = [W/\mathfrak{P}_j : k].$$

Assume throughout this theorem that A and W are finite free V-modules of rank $[L : K]$, that W/A has finite V-length, and that both discriminants $\Delta_{A/V}$ and $\Delta_{W/V}$ are computed with respect to the trace pairing

$$(x, y) \longmapsto \mathrm{Tr}_{L/K}(xy).$$

Write

$$d := v(\Delta_{A/V}).$$

Assume that the maximal order discriminant is tame at every prime $\mathfrak{P}_j$ above V, so that

$$v(\Delta_{W/V}) = \sum_{j=1}^{g} f_j(e_j - 1).$$

Then the V-length of the normalization defect is

$$\ell_V(W/A) = \frac{d - \sum_{j=1}^{g} f_j(e_j - 1)}{2}.$$

In the special case where W is a DVR, equivalently where there is a single prolongation of V to L, this reduces to

$$\ell_V(W/A) = \frac{d - f(e - 1)}{2}.$$

Consequently, the discriminant determines the total normalization-defect length, not the full π-primary torsion profile of W/A.

If, in addition, the cyclic normalization condition is imposed in the length-normalized form

$$W/A \cong A/(\pi^r) \qquad \text{and} \qquad \ell_V(A/(\pi^r)) = r,$$

then

$$r = \ell_V(W/A) = \frac{d - \sum_{j=1}^{g} f_j(e_j - 1)}{2},$$

and the cyclic formula of Theorem 6.14 gives

$$\ell_V\big(\mathrm{DIC}_1(A, W; (\pi^n))\big) = \min\{n, \ell_V(W/A)\}.$$

Proof. Under the DVR-order finiteness hypotheses imposed in the statement, namely that A and W are finite free V-modules of the same rank and that W/A has finite V-length, the classical discriminant-index relation gives

$$v(\Delta_{A/V}) = v(\Delta_{W/V}) + 2\,\ell_V(W/A).$$

Since the maximal order is assumed tame at every prime $\mathfrak{P}_j$ lying over V, the tame discriminant formula gives

$$v(\Delta_{W/V}) = \sum_{j=1}^{g} f_j(e_j - 1).$$

Thus

$$d = \sum_{j=1}^{g} f_j(e_j - 1) + 2\,\ell_V(W/A),$$

and therefore

$$\ell_V(W/A) = \frac{d - \sum_{j=1}^{g} f_j(e_j - 1)}{2}.$$

This identity determines only the total V-length of the normalization quotient W/A. It does not by itself determine the full elementary-divisor or π-primary torsion profile of W/A, and hence it does not, in general, determine every Tor-defect length $\ell_V(\mathrm{DIC}_1(A, W; (\pi^n)))$.

Under the additional cyclic normalization condition $W/A \cong A/(\pi^r)$, together with the explicit length normalization $\ell_V(A/(\pi^r)) = r$, one has

$$r = \ell_V(W/A).$$

The stated formula for $\ell_V(\mathrm{DIC}_1(A, W; (\pi^n)))$ then follows from Theorem 6.14. ☐

Corollary 6.17 (Different-index length formula for the normalization defect)**.** *Let V be a Henselian DVR with uniformizer π, fraction field K, and residue field k. Let L/K be a finite separable extension such that the integral closure W of V in L is a DVR. Let*

$$\mathfrak{D}_{W/V}$$

denote the different ideal, and write

$$\mathfrak{D}_{W/V} = \mathfrak{p}_W^{\delta}.$$

Let $f = [k_W : k]$, and let $A = V[\alpha] \subseteq W$ be a finite V-order. Put

$$d_A := v(\Delta_{A/V}).$$

Then

$$\ell_V(W/A) = \frac{d_A - f\delta}{2}.$$

If, in addition,

$$W/A \cong A/(\pi^r) \qquad \text{and} \qquad \ell_V(A/(\pi^r)) = r,$$

then

$$r = \frac{d_A - f\delta}{2},$$

and for every $n \geq 1$,

$$\ell_V\big(\mathrm{DIC}_1(A, W; (\pi^n))\big) = \min\{n, \ell_V(W/A)\}.$$

Proof. For a finite separable extension of DVRs, the discriminant of the maximal order is the norm of the different:

$$\Delta_{W/V} = N_{W/V}(\mathfrak{D}_{W/V}).$$

Since

$$\mathfrak{D}_{W/V} = \mathfrak{p}_W^{\delta},$$

we obtain

$$v(\Delta_{W/V}) = f\delta.$$

The discriminant-index relation for the finite order $A \subseteq W$ gives

$$v(\Delta_{A/V}) = v(\Delta_{W/V}) + 2\,\ell_V(W/A).$$

Therefore

$$d_A = f\delta + 2\,\ell_V(W/A),$$

and hence

$$\ell_V(W/A) = \frac{d_A - f\delta}{2}.$$

The final Tor formula is not a consequence of the different alone. It follows only after imposing the additional cyclic normalization condition and the explicit length normalization, in which case it is exactly the cyclic specialization of Theorem 6.14. □

7 Conditional collapse in Henselian defectless settings

Definition 7.1 (Defectless extension)**.** A finite extension L/K of valued fields is called defectless if the usual equality

$$[L : K] = \sum_j e_j f_j$$

holds over the extensions of the valuation from K to L ([6, Ch. 6, §6.1–§6.2, pp. 151–171]; [7, Ch. III, §18, pp. 136–143]).

Remark 7.2 (Conditional role of defectless Henselian hypotheses)*.* Let V be a Henselian valuation ring and let $L = K(\alpha)$ be a finite defectless simple extension. Let $A = V[\alpha]$, and let W be the integral closure of A in L. The defectless and Henselian hypotheses are used here only as valuation-theoretic background conditions controlling the classical extension of valuations. They do not, by themselves, imply that the normalization quotient W/A is flat, nor do they imply

$$\mathrm{Tor}_i^A(W/A, A/I_\gamma) = 0 \qquad (i \geq 1).$$

Consequently, any vanishing statement for the valuation-indexed derived defects

$$\mathrm{Def}_i(\gamma) = \mathrm{Tor}_i^A(W/A, A/I_\gamma)$$

must be read as conditional on an independent homological hypothesis, such as Tor-independence of W/A from A/I_γ, or flatness of W/A along the chosen valuation reductions.

Corollary 7.3 (Conditional flat-reduction collapse)**.** *In the situation of Remark 7.2, assume additionally that, for a fixed valuation depth γ,*

$$\mathrm{Tor}_i^A(W/A, A/I_\gamma) = 0 \qquad \text{for all } i \geq 1.$$

Then

$$\mathrm{Def}_i(\gamma) = 0 \qquad \text{for all } i \geq 1.$$

In particular, if this Tor-independence holds for all sufficiently large γ, then the valuation-indexed derived defect system becomes concentrated in degree zero beyond that valuation depth.

Proof. This is immediate from the definition

$$\mathrm{Def}_i(\gamma) = \mathrm{Tor}_i^A(W/A, A/I_\gamma).$$

The conclusion is therefore not a new consequence of defectlessness alone, but only the homological collapse obtained after imposing the stated Tor-independence condition. □

8 Examples and computations

Example 8.1 (Orientation: discrete valuation ring case)**.** Let V be a DVR with uniformizer π, fraction field K, and let $L = K(\alpha)$ be a finite separable extension. Put $A = V[\alpha]$, and let W be the integral closure of A in L.

For $I = (\pi^n)$, the derived defect is

$$\mathrm{DIC}_i(A, W; (\pi^n)) = \mathrm{Tor}_i^A(W/A, A/(\pi^n)).$$

If W/A is π-torsion-free and flat over the relevant localization of A, then the higher defects vanish.

Example 8.2 (Orientation: ramified quadratic extension)**.** Let V be a DVR with uniformizer π, and consider

$$L = K(\sqrt{\pi}).$$

Let $A = V[\sqrt{\pi}]$. If A is already integrally closed, then $W = A$, and all derived defects vanish:

$$\mathrm{DIC}_i(A, W; I) = 0 \quad \text{for all } i.$$

If the chosen generator gives a non-normal order $A \subsetneq W$, then W/A may contribute nonzero derived defects.

Example 8.3 (Orientation: conductor-detected defect)**.** Assume $A \subsetneq W$ and let $\mathfrak{c} = (A : W)$. For $I = \mathfrak{c}$, one obtains

$$\mathrm{DIC}_1(A, W; \mathfrak{c}) = \mathrm{Tor}_1^A(W/A, A/\mathfrak{c}).$$

A nonzero connecting image

$$\mathrm{im}\big(\mathrm{Tor}_1^A(W/A, A/\mathfrak{c}) \longrightarrow A/\mathfrak{c}\big) \neq 0$$

detects failure of exact reduction modulo the conductor. Thus the relevant obstruction is not merely the nonvanishing of $\mathrm{DIC}_1(A, W; \mathfrak{c})$, but the nonvanishing of its image under the connecting homomorphism in the long exact Tor sequence.

Example 8.4 (Template computation: DVR-order normalization thickness)**.** Let V be a DVR with uniformizer π, and suppose that $A \subsetneq W$ is an order in a finite extension L/K. Assume for simplicity that W is finite over A, that π is a non-zero-divisor on A, and that the normalization quotient is cyclic as an A-module:

$$W/A \cong A/(\pi^r)$$

for some $r \geq 1$. This is an A-cyclic template computation. It is not an elementary-divisor decomposition over V; the elementary-divisor formulas in Theorem 6.13 are stated over the DVR V.

Then for $I = (\pi^n)$, one has

$$\mathrm{DIC}_1(A, W; I) = \mathrm{Tor}_1^A(A/(\pi^r), A/(\pi^n)).$$

Using the two-term free resolution of $A/(\pi^r)$, valid because π is a non-zero-divisor on A, we obtain

$$\mathrm{Tor}_1^A(A/(\pi^r), A/(\pi^n)) \cong \{x \in A/(\pi^n) : \pi^r x = 0\}.$$

Hence, in this cyclic A-module situation,

$$\mathrm{DIC}_1(A, W; (\pi^n)) \cong A/(\pi^{\min(r,n)}).$$

Thus this example computes a cyclic A-module normalization thickness. It should be read separately from the V-module elementary-divisor formulas of Theorem 6.13.

Example 8.5 (Main computed example: non-normal DVR order)**.** Let k be a field with $\mathrm{char}\, k \neq 2$, let $V = k[[\pi]]$, and put $K = k((\pi))$. Let $L = K(u)$ with $u^2 = \pi$. Then $W = k[[u]]$ is the integral closure of V in L.

Set

$$A = V[u^3] = k[[u^2, u^3]] \subsetneq W.$$

Since $u = u^3/u^2$, one has $L = K(u^3)$, so this is still a simple extension order. The quotient is concrete:

$$W/A \cong k \cdot \overline{u}.$$

Its annihilator in A is the maximal ideal

$$\mathfrak{m} = (u^2, u^3),$$

so

$$W/A \cong A/\mathfrak{m}.$$

Indeed, every element of $W = k[[u]]$ has a unique expansion

$$a_0 + a_1 u + a_2 u^2 + \cdots .$$

Since $A = k[[u^2, u^3]]$ contains 1 and all powers u^n for $n \geq 2$, the only missing monomial modulo A is u. Hence

$$W/A \cong k \cdot u$$

as a k-vector space. Moreover $u^2 u = u^3 \in A$ and $u^3 u = u^4 \in A$, so $\mathfrak{m} = (u^2, u^3)$ annihilates the class of u. Conversely, if $a \in A$ has nonzero constant term, then $au \notin A$. Hence the annihilator is exactly $\mathfrak{m}$, and therefore $W/A \cong A/\mathfrak{m}$.

Therefore

$$\mathrm{DIC}_1(A, W; \mathfrak{m}) = \mathrm{Tor}_1^A(A/\mathfrak{m}, A/\mathfrak{m}) \cong \mathfrak{m}/\mathfrak{m}^2,$$

which is nonzero. In fact, $\mathfrak{m}/\mathfrak{m}^2$ is generated by the classes of u^2 and u^3.

Example 8.6 (Main computed example: two elementary-divisor normalization defect)**.** Let V be a DVR with uniformizer π, and let $A = V[\alpha] \subseteq W$ be a finite DVR-order normalization extension such that π is a non-zero-divisor on A. Assume that the normalization quotient has two elementary divisors:

$$W/A \cong V/(\pi^2) \oplus V/(\pi^5)$$

as a finite-length π-primary V-module.

For $I = (\pi^n)$, the first derived normalization defect is

$$\mathrm{DIC}_1(A, W; (\pi^n)) = \mathrm{Tor}_1^A(W/A, A/(\pi^n)).$$

By the general π-primary torsion-profile formula of Theorem 6.13, we obtain

$$\ell_V\big(\mathrm{DIC}_1(A, W; (\pi^n))\big) = \min(2, n) + \min(5, n).$$

Equivalently, the values are

$$\ell_V\big(\mathrm{DIC}_1(A, W; (\pi^n))\big) = \begin{cases} 2n, & 1 \leq n \leq 2, \\ n + 2, & 3 \leq n \leq 5, \\ 7, & n \geq 5. \end{cases}$$

Thus the first derived defect detects two distinct normalization thickness layers, not merely the total length $\ell_V(W/A) = 7$. In particular, this example shows explicitly why the derived defect profile contains more information than the single discriminant-index length alone.

Example 8.7 (Orientation: ramified cubic extension with conditional defect)**.** Let $V = \mathbb{Z}_{(3)}$, $K = \mathbb{Q}$, and $L = \mathbb{Q}(\theta)$, where $\theta^3 = 3$. The polynomial $x^3 - 3$ is Eisenstein at 3, so the extension is totally ramified at 3. Put $A = V[\theta]$, and let B be the integral closure of A in L.

If $A = B$, then all derived defects vanish. If instead one replaces A by a non-normal V-order $A' \subsetneq B$, then

$$\mathrm{DIC}_1(A', B; (3)) = \mathrm{Tor}_1^{A'}(B/A', A'/(3))$$

measures the failure of exact reduction at the ramified prime.

Thus the computational content of the section is concentrated in Examples 8.5 and 8.6. The preceding and following orientation examples are included only to situate these computations within the valuation-theoretic normalization framework and should not be read as independent non-conditional computations.

The examples in this section have two different roles. Examples 8.1 to 8.4 and 8.7 are orientation examples: they indicate how the general formalism specializes under standard hypotheses or how a conditional defect would be detected. The genuinely computed model examples are Examples 8.5 and 8.6, where the normalization quotient and the resulting first Tor defect are computed explicitly.

9 Main theorem

Theorem 9.1 (Summary theorem for derived normalization defects)**.** *Let V be a valuation ring with fraction field K, let $L = K(\alpha)$ be a finite simple extension, put $A = V[\alpha]$, and let W be the integral closure of A in L. For every valuation-compatible ideal $I \subseteq A$, the assignment*

$$I \longmapsto \left\{\operatorname{Tor}_i^A(W/A, A/I)\right\}_{i\geq 0}$$

defines a functorial system of derived normalization-defect modules.

The content is the resulting valuation-theoretic package: ordinary Tor groups are used to produce normalization-defect modules whose conductor support, connecting-image obstruction, and DVR-order torsion profile are explicitly identified. In this sense the theorem gives a computable homological diagnostic for normalization defects, rather than a new abstract Tor formalism.

Its higher terms vanish under flat reduction and, under eventual Tor-independence, vanish asymptotically along deep valuation ideals. In the DVR-order case, the first derived defect is controlled in general by the π-primary torsion profile formula of Theorem 6.13. The cyclic case gives the π-primary normalization-thickness formula of Theorem 6.14, while Theorem 6.16 and Corollary 6.17 determine the total V-length $\ell_V(W/A)$, not the full π-primary torsion profile unless the extra cyclic length-normalized hypothesis is imposed. Moreover, the image of

$$\operatorname{Tor}_1^A(W/A, A/I) \longrightarrow A/I$$

under the connecting homomorphism is precisely the first obstruction to preservation of injectivity after reduction modulo I.

Proof. The proof has three parts: construction and functoriality, collapse under flat reduction, and detection of the first obstruction.

First, the inclusion $A \subseteq W$ gives the canonical short exact sequence

$$0 \longrightarrow A \longrightarrow W \longrightarrow W/A \longrightarrow 0.$$

The quotient W/A is the normalization defect module. It measures the failure of $A = V[\alpha]$ to be integrally closed in L. For an ideal $I \subseteq A$, reduction modulo I is represented by tensoring with A/I. Therefore the natural homological objects measuring exactness of this reduction are

$$\operatorname{Tor}_i^A(W/A, A/I), \qquad i \geq 0.$$

This gives the assignment

$$I \longmapsto \left\{\operatorname{Tor}_i^A(W/A, A/I)\right\}_{i\geq 0}.$$

We now verify functoriality. Suppose that $I \subseteq J$ are valuation-compatible ideals of A. Then there is a natural quotient map

$$A/I \longrightarrow A/J.$$

Since $\operatorname{Tor}_i^A(W/A, -)$ is functorial in its second variable, this map induces natural homomorphisms

$$\operatorname{Tor}_i^A(W/A, A/I) \longrightarrow \operatorname{Tor}_i^A(W/A, A/J)$$

for every $i \geq 0$. Hence the family

$$\left\{\operatorname{Tor}_i^A(W/A, A/I)\right\}_{i\geq 0}$$

varies functorially with the valuation-compatible ideal I. In particular, if I_γ is a valuation-indexed family and $\gamma' \geq \gamma$, so that $I_{\gamma'} \subseteq I_\gamma$, then the natural map

$$A/I_{\gamma'} \longrightarrow A/I_\gamma$$

induces transition maps

$$\operatorname{Tor}_i^A(W/A, A/I_{\gamma'}) \longrightarrow \operatorname{Tor}_i^A(W/A, A/I_\gamma).$$

Thus the construction forms a functorial valuation-indexed system of Tor-modules attached to the chosen valuation ideals.

The degree-zero term recovers the classical reduction of the normalization defect. Indeed,

$$\operatorname{Tor}_0^A(W/A, A/I) \cong (W/A) \otimes_A A/I.$$

Thus the new construction agrees in degree zero with the ordinary quotient W/A after reduction modulo I, while the terms with $i \geq 1$ record the failure of this reduction to be exact.

Next assume that the reduction is flat in the sense that W/A is flat against the relevant quotients A/I, or more strongly that W/A is flat over A. Flatness implies vanishing of higher Tor:

$$\operatorname{Tor}_i^A(W/A, A/I) = 0 \qquad \text{for all } i \geq 1.$$

Therefore all higher derived terms vanish, and the derived defect system becomes concentrated in its degree-zero part

$$(W/A) \otimes_A A/I.$$

This proves the flat-reduction collapse assertion.

It remains to explain the obstruction statement. Apply the functor $- \otimes_A A/I$ to the canonical exact sequence

$$0 \longrightarrow A \longrightarrow W \longrightarrow W/A \longrightarrow 0.$$

The associated long exact Tor sequence contains

$$\operatorname{Tor}_1^A(W, A/I) \longrightarrow \operatorname{Tor}_1^A(W/A, A/I) \xrightarrow{\delta} A \otimes_A A/I \longrightarrow W \otimes_A A/I \longrightarrow (W/A) \otimes_A A/I \longrightarrow 0.$$

Using the natural identifications

$$A \otimes_A A/I \cong A/I, \qquad W \otimes_A A/I \cong W/IW,$$

we obtain the exact sequence

$$\operatorname{Tor}_1^A(W, A/I) \longrightarrow \operatorname{Tor}_1^A(W/A, A/I) \xrightarrow{\delta} A/I \longrightarrow W/IW \longrightarrow (W/A) \otimes_A A/I \longrightarrow 0.$$

The map

$$A/I \longrightarrow W/IW$$

is exactly the reduction modulo I of the normalization inclusion $A \hookrightarrow W$. Hence the reduced sequence

$$0 \longrightarrow A/I \longrightarrow W/IW \longrightarrow (W/A) \otimes_A A/I \longrightarrow 0$$

is exact at A/I precisely when the connecting image

$$\operatorname{im}(\delta) \subseteq A/I$$

is zero. Thus any nonzero element in the image of

$$\delta : \operatorname{Tor}_1^A(W/A, A/I) \longrightarrow A/I$$

gives an explicit obstruction to preserving left exactness after reduction modulo I.

Consequently, the first nonzero higher derived term

$$\mathrm{Tor}_1^A(W/A, A/I)$$

is the first Tor term from which the actual obstruction to injectivity after reduction can arise, namely through the connecting image

$$\mathrm{im}\big(\mathrm{Tor}_1^A(W/A, A/I) \longrightarrow A/I\big)\,.$$

Higher groups

$$\mathrm{Tor}_i^A(W/A, A/I), \qquad i \geq 2,$$

are homological diagnostic modules attached to the same reduction problem. They should not be described as actual obstruction modules unless an additional obstruction tower, spectral sequence, or comparable structure is constructed. Therefore the assignment

$$I \longmapsto \big\{\mathrm{Tor}_i^A(W/A, A/I)\big\}_{i\geq 0}$$

defines a functorial system of derived normalization-defect modules, collapses under flat reduction, and detects the first possible obstruction to exact reduction through its first higher Tor term. ☐

10 Comparison with classical integral closure

Proposition 10.1 (Classical shadow)**.** *The degree-zero part of the derived normalization-defect system is*

$$\mathrm{DIC}_0(A, W; I) = (W/A) \otimes_A A/I.$$

Thus the derived normalization-defect system supplements the classical quotient W/A with higher ordinary Tor defect modules.

Proof. By definition, the derived integral closure defect is

$$\mathrm{DIC}_i(A, W; I) = \mathrm{Tor}_i^A(W/A, A/I).$$

For $i = 0$, the zeroth Tor functor is the ordinary tensor product. Hence

$$\mathrm{DIC}_0(A, W; I) = \mathrm{Tor}_0^A(W/A, A/I) \cong (W/A) \otimes_A A/I.$$

This tensor product is precisely the reduction of the normalization defect module W/A modulo I. In other words, it records what remains of the failure of A to be integrally closed after passing from A to A/I.

The higher terms

$$\mathrm{DIC}_i(A, W; I) = \mathrm{Tor}_i^A(W/A, A/I), \qquad i \geq 1,$$

diagnose the failure of this reduction process to be exact. Therefore the derived normalization-defect system has the classical reduction of W/A as its degree-zero shadow, while the higher Tor groups add homological defect information not visible in the ordinary quotient. The actual obstruction to injectivity after reduction is still the connecting image of the first Tor group. ☐

Corollary 10.2 (No artificial derived contribution)**.** *If $W = A$, then*

$$\mathrm{DIC}_i(A, W; I) = 0 \qquad \textit{for all } i \geq 0.$$

Proof. If $W = A$, then the normalization quotient vanishes:

$$W/A = 0.$$

Therefore, for every ideal $I \subseteq A$ and every $i \geq 0$, we have

$$\mathrm{DIC}_i(A, W; I) = \mathrm{Tor}_i^A(W/A, A/I) = \mathrm{Tor}_i^A(0, A/I).$$

Since the zero module has the zero projective resolution and tensoring it with any A-module remains zero, all its Tor groups vanish:

$$\mathrm{Tor}_i^A(0, A/I) = 0 \qquad \text{for all } i \geq 0.$$

Hence

$$\mathrm{DIC}_i(A, W; I) = 0 \qquad \text{for all } i \geq 0.$$

This confirms that the derived defect system introduces no artificial homological defect when there is no normalization defect to begin with. In particular, if A is already integrally closed in L, then both the classical quotient W/A and all higher derived defect modules vanish.

□

Remark 10.3 (Scope of the converse). The converse direction

$$\mathrm{DIC}_1(A, W; I) = 0 \Longrightarrow W = A$$

is not valid for an arbitrary ideal I without additional hypotheses, because a single reduction may fail to see the whole normalization quotient. The precise converse proved in Corollary 6.15 is therefore intentionally restricted to the cyclic π-primary DVR-order case, where W/A has a single measurable thickness parameter r. In that setting the first derived defect detects exactly whether this thickness is positive, and hence whether the normalization quotient is nonzero.

11 Concluding remarks

The construction introduced here gives a homological refinement of integral closure in simple extensions of valuation rings. Its strength is that it remains close to classical valuation theory while detecting additional failure of exactness through ordinary derived functors. This makes the framework suitable for valuation-theoretic readers while still allowing derived techniques to enter in a controlled and computational form.

Future work may include:

(i) extension of the discriminant-index comparison from the π-primary torsion-profile formula to non-π-primary and mixed primary normalization defects;

(ii) comparison, in examples where additional ramification data are present, of the resulting normalization-defect torsion profiles with higher ramification filtrations, Newton polygons, and higher different ideals;

(iii) extension to towers of simple extensions;

(iv) interaction with Henselian lifting and defect extensions.

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