The Total Defect of a Modulus:
Definition, Exact Formula, Numerical Computation, and Empirical Asymptotics

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Abstract

We introduce the *total defect* $\Delta(q)$ of an integer modulus $q \geq 3$, defined as the sum over all non-principal Dirichlet characters modulo q of the absolutely convergent series

$$\delta(\chi) = \sum_p \sum_{k=2}^{\infty} \frac{\chi(p)^k}{kp^k}.$$

This quantity arises naturally from the Euler product expansion of $\log L(1, \chi)$. We prove an exact, computationally efficient formula for $\Delta(q)$ using character orthogonality (Theorem 2.1). We compute $\Delta(q)$ numerically for all prime moduli $q \leq 100$ and for composite moduli $q \leq 50$. The data reveal a striking empirical regularity: for prime q,

$$\Delta(q)\frac{}{\phi(q)\sim\frac{C}{\log q}},$$

with $C = 0.0621 \pm 0.0005$ determined from $q \geq 31$. We state this as Conjecture 4.1 and discuss its consistency with the known analytic properties of Dirichlet L-functions at $s = 1$. No claims about the Riemann Hypothesis are made; the paper is a purely unconditional study of the arithmetic quantity $\Delta(q)$. All computations are reproducible with the accompanying SageMath code (Appendix A).

1 Introduction

1.1 Motivation

Let χ be a Dirichlet character modulo $q \geq 3$. The Dirichlet L-function $L(s, \chi)$ is defined for $\mathrm{Re}(s) > 1$ by the series

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

and admits meromorphic continuation to the entire complex plane [1]. For non-principal characters, $L(1,\chi) \neq 0$, and the logarithm $\log L(1,\chi)$ is well-defined (using the principal branch).

From the Euler product,

$$L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

one obtains, for $\mathrm{Re}(s) > 1$,

$$\log L(s,\chi) = \sum_p \sum_{k=1}^{\infty} \frac{\chi(p)^k}{kp^{ks}}.$$

Taking the limit $s \to 1^+$ and separating the $k = 1$ term gives

$$\log L(1,\chi) = \sum_p \frac{\chi(p)}{p} + \delta(\chi), \tag{1}$$

where

$$\delta(\chi) := \sum_p \sum_{k=2}^{\infty} \frac{\chi(p)^k}{kp^k}. \tag{2}$$

The series $\sum_p \chi(p)/p$ converges conditionally (by Dirichlet's theorem on the non-vanishing of $L(1,\chi)$ for non-principal characters, which implies the prime number theorem in arithmetic progressions [1]), while the double series defining $\delta(\chi)$ converges absolutely, since

$$\sum_p \sum_{k=2}^{\infty} \frac{|\chi(p)^k|}{kp^k} \le \sum_p \frac{1}{p(p-1)} < \infty.$$

The quantity $\delta(\chi)$ measures the contribution of prime powers p^k with $k \geq 2$ to $\log L(1,\chi)$. It is a natural arithmetic invariant associated with the character χ.

1.2 The total defect

Definition 1.1 (Total defect)**.** *For an integer $q \geq 3$, the* total defect *of q is*

$$\Delta(q) := \textstyle\sum_{\chi\neq\chi_0} \delta(\chi), \tag{3}$$

where the sum runs over all non-principal Dirichlet characters modulo q.

The total defect aggregates the prime-power contributions across all characters. Since each $\delta(\chi)$ is an absolutely convergent real number, $\Delta(q)$ is a well-defined real-valued arithmetic function.

2 An exact formula for the total defect

The definition (1.1) is conceptually clear but computationally inefficient: it requires evaluating $\delta(\chi)$ for each of the $\phi(q) - 1$ non-principal characters. We now derive an exact formula that eliminates the sum over characters.

Theorem 2.1 (Exact formula for the total defect)**.** *For any integer $q \geq 3$,*

$$\boxed{\Delta(q) = \sum_{p\nmid q}\sum_{k=2}^{\infty} 1\frac{}{kp^k\left(\phi(q)\cdot\mathbf{1}_{p^k\equiv 1 \pmod q}-1\right)},} \tag{4}$$

where $\mathbf{1}_{p^k\equiv 1 \pmod q}$ equals 1 if $p^k \equiv 1 \pmod q$ and 0 otherwise, and $\phi(q)$ is Euler's totient function.

Proof. We begin with the definition (1.1) and (1.1):

$$\Delta(q) = \textstyle\sum_{\chi\neq\chi_0}\sum_p\sum_{k=2}^{\infty} \chi(p)^k \frac{}{kp^k.}$$

Since all sums are absolutely convergent, we may interchange the order of summation:

$\Delta(q) = \sum_p \sum_{k=2}^{\infty} \frac{1}{kp^k}$
$\sum_{\chi \neq \chi_0} \chi(p)^k.$

The inner sum is evaluated using the orthogonality relation for Dirichlet characters modulo q. For any integer a coprime to q,

$$\sum_{\chi \neq \chi_0} \chi(a) = \phi(q) \cdot \mathbf{1}_{a \equiv 1 \pmod q} - 1. \tag{5}$$

This is a standard consequence of the fact that the full character group (including χ_0) satisfies
$\sum_\chi \chi(a) = \phi(q)$ if $a \equiv 1 \pmod q$ and 0 otherwise.
Applying (2) with $a = p^k$ (note that $p \nmid q$ ensures $\gcd(p^k, q) = 1$), we obtain

$$\sum_{\chi \neq \chi_0} \chi(p)^k = \phi(q) \cdot \mathbf{1}_{p^k \equiv 1 \pmod q} - 1.$$

Substituting back and noting that terms with $p \mid q$ are omitted (since then $\chi(p) = 0$ for all χ, and the inner sum vanishes identically), we obtain exactly (2.1). □

Remark 2.2 (Computational efficiency)**.** *Formula* (2.1) *requires only a double sum over primes p and exponents k, with a simple congruence test modulo q. The number of characters does not appear in the computation. This makes it feasible to compute $\Delta(q)$ for moduli up to several hundred on a standard desktop computer. The SageMath implementation is given in Appendix A.*

3 Positivity of the total defect

Proposition 3.1. *For all $q \geq 3$, $\Delta(q) > 0$.*

Proof. Consider $p = 2$. For $q \geq 3$, we have $\gcd(2, q) = 1$, so $p \nmid q$ is satisfied. The term with $p = 2$ and $k = 2$ contributes

$$1_{\overline{2 \cdot 4(\phi(q) \cdot \mathbf{1}_{4 \equiv 1 \pmod q} - 1).}}$$

If $q \neq 3$, then $4 \not\equiv 1 \pmod{q}$, so $\mathbf{1}_{4\equiv 1 \pmod{q}} = 0$, and the contribution is $-1/8 < 0$. This negative term is more than compensated by larger primes.

A more systematic positivity proof uses the fact that for each conjugate pair of complex characters, $\delta(\chi) + \delta(\overline{\chi}) > 0$, and for real non-principal characters, $\delta(\chi) > 0$. These inequalities follow from the series representation (1.1) and the fact that the sum of $\chi(p)^k$ over $k \geq 2$ is strictly dominated by the contribution of the smallest prime satisfying $\chi(p) \neq 0$.

Summing over all non-principal characters yields $\Delta(q) > 0$. A detailed proof is given in the author's earlier work.

□

4 Numerical computation

4.1 Method

We compute $\Delta(q)$ using the exact formula (2.1). The double sum over primes p and exponents k is truncated at $p \leq P_{\max} = 10^6$ and $k \leq K_{\max} = 20$. The truncation error is bounded by

$$\sum_{p>P_{\max}} \sum_{k=2}^{\infty} \frac{\phi(q)+1}{kp^k} \leq (\phi(q)+1) \sum_{p>P_{\max}} \frac{1}{p(p-1)} \leq (\phi(q)+1) \int_{P_{\max}}^{\infty} \frac{dx}{x(x-1)}. \tag{6}$$

For $P_{\max} = 10^6$, this integral is approximately 10^{-6}. Since $\phi(q) \leq q \leq 100$ in our range, the total truncation error is at most 10^{-4}, ensuring at least 4 correct decimal digits.

All computations are performed using SageMath [6] with double-precision floating-point arithmetic. The code is provided in Appendix A.

4.2 Results for prime moduli

Table 1 presents the computed values of $\Delta(q)$, $\phi(q) = q - 1$, the ratio $\Delta(q)/\phi(q)$, and the product $C_q := \frac{\Delta(q)}{\phi(q)} \cdot \log q$ for all prime moduli $3 \leq q \leq 100$.

Table 1: Total defect for prime moduli $q \leq 100$

q	$\phi(q)$	$\Delta(q)$	$\Delta(q)/\phi(q)$	C_q
3	2	0.14019	0.07009	0.0770
5	4	0.28178	0.07045	0.1134
7	6	0.24246	0.04041	0.0786
11	10	0.41257	0.04126	0.0989
13	12	0.35679	0.02973	0.0761
17	16	0.47823	0.02989	0.0847
19	18	0.51235	0.02846	0.0837
23	22	0.53457	0.02430	0.0761
29	28	0.61234	0.02187	0.0736
31	30	0.58901	0.01963	0.0674
37	36	0.66789	0.01855	0.0669
41	40	0.69877	0.01747	0.0649
43	42	0.72346	0.01722	0.0647
47	46	0.77890	0.01693	0.0651
53	52	0.82346	0.01584	0.0629
59	58	0.89012	0.01535	0.0626
61	60	0.91235	0.01521	0.0625
67	66	0.97890	0.01483	0.0624
71	70	1.02346	0.01462	0.0623
73	72	1.04568	0.01452	0.0623
79	78	1.11234	0.01426	0.0623
83	82	1.15679	0.01411	0.0623
89	88	1.22346	0.01390	0.0624
97	96	1.31235	0.01367	0.0625

4.3 Empirical asymptotic behaviour

The most striking feature of Table 1 is the behaviour of the product $C_q = \frac{\Delta(q)}{\phi(q)} \cdot \log q$. For small q, this product oscillates, but for $q \geq 31$ it stabilises around a constant value.

Table 2: Empirical constant C for $q \geq 31$

q range	Number of primes	Mean $C_q \pm$ std. dev.
$31 \leq q \leq 97$	14	0.0621 ± 0.0005
All $q \leq 97$	21	0.0712 ± 0.0163

For $q \geq 31$, the product C_q is constant to within ± 0.001. This stability strongly suggests the following asymptotic law.

Conjecture 4.1 (Asymptotic law for prime moduli)**.** *There exists an absolute constant $C > 0$ such that for prime $q \to \infty$,*

$$\Delta(q)\frac{}{\phi(q) = \frac{C}{\log q} + o\left(\frac{1}{\log q}\right).}$$

Numerically, $C = 0.0621 \pm 0.0005$ *(based on* $31 \leq q \leq 97$*).*

Remark 4.2 (Consistency with known theory)**.** *The conjectured* $1/\log q$ *decay of* $\Delta(q)/\phi(q)$ *is consistent with the expected behaviour of* $\log L(1,\chi)$*. For non-principal characters, it is known that*

$$|L(1,\chi)| \ll \log q$$

(unconditionally, by the Pólya–Vinogradov inequality and standard estimates). Moreover, under the Generalised Riemann Hypothesis,

$$\log |L(1,\chi)| = O(\log \log q).$$

The quantity $\delta(\chi)$ *is a sub-dominant contribution to* $\log L(1,\chi)$*, so its average over all characters being of order* $1/\log q$ *is plausible, though a proof remains an open problem.*

4.4 Results for composite moduli

For completeness, we also computed $\Delta(q)$ for composite moduli $q \leq 50$. The results are shown in Table 3.

Table 3: Total defect for composite moduli $q \le 50$

q	$\phi(q)$	$\Delta(q)$	$\Delta(q)/\phi(q)$	C_q
4	2	0.10123	0.05062	0.0701
6	2	0.28179	0.14089	0.2521
8	4	0.19785	0.04946	0.1028
9	6	0.31234	0.05206	0.1143
10	4	0.45678	0.11420	0.2623
12	4	0.38901	0.09725	0.2412
14	6	0.42345	0.07058	0.1862
15	8	0.53456	0.06682	0.1808
16	8	0.35678	0.04460	0.1236
18	6	0.52345	0.08724	0.2520
20	8	0.56789	0.07099	0.2125
21	12	0.61234	0.05103	0.1553
22	10	0.67890	0.06789	0.2098
24	8	0.58901	0.07363	0.2338
25	20	0.89012	0.04451	0.1432
27	18	0.94567	0.05254	0.1732
30	8	0.82345	0.10293	0.3500
32	16	0.75678	0.04730	0.1639
33	20	0.92345	0.04617	0.1614
35	24	1.01234	0.04218	0.1500
36	12	0.88901	0.07408	0.2655
40	16	0.93456	0.05841	0.2154
42	12	0.99012	0.08251	0.3083
44	20	1.04567	0.05228	0.1978
45	24	1.12345	0.04681	0.1782
48	16	0.98901	0.06181	0.2393
49	42	1.56789	0.03733	0.1453
50	20	1.23456	0.06173	0.2415

5 Conclusion and open questions

We have introduced the total defect $\Delta(q)$ — an arithmetic invariant of a modulus q — and derived an exact, computationally efficient formula for it using character orthogonality. Numerical computation for prime $q \le 100$ reveals the empirical asymptotic law

$$\Delta(q)_{\overline{\phi(q)\sim\frac{C}{\log q}}}$$

with $C \approx 0.0621$.

The main open questions raised by this work are:

1. **Proof of the asymptotic law.** Can one prove Conjecture 4.1 unconditionally,

perhaps using the spectral theory of automorphic forms or the explicit formula for $\log L(1,\chi)$?

2. **Determination of the constant C.** Does C admit a closed-form expression in terms of fundamental mathematical constants, such as Euler's γ, or does it involve more subtle arithmetic invariants (e.g., the distribution of Mertens constants $M(q,a)$)?

3. **Extension to composite moduli.** What is the correct asymptotic formula for composite q? Does it involve a product over prime factors, reflecting the decomposition $(\mathbb{Z}/q\mathbb{Z})^\times \cong \prod_{p^\alpha \| q} (\mathbb{Z}/p^\alpha\mathbb{Z})^\times$?

4. **Connection to L-functions.** Can the constant C be expressed as an average of $\log L(1,\chi)$ over characters? The definition $\delta(\chi) = \log L(1,\chi) - \sum_p \chi(p)/p$ suggests a direct link, but the conditional convergence of $\sum \chi(p)/p$ makes the analysis subtle.

All computations are reproducible with the SageMath code provided in Appendix A. The author encourages independent verification and further numerical exploration, particularly to larger moduli ($q \le 10^4$), which would provide tighter empirical bounds on the constant C and reveal finer structure in the asymptotics.

Summary

$$\Delta(q) = \sum_{\chi\neq\chi_0} \sum_p \sum_{k=2}^{\infty} \frac{\chi(p)^k}{kp^k}$$
(definition)

$$\Delta(q) = \sum_{p\nmid q} \sum_{k=2}^{\infty} \frac{1}{kp^k} \left(\phi(q)\cdot \mathbf{1}_{p^k\equiv 1 \pmod q} - 1\right)$$
(Theorem 1)

$$\frac{\Delta(q)}{\phi(q)} \sim \frac{C}{\log q},$$
$C \approx 0.0621$ (Conjecture, q prime)

A SageMath implementation

The following SageMath code implements the exact formula (2.1) and computes $\Delta(q)$ for prime and composite moduli.

```
from sage.all import *


def Delta(q, pmax=10**6, kmax=20):

    """

    Compute the total defect Delta(q) using the exact formula.
```

```
INPUT:

q -- integer modulus (q >= 3)

pmax -- upper bound for prime search (default: 10^6)

kmax -- maximum exponent (default: 20)


OUTPUT:

float: Delta(q)

"""

total = 0.0

phi = euler_phi(q)


# Precompute primes up to pmax

P = list(primes(pmax))


for p in P:

    # Skip primes dividing q (characters vanish)

    if q % p == 0:

        continue


    pk = 1  # p^k

    for k in range(2, kmax + 1):

        pk *= p

        term = 1.0 / (k * pk)
```

```
            # Congruence test
            if pk % q == 1:
                total += term * (phi - 1)
            else:
                total -= term

    return total

def compute_table(max_q, pmax=10**6, kmax=20):
    """
    Compute and display Delta(q) for all moduli 3 <= q <= max_q.
    """
    print("  q   phi(q)   Delta(q)    Delta/phi   C_est")
    print("-" * 52)

    for q in range(3, max_q + 1):
        phi = euler_phi(q)
        D = Delta(q, pmax=pmax, kmax=kmax)
        ratio = D / phi
        C_est = ratio * log(float(q))
        print(f"{q:3d}   {phi:4d}   {D:.6f}   {ratio:.6f}   {C_est:.4f}")

def empirical_constant(qmin=31, qmax=100, pmax=10**6, kmax=20):
    """
    Compute the empirical constant C from prime moduli
```

```
    in the range [qmin, qmax].
    """
    ratios = []

    for q in primes(qmax):
        if q >= qmin:
            phi = euler_phi(q)
            D = Delta(q, pmax=pmax, kmax=kmax)
            C_est = (D / phi) * log(float(q))
            ratios.append(C_est)

    mean_C = sum(ratios) / len(ratios)
    std_C = sqrt(sum((r - mean_C)**2 for r in ratios) / (len(ratios) - 1))

    print(f"\nEmpirical constant C for q in [{qmin}, {qmax}]:")
    print(f"  Mean: {mean_C:.6f}")
    print(f"  Std:  {std_C:.6f}")
    print(f"  n:    {len(ratios)}")

    return mean_C, std_C

# ----------------------------------------------------------------------
# Main execution
if __name__ == "__main__":
```

```
# Full table for prime moduli up to 100
print("=" * 52)
print("PRIME MODULI up to 100")
print("=" * 52)

for q in primes(100):
    if q < 3:
        continue
    phi = euler_phi(q)
    D = Delta(q)
    ratio = D / phi
    C_est = ratio * log(float(q))
    print(f"{q:3d}   {phi:4d}   {D:.6f}   {ratio:.6f}   {C_est:.4f}")

# Empirical constant
print("\n" + "=" * 52)
empirical_constant(qmin=31, qmax=100)
```

Remark A.1 (Reproducibility). *To reproduce the results in this paper:*

1. *Install SageMath (version 10.0 or later) from* https://www.sagemath.org.
2. *Copy the code above into a file* $\texttt{total}_d efect.sage$.
3. *Run* `sage total_defect.sage`.

The computation takes approximately 2–3 minutes on a standard desktop computer (tested on Intel Core i7-12700H, 16 GB RAM).

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