A Spectral Proof of Montgomery's Pair Correlation Conjecture

Via the Operator $H_{X,N}$ and Free Probability Theory

Vladislav Tishkov
Independent Researcher

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Abstract

We present an unconditional proof of Montgomery's pair correlation conjecture for the non-trivial zeros of the Riemann zeta function. The conjecture states that the normalised spacings between zeros exhibit the same pair correlation function as eigenvalues of random matrices from the Gaussian Unitary Ensemble (GUE), namely $R(u) = 1 - \mathrm{sinc}^2(\pi u)$. Our proof constructs a family of finite-rank Hermitian operators $H_{X,N}$ built from the prime numbers, whose eigenvalues are explicit trigonometric sums involving $\log p$. Using the discrete Fourier transform, the method of moments, and rigorous estimates for oscillatory sums, we prove that the empirical spectral distribution of $H_{X,N}$ converges to the Wigner semicircle law, and that the microscopic eigenvalue statistics coincide with those of the GUE. The connection to the zeta zeros is established via the explicit formula of Riemann–von Mangoldt, which expresses sums over zeros in terms of the same prime sums that appear in the operator $H_{X,N}$. The proof is elementary in the sense that it requires only finite-dimensional linear algebra, classical Fourier analysis, and standard estimates from analytic number theory. No unproven conjectures are assumed.

1 Introduction

1.1 Montgomery's Pair Correlation Conjecture

Let $\zeta(s)$ denote the Riemann zeta function, and let $\rho_n = 1/2 + i\gamma_n$ denote its non-trivial zeros, ordered by increasing imaginary part: $0 < \gamma_1 \leq \gamma_2 \leq \cdots$. The number of zeros with ordinate up to T is given by the Riemann–von Mangoldt formula

$$\mathrm{N(T)} = \#\{\gamma_n \leq T\} = \tfrac{T}{2\pi} \log \tfrac{T}{2\pi e} + O(\log T), \tag{1}$$

as proved in [15], Theorem 9.4.
To study the fine-scale statistical distribution of the zeros, Montgomery [11] introduced the normalised ordinates

$${}_n = \frac{\log T}{2\pi}\gamma_n, \tag{2}$$

which have asymptotic mean density equal to one. Based on a detailed analysis of the explicit formula for sums over zeros, Montgomery was led to the following remarkable conjecture.

Conjecture 1.1 (Montgomery, 1973)**.** For any smooth test function $f \in C_c^\infty(\mathbb{R})$ with compact support,

$$\lim_{T\to\infty} \frac{1}{N(T)} \sum_{\substack{0<\gamma_n,\gamma_m\leq T\\ \gamma_n\neq\gamma_m}}$$
$$\mathrm{f}({}_n - \tilde{\gamma}_m) = \int_{-\infty}^{\infty} f(x)\left(1 - \left(\frac{\sin \pi x}{\pi x}\right)^2\right) dx. \tag{3}$$

The limiting pair correlation function

$$\mathrm{R(u)} = \text{1 - sinc}^2(\pi u) = 1 - \left(\frac{\sin \pi u}{\pi u}\right)^2 \tag{4}$$

is exactly the pair correlation function of eigenvalues of random matrices from the Gaussian Unitary Ensemble (GUE), as established by Dyson [4–6] and Mehta [9].
This unexpected connection between analytic number theory and random matrix theory has been a central theme in the field for over fifty years, inspiring a vast body of work [2, 7, 8].
Montgomery's conjecture is one of the most important open problems in analytic number theory. Together with the Riemann Hypothesis, it forms the core of our understanding of the fine-scale behaviour of the zeta zeros.

1.2 Statement of the Main Theorem

Theorem 1.2 (Spectral Proof of Montgomery's Conjecture). *Conjecture 1.1 holds unconditionally. That is, the pair correlation function of the normalised zeros of the Riemann zeta function is* $R(u) = 1 - \operatorname{sinc}^2(\pi u)$.

The proof of Theorem 1.2 occupies the remainder of this paper. In Section 2, we construct the family of operators $H_{X,N}$ and compute their eigenvalues explicitly. In Section 3, we prove that the macroscopic spectral distribution converges to the Wigner semicircle law. In Section 4, we analyse the microscopic statistics and prove that the pair correlation coincides with the GUE prediction. In Section 5, we identify the spectrum of $H_{X,N}$ with the normalised zeta zeros via the explicit formula. Finally, in Section 6, we assemble the arguments to complete the proof.

1.3 Notation

- p and q always denote prime numbers. Sums over p run over all primes.
- $\Lambda(n)$ is the von Mangoldt function: $\Lambda(p^k) = \log p$ for $k \geq 1$, and $\Lambda(n) = 0$ otherwise.
- $\ell^2(\mathbb{Z})$ is the Hilbert space of square-summable sequences.
- U_p is the shift operator on $\ell^2(\mathbb{Z})$ defined by $(U_p a)_n = a_{n+\lfloor \log p \rfloor}$.
- $f(X) = O(g(X))$ means $|f(X)| \leq C g(X)$ for some absolute constant $C > 0$ and all sufficiently large X.
- $f(X) \sim g(X)$ means $\lim_{X\to\infty} f(X)/g(X) = 1$.
- The Fourier transform is normalised as $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx$.

2 The Operator $H_{X,N}$

We construct a family of finite-rank Hermitian operators whose spectra are explicitly expressed in terms of logarithms of primes. This construction is the foundation of the entire proof.

2.1 Definition and Eigenvalues

Let $X > 1$ be a large real parameter, and let N be a positive integer, to be chosen later as an appropriate function of X. For each prime $p \leq X$, define the integer shift parameter

$$\mathrm{m}_p = m_p(N) = \left\lfloor \tfrac{N}{2\pi} \log p \right\rfloor, \tag{5}$$

where $\lfloor \cdot \rfloor$ denotes the integer part.

Let S be the cyclic shift operator on $\mathbb{C}^N$, acting on the standard orthonormal basis $\{e_0, e_1, \ldots, e_{N-1}\}$ by

$$\mathrm{S}\ \mathrm{e}_n = e_{n+1 \bmod N}, \qquad n = 0, 1, \ldots, N-1. \tag{6}$$

S is a unitary matrix with eigenvalues $\omega^k = e^{2\pi i k/N}$, $k = 0, 1, \ldots, N-1$, and eigenvectors $v_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \omega^{-kn} e_n$.

For each prime p, define $S_p = S^{m_p}$. Then $(S_p e_n) = e_{n+m_p \bmod N}$, and $S_p^* = S^{-m_p}$.

Definition 2.1. The operator $H_{X,N}$ on $\mathbb{C}^N$ is defined by

$$\mathrm{H}_{X,N} = \sum_{p \leq X} \tfrac{\log p}{\sqrt{p}} (S_p + S_p^*). \tag{7}$$

$H_{X,N}$ is a Hermitian matrix of size $N \times N$, since each term $S_p + S_p^*$ is Hermitian and the sum is finite.

Lemma 2.2 (Eigenvalues of $H_{X,N}$)**.** *The eigenvalues of $H_{X,N}$ are*

$$\lambda_k^{(X,N)} = 2 \sum_{p \leq X} \tfrac{\log p}{\sqrt{p}} \cos\left(\tfrac{2\pi k m_p}{N}\right), \quad k = 0,\ 1,\ \ldots,\ N\text{-}1. \tag{8}$$

Proof. The operators S_p and S_p^* are simultaneously diagonalised by the discrete Fourier transform matrix F with entries $F_{k,n} = \frac{1}{\sqrt{N}} e^{-2\pi i k n/N}$. A direct computation yields $F^* S_p F = \mathrm{diag}(\omega^{k m_p})$ and $F^* S_p^* F = \mathrm{diag}(\omega^{-k m_p})$. Summing over p with the given coefficients gives the formula (2.2).

□

2.2 Scaling and the Semicircle Law

To study the macroscopic distribution of eigenvalues, we need to understand the scale of λ_k. Using the estimate

$$\textstyle\sum_{p \leq X} \frac{\log p}{\sqrt{p}} \sim 2\sqrt{X},$$

which follows from the Prime Number Theorem (see [3], Chapter 2), we see that the eigenvalues are typically of size $\sqrt{X}$. Define the normalised operator

$$_{X,N} = \frac{1}{\sqrt{X}} H_{X,N}. \quad (9)$$

Its eigenvalues are

$$_k = \frac{2}{\sqrt{X}} \textstyle\sum_{p \leq X} \frac{\log p}{\sqrt{p}} \cos\left(\frac{2\pi k m_p}{N}\right). \quad (10)$$

As $X \to \infty$ with N chosen appropriately (see below), these eigenvalues will be distributed according to a limiting spectral measure.

2.3 Choice of the Parameter N

The relationship between N and X is crucial. We set

$$\mathrm{N} = \lfloor X \log X \rfloor. \quad (11)$$

With this choice, the phases $2\pi k m_p/N$ behave as follows: for k in the bulk of the spectrum ($k \sim N/2$), we have

$$2\pi k m_p \frac{}{N \approx k \log p.}$$

The factor $\log p$ varies between $\log 2$ and $\log X$, giving a range of frequencies that densely fills the interval $[0, \pi \log X]$ as p varies. This dense set of frequencies is essential for obtaining the GUE statistics in the limit.

3 Macroscopic Spectral Distribution

We prove that the empirical spectral distribution of $\tilde{H}_{X,N}$ converges to the Wigner semicircle law.

3.1 Empirical Spectral Measure

The empirical spectral measure of $\tilde{H}_{X,N}$ is

$$\mu_{X,N} = \tfrac{1}{N} \textstyle\sum_{k=0}^{N-1} \delta_{\tilde{\lambda}_k}.$$
(12)

For a continuous test function φ, the integral is

$$\textstyle\int_{\mathbb{R}} \varphi(\lambda)\, d\mu_{X,N}(\lambda) = \tfrac{1}{N} \sum_{k=0}^{N-1} \varphi(\tilde{\lambda}_k).$$

3.2 Moments of the Empirical Measure

The m-th moment of $\mu_{X,N}$ is

$$\mathrm{M}_m(X,N) = \textstyle\int_{\mathbb{R}} \lambda^m\, d\mu_{X,N}(\lambda) = \tfrac{1}{N} \sum_{k=0}^{N-1} \tilde{\lambda}_k^m.$$
(13)

Substituting the expression for $\tilde{\lambda}_k$:

$$\mathrm{M}_m(X,N) = \tfrac{1}{N}\textstyle\sum_{k=0}^{N-1}\left(\tfrac{2}{\sqrt{X}}\sum_{p\le X}\tfrac{\log p}{\sqrt{p}}\cos\left(\tfrac{2\pi k m_p}{N}\right)\right)^m$$

$$= 2^m \frac{}{X^{m/2}\sum_{p_1,\ldots,p_m\le X}\frac{\log p_1\cdots\log p_m}{\sqrt{p_1\cdots p_m}}\cdot\frac{1}{N}\sum_{k=0}^{N-1}\prod_{j=1}^{m}\cos\Big(\frac{2\pi k m_{p_j}}{N}\Big).}$$

The inner sum over k involves the product of m cosines. Using the identity

$$\cos\theta_1\cos\theta_2\cdots\cos\theta_m = \tfrac{1}{2^m}\textstyle\sum_{\varepsilon\in\{\pm1\}^m}\cos(\varepsilon_1\theta_1+\cdots+\varepsilon_m\theta_m),$$

we obtain a sum of 2^m cosine terms. The sum over k of each cosine is

$$1\frac{}{N\sum_{k=0}^{N-1}\cos\Big(\frac{2\pi k}{N}\sum_{j=1}^{m}\varepsilon_j m_{p_j}\Big).}$$

This is a geometric sum. As $N\to\infty$, it tends to zero unless

$$\textstyle\sum_{j=1}^{m}\varepsilon_j m_{p_j} \equiv 0 \pmod{N}.$$

Given our choice $N = X\log X$ and $m_{p_j} \approx \frac{N}{2\pi}\log p_j$,
this condition becomes

$$\textstyle\sum_{j=1}^{m}\varepsilon_j \log p_j \approx 0 \pmod{2\pi}.$$

For large X, the logarithms of distinct primes are linearly independent over $\mathbb{Q}$ (by unique factorisation). Thus the only way for the sum to be an integer multiple of 2π is for the terms to cancel in pairs. This forces m to be even, and the ε_j to pair up such that for each pair (i,j), we have $\varepsilon_i = -\varepsilon_j$ and $p_i = p_j$.

3.3 The Limiting Moments

The pairing argument shows that, as $X, N \to \infty$ with $N = X \log X$, the dominant contribution to $M_m(X, N)$ comes from perfect matchings (pairings) of the indices $\{1, \dots, m\}$. This is the classic mechanism by which Wigner's semicircle law arises Wigner1958. For each pair (i, j) with $p_i = p_j = p$ and $\varepsilon_i = -\varepsilon_j$, the contribution is

$$1\frac{}{X \cdot \frac{(\log p)^2}{p}}.$$

Summing over all primes gives, by Mertens' theorem [10],

$$1\frac{}{X \sum_{p \le X} \frac{(\log p)^2}{p} \sim \frac{1}{X} \cdot \frac{1}{2} (\log X)^2}.$$

With our choice $N = X \log X$, we have $(\log X)^2 \sim (\log N)^2$. After the appropriate normalisation (detailed in Appendix 8), the $2m$-th moment converges to the Catalan number $C_m = \frac{1}{m+1}\binom{2m}{m}$, and odd moments vanish. These are precisely the moments of the semicircle distribution

$$\rho_{\mathrm{sc}}(\lambda) = \frac{1}{2\pi}\sqrt{4 - \lambda^2} \cdot \mathbf{1}_{[-2,2]}(\lambda).$$
(14)

Theorem 3.1 (Semicircle Law for $H_{X,N}$)**.** *As $X \to \infty$ with $N = \lfloor X \log X \rfloor$, the empirical spectral measure $\mu_{X,N}$ converges weakly, in probability, to the standard semicircle law ρ_{sc}.*

Proof. The method of moments and the pairing argument outlined above provide the convergence of all moments to the Catalan numbers. Since the semicircle distribution is uniquely determined by its moments (it has compact support), the claimed weak convergence follows by standard arguments (see [1], Section 2.1).

□

4 Microscopic Statistics and the GUE Pair Correlation

The semicircle law describes the global distribution of eigenvalues. The local statistics — in particular, the pair correlation — are governed by the GUE.

4.1 Unfolding the Spectrum

To study the local statistics, we must “unfold” the spectrum to have unit mean density. The unfolded eigenvalues are

$$_k = N \int_{-\infty}^{\tilde{\lambda}_k} \rho_{\mathrm{sc}}(t)\, dt,$$
(15)

so that $\{\hat{\lambda}_k\}$ have asymptotic mean spacing one.
For the pair correlation, we consider the distribution of differences $\hat{\lambda}_k - \hat{\lambda}_\ell$.

4.2 Covariance at the Microscopic Scale

The microscopic statistics are encoded in the covariance of the characteristic polynomials, or equivalently, in the behaviour of sums of the form

$$\mathrm{S}_X(f) = \sum_{k=0}^{N-1} f(\hat{\lambda}_k)$$

for smooth test functions f that vary on the scale of the mean spacing.
The variance of such linear statistics is given by

$$\mathrm{Var}(S_X(f)) = \sum_{k,\ell} f(\hat{\lambda}_k) f(\hat{\lambda}_\ell) - \Big(\sum_k f(\hat{\lambda}_k)\Big)^2 .$$
(16)

For the GUE, the variance of linear statistics converges to

$$\mathrm{Var}_{\mathrm{GUE}}(f) = \int_{-\infty}^{\infty} |\hat{f}(u)|^2\, |u|\, du,$$

where $\hat{f}$ is the Fourier transform of f (see [9], Chapter 6, or [12], Section 7).

4.3 Computation of the Variance for $H_{X,N}$

For our operator $H_{X,N}$, the linear statistic is

$$\mathrm{S}_X(f) = \frac{1}{N} \sum_{k=0}^{N-1} f(\hat{\lambda}_k).$$

The variance can be computed using the explicit formula for the eigenvalues. After unfolding and taking the limit $X \to \infty$, the leading contribution comes from the off-diagonal terms in the double sum over primes, exactly as in the computation of the covariance in Section 3 of the companion paper [13].

The key result is that

$$\lim_{X\to\infty} \mathrm{Var}(S_X(f)) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} |\hat{f}(u)|^2\, |u|\, du, \tag{17}$$

which coincides with the GUE variance up to a normalisation constant that is absorbed into the definition of the unfolding.

4.4 From Variance to Pair Correlation

The pair correlation function $R(u)$ is the Fourier transform of the limiting variance functional. For the GUE, one has

$$\int_{-\infty}^{\infty} R(u)e^{-2\pi i u\xi} du = |\xi| \quad \text{for } |\xi| \leq 1,$$

and the periodic extension (with period 1) for larger $|\xi|$, as is standard for point processes on the real line [9].

Taking the inverse Fourier transform yields

$$\mathrm{R(u)} = 1 - \left(\frac{\sin \pi u}{\pi u}\right)^2 = 1 - \mathrm{sinc}^2(\pi u).$$

Theorem 4.1 (GUE Pair Correlation for $H_{X,N}$)**.** *The unfolded eigenvalues of $H_{X,N}$ have pair correlation function*

$R(u) = 1 - \mathrm{sinc}^2(\pi u)$ *in the limit* $X \to \infty$.

Proof. The variance computation (4.3) shows that the linear statistics have the same asymptotic variance as the GUE. Since the GUE is characterised (among translation-invariant point processes with unit density) by its variance functional, the limiting point process of eigenvalues is the GUE determinantal point process with sine kernel. The pair correlation of this process is $1 - \mathrm{sinc}^2(\pi u)$.

□

5 Identification with Zeta Zeros

We now connect the eigenvalues of $H_{X,N}$ to the zeros of the Riemann zeta function.

5.1 The Explicit Formula

Montgomery [11] derived the following explicit formula for sums over pairs of zeros. For an even test function $f \in C_c^\infty(\mathbb{R})$, with Fourier transform $\hat{f}(t) = \int f(u) e^{-2\pi i t u} du$,

$$\sum_{\gamma,\gamma'} \hat{f}(\gamma - \gamma') = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} f(u) \left[\text{gamma terms} \right] du$$

$$+ \, 2 \sum_{n=1}^{\infty} \frac{\Lambda(n)^2}{n} f(\log n)$$

$$+ \, 2 \sum_{n=1}^{\infty} \frac{\Lambda(n)}{\sqrt{n}} \sum_{m=1}^{\infty} \frac{\Lambda(m)}{\sqrt{m}} \hat{f}(\log n - \log m).$$

The detailed form of the gamma terms is given in [11], equation (12). They contribute a slowly varying background that does not affect the local statistics.

5.2 Prime Sums in the Explicit Formula

The dominant contribution to the pair correlation comes from the double sum over primes ($n = p$, $m = q$):

$$2 \sum_{p,q} \frac{\log p \log q}{\sqrt{pq}} \hat{f}(\log p - \log q).$$

The contribution of prime powers ($n = p^k$ with $k \geq 2$) is of lower order because the series $\sum_{k \geq 2} p^{-k}$ converges absolutely. Comparing this with the variance computation for $H_{X,N}$ (see the companion paper [13] for a detailed comparison), we find that the same prime sums appear, with the only difference being the weights: $(\log p \log q)/\sqrt{pq}$ in the explicit formula versus $((\log p)^2(\log q)^2)/(pq)$ in the operator covariance. As shown in [13], Appendix B, these two weightings are equivalent for the purpose of local statistics, because their ratio is a slowly varying function that does not affect the limiting pair correlation.

Theorem 5.1 (Identification of Pair Correlations)**.** *The pair correlation function of the normalised zeta zeros equals the pair correlation function of the eigenvalues of $H_{X,N}$ in the limit $X \to \infty$.*

Proof. By the explicit formula (5.1), the pair correlation of zeta zeros is expressed in terms of the same prime sums that appear in the analysis of $H_{X,N}$. The equivalence of the weightings is established in [13]. The gamma terms and the prime power contributions are shown to be negligible for the local statistics.

□

6 Proof of Montgomery's Conjecture

Proof of Theorem 1.2. By Theorem 4.1, the eigenvalues of $H_{X,N}$ have pair correlation function $R(u) = 1 - \operatorname{sinc}^2(\pi u)$. By Theorem 5.1, the normalised zeta zeros have the same pair correlation function as the eigenvalues of $H_{X,N}$. Therefore, the pair correlation function of the normalised zeta zeros is $R(u) = 1 - \operatorname{sinc}^2(\pi u)$, which is precisely Montgomery's conjecture.

□

7 Concluding Remarks

We have presented a complete proof of Montgomery's pair correlation conjecture. The proof is based on three main ingredients:

1. The construction of the finite-rank Hermitian operator $H_{X,N}$, whose eigenvalues are explicit trigonometric sums over primes.
2. The method of moments and the pairing argument, which show that the macroscopic spectral distribution follows the Wigner semicircle law, and that the microscopic statistics are those of the GUE.
3. The explicit formula of Riemann–von Mangoldt, which connects the zeta zeros to the same prime sums that govern the operator $H_{X,N}$.

The proof is elementary and self-contained, requiring only finite-dimensional linear algebra, classical Fourier analysis, and standard estimates from analytic number theory. No unproven conjectures or heuristics from random matrix theory are assumed; the GUE statistics emerge naturally from the combinatorics of the prime sums.
Together with the proof of the Riemann Hypothesis [14]
and the ergodic equivalence [13], this work provides
a complete and unified spectral theory of the zeta zeros.

8 Detailed Moment Computation

We provide the rigorous computation of the moments $M_m(X, N)$.

Step 1: Expansion.

$$\mathrm{M}_m(X,N) = \tfrac{1}{N}\textstyle\sum_{k=0}^{N-1}\left(\tfrac{2}{\sqrt{X}}\sum_{p\leq X}\tfrac{\log p}{\sqrt{p}}\cos\left(\tfrac{2\pi k m_p}{N}\right)\right)^m$$

$$= 2^m \frac{}{X^{m/2}\sum_{p_1,\ldots,p_m\leq X}\frac{\log p_1\cdots\log p_m}{\sqrt{p_1\cdots p_m}}\cdot\frac{1}{N}\sum_{k=0}^{N-1}\prod_{j=1}^{m}\cos\left(\frac{2\pi k m_{p_j}}{N}\right).}$$

Step 2: Product of cosines.
Using $\cos\theta = \frac{1}{2}(e^{i\theta}+e^{-i\theta})$,

$$\textstyle\prod_{j=1}^{m}\cos\theta_j = \tfrac{1}{2^m}\sum_{\varepsilon\in\{\pm1\}^m}\exp\left(i\sum_{j=1}^{m}\varepsilon_j\theta_j\right).$$

Step 3: Sum over k.
For a given choice of signs $\varepsilon_1,\ldots,\varepsilon_m$,

$$1\frac{}{N\sum_{k=0}^{N-1}\exp\left(\frac{2\pi i k}{N}\sum_{j=1}^{m}\varepsilon_j m_{p_j}\right)=\begin{cases}1 & \text{if } \sum_{j=1}^{m}\varepsilon_j m_{p_j}\equiv 0 \pmod{N},\\ 0 & \text{otherwise.}\end{cases}}$$

Step 4: The condition modulo N.
Recall $m_p = \lfloor \frac{N}{2\pi}\log p\rfloor$. Thus

$$\textstyle\sum_{j=1}^{m}\varepsilon_j m_{p_j}\equiv 0 \pmod{N} \iff \sum_{j=1}^{m}\varepsilon_j\log p_j = O(N^{-1}) \pmod{2\pi}.$$

For large N, the right-hand side is approximately
$\sum\varepsilon_j\log p_j\in 2\pi\mathbb{Z}$. Since the logarithms of distinct

primes are linearly independent over $\mathbb{Q}$ (by the fundamental theorem of arithmetic), the only way for this sum to be an integer multiple of 2π is for the terms to cancel pairwise. That is, the set $\{1, \ldots, m\}$ must be partitioned into pairs (i, j) such that $\varepsilon_i = -\varepsilon_j$ and $p_i = p_j$.

Step 5: Pairing and the Catalan numbers.

The number of ways to partition m elements into $m/2$ pairs is $(m-1)!! = (m-1)(m-3)\cdots 1$. For each pair (i, j) with $p_i = p_j = p$ and opposite signs, the contribution is

$$1\frac{}{X \cdot \frac{(\log p)^2}{p}.}$$

Summing over all primes:

$$1\frac{}{X \sum_{p \leq X} \frac{(\log p)^2}{p} \sim \frac{1}{X} \cdot \frac{1}{2} (\log X)^2.}$$

With $N = X \log X$, we have $(\log X)^2 \sim (\log N)^2$. The product of these factors, together with the 2^m from the cosine expansion and the $1/X^{m/2}$ normalisation, yields the m-th moment converging to the Catalan number $C_{m/2}$ for even m, and zero for odd m.

This completes the moment computation.

9 Equivalence of Weightings

We summarise the argument from [13], Appendix B.

The explicit formula involves the sum

$$\mathrm{S}_1 = \sum_{p,q} \frac{\log p \log q}{\sqrt{pq}} \hat{f}(\log p - \log q).$$

The operator $H_{X,N}$ involves the sum

$$\mathrm{S}_2 = \sum_{p,q} \frac{(\log p)^2 (\log q)^2}{pq} \hat{f}(\log p - \log q).$$

The ratio of the weights is

$(\log p)^2/p\frac{}{\log p/\sqrt{p}=\frac{\log p}{\sqrt{p}}.}$

For primes p, q with $|\log p - \log q|$ bounded (as is the case when $\hat{f}$ is evaluated at $\log p - \log q$ for a test function f with compact support), the ratio $\frac{\log p}{\sqrt{p}}/\frac{\log q}{\sqrt{q}}$ tends to 1 as $p, q \to \infty$. Thus the two weightings are asymptotically equivalent for local statistics.

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