# Quantum Reference Frames Beyond Ideality: Complete-Order Quotients, Resource Lattices, and Certified Transformations

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Quantum reference frame (QRF) transformations are usually presented as unitary or gauge-fixing changes of coordinates. That picture is exact for ideal frames but fails for finite clocks, gyroscopes, POVM-defined tokens, observers correlated with the targets they describe, and relativistic frames with quantum boost and clock degrees of freedom; the obstruction is loss of distinguishability. We model a non-ideal frame by a normal covariant perspective channel $\mathsf{P}_R : \mathcal{T}_{\mathrm{phys}} \to \mathcal{T}_R$. A deterministic transformation $A \to B$ then exists exactly when $\operatorname{Ker} \mathsf{P}_A \subseteq \operatorname{Ker} \mathsf{P}_B$, and is unique on the preparable image; ancilla-stable transformations are complete-order quotient morphisms, while a laboratory implementation on a chosen output space is a separate Choi–Arveson extension problem. The common invariant content of a family of frames is a Heisenberg operator system $\mathcal{I}_{\mathcal{F}} = \bigcap_R (\operatorname{Ker} \mathsf{P}_R)^{\perp}$, carrying an algebraic product only on its product-compatible core. For finite frames in the regular representation, concrete covariant degradability $A \to B$ is decided by a group-convolution linear program with explicit dual witnesses; for Abelian groups this becomes a Fourier mode-support resource lattice. Our main result is that for *every* finite group this convolution linear program already equals the full diamond-norm deficiency, $\delta(B|A) = \frac{1}{2} \min_r \|p_B - r * p_A\|_1$: coherent, multiplicity-exploiting post-processings never outperform random translations, so the resource theory of frame degradation is entirely classical. The proof uses an augmented entangled witness that charges discarded probability mass at the optimal-translation rate; we supply central and non-central $S_3$ certificates and a commutant-reduced verification through $A_4$. The same quotient principle governs process tensors, protected relational sectors, entangled observers, relativistic covariance – the boosted-spin perspective being an explicit $U(1)$ dephasing channel whose multiplier is the characteristic function of the Thomas–Wigner angle – and observer networks. Reproducible numerical certificates accompany every analytic claim.

## I. INTRODUCTION

A reference frame is usually treated as a classical background convention: an origin, orientation, clock, tetrad, laboratory, or external coordinate system with respect to which states and observables are written. Relational quantum mechanics and early quantum-reference-system work already show that this convention cannot be fundamental in a closed quantum description [1–3]. Quantum theory does not permit this idealization to be fundamental. Any real frame is a physical system with a finite mass, finite energy, finite angular momentum, finite clock bandwidth, finite localization and possible entanglement with the system observed. In relativistic physics the frame may also carry a momentum wave packet or a superposition of velocities, so that boosts, Wigner rotations and simultaneity conventions become frame degrees of freedom.

The modern theory of quantum reference frames (QRFs) has several well-developed but not fully unified branches. Early work by Aharonov and Kaufherr treated frames attached to quantum objects of finite mass [3]. The resource-theoretic tradition identifies the absence of a reference frame with a superselection restriction and studies reference tokens as consumable resources [6, 7, 14]. The recent QRF program develops explicit transformations between quantum perspectives, including perspective-neutral, operational and relative-subsystem formulations [8, 9, 11–13, 15, 16, 33]. The relativistic branch constructs superpositions of Lorentz boosts and spin transformations [10, 17], while the constrained-systems and algebraic branches relate QRFs to Dirac quantization, gauge fixing, relational observables and crossed products [22, 37, 38]. There are also important Russian contributions, particularly S. N. Mayburov's work on quantum reference frames, time operators and mass-dependent corrections to spacetime transformations [4, 5, 31, 32].

Despite this progress, there remains a sharp structural problem.

> *Problem.* Give a unique, invariant and relativistically covariant notion of transformation between quantum reference frames when the frames are non-ideal, described by POVMs or instruments rather than sharp coordinates, possibly entangled with the systems they describe, and possibly relativistic observers with quantum boosts, clocks and tetrads.

The common hidden assumption behind the difficulty is that a change of QRF must be a unitary map between Hilbert-space descriptions. This assumption is too strong. It is correct only when both frames preserve all relational information needed by the other. A finite clock loses temporal Fourier modes; a bounded-size phase reference damps asymmetry modes; an unsharp position frame convolves relational wave functions; a relativistic wave-packet observer smears the boost rapidity and hence the spin-momentum relation. No unitary inverse can reconstruct modes that the first frame cannot operationally distinguish.

The correct object is therefore not a universal unitary but a channel on operational equivalence classes. The analysis below makes this precise and derives the associated uniqueness theorem. The result has three parts.

(i) **Transformation.** A perspective is a channel $\mathsf{P}_R$ from the invariant physical theory to the description accessible to frame $R$. An $A \to B$ transformation exists exactly when every physical distinction erased by $A$ is also erased by $B$:

$$\mathrm{Ker}\,\mathsf{P}_A \subseteq \mathrm{Ker}\,\mathsf{P}_B.$$

It is then unique on $\mathrm{Ran}\,\mathsf{P}_A$. Reversibility is equivalent to equality of kernels. Complete positivity is the only remaining physical implementability condition.

(ii) **Invariants.** In the Heisenberg picture, the complete common invariant object for a family $\mathcal{F}$ of noisy frames is the annihilator-defined operator system

$$\mathcal{I}_{\mathcal{F}} = \bigcap_{R\in\mathcal{F}} (\mathrm{Ker}\,\mathsf{P}_R)^{\perp}.$$

For closed Heisenberg ranges this is the intersection of the pullback ranges. All frame-independent probabilities are functions of states restricted to this operator system. Algebraic invariants are recovered from its multiplicative core or from its $C^*$-envelope; this extra step is necessary because a CP image is not generally closed under multiplication.

(iii) **Non-ideality and relativity.** For compact Abelian groups, non-ideality is exactly a Fourier multiplier on relational charge modes. For Poincaré and Lorentz frames, the same theorem holds sectorwise in the representation decomposition, with Wigner rotations and superposed boosts included in $\mathsf{P}_R$.

The central claim is not that all non-ideal frames are secretly invertible. The central claim is the opposite and is mathematically stronger: *there is a unique transformation exactly on the quotient where a transformation can physically exist, and the obstruction to stronger transformations is completely classified by kernels or, equivalently, by lost invariant modes.* This turns the problem of uniqueness into a theorem and the problem of invariants into an algebraic intersection.

## II. STATE OF THE ART AND THE UNRESOLVED GAP

The current literature contains several compatible insights but no single theorem that covers ideal, non-ideal, entangled and relativistic frames simultaneously.

### A. Ideal and operational QRF transformations

The operational QRF program introduced transformations that describe states, dynamics and measurements from the viewpoint of quantum systems rather than an external classical background. In the canonical examples, a particle or laboratory may be in a superposition of relative positions or velocities. The transformation to its perspective is a controlled translation, parity-swap, or a coherent group transformation. These transformations display the frame dependence of superposition and entanglement and suggest a quantum generalization of covariance.

The perspective-neutral program embeds several frame perspectives into a constrained physical Hilbert space. Taking a perspective is analogous to gauge fixing; changing perspective is a symmetry transformation followed by another gauge fixing. This is powerful for ideal clocks, rods and finite-dimensional models with sufficiently sharp internal coordinates. General symmetry groups were treated by constructing relational operators and changes of frame associated with a group $G$.

Operational approaches identify relative observables as invariants on the composite of system and frame and quotient states by operational indistinguishability [16, 33]. This is already close to the solution developed below. However, the standard presentation often constructs explicit invertible maps only under localizability assumptions on the frames. Once a frame is not sharply localizable, the map need not be invertible, and uniqueness has to be understood on images or quotients rather than on a full Hilbert space.

### B. Non-ideal frames and finite resources

A real reference frame is a token with finite resources. In the resource theory of asymmetry, a phase reference, gyroscope or clock is a quantum system carrying a group representation. If it is finite, it cannot encode all group elements sharply. Its POVM has overlapping effects and the corresponding relational description is noisy. The absence or degradation of the frame appears as dephasing, superselection, or loss of asymmetry modes [6, 7]. Recent non-ideal QRF work obtains reversible perspective transformations by retaining the non-ideal frame degrees of freedom explicitly and shows that finite resources impose superselection and back-reaction effects [34].

Finite clocks provide the clearest example. Time is measured by a covariant POVM, and finite energy bandwidth limits the distinguishability of different clock readings. The ideal Page–Wootters clock is recovered only in an infinite-resource limit. For finite clocks, the effective evolution relative to the clock can become non-unitary or temporally non-local. This is not a paradox; it is precisely the statement that the clock perspective channel has a nontrivial kernel.

The missing general statement is: when can one transform from the description supplied by one finite resource token to that supplied by another? The answer cannot be "always", because a poor clock cannot reconstruct temporal coherences that it never records. It also cannot be "never", because one noisy token can be sufficient for another less informative token. The correct condition is the kernel inclusion theorem below.

### C. Entangled observers

In many QRF examples, changing perspective changes which subsystems appear entangled. This is not a contradiction because a perspective change changes the tensor factorization used to define subsystems. Ordinary entanglement entropy of a tensor factor is therefore not a frame invariant; recent work instead identifies constrained observer-dependent and diagonal Rényi invariants [21]. Recent results identify invariant combinations, such as sums involving entanglement and subsystem coherence or determinant-like covariance quantities in specific models [19, 20, 41]. A complementary no-go result shows that passive QRF transformations cannot create entanglement between fixed physical systems [35].

A fully general solution should not assume that the frame is initially uncorrelated with the system. A laboratory, clock, gyroscope or relativistic particle can be entangled with the observed degrees of freedom. The perspective map must therefore act on joint density operators, not on product states. The invariant notion of correlation must be algebraic: correlations between subalgebras of the physical invariant operator algebra are meaningful, while correlations between perspective-dependent tensor factors are not generally invariant.

### D. Relativistic observers

Relativistic QRFs introduce additional structure. A spin qubit is operationally defined through a measurement arrangement, but Lorentz boosts entangle spin and momentum. For a particle moving in a superposition of velocities, one needs a superposition of boosts to move to its rest frame [10]. More recent work extends QRF transformations to Lorentz symmetry using spacetime wave functions without a preferred slicing and identifies superpositions of time dilation and length contraction [17]; a related quantum-group line connects QRF transformations with first-order quantum-deformed Galilei transformations [36].

The gap is that relativistic constructions are usually presented in ideal or controlled-coherent form. A relativistic observer is also non-ideal: its momentum wave packet has finite spread, its tetrad is uncertain, and its clock has finite energy. The present formalism treats Lorentz and Poincaré frames as covariant POVM frames on homogeneous spaces and applies the same uniqueness and invariant operator-system theorem sectorwise in the unitary representation decomposition.

Recent 2025–2026 developments reinforce this diagnosis from complementary directions: operational transformations identify states by relative observables and localizability rather than by a universal unitary [16]; finite-resource QRFs exhibit superselection and back-reaction [34]; circuit implementations make the entangling cost of a gate perspective dependent [42]; temporal QRFs clarify measurements relative to non-ideal clocks [18]; phase-space QRFs produce frame-relative uncertainty relations [43], and many-body studies explicitly warn that purely unitary frame changes do not exhaust relational predictions [44]. These results all point to the same mathematical structure: the physical object is an operational quotient equipped with a complete order, not a coordinate change on a fixed Hilbert space.

**Remark II.1** (Scope of the main theorems)**.** *The exact kernel and complete-kernel theorems are finite-dimensional as written. Their normal von Neumann analogues require weak-closed kernels, normal instruments, normal completely positive maps, and the usual domain restrictions for unbounded generators and noncompact groups. Relativistic* *examples in later sections should therefore be read as structured applications of the quotient principle, not as a full construction of a generally covariant quantum-gravity theory.*

## III. OPERATIONAL AXIOMS

We work in finite dimensions or, more generally, with normal states on von Neumann algebras. Compact groups are treated first. Locally compact and relativistic groups require standard domain restrictions; these are stated in Section XI. The finite-dimensional case already contains the logical core.

Let $G$ be a second countable unimodular group with Haar measure $\mathrm{d}g$. A physical collection consists of frame systems $R_1, \dots, R_n$ and other systems $S$ with Hilbert spaces $\mathcal{H}_X$ and strongly continuous unitary or projective unitary representations $U_X(g)$. The total Hilbert space is

$$\mathcal{H} = \bigotimes_X \mathcal{H}_X, \qquad U(g) = \bigotimes_X U_X(g).$$

The invariant algebra is

$$\mathfrak{M}^G = \{A \in \mathcal{B}(\mathcal{H}) : U(g) A U(g)^\dagger = A,\ \forall g \in G\}.$$

Physical states are normal positive functionals on $\mathfrak{M}^G$, equivalently density operators modulo the annihilator of $\mathfrak{M}^G$. In finite dimensions one may take $G$-invariant density operators without loss of operational content.

**Axiom III.1** (Relational operationality)**.** *Only invariant probabilities are primitive. Two mathematical states that agree on $\mathfrak{M}^G$ are physically identical.*

A frame is not assumed to be ideal. The technically safest primitive is not a square root of a POVM density, but a normal covariant instrument in the operational sense of quantum measurement theory [24, 25]. This avoids a common pathology: for a continuous POVM the symbol $E(\mathrm{d}x)^{1/2}$ need not define a countably additive completely positive operation without an instrument or a chosen density.

**Definition III.2** (Covariant operational quantum frame)**.** *A quantum frame $R$ for a homogeneous space $X_R = G/H_R$ consists of the representation $U_R$ together with a normal completely positive instrument*

$$\mathfrak{I}_R(\Delta) : \mathcal{T}(\mathcal{H}) \longrightarrow \mathcal{T}(\mathcal{H}_{\bar{R}}), \qquad \Delta \in \mathcal{B}(X_R),$$

*where $\bar{R}$ denotes all degrees of freedom except $R$. The instrument is normalized by*

$$\mathrm{Tr}\ \mathfrak{I}_R(X_R)(\rho) = \mathrm{Tr}\ \rho$$

*and covariant in the sense that*

$$\mathfrak{I}_R(g\Delta)\big(U(g)\rho U(g)^\dagger\big) = U_{\bar{R}}(g)\, \mathfrak{I}_R(\Delta)(\rho)\, U_{\bar{R}}(g)^\dagger.$$

*Its associated POVM is the normal effect measure*

$$E_R(\Delta) = \mathfrak{I}_R(\Delta)^*(\mathbf{1}_{\bar{R}}).$$

*A local frame measurement with Kraus density $M_\alpha(x)$ is the special case*

$$\mathfrak{I}_R(\mathrm{d}x)(\rho) = \sum_\alpha \mathrm{Tr}_R\big[(M_\alpha(x) \otimes \mathbf{1}_{\bar{R}})\rho(M_\alpha(x)^\dagger \otimes \mathbf{1}_{\bar{R}})\big]\, \mathrm{d}x.$$

*The frame is ideal only when the corresponding localization observable is sharp and admits perfectly localized states on the sector considered.*

Choose a measurable section $s_R : X_R \to G$. If $H_R$ is nontrivial, the output algebra is restricted to the $H_R$-invariant algebra of the remaining systems; this is the residual gauge.

**Definition III.3** (Perspective channel)**.** *The perspective channel of $R$ is*

$$\boxed{\mathsf{P}_R(\rho) = \int_{X_R} U_{\bar{R}}(s_R(x))^\dagger\, \mathfrak{I}_R(\mathrm{d}x)(\rho)\, U_{\bar{R}}(s_R(x)).} \tag{1}$$

*The integral is understood in the trace-norm sense in finite dimension and in the normal weak sense for von Neumann preduals. For a discrete outcome space it is a sum. The Heisenberg dual is*

$$\boxed{\mathsf{P}_R^*(A) = \int_{X_R} \mathfrak{J}_R(\mathrm{d}x)^* \big(U_{\bar{R}}(s_R(x)) A U_{\bar{R}}(s_R(x))^\dagger\big)\,.} \tag{2}$$

Operationally, $\mathsf{P}_R$ means: measure the frame coordinate $x$, transform the rest of the universe by the inverse coordinate so that $R$ is placed at the reference value, and discard the frame degrees of freedom. This is a quantum analogue of gauge fixing with a finite-resolution gauge condition. If $E_R$ is sharp, (1) reduces to ideal conditionalization. If $E_R$ is unsharp, it is a noisy conditionalization.

**Lemma III.4** (Basic properties)**.** *$\mathsf{P}_R$ is completely positive and trace preserving. Its dual $\mathsf{P}_R^*$ is unital and completely positive. If $\rho$ is $G$-invariant and $A$ is invariant under the residual subgroup $H_R$, then $\mathsf{P}_R^*(A) \in \mathfrak{M}^G$. Changing the measurable section changes $\mathsf{P}_R$ only by a residual $H_R$-gauge channel on the output.*

*Proof.* Each $\mathfrak{J}_R(\mathrm{d}x)$ is completely positive, conjugation by $U_{\bar{R}}(s_R(x))$ is a $^*$-automorphism, and normal integration preserves complete positivity. Normalization of the instrument gives

$$\mathrm{Tr}\ \mathsf{P}_R(\rho) = \int_{X_R} \mathrm{Tr}\ \mathfrak{J}_R(\mathrm{d}x)(\rho) = \mathrm{Tr}\ \rho,$$

so $\mathsf{P}_R$ is trace preserving and $\mathsf{P}_R^*$ is unital completely positive. Covariance of $\mathfrak{J}_R$, the section identity $gs_R(x) = s_R(gx)h(x,g)$ with $h(x,g) \in H_R$, and $H_R$-invariance of the output observable imply

$$U(g)\mathsf{P}_R^*(A)U(g)^\dagger = \mathsf{P}_R^*(A),$$

so the pullback is a physical invariant observable. Replacing $s_R(x)$ by $s_R(x)h(x)$ conjugates the output by a residual $H_R$-gauge operation and therefore leaves the quotient perspective unchanged after restriction to the residual invariant algebra. ☐

**Definition III.5** (Operational perspective)**.** *The $R$-perspective state space is the image*

$$\mathsf{St}_R = \mathrm{Ran}\,\mathsf{P}_R.$$

*Two physical states $\rho, \sigma$ are indistinguishable from perspective $R$ if*

$$\mathsf{P}_R(\rho) = \mathsf{P}_R(\sigma), \qquad \textit{equivalently} \qquad \rho - \sigma \in \mathrm{Ker}\,\mathsf{P}_R.$$

This definition is deliberately austere. It does not assume that $R$ is unentangled from $\bar{R}$, does not assume a sharp coordinate, and does not assume a nonrelativistic group.

## IV. THE UNIQUENESS THEOREM

The following theorem is the core result. It is a finite-dimensional theorem as stated. The normal von Neumann case is obtained by replacing vector spaces with preduals and by requiring normal completely positive maps.

**Theorem IV.1** (Uniqueness and existence of frame transformations)**.** *Let $\mathsf{V}$ be the real vector space of physical trace-class states or signed states on the invariant theory, and let*

$$\mathsf{P}_A : \mathsf{V} \to \mathsf{V}_A, \qquad \mathsf{P}_B : \mathsf{V} \to \mathsf{V}_B$$

*be two perspective channels restricted to $\mathsf{V}$. A deterministic linear transformation*

$$T_{A\to B} : \mathrm{Ran}\,\mathsf{P}_A \to \mathrm{Ran}\,\mathsf{P}_B$$

*satisfying*

$$T_{A\to B}\mathsf{P}_A(\rho) = \mathsf{P}_B(\rho) \qquad \forall \rho \in \mathsf{V} \tag{3}$$

*exists if and only if*

$$\boxed{\mathrm{Ker}\,\mathsf{P}_A \subseteq \mathrm{Ker}\,\mathsf{P}_B.} \tag{4}$$

*When it exists, it is unique on* $\operatorname{Ran}\mathsf{P}_A$ *and is explicitly*

$$\boxed{T_{A\to B}\big(\mathsf{P}_A(\rho)\big) = \mathsf{P}_B(\rho).} \tag{5}$$

*It is reversible on the operational images if and only if*

$$\operatorname{Ker}\mathsf{P}_A = \operatorname{Ker}\mathsf{P}_B.$$

*It admits a completely positive trace-preserving extension to the full output operator space of* $A$ *if and only if the quotient map* (5) *is completely positive on the operator system* $\operatorname{Ran}\mathsf{P}_A$*.*

*Proof.* Assume first that $T_{A\to B}$ exists. If $X \in \operatorname{Ker}\mathsf{P}_A$, then $\mathsf{P}_A(X) = 0$, hence

$$\mathsf{P}_B(X) = T_{A\to B}\mathsf{P}_A(X) = 0.$$

Thus $\operatorname{Ker}\mathsf{P}_A \subseteq \operatorname{Ker}\mathsf{P}_B$.

Conversely, suppose (4) holds. Define $T$ on $\operatorname{Ran}\mathsf{P}_A$ by $T(\mathsf{P}_A(\rho)) = \mathsf{P}_B(\rho)$. This is well-defined: if $\mathsf{P}_A(\rho) = \mathsf{P}_A(\sigma)$, then $\rho - \sigma \in \operatorname{Ker}\mathsf{P}_A \subseteq \operatorname{Ker}\mathsf{P}_B$, so $\mathsf{P}_B(\rho) = \mathsf{P}_B(\sigma)$. Linearity is immediate. Uniqueness follows because every element of $\operatorname{Ran}\mathsf{P}_A$ has the form $\mathsf{P}_A(\rho)$ and (3) fixes its image.

For reversibility, an inverse $T_{B\to A}$ exists exactly when $\operatorname{Ker}\mathsf{P}_B \subseteq \operatorname{Ker}\mathsf{P}_A$ as well. Thus both directions exist exactly when the kernels are equal. Complete positivity is not implied by linear well-definedness. It is precisely the condition that the induced map on the operator system $\operatorname{Ran}\mathsf{P}_A$ be completely positive; Arveson's extension theorem supplies a CP extension when the domain is an operator system, and trace preservation is the dual unitality condition on the span containing the identity. □

**Theorem IV.2** (Complete-kernel quotient theorem)**.** *Let* $\mathsf{P}_A, \mathsf{P}_B$ *be finite-dimensional perspective channels on the trace-class operators of the physical theory. For each matrix level define*

$$\mathcal{K}_R^{(n)} = \operatorname{Ker}(\mathbf{1}_n \otimes \mathsf{P}_R).$$

*The following statements hold.*

*(i) The ordinary affine transformation* $A \to B$ *on unassisted states exists and is unique on* $\operatorname{Ran}\mathsf{P}_A$ *if and only if*

$$\mathcal{K}_A^{(1)} \subseteq \mathcal{K}_B^{(1)}.$$

*(ii) The intrinsically completely positive transformation between the matrix-ordered quotient perspectives exists if and only if the quotient map is well-defined at every matrix level,*

$$\boxed{\mathcal{K}_A^{(n)} \subseteq \mathcal{K}_B^{(n)} \quad \text{for every } n,}$$

*with both quotient cones defined as images, up to Archimedean closure, of the physical positive cones. Equivalently, kernel inclusion gives the unique linear quotient map, and the quotient-cone construction makes that map completely positive. It is determined at every matrix level by*

$$\tau_{A\to B}^{(n)}\big((\mathbf{1}_n \otimes \mathsf{P}_A)(X)\big) = (\mathbf{1}_n \otimes \mathsf{P}_B)(X).$$

*For concrete output matrix algebras with inherited cones, this item must be replaced by the explicit complete-positivity or Choi-interpolation condition of Theorem IV.5 and Corollary IV.6.*

*(iii) The transformation is reversible as a completely positive perspective change if and only if*

$$\mathcal{K}_A^{(n)} = \mathcal{K}_B^{(n)} \qquad \forall n.$$

*(iv) A concrete CPTP channel on a chosen output Hilbert space is an additional representation theorem: it exists exactly when the intrinsic quotient map has a completely positive trace-preserving extension to the ambient matrix algebra.*

*Proof.* The first item is Theorem IV.1. For the matrix levels, suppose ((ii)) holds. Define $\tau^{(n)}_{A\to B}$ by the displayed formula. If two representatives $X, Y$ have the same $A$-image, then $X - Y \in \mathcal{K}^{(n)}_A \subseteq \mathcal{K}^{(n)}_B$, hence they have the same $B$-image; the map is therefore well-defined and unique.

The positive cone of the intrinsic quotient perspective at level $n$ is the image under $\mathbf{1}_n \otimes \mathsf{P}_R$ of the positive cone of the physical matrix-ordered trace-class space, followed by Archimedean closure. If an element is positive in the $A$ quotient, it has positive representatives up to the usual Archimedean approximation; applying the completely positive map $\mathbf{1}_n \otimes \mathsf{P}_B$ gives a positive element in the $B$ quotient. Thus $\tau^{(n)}_{A\to B}$ is positive for all $n$, which is precisely complete positivity of the quotient map. If complete kernels are equal, the same construction in the reverse direction gives a completely positive inverse, hence a complete order isomorphism. Conversely, any completely positive perspective transformation tensored with $\mathbf{1}_n$ sends zero $A$-data to zero $B$-data, so the inclusions of complete kernels are necessary. The final statement is the finite-dimensional Arveson–Choi extension problem for the chosen concrete embedding of the quotient operator system. ☐

**Lemma IV.3** (Algebraic kernels versus complete order)**.** *For finite-dimensional vector spaces,*

$$\mathrm{Ker}(\mathbf{1}_n \otimes \mathsf{P}_R) = M_n(\mathrm{Ker}\,\mathsf{P}_R).$$

*Hence the phrase "complete kernel" is not meant to introduce a new algebraic kernel beyond the ordinary kernel. Its role is to keep track of the matrix quotient cones. Kernel inclusion gives a unique linear factorization; the quotient-cone construction makes the intrinsic factorization completely positive; and a concrete laboratory channel is a further extension problem.*

*Proof.* Choose a basis of $M_n$. An element of $M_n \otimes \mathcal{V}$ belongs to $\mathrm{Ker}(\mathbf{1}_n \otimes \mathsf{P}_R)$ exactly when each coefficient in $\mathcal{V}$ is mapped to zero by $\mathsf{P}_R$. Thus the kernel is $M_n(\mathrm{Ker}\,\mathsf{P}_R)$. Positivity is not encoded by this linear identity; it is encoded by the cones assigned to the quotient at every matrix level. ☐

**Remark IV.4** (Why the complete-kernel condition matters)**.** *The single-level condition* $\mathrm{Ker}\,\mathsf{P}_A \subseteq \mathrm{Ker}\,\mathsf{P}_B$ *is the correct criterion when no inaccessible reference ancilla is allowed. Quantum laboratories, however, are normally tested by entangling the input with an ancilla and asking whether the same operational transformation still exists. Complete positivity is exactly stability under all such ancillary extensions. Thus* ((ii)) *is the non-ideal QRF analogue of complete sufficiency in quantum statistical comparison, while the Choi condition in* Section XVI *is its finite experimental certificate.*

### A. Intrinsic quotient versus concrete implementation

There is a subtle but important distinction. The intrinsic perspective of a non-ideal frame is not, in general, the whole matrix algebra in which an experimentalist chooses to encode its readout. It is the matrix-ordered state quotient, or equivalently the dual Heisenberg operator-system quotient, determined by operational equivalence. A chosen Hilbert-space representation may contain states and observables that are not in the operational image of the frame. Demanding a channel on that larger ambient algebra is therefore an extension problem, not part of the quotient definition.

**Theorem IV.5** (Complete-order implementation theorem)**.** *Let* $\mathsf{Q}_R = \mathcal{T}_{\mathrm{phys}}/\,\mathrm{Ker}\,\mathsf{P}_R$ *denote the intrinsic Schrödinger-picture ordered predual quotient of a frame $R$, equipped with matrix quotient cones and dual to the Heisenberg operator system* $(\mathrm{Ker}\,\mathsf{P}_R)^{\perp}$*. If*

$$\mathrm{Ker}(\mathbf{1}_n \otimes \mathsf{P}_A) \subseteq \mathrm{Ker}(\mathbf{1}_n \otimes \mathsf{P}_B) \qquad \forall n,$$

*then there is a unique completely positive trace-preserving intrinsic transformation*

$$\tau_{A\to B} : \mathsf{Q}_A \to \mathsf{Q}_B$$

*satisfying* $\tau_{A\to B}[\mathsf{P}_A(\rho)] = [\mathsf{P}_B(\rho)]$*. Now choose concrete complete order embeddings*

$$\iota_A : \mathsf{Q}_A \hookrightarrow \mathcal{T}(\mathcal{H}_A), \qquad \iota_B : \mathsf{Q}_B \hookrightarrow \mathcal{T}(\mathcal{H}_B).$$

*A laboratory channel* $\Lambda_{A\to B} : \mathcal{T}(\mathcal{H}_A) \to \mathcal{T}(\mathcal{H}_B)$ *realizing the same perspective change exists if and only if the partially defined map*

$$\iota_B \tau_{A\to B} \iota_A^{-1} : \iota_A(\mathsf{Q}_A) \longrightarrow \iota_B(\mathsf{Q}_B)$$

*has a CPTP extension to all of* $\mathcal{T}(\mathcal{H}_A)$*.*

*Proof.* The first statement is Theorem IV.2. The second is necessity by restriction. For sufficiency, a CPTP extension is exactly a physical channel whose restriction to the embedded quotient agrees with the intrinsic map. In Heisenberg picture this is the usual unital completely positive extension problem for operator systems; in finite dimensions it is equivalent to Choi positivity plus the linear interpolation constraints imposed on the embedded quotient. The theorem therefore identifies the only additional assumption needed to pass from the coordinate-free QRF transformation to a concrete device acting on a selected Hilbert space. ☐

**Corollary IV.6** (Finite Choi interpolation form)**.** *Let $X_j$ be a basis of the concrete operator system $\iota_A(\mathsf{Q}_A)$ and let $Y_j = \iota_B \tau_{A\to B} \iota_A^{-1}(X_j)$. A concrete channel exists iff there is a Choi matrix $J_\Lambda \geq 0$ such that*

$$\mathrm{Tr}_{\mathcal{H}_B} J_\Lambda = \mathbf{1}_{\mathcal{H}_A}, \qquad \mathrm{Tr}_{\mathcal{H}_A}\big[(X_j^T \otimes \mathbf{1}_{\mathcal{H}_B})J_\Lambda\big] = Y_j \quad \forall j.$$

*If this semidefinite feasibility problem is infeasible, every separating hyperplane of the SDP dual is a finite experimental witness against the proposed concrete QRF transformation.*

**Remark IV.7** (Practical consequence)**.** *Many apparent contradictions in the QRF literature come from silently replacing the intrinsic quotient cone by the positive cone of a larger Hilbert space. Kernel inclusion classifies the unique informational transformation. Choi interpolation classifies whether a chosen piece of laboratory hardware can implement it without postselection or hidden non-positive extension.*

**Corollary IV.8** (No hidden exact unitary for generic non-ideal frames)**.** *If $\mathsf{P}_A$ has a nontrivial kernel and $\mathsf{P}_B$ distinguishes at least one mode in that kernel, no deterministic transformation from the $A$-perspective to the $B$-perspective exists. In particular, no unitary QRF transformation can recover relational information erased by a non-ideal frame.*

*Proof.* This is the contrapositive of Theorem IV.1. A unitary transformation would be a special reversible deterministic transformation and hence would require kernel equality, in particular no loss relative to the target perspective. ☐

**Corollary IV.9** (Canonical finite-dimensional representative)**.** *In finite dimensions, when* (4) *holds, the unique linear transformation on the image is represented by*

$$T^{\mathrm{can}}_{A\to B} = \mathsf{P}_B\, \mathsf{P}_A^{+}$$

*on* $\mathrm{Ran}\,\mathsf{P}_A$*, where* $\mathsf{P}_A^{+}$ *is the Moore–Penrose inverse relative to any chosen Hilbert–Schmidt inner product. The representative is independent of the chosen inverse on* $\mathrm{Ran}\,\mathsf{P}_A$*.*

*Proof.* For $y = \mathsf{P}_A(\rho)$, the Moore–Penrose inverse picks a representative $\rho_0 = \mathsf{P}_A^{+}y$ satisfying $\mathsf{P}_A\rho_0 = y$. Since $\rho - \rho_0 \in \mathrm{Ker}\,\mathsf{P}_A \subseteq \mathrm{Ker}\,\mathsf{P}_B$, $\mathsf{P}_B\rho_0 = \mathsf{P}_B\rho$. Hence $\mathsf{P}_B\mathsf{P}_A^{+}y$ is the unique quotient value. ☐

## B. Interpretation

The theorem has a direct physical reading. A perspective is a statistic of the invariant physical state. A change of reference frame is possible exactly when the source statistic is sufficient for the target statistic. The ambiguity in QRF transformations is not an ambiguity of physics; it is an attempt to define a function on representatives when only equivalence classes have operational meaning.

There are three regimes.

**R1. Ideal reversible regime.** $\mathrm{Ker}\,\mathsf{P}_A = \mathrm{Ker}\,\mathsf{P}_B$ on the relevant sector. Standard unitary or isometric QRF transformations live here.

**R2. Non-ideal but deterministic regime.** $\mathrm{Ker}\,\mathsf{P}_A \subsetneq \mathrm{Ker}\,\mathsf{P}_B$. The $B$-perspective is a coarse-graining of the $A$-perspective. There is a unique channel-like transformation from $A$ to $B$ but not back.

**R3. Incompatible information regime.** Neither kernel includes the other. Each frame sees some relational distinctions the other loses. There is no deterministic transformation without extra information, a prior, or a physical interaction with the original system.

This trichotomy is the promised solution to the uniqueness problem.

## V. COMPLETE INVARIANT OPERATOR SYSTEM

A frame-independent statement is one that can be written equivalently in every perspective. The Heisenberg picture makes this exact, but it also exposes a technical point that is usually hidden in ideal QRF models: ranges of unital completely positive maps are operator systems, not necessarily algebras. Hence the mathematically correct universal object for non-ideal frames is an operator system. Algebras appear only after one passes to a multiplicative core or an envelope.

**Definition V.1** (Common invariant operator system). *For a family of frames $\mathcal{F}$, define the common invariant operator system by the annihilator formula*

$$\boxed{\mathcal{I}_{\mathcal{F}} = \bigcap_{R\in\mathcal{F}} (\operatorname{Ker}\mathsf{P}_R)^{\perp} \subseteq \mathfrak{M}^G,} \tag{6}$$

*where $(\operatorname{Ker}\mathsf{P}_R)^{\perp} = \{X \in \mathfrak{M}^G : \operatorname{Tr}[\rho X] = 0\ \forall \rho \in \operatorname{Ker}\mathsf{P}_R\}$. In finite dimensions, and more generally whenever the relevant Heisenberg ranges are closed,*

$$(\operatorname{Ker}\mathsf{P}_R)^{\perp} = \operatorname{Ran}\mathsf{P}_R^{*}, \qquad \mathcal{I}_{\mathcal{F}} = \bigcap_{R\in\mathcal{F}} \operatorname{Ran}\mathsf{P}_R^{*}.$$

*This annihilator definition is the stable one in normal infinite-dimensional settings, where the raw range of a normal completely positive map need not be closed. Each $(\operatorname{Ker}\mathsf{P}_R)^{\perp}$ is a unital self-adjoint subspace, hence $\mathcal{I}_{\mathcal{F}}$ is an operator system. Its $C^*$-envelope is denoted $C^*_{\mathrm{env}}(\mathcal{I}_{\mathcal{F}})$ [45–48, 53].*

**Remark V.2** (Why not an algebra?). *If a perspective channel is a conditional expectation or a $*$-homomorphism, its Heisenberg range may be a von Neumann algebra. A generic POVM/instrument perspective is only completely positive. The product of two pulled-back observables can fail to be the pullback of any observable in the same perspective. Calling (6) an algebra would therefore assert a false closure property. The operator-system formulation keeps all probabilities and all effects, and it is the correct level for non-ideal observers.*

**Theorem V.3** (Completeness of common affine invariants). *Let $\mathcal{F}$ be any set of quantum frames. Two perspective observables $A_R$ and $A_{R'}$ represent the same physical invariant effect if and only if*

$$\mathsf{P}_R^{*}(A_R) = \mathsf{P}_{R'}^{*}(A_{R'}).$$

*The set of all effects and observables representable in every frame is exactly $\mathcal{I}_{\mathcal{F}}$. Consequently every affine scalar invariant accessible in all frames is the expectation value of an element of $\mathcal{I}_{\mathcal{F}}$, and two physical states are indistinguishable by all common-invariant experiments iff their restrictions to $\mathcal{I}_{\mathcal{F}}$ coincide.*

*Proof.* A perspective observable $A_R$ is operationally measured on a physical state $\rho$ with expectation

$$\operatorname{Tr}[\mathsf{P}_R(\rho)A_R] = \operatorname{Tr}[\rho\,\mathsf{P}_R^{*}(A_R)].$$

Thus two observables in two perspectives agree on all physical states exactly when their Heisenberg pullbacks agree. An observable can be represented in every frame exactly when this common pullback annihilates every operationally invisible direction of every frame, i.e. when it lies in every $(\operatorname{Ker}\mathsf{P}_R)^{\perp}$. In closed-range cases this is the same as lying in every range $\operatorname{Ran}\mathsf{P}_R^{*}$. Affine functionals on the state space are represented by observables, giving the final statement. ☐

**Definition V.4** (Product-compatible invariant core). *Let $\mathcal{I}_{\mathcal{F}}$ be the common invariant operator system. Its product-compatible core*

$$\mathcal{A}_{\mathcal{F}} \subseteq \mathcal{I}_{\mathcal{F}}$$

*is the largest self-adjoint unital subspace with the following property: for all $X, Y \in \mathcal{A}_{\mathcal{F}}$ and for every pair of frames $R, R' \in \mathcal{F}$, whenever*

$$X = \mathsf{P}_R^{*}(X_R) = \mathsf{P}_{R'}^{*}(X_{R'}), \qquad Y = \mathsf{P}_R^{*}(Y_R) = \mathsf{P}_{R'}^{*}(Y_{R'}),$$

*the physical pullback of the product is representation-independent,*

$$\mathsf{P}_R^{*}(X_R Y_R) = \mathsf{P}_{R'}^{*}(X_{R'} Y_{R'}),$$

*and this common product again belongs to $\mathcal{A}_{\mathcal{F}}$. Equivalently, after choosing complete-order retractions onto the common quotient, $\mathcal{A}_{\mathcal{F}}$ is the common multiplicative domain on which all induced products coincide. Outside $\mathcal{A}_{\mathcal{F}}$ the invariant object is only an operator system; multiplying two common effects is then not an invariantly defined operation.*

**Proposition V.5** (Algebraic invariants and the $C^*$-envelope)**.** *The operator system $\mathcal{I}_{\mathcal{F}}$ has a canonical abstract $C^*$-envelope $C^*_{\mathrm{env}}(\mathcal{I}_{\mathcal{F}})$. This envelope is a representation-independent completion of the order structure, but it should not be confused with a physical product of arbitrary perspective effects. The latter is invariantly meaningful exactly on the product-compatible core $\mathcal{A}_{\mathcal{F}}$. If every $\mathsf{P}_R^*$ is a complete order embedding on $\mathcal{I}_{\mathcal{F}}$ and the products of representatives are compatible, then $\mathcal{A}_{\mathcal{F}}$ is a concrete $C^*$-subalgebra of $\mathfrak{M}^G$ and coincides with the product-closed part of $\mathcal{I}_{\mathcal{F}}$.*

*Proof.* The $C^*$-envelope is the minimal $C^*$-algebra generated by a completely isometric copy of an operator system and is unique up to $*$-isomorphism. It therefore canonically records the noncommutative boundary of the common order data. However, the envelope product is an abstract completion product. It becomes the physical product of perspective observables only when multiplication of representatives has the same Heisenberg pullback in every frame. This is precisely the defining condition of $\mathcal{A}_{\mathcal{F}}$; under complete-order embeddings and product compatibility, closure under multiplication, adjoint and the $C^*$ identity are inherited from $\mathfrak{M}^G$. □

**Corollary V.6** (Invariant state data)**.** *The complete affine state invariant shared by all frames in $\mathcal{F}$ is the equivalence class*

$$\rho \sim_{\mathcal{F}} \sigma \quad \Longleftrightarrow \quad \mathrm{Tr}[\rho X] = \mathrm{Tr}[\sigma X] \quad \forall X \in \mathcal{I}_{\mathcal{F}}.$$

*In finite dimensions this is the restriction of $\rho$ to the ordered vector space $\mathcal{I}_{\mathcal{F}}$. Spectral invariants are well-defined on the algebraic core $\mathcal{A}_{\mathcal{F}}$ or on $C^*_{\mathrm{env}}(\mathcal{I}_{\mathcal{F}})$, not on an arbitrary operator-system element without a chosen product.*

**Theorem V.7** (Maximal common reversible quotient)**.** *Let $\mathcal{S}_*$ denote the finite-dimensional physical ordered state space, or the normal predual in the von Neumann setting with norm-closed predual kernels and weak-$*$-closed annihilators in the observable algebra. For a family of perspective maps $\{\mathsf{P}_R : \mathcal{S}_* \to \mathcal{S}_{R,*}\}_{R\in\mathcal{F}}$, define*

$$K_{\mathcal{F}} = \overline{\sum_{R\in\mathcal{F}} \mathrm{Ker}\,\mathsf{P}_R}, \qquad \mathcal{Q}_{\mathcal{F}} = \mathcal{S}_*/K_{\mathcal{F}}. \tag{7}$$

*Then $\mathcal{Q}_{\mathcal{F}}$ is the maximal state-space quotient that can be reconstructed from every perspective in $\mathcal{F}$. More precisely, if $L : \mathcal{S}_* \to V$ is any affine-linear statistic for which $L = L_R\mathsf{P}_R$ for every $R$, then $L$ factors uniquely through $\mathcal{Q}_{\mathcal{F}}$. In the Heisenberg picture,*

$$\mathcal{Q}_{\mathcal{F}}^* \simeq \bigcap_{R\in\mathcal{F}} (\mathrm{Ker}\,\mathsf{P}_R)^{\perp} = \mathcal{I}_{\mathcal{F}}, \tag{8}$$

*with $\bigcap_R \mathrm{Ran}\,\mathsf{P}_R^*$ substituted for the right-hand side whenever the Heisenberg ranges are closed. The identification is an isomorphism of ordered vector spaces, and at matrix levels of operator systems whenever complete kernels are used. Consequently all perspectives in $\mathcal{F}$ are exactly mutually reversible on their full images if and only if their complete kernels coincide. If the complete kernels do not coincide, $\mathcal{Q}_{\mathcal{F}}$ is the largest exact observer-independent reversible core.*

*Proof.* A statistic $L$ is reconstructible from perspective $R$ exactly when it is constant on the equivalence classes of $\mathsf{P}_R$, i.e. when $\mathrm{Ker}\,\mathsf{P}_R \subseteq \mathrm{Ker}\,L$. Reconstructibility from every perspective is therefore equivalent to

$$\sum_{R\in\mathcal{F}} \mathrm{Ker}\,\mathsf{P}_R \subseteq \mathrm{Ker}\,L,$$

with closure in the normal infinite-dimensional case. This is precisely the universal property of the quotient $\mathcal{S}_*/K_{\mathcal{F}}$. Dualizing gives

$$(\mathcal{S}_*/K_{\mathcal{F}})^* = K_{\mathcal{F}}^{\perp} = \bigcap_{R\in\mathcal{F}} (\mathrm{Ker}\,\mathsf{P}_R)^{\perp}.$$

For finite-dimensional surjective maps, $(\mathrm{Ker}\,\mathsf{P}_R)^{\perp} = \mathrm{Ran}\,\mathsf{P}_R^*$; the same statement holds for normal maps after taking weak-$*$ closures of Heisenberg ranges. The matrix-ordered statement follows by applying the same argument to $M_n(\mathcal{S}_*)$ and to $\mathbf{1}_n \otimes \mathsf{P}_R$. Finally, exact transformations $R \to R'$ and $R' \to R$ imply the two complete-kernel inclusions in opposite directions, hence equality. Equality of complete kernels gives the inverse complete-order quotient identifications on the images. □

**Corollary V.8** (Finite-Abelian reversible core)**.** *For finite Abelian convolution frames, let*

$$S_R = \{\chi \in \widehat{G} : \widehat{\mu}_R(\chi) \neq 0\}$$

*be the surviving Fourier support of frame $R$. The exact common reversible core is the span of the modes in $\bigcap_R S_R$. Thus*

$$\dim \mathcal{Q}_{\mathcal{F}} = \sum_{\chi \in \cap_R S_R} m_\chi,$$

*where $m_\chi$ is the multiplicity of the charge-difference block. A directed physical transformation $A \to B$ requires $S_B \subseteq S_A$ and the Bochner positivity condition of Theorem VI.2; bidirectional physical reversibility requires equality of supports and deterministic phases on the active sector.*

### A. Universal invariants

Several familiar invariants appear as special cases.

(a) **Casimirs and superselection labels.** For a group representation, central operators in the commutant of $U(G)$ belong to $\mathfrak{M}^G$ and survive every perspective that does not erase the corresponding sector. For Poincaré symmetry these include $P^\mu P_\mu$ and the Pauli–Lubanski invariant $W^\mu W_\mu$ on one-particle sectors.

(b) **Relational probabilities.** The probability that system $S$ has property $O$ when frame $R$ reads $x$ is the expectation of $\mathsf{P}_R^*(O)$ and is invariant by construction.

(c) **Algebraic spectra.** Spectra of states on a common finite-dimensional invariant algebra are frame-independent; for genuinely noisy perspectives the correct replacement is the ordered-state restriction to $\mathcal{I}_{\mathcal{F}}$ plus spectra on $\mathcal{A}_{\mathcal{F}}$.

(d) **Diagonal charge entropies.** When frames are associated with a group decomposition into charge sectors, functions of the diagonal charge distribution are invariant under frame changes that preserve the charge-sector center.

(e) **Scattering probabilities.** S-matrix elements between invariant in/out relational states are invariant because the scattering operator commutes with the symmetry and is represented in the physical invariant theory.

Frame-dependent quantities are equally characterized: an observable or entanglement measure is frame-dependent exactly when it is not a function of the restriction to $\mathcal{I}_{\mathcal{F}}$ and its product-compatible core.

## VI. COMPACT ABELIAN AND FINITE-GROUP SOLUTION

The abstract kernel theorem becomes explicit for compact Abelian groups. This section gives a complete classification of non-ideal transformations in terms of Fourier modes. It covers phase references, finite clocks with periodic time, orientation references for finite Abelian subgroups, and discrete translational frames.

Let $G$ be compact Abelian with Pontryagin dual $\widehat{G}$. Let $\mu_R$ be the response probability measure of frame $R$: when the true relational group element is $g$, the frame reports $x$ with distribution $\mu_R(xg^{-1})$. The corresponding perspective channel on the ideal relational state is a group-covariant noise channel

$$\mathsf{P}_R(\rho) = \int_G U(g)\rho U(g)^\dagger \, \mu_R(\mathrm{d}g). \tag{9}$$

Let $\rho_\chi$ denote the component of $\rho$ transforming according to character $\chi \in \widehat{G}$:

$$\mathrm{Ad}_{U(g)}(\rho_\chi) = \chi(g)\rho_\chi.$$

The Fourier transform of $\mu_R$ is

$$\widehat{\mu}_R(\chi) = \int_G \overline{\chi(g)} \, \mu_R(\mathrm{d}g).$$

**Lemma VI.1** (Mode damping). *For every charge mode $\chi$,*

$$\mathsf{P}_R(\rho_\chi) = \widehat{\mu}_R(\chi)\rho_\chi.$$

*Thus* $\mathrm{Ker}\,\mathsf{P}_R$ *is the span of all modes with $\widehat{\mu}_R(\chi) = 0$ plus any intrinsic physical kernel unrelated to the frame.*

*Proof.* Using (9),

$$\begin{aligned}\mathsf{P}_R(\rho_\chi) &= \int_G \mathrm{Ad}_{U(g)}(\rho_\chi)\,\mu_R(\mathrm{d}g)\\ &= \left(\int_G \chi(g)\,\mu_R(\mathrm{d}g)\right)\rho_\chi.\end{aligned}$$

The complex conjugation convention only changes whether $\chi$ or $\overline{\chi}$ appears. Vanishing Fourier coefficients are exactly lost modes. ☐

**Theorem VI.2** (Fourier classification of imperfect frame changes)**.** *Let $A$ and $B$ be compact Abelian non-ideal frames with response measures $\mu_A, \mu_B$. A covariant linear transformation $T_{A\to B}$ satisfying $T_{A\to B}\mathsf{P}_A = \mathsf{P}_B$ exists if and only if*

$$\widehat{\mu}_A(\chi) = 0 \quad\Longrightarrow\quad \widehat{\mu}_B(\chi) = 0 \qquad \forall \chi \in \widehat{G}. \tag{10}$$

*On active modes it is unique and given by*

$$T_{A\to B}(\rho_\chi) = q_\chi\,\rho_\chi, \qquad q_\chi = \frac{\widehat{\mu}_B(\chi)}{\widehat{\mu}_A(\chi)}. \tag{11}$$

*It is completely positive and trace preserving as a covariant noise channel if and only if $q_\chi$ is positive definite, equivalently if there exists a probability measure $\nu$ on $G$ such that*

$$q_\chi = \widehat{\nu}(\chi), \qquad \mu_B = \nu * \mu_A.$$

*It is reversibly CPTP in both directions only when both response measures differ by a deterministic group translation on the active sector; in particular, exact bidirectional recovery of a genuinely smeared frame is impossible.*

*Proof.* The existence criterion is Theorem IV.1 written in the diagonal Fourier basis. The unique map must multiply the $A$-output mode $\widehat{\mu}_A(\chi)\rho_\chi$ by $\widehat{\mu}_B(\chi)/\widehat{\mu}_A(\chi)$, giving (11). A covariant channel diagonal in the character basis is a random unitary convolution channel exactly when its multiplier sequence is the Fourier transform of a probability measure; this is Bochner's theorem for compact Abelian groups. Such a measure $\nu$ exists precisely when $\mu_B = \nu * \mu_A$. If both directions are CPTP, then $\mu_B = \nu * \mu_A$ and $\mu_A = \eta * \mu_B$, so $\mu_A = (\eta * \nu) * \mu_A$. On all active modes,

$$\widehat{\eta}(\chi)\widehat{\nu}(\chi) = 1.$$

Fourier transforms of probability measures have modulus at most one. Equality on active modes requires deterministic phases; hence the measures are translations on the active sector. Genuine smearing has some multiplier of modulus strictly less than one and cannot be inverted by a probability convolution. ☐

**Theorem VI.3** (Finite Abelian mode-support lattice)**.** *Fix a finite Abelian group and restrict attention to a finite active Fourier set $X \subseteq \widehat{G}$. For a frame $R$, define its surviving mode support*

$$S_R = \{\chi \in X : \widehat{\mu}_R(\chi) \neq 0\}, \qquad Z_R = X \setminus S_R.$$

*Then:*

*(a) the intrinsic deterministic transformation $A \to B$ exists iff $S_B \subseteq S_A$, equivalently $Z_A \subseteq Z_B$;*

*(b) $A$ and $B$ are intrinsically reversible iff $S_A = S_B$;*

*(c) for a family $\mathcal{F}$, the largest Fourier sector reconstructible from every member has support*

$$S^{\mathrm{core}}_{\mathcal{F}} = \bigcap_{R\in\mathcal{F}} S_R;$$

*(d) after quotienting frames by reversible equivalence, the assignment $R \mapsto S_R$ embeds the intrinsic information order into the inclusion order of subsets of $X$. Intersections give intrinsic common cores. Unions give least upper bounds precisely when the selected physical class contains a frame whose response has the corresponding zero set.*

*Proof.* The Fourier criterion in Theorem VI.2 says $A \to B$ iff every mode killed by $A$ is also killed by $B$. On $X$ this is $Z_A \subseteq Z_B$, equivalently $S_B \subseteq S_A$. Reversibility requires the criterion in both directions and hence equality of supports. A mode is reconstructible from every frame in $\mathcal{F}$ iff it survives in each $S_R$, giving the intersection formula. The final statement is the same order relation after identifying frames with equal surviving supports. The qualification on joins is necessary: positivity of the underlying response measure can forbid an arbitrary prescribed Fourier zero set. ☐

**Proposition VI.4** (Finite convolution certificate and dual witness)**.** *Let $G$ be finite Abelian and let $p_A, p_B \in \mathrm{Prob}(G)$ be classical response distributions. The optimal covariant token-level degradation error is*

$$\Delta_{\mathrm{conv}}(B|A) = \frac{1}{2} \min_{r \in \mathrm{Prob}(G)} \|p_B - p_A * r\|_1.$$

*Writing $C_A r = p_A * r$, it has the dual form*

$$2\Delta_{\mathrm{conv}}(B|A) = \max_{\|y\|_\infty \le 1} \left[ \langle y, p_B \rangle - \max_{g \in G} \langle y, \tau_g p_A \rangle \right],$$

*where $(\tau_g p_A)(h) = p_A(hg^{-1})$. Thus any vector $y$ satisfying $\|y\|_\infty \le 1$ gives a rigorous lower bound, and an optimal $y$ is a finite experimental no-go certificate for covariant post-processing $A \to B$.*

*Proof.* The set of covariant degradations of $A$ is the convex hull of the translates of $p_A$:

$$K_A = \{C_A r : r \in \mathrm{Prob}(G)\}.$$

Hence $2\Delta_{\mathrm{conv}}(B|A)$ is the $\ell_1$-distance from $p_B$ to $K_A$. The standard dual representation of distance from a point to a compact convex set gives

$$\mathrm{dist}_{\ell_1}(p_B, K_A) = \max_{\|y\|_\infty \le 1} \left( \langle y, p_B \rangle - \sup_{x \in K_A} \langle y, x \rangle \right).$$

Since the support function of $K_A$ is

$$\sup_{r \in \mathrm{Prob}(G)} \langle y, C_A r \rangle = \max_{g \in G} \langle y, \tau_g p_A \rangle,$$

the stated formula follows. Strong duality holds because the feasible simplex is compact and the finite-dimensional linear program is feasible. When the optimum is positive, the displayed functional separates $p_B$ from every covariant degradation of $A$.

☐

**Theorem VI.5** (Robust dual witness under calibration uncertainty)**.** *Let $G$ be finite Abelian, let $p_A, p_B \in \mathrm{Prob}(G)$ be the true token responses, and let $\widehat{p}_A, \widehat{p}_B$ be calibrated estimates satisfying*

$$\|\widehat{p}_A - p_A\|_1 \le \varepsilon_A, \qquad \|\widehat{p}_B - p_B\|_1 \le \varepsilon_B.$$

*For any vector $y \in \mathbb{R}^G$ with $\|y\|_\infty \le 1$, define the empirical dual margin*

$$m_y(\widehat{p}_B | \widehat{p}_A) = \langle y, \widehat{p}_B \rangle - \max_{g \in G} \langle y, \tau_g \widehat{p}_A \rangle.$$

*Then the true covariant degradation error obeys*

$$\Delta_{\mathrm{conv}}(B|A) \ge \frac{1}{2} \big[ m_y(\widehat{p}_B | \widehat{p}_A) - \varepsilon_A - \varepsilon_B \big]. \tag{12}$$

*Thus a single published vector $y$, together with confidence radii for the two response histograms, is a finite-data certificate excluding the covariant simulation $A \to B$ whenever the right-hand side is positive.*

*Proof.* By the dual formula of Proposition VI.4,

$$2\Delta_{\mathrm{conv}}(B|A) \ge \langle y, p_B \rangle - \max_{g \in G} \langle y, \tau_g p_A \rangle.$$

The first term changes by at most $\varepsilon_B$. For the support term,

$$\begin{aligned} \left| \max_g \langle y, \tau_g p_A \rangle - \max_g \langle y, \tau_g \widehat{p}_A \rangle \right| &\le \max_g |\langle y, \tau_g (p_A - \widehat{p}_A) \rangle| \\ &\le \|y\|_\infty \, \|p_A - \widehat{p}_A\|_1 \le \varepsilon_A. \end{aligned}$$

Substituting these two Lipschitz estimates gives (12). ☐

**Proposition VI.6** (Fourier sharpness monotones)**.** *Let $G$ be finite Abelian and suppose $p_B = p_A * r$ for some $r \in \mathrm{Prob}(G)$. Then for every character $\chi \in \widehat{G}$,*

$$|\widehat{p}_B(\chi)| \le |\widehat{p}_A(\chi)|.$$

*Consequently, for every non-negative weight family $w_\chi \ge 0$ supported away from the trivial character,*

$$M_w(p) := \sum_{\chi \ne \mathbf{1}} w_\chi |\widehat{p}(\chi)|^2$$

*is a monotone under covariant degradation:*

$$M_w(p_B) \le M_w(p_A).$$

*Violation of any of these inequalities is an immediate necessary no-go test. Satisfaction of all of them is not sufficient for degradability; the complete finite-Abelian certificate remains the convolution LP and its dual witness.*

*Proof.* Convolution gives $\widehat{p}_B(\chi) = \widehat{p}_A(\chi)\widehat{r}(\chi)$. Since $r$ is a probability distribution,

$$|\widehat{r}(\chi)| = \left| \sum_{g \in G} r(g)\overline{\chi(g)} \right| \le \sum_{g \in G} r(g) = 1.$$

The pointwise inequality follows, and summation with non-negative weights gives the monotonicity of $M_w$. Non-sufficiency is the usual distinction between coordinatewise modulus constraints and positive definiteness of the full Fourier ratio. □

**Proposition VI.7** (Common degraded finite-Abelian observer)**.** *Let $p_1, \ldots, p_m \in \mathrm{Prob}(G)$ be finite-Abelian response distributions. A token response $q \in \mathrm{Prob}(G)$ is an exact common descendant of all $p_i$ iff*

$$q = p_i * r_i, \qquad r_i \in \mathrm{Prob}(G), \qquad i = 1, \ldots, m.$$

*Equivalently,*

$$q \in \bigcap_{i=1}^{m} K_{p_i}, \qquad K_{p_i} := \{p_i * r : r \in \mathrm{Prob}(G)\}.$$

*For any linear sharpness score $w \in \mathbb{R}^G$, the sharpest common descendant for that score is obtained by the finite linear program*

$$\begin{aligned} \text{maximize} \quad & \langle w, q \rangle \\ \text{subject to} \quad & q = C(p_i) r_i, \quad r_i \in \mathrm{Prob}(G), \quad i = 1, \ldots, m, \\ & q \in \mathrm{Prob}(G). \end{aligned}$$

*If the optimum is $q_*$, then every frame $p_i$ physically simulates $q_*$ by the reported kernel $r_i$. Conversely, infeasibility of any additional linear specification on $q$ has a dual separating witness. Thus finite-Abelian observer-web consensus is not a convention: it is a reproducible convex feasibility problem.*

*Proof.* The first equivalence is the definition of concrete covariant degradation for finite-Abelian token frames. The set $K_{p_i}$ is the image of a simplex under the linear convolution map $C(p_i)$, hence is a compact convex polytope. Membership in all $K_{p_i}$ is exactly the existence of probability kernels $r_i$ with the displayed equations. Adding a linear objective and the simplex constraints gives a finite linear program. Linear-programming duality supplies separating hyperplanes for infeasible affine specifications. □

**Theorem VI.8** (Finite-sample certificate for Abelian token frames)**.** *Let $G$ be finite with $d = |G|$. Let $\widehat{p}_A, \widehat{p}_B$ be empirical response distributions obtained from independent calibrations of $p_A, p_B$, and define*

$$\widehat{\Delta}_{\mathrm{conv}}(B|A) = \frac{1}{2} \min_{r \in \mathrm{Prob}(G)} \|\widehat{p}_B - \widehat{p}_A * r\|_1.$$

*If*

$$\|\widehat{p}_A - p_A\|_1 \le \varepsilon_A, \qquad \|\widehat{p}_B - p_B\|_1 \le \varepsilon_B,$$

*then*

$$\Delta_{\mathrm{conv}}(B|A) \ge \widehat{\Delta}_{\mathrm{conv}}(B|A) - \frac{\varepsilon_A + \varepsilon_B}{2}. \tag{13}$$

*Moreover, if $m_A, m_B$ independent token samples are used, then with probability at least $1-\alpha$ the choice*

$$\varepsilon_R^{\mathrm{W}} = \sqrt{\frac{2}{m_R}\log\left(\frac{2(2^d-2)}{\alpha}\right)}, \qquad R \in \{A, B\}, \tag{14}$$

*is admissible in (13). This uses the sharp finite-alphabet $\ell_1$ concentration inequality of Weissman–Ordentlich–Seroussi–Verdú–Weinberger [61]. A fully elementary, but usually looser, admissible alternative is the coordinatewise Hoeffding radius*

$$\varepsilon_R^{\mathrm{H}} = d\sqrt{\frac{\log(4d/\alpha)}{2m_R}}.$$

*Thus a positive right-hand side in (13) is a finite-data no-go certificate for the covariant degradation $A \to B$.*

*Proof.* For any $r \in \mathrm{Prob}(G)$, convolution by $r$ is an $\ell_1$-contraction. Hence

$$\begin{aligned} \|p_B - p_A * r\|_1 &\ge \|\widehat{p}_B - \widehat{p}_A * r\|_1 \\ &\quad - \|p_B - \widehat{p}_B\|_1 - \|(p_A - \widehat{p}_A) * r\|_1 \\ &\ge \|\widehat{p}_B - \widehat{p}_A * r\|_1 - \varepsilon_B - \varepsilon_A. \end{aligned}$$

Taking the minimum over $r$ and multiplying by $1/2$ gives (13). For the sampling statement, the Weissman inequality gives

$$\Pr\{\|\widehat{p}_R - p_R\|_1 > \varepsilon\} \le (2^d - 2)\exp(-m_R\varepsilon^2/2).$$

Using failure probability $\alpha/2$ for each of $R = A, B$ yields (14) and a union bound gives joint coverage $1-\alpha$. The Hoeffding formula follows by bounding each coordinate and summing the coordinate errors. □

**Proposition VI.9** (Robust Fourier support from calibration data)**.** *Let $G$ be finite Abelian, let $d = |G|$, and let $\widehat{p}$ be an empirical response distribution for an unknown token response $p$. If*

$$\|\widehat{p} - p\|_1 \le \varepsilon,$$

*then every character $\chi \in \widehat{G}$ obeys*

$$\left|\widehat{\widehat{p}}(\chi) - \widehat{p}_{\mathrm{true}}(\chi)\right| \le \varepsilon, \tag{15}$$

*where the left hat denotes the Fourier coefficient of the empirical distribution and $\widehat{p}_{\mathrm{true}}$ the coefficient of $p$. Consequently*

$$|\widehat{\widehat{p}}(\chi)| > \varepsilon \quad \Longrightarrow \quad \widehat{p}_{\mathrm{true}}(\chi) \neq 0.$$

*Modes with $|\widehat{\widehat{p}}(\chi)| \le \varepsilon$ are not certified zero by finite data alone; they are statistically ambiguous unless an analytic response model, a symmetry constraint, or an independent structural calibration is supplied. Exact mode-support inclusions in Theorem VI.3 are therefore mathematical or model-calibrated statements, whereas finite-data impossibility claims should be reported through the total-variation LP certificate of Proposition VI.4 and Theorem VI.8.*

*Proof.* Since $|\chi(g)| = 1$,

$$\begin{aligned} \left|\widehat{\widehat{p}}(\chi) - \widehat{p}_{\mathrm{true}}(\chi)\right| &= \left|\sum_{g\in G} \overline{\chi(g)}\big(\widehat{p}(g) - p(g)\big)\right| \\ &\le \sum_{g \in G} |\widehat{p}(g) - p(g)| = \|\widehat{p} - p\|_1. \end{aligned}$$

The nonzero certificate follows from the reverse triangle inequality. If the empirical magnitude is at most $\varepsilon$, the confidence interval for the true Fourier coefficient contains the origin, so the calibration data do not exclude a true zero. The final statement is the operational prescription forced by this interval: exact kernel supports require model-level information; finite-data concrete no-go statements are made by the stable convex separation bound. □

The same data also yield a witness-level bound. For every dual vector $y$ with $\|y\|_\infty \le 1$,

$$2\Delta_{\text{conv}}(B|A) \ge \langle y, \widehat{p}_B\rangle - \max_{g\in G}\langle y, \tau_g \widehat{p}_A\rangle \\ - \varepsilon_B - \varepsilon_A.$$

Thus a published finite-Abelian QRF comparison can be made fully reproducible by reporting the empirical histograms, the primal degradation kernel, the dual vector and the confidence radii.

### A. Phase reference and finite clock

For $G = U(1)$, write $n \in \mathbb{Z}$ for the charge-difference mode. A phase or clock frame with response density $\mu_R(\theta)$ damps coherences by

$$\rho_{mn} \longmapsto \widehat{\mu}_R(m-n)\rho_{mn}.$$

If a finite clock has bandwidth $|n| \le N$, then all modes outside the accessible band are absent. A transformation from clock $A$ to clock $B$ exists only if $B$ has no temporal Fourier component that $A$ lacks. A sharp ideal clock has $\widehat{\mu}(n) = 1$ for all $n$; a Gaussian-like phase packet has approximately $\widehat{\mu}(n) = e^{-\sigma^2 n^2/2}$; a completely phase-free token has $\widehat{\mu}(n) = 0$ for $n \neq 0$ and destroys all off-diagonal charge coherences.

This single formula explains why finite temporal reference frames can yield non-unitary relational dynamics. The clock does not merely re-label a unitary history; it applies a frequency filter. Only on the surviving band is there a unique transformation to another clock.

### B. Finite group example

Let $G = \mathbb{Z}_N$ and let $\omega = e^{2\pi i/N}$. A response distribution $p_R(k)$ has Fourier coefficients

$$\widehat{p}_R(\ell) = \sum_{k=0}^{N-1} p_R(k)\omega^{-k\ell}.$$

The frame erases precisely the relational modes $\ell$ with $\widehat{p}_R(\ell) = 0$. If

$$p_A = (1-\epsilon)\delta_0 + \epsilon\frac{1}{N}\mathbf{1},$$

then $\widehat{p}_A(0) = 1$ and $\widehat{p}_A(\ell) = 1-\epsilon$ for $\ell \neq 0$. The frame is invertible as a linear map for $\epsilon < 1$ but its inverse multiplies nonzero modes by $(1-\epsilon)^{-1}$ and is not completely positive for $\epsilon > 0$. Hence mathematical inversion exists, but physical deterministic inversion does not. This distinction is essential for non-ideal QRFs.

## VII. MEASUREMENT UPDATE AND INSTRUMENTS

A POVM determines probabilities but not post-measurement states. If the frame is itself measured, a full instrument $\mathcal{J}_R(\mathrm{d}x)$ should replace $E_R(\mathrm{d}x)^{1/2}\rho E_R(\mathrm{d}x)^{1/2}$. The perspective channel becomes

$$\mathsf{P}_R^{\mathcal{J}}(\rho) = \int_{X_R} U_{\bar{R}}(s_R(x))^\dagger \; \mathrm{Tr}_R[\mathcal{J}_R(\mathrm{d}x)(\rho)] \, U_{\bar{R}}(s_R(x)).$$

All theorems above remain true with $\mathsf{P}_R$ replaced by $\mathsf{P}_R^{\mathcal{J}}$. The kernel condition then includes both finite resolution and measurement disturbance. This is the correct language for Wigner-friend-like frame changes and for observers whose records are physical memory systems.

The distinction also resolves an apparent ambiguity in “collapse relative to a QRF”. Projection or update is not a primitive frame transformation; it is part of the chosen instrument. Different instruments with the same POVM can define different post-measurement perspectives, but their probability invariants are the same. The uniqueness theorem applies after the operational instrument has been specified.

## VIII. DYNAMICAL QRFS: PROCESS TENSORS AND COMB QUOTIENTS

A reference frame is rarely used only once. A clock is read repeatedly, a laboratory frame defines a sequence of interventions, and an accelerated or gravitational observer assigns probabilities to temporally ordered events. The state-level theorem must therefore lift to dynamical objects. The correct finite-dimensional object is an $m$-step process tensor, equivalently a quantum comb [54–56].

Let $\mathsf{Comb}_m$ denote the cone of positive Choi operators $W$ on the input-output Hilbert spaces of $m$ interventions satisfying the usual deterministic comb trace constraints

$$\mathrm{Tr}_{O_m} W^{(m)} = I_{I_m} \otimes W^{(m-1)}, \quad \dots, \quad \mathrm{Tr}_{O_1} W^{(1)} = I_{I_1}.$$

Let $\mathcal{L}_m = \operatorname{span} \mathsf{Comb}_m$ be the associated ordered operator system. A dynamical perspective of a frame $R$ is a completely positive linear map

$$\mathbb{P}_R^{(m)} : \mathcal{L}_m \longrightarrow \mathcal{L}_{m,R}$$

which maps deterministic combs to deterministic relative combs and maps subnormalized combs to subnormalized relative combs. Memory in the frame is allowed: $\mathbb{P}_R^{(m)}$ need not factorize over time.

**Theorem VIII.1** (Dynamical complete-order quotient theorem)**.** *For two $m$-step dynamical perspectives $A$ and $B$, the formula*

$$\mathbb{T}^{(m)}_{A\to B}\big(\mathbb{P}_A^{(m)} W\big) = \mathbb{P}_B^{(m)} W$$

*defines a unique affine transformation on deterministic relative combs iff*

$$\operatorname{Ker} \mathbb{P}_A^{(m)} \subseteq \operatorname{Ker} \mathbb{P}_B^{(m)} \quad \textit{on } \mathcal{L}_m.$$

*It defines a physical deterministic superchannel on all ancilla-extended process experiments iff the quotient is completely positive at every matrix level,*

$$\operatorname{Ker}(\mathbf{1}_n \otimes \mathbb{P}_A^{(m)}) \subseteq \operatorname{Ker}(\mathbf{1}_n \otimes \mathbb{P}_B^{(m)}) \quad \forall n,$$

*and the induced map preserves the comb normalization hyperplanes. In finite dimensions this is equivalent to positivity of the Choi operator of the quotient together with the linear trace constraints for deterministic combs.*

*Proof.* The first assertion is the first isomorphism theorem for the linear map $\mathbb{P}_A^{(m)}$ restricted to the comb span. If two neutral combs have the same $A$-perspective, their difference lies in $\operatorname{Ker} \mathbb{P}_A^{(m)}$; (VIII.1) is exactly the condition that they also have the same $B$-perspective. Uniqueness follows because the image of $\mathbb{P}_A^{(m)}$ is generated by elements of the form $\mathbb{P}_A^{(m)} W$.

Physical process transformations are deterministic superchannels. In the Choi representation these are precisely completely positive maps between the relevant matrix-ordered operator systems satisfying the affine causality constraints. Complete positivity must hold after tensoring the whole experiment with an arbitrary inaccessible ancilla, which gives (VIII.1). Conversely, a completely positive quotient preserving the trace hyperplanes maps deterministic combs to deterministic combs and subnormalized combs to subnormalized combs, hence is a valid superchannel on the process perspectives. Finite-dimensional complete positivity and causality preservation are Choi positivity plus linear partial-trace equations. □

**Corollary VIII.2** (Dynamical Blackwell monotonicity)**.** *If $A \succeq B$ at the process level, then every finite adaptive discrimination, control, estimation or memory-witnessing task performed in the $B$-perspective is simulable from the $A$-perspective with the same payoff. If no such superchannel exists, a finite comb decision problem separates the two perspectives.*

*Proof.* An adaptive decision problem is a tester, i.e. a positive dual comb paired linearly with the process tensor. Composition with the deterministic superchannel from $A$ to $B$ pulls every $B$-tester back to an $A$-tester with identical pairing. Separation in the converse direction follows from finite-dimensional convex separation of the image of all $A$-simulable $B$-processes from the target $B$-process. □

**Proposition VIII.3** (Repeated finite-Abelian clock blur)**.** *Let $G$ be finite Abelian and suppose that the one-use perspectives of two clocks satisfy*

$$\mathsf{P}_B = \mathcal{D}_\nu \mathsf{P}_A$$

*with $\nu$ a probability distribution on $G$. If the same independent frame blur acts at each of $m$ interventions, then the dynamical quotient is the $m$-fold convolution*

$$\mathbb{T}^{(m)}_{A\to B} = \mathcal{D}_\nu^{\otimes m}, \qquad q^{(m)}_\chi = \widehat{\nu}(\chi)^m$$

*on the temporal Fourier mode $\chi$ of an $m$-step coherence. Any formal inverse on an active mode has multiplier $\widehat{\nu}(\chi)^{-m}$; hence inverse frame changes are exponentially ill-conditioned whenever $|\widehat{\nu}(\chi)| < 1$.*

*Proof.* Independent blur composes multiplicatively in Fourier space. A one-use quotient multiplies mode $\chi$ by $q_\chi = \widehat{\nu}(\chi)$. An $m$-use coherence carrying the same temporal charge through all uses receives the product $q_\chi^m$. The group-domain kernel corresponding to this product is $\nu^{*m}$. The inverse multiplier is $q_\chi^{-m}$ on active modes, so its norm grows exponentially if the blur is nontrivial. ☐

The dynamical theorem is important for relativistic observers. A Lorentz or gravitational frame transformation that works for isolated density matrices but fails on a sequence of localized operations is not a physical change of observer. Causality, memory and local instrument structure must survive at the comb level.

## IX. PROTECTED RELATIONAL SECTORS AND ERROR-CORRECTED QRFS

The complete-order quotient theorem says when one imperfect perspective contains enough information to simulate another. It also shows how an imperfect frame can be made effectively ideal on a deliberately chosen sector. This is not an additional postulate: it is exactly quantum error correction applied to the perspective channel [58, 60], and it is compatible with the emerging dictionary between QRF gauge descriptions and quantum error-correcting encodings [39].

Let

$$\mathsf{P}_R(\rho) = \sum_\mu E_\mu \rho E_\mu^\dagger$$

be a finite-dimensional perspective channel and let $\Pi_\mathcal{C}$ project onto a physical code sector $\mathcal{C} \subset \mathcal{H}_{\text{phys}}$. The intended ideal relational description on the code is the identity channel on states supported in $\mathcal{C}$, or more generally a faithful representation of the relational algebra on $\mathcal{C}$.

**Theorem IX.1** (Error-corrected quantum reference frame sector)**.** *There exists a CPTP recovery channel $\mathcal{R}_R$ such that*

$$\mathcal{R}_R \mathsf{P}_R(\rho) = \rho \qquad \textit{for all } \rho = \Pi_\mathcal{C} \rho \Pi_\mathcal{C}$$

*if and only if the perspective Kraus operators obey*

$$\Pi_\mathcal{C} E_\mu^\dagger E_\nu \Pi_\mathcal{C} = \alpha_{\mu\nu} \Pi_\mathcal{C} \qquad \forall \mu, \nu \tag{16}$$

*for a positive matrix $(\alpha_{\mu\nu})$. In this case the complete kernel of $\mathsf{P}_R$ restricted to the code is trivial, the induced perspective quotient is completely order-isomorphic to $\mathcal{B}(\mathcal{C})$, and all frame-relative expectation values on the code are recoverable from the non-ideal frame output.*

*Proof.* This is the exact Knill–Laflamme condition applied to the channel $\mathsf{P}_R$ restricted to $\mathcal{B}(\mathcal{C})$. If a recovery exists, the complementary channel of $\mathsf{P}_R$ is constant on the code; equivalently all environment matrix elements $E_\mu^\dagger E_\nu$ compress to scalars on $\mathcal{C}$, giving (16). Conversely, (16) implies that the error syndrome can be coherently measured without revealing logical information and then corrected by a conditional partial isometry. Since $\mathcal{R}_R \mathsf{P}_R$ is the identity on $\mathcal{B}(\mathcal{C})$, the restricted channel is completely isometric and has zero complete kernel on the code. ☐

The theorem converts a conceptual problem into an engineering principle. A clock, gyroscope or rod with finite spread generally imposes dephasing, coarse graining or recoil on relational degrees of freedom. A relational code sector cancels this loss exactly when the frame-induced errors are syndrome-like and carry no logical information. The result is stronger than approximate classical calibration: it supplies a physical recovery map and remains valid with arbitrary inaccessible ancillas.

**Corollary IX.2** (Decoherence-free relational algebra)**.** *Let* $\mathfrak{A}_{\rm rel} \subseteq \mathcal{B}(\mathcal{C})$ *be a relational operator algebra. If every Kraus product* $E_\mu^\dagger E_\nu$ *lies in the commutant of* $\mathfrak{A}_{\rm rel}$ *after compression to* $\mathcal{C}$*, and is scalar on each irreducible logical block, then* $\mathfrak{A}_{\rm rel}$ *is contained in the multiplicative core of* $\mathsf{P}_R^* \mathcal{R}_R^*$*. Hence all observables in* $\mathfrak{A}_{\rm rel}$ *are sharp in the recovered perspective even though the raw frame is non-ideal.*

*Proof.* The block-scalar condition is the subsystem analogue of (16). The corrected Heisenberg channel fixes $\mathfrak{A}_{\rm rel}$ pointwise. Fixed points of a unital completely positive map on which the Schwarz inequality is saturated lie in its multiplicative domain, which proves the claim. □

For non-ideal QRFs this gives a useful distinction. A frame may be globally irreversible, because $\mathrm{Ker}(\mathbf{1}_n \otimes \mathsf{P}_R)$ is nonzero, but still be perfectly reversible on a relational subsystem. Therefore "finite resource" does not mean "unavoidably lossy"; it means that the recoverable relational algebra is selected by the complete-kernel and Knill–Laflamme conditions.

## X. ENTANGLED OBSERVERS AND ALGEBRAIC CORRELATIONS

The perspective channel (1) acts on an arbitrary joint density operator $\rho_{R\bar{R}}$. No product assumption is used. If the frame is entangled with the system, the conditional output is affected exactly as any quantum instrument output is affected by entanglement.

**Theorem X.1** (Entangled-frame data processing)**.** *Suppose* $T_{A\to B}$ *exists as a CPTP map, so that* $\mathsf{P}_B = T_{A\to B}\mathsf{P}_A$*. For any two physical states* $\rho, \sigma$ *and any contractive distinguishability* $D$*,*

$$D(\mathsf{P}_B\rho, \mathsf{P}_B\sigma) \le D(\mathsf{P}_A\rho, \mathsf{P}_A\sigma).$$

*For any entanglement or correlation monotone* $E$ *defined relative to a fixed output algebraic split and monotone under the relevant local channels,*

$$E_B(\rho) \le E_A(\rho)$$

*whenever* $T_{A\to B}$ *is local with respect to that split. If the frame transformation is a* $^*$*-isomorphism of the relevant output algebras, the corresponding algebraic entanglement is invariant.*

*Proof.* The first inequality is the data-processing inequality for the CPTP map $T_{A\to B}$. The second is the defining monotonicity property of $E$. A $^*$-isomorphism has a $^*$-inverse, so the monotonicity inequality holds in both directions and becomes equality. □

**Theorem X.2** (Passive QRF transformations are separable across fixed physical systems)**.** *Let a passive transformation between two descriptions of fixed physical systems* $X$ *and* $Y$ *be generated by a classical or quantum uncertainty over a common group action,*

$$\Lambda(\rho_{XY}) = \int_G \mathrm{Ad}_{U_X(g)\otimes U_Y(g)}(\rho_{XY})\, \nu(\mathrm{d}g),$$

*possibly followed by local coarse grainings on* $X$ *and* $Y$*. Then* $\Lambda$ *is a separable channel across the fixed split* $X|Y$*. Consequently it cannot create entanglement between* $X$ *and* $Y$ *from a separable input and cannot increase any entanglement monotone across that fixed split.*

*Proof.* The displayed map is a convex mixture of product unitaries. Adding local coarse grainings gives a convex mixture of product completely positive maps. Such maps are separable channels. If $\rho_{XY} = \sum_i p_i \rho_X^i \otimes \rho_Y^i$, then

$$\Lambda(\rho_{XY}) = \sum_i p_i \int_G \rho_X^i(g) \otimes \rho_Y^i(g)\, \nu(\mathrm{d}g),$$

where $\rho_X^i(g) = U_X(g)\rho_X^i U_X(g)^\dagger$ and similarly for $Y$. is still a convex mixture of product states, and the same argument with local post-processings proves monotonicity for entanglement monotones. Apparent entanglement creation under active QRF changes therefore comes from a change of subsystem algebra, not from a passive channel acting on a fixed split. □

This theorem separates three notions that are often conflated.

(a) **Perspective tensor entanglement.** Entanglement of tensor factors in a chosen perspective can change because the tensor factors themselves are perspective-dependent.

(b) **Accessible entanglement.** Entanglement computed after a non-ideal perspective channel can decrease by data processing.

(c) **Relational algebraic entanglement.** Correlation between subalgebras of $\mathfrak{M}^G$ is invariant under changes of representation and is the correct frame-independent object.

**Definition X.3** (Relational algebraic mutual information)**.** *Let $\mathfrak{A}, \mathfrak{B} \subseteq \mathfrak{M}^G$ be commuting invariant von Neumann subalgebras and let $\omega$ be a normal physical state. The relational mutual information is*

$$I_\omega(\mathfrak{A} : \mathfrak{B}) = S(\omega_{\mathfrak{A}\vee\mathfrak{B}} \,\|\, \omega_{\mathfrak{A}} \otimes \omega_{\mathfrak{B}})$$

*whenever the product state is defined. In finite dimensions this is*

$$I_\omega(\mathfrak{A} : \mathfrak{B}) = S(\omega_{\mathfrak{A}}) + S(\omega_{\mathfrak{B}}) - S(\omega_{\mathfrak{A}\vee\mathfrak{B}}).$$

Because $\mathfrak{A}$ and $\mathfrak{B}$ are invariant subalgebras, this quantity belongs to the invariant theory. Perspective-dependent entropies are recovered by choosing perspective-dependent embedded subalgebras $\mathsf{P}_R^*(\mathcal{B}(\mathcal{H}_{S_R}))$, and their differences are controlled by the size and kernel of $\mathsf{P}_R$.

## XI. RELATIVISTIC QUANTUM REFERENCE FRAMES

The preceding results are algebraic and therefore do not depend on nonrelativistic kinematics. Relativistic frames require care because the Lorentz and Poincaré groups are noncompact, physical Hilbert spaces decompose into direct integrals, and localization POVMs may be unbounded or only covariant on dense domains. The solution is to apply the theorem on finite-energy, finite-bandwidth operational domains and then take controlled limits.

The sharpened complete-kernel condition is especially important here. A relativistic observer may be entangled with an inaccessible spin, momentum or field ancilla; a transformation that works only on reduced one-particle density matrices but fails after tensoring with that ancilla is not a physical QRF transformation. Thus relativistic covariance must be imposed simultaneously with complete positivity.

### A. Poincaré-covariant perspective channels

Let $G = \mathcal{P}_+^\uparrow = \mathbb{R}^{1,3} \rtimes SO^+(1,3)$ or a relevant subgroup. A relativistic frame may specify an origin and tetrad, so its outcome space is a homogeneous space $X_R = G/H_R$. A covariant POVM satisfies

$$U_R(a,\Lambda)E_R(\Delta)U_R(a,\Lambda)^\dagger = E_R((a,\Lambda)\Delta).$$

The perspective channel is still (1), with $s_R(x)$ a Poincaré transformation carrying the reported origin/tetrad to the standard one. The unitary $U_{\bar{R}}(s_R(x))$ includes translations, boosts, rotations, and therefore Wigner rotations on spin.

For a one-particle sector,

$$\mathcal{H}_{m,s} = L^2(\mathcal{O}_m, \mathrm{d}\mu_m) \otimes \mathbb{C}^{2s+1},$$

where $\mathcal{O}_m$ is the mass shell. A Lorentz transformation acts by

$$(U(\Lambda)\psi)_\sigma(p) = \sum_{\sigma'} D^{(s)}_{\sigma\sigma'}(W(\Lambda, \Lambda^{-1}p))\psi_{\sigma'}(\Lambda^{-1}p),$$

with Wigner rotation $W$. Hence the perspective channel automatically entangles or disentangles spin and momentum according to the frame's boost POVM.

**Theorem XI.1** (Relativistic sector theorem)**.** *Let $\mathcal{D}$ be a finite-energy operational domain invariant under the relevant Poincaré representation and stable under the POVM instruments of frames $A$ and $B$. The restrictions*

$$\mathsf{P}_A, \mathsf{P}_B : \mathcal{T}_1(\mathcal{D})^G \to \mathcal{T}_1$$

*satisfy Theorems IV.1 and V.3. In the direct-integral decomposition into Wigner sectors,*

$$\mathcal{H} = \int^{\oplus} \mathcal{H}_{m,s,\lambda}\, \mathrm{d}\mu(m,s,\lambda),$$

*the transformation condition is sectorwise kernel inclusion. The Poincaré Casimirs $P^\mu P_\mu$ and $W^\mu W_\mu$ lie in the center of the invariant operator system and are universal invariants on sectors not erased by the frame POVMs.*

*Proof.* On $\mathcal{D}$, all maps are normal CP maps between trace-class spaces. The proof of Theorem IV.1 is purely linear and carries over. The Heisenberg proof of Theorem V.3 also carries over. The Wigner decomposition block-diagonalizes the group action; central Casimirs act as multiplication by sector labels and commute with all Poincaré transformations, hence lie in $\mathfrak{M}^G$. □

### B. Spin in a non-ideal relativistic frame

An ideal rest-frame spin observable is pulled back to the laboratory by a boost depending on the particle momentum. If the frame has a boost wave packet rather than a sharp boost, the observed spin POVM is a boost-smeared observable:

$$S^{(R)}_{\mathbf{n}} = \int \mathrm{d}\Lambda\, E_R(\mathrm{d}\Lambda) \otimes U(\Lambda)^\dagger S^{\mathrm{rest}}_{\mathbf{n}} U(\Lambda).$$

The non-ideal frame damps spin-momentum coherences according to the boost distribution. Transforming from frame $A$ to frame $B$ is possible exactly when the boost-spin modes erased by $A$ are also erased by $B$. Otherwise the requested $B$-spin statistics are not functions of the $A$-perspective state.

This gives a precise answer to the operational spin problem. The relativistic spin observable is not an observer-independent tensor factor; it is a relational observable in $\mathfrak{M}^G$. Its non-ideal formulations form an operator system, and the common invariant content is the intersection of their pullback ranges.

#### *A concrete relativistic perspective: boosted spin as a $U(1)$ multiplier*

The qualitative discussion above becomes a precise compact-Abelian statement for a boosted spin. Consider a spin-$\frac{1}{2}$ particle of fixed rapidity $\zeta$ along $\mathbf{x}$, observed by a frame boosted along $\mathbf{y}$ with rapidity $\xi$. The induced Thomas–Wigner rotation is about $\mathbf{z}$ by the angle

$$\Omega(\zeta,\xi), \qquad \tan\frac{\Omega}{2} = \frac{\sinh\zeta\,\sinh\xi}{\cosh\zeta+\cosh\xi},$$

the standard spin–momentum coupling of relativistic quantum information [10]. If the observer's boost rapidity is uncertain with a symmetric law $\mu_R(\xi)$, the spin perspective is the rotation-covariant channel

$$\mathsf{P}_R(\rho) = \mathbb{E}_{\xi\sim\mu_R}\big[R_{\mathbf{z}}(\Omega(\zeta,\xi))\,\rho\, R_{\mathbf{z}}(\Omega(\zeta,\xi))^\dagger\big].$$

**Theorem XI.2** (Boosted spin is a relativistic phase-reference channel)**.** *The boosted-spin perspective $\mathsf{P}_R$ is exactly a $U(1)$ Fourier-multiplier (dephasing) channel about $\mathbf{z}$ with off-diagonal multiplier*

$$\lambda_R = \mathbb{E}_{\xi\sim\mu_R}\big[e^{i\Omega(\zeta,\xi)}\big],$$

*the characteristic function of the Thomas–Wigner angle; equivalently it is the charge-$\chi = 1$ clock channel of Theorem VI.2 in Section VI. Consequently, for two observers with real coaxial multipliers $\lambda_A \geq \lambda_B \geq 0$ the forward simulation $A \to B$ always exists with $\delta(B|A) = 0$, realized by an explicit additional dephasing, whereas the reverse is irreversible with*

$$\delta(A|B) = \tfrac{1}{2}\big(\lambda_A - \lambda_B\big)_+.$$

*Proof.* A rotation $R_{\mathbf{z}}(\Omega)$ acts on the spin coherence $|0\rangle\langle 1|$ by the phase $e^{i\Omega}$ and fixes the populations; averaging over $\mu_R$ multiplies the coherence by $\mathbb{E}[e^{i\Omega}] = \lambda_R$ and preserves the diagonal, which is precisely the action of the $U(1)$ multiplier channel $D_{\mu_R}$ on the single active charge mode $\chi = 1$. The whole compact-Abelian apparatus therefore applies. For coaxial real multipliers the comparison is between two dephasing channels with the same axis: when $\lambda_B \leq \lambda_A$ the map $\rho \mapsto \rho$ followed by dephasing with ratio $\lambda_B/\lambda_A \in [0,1]$ is CPTP and sends $\mathsf{P}_A$ to $\mathsf{P}_B$, so $\delta(B|A) = 0$; when $\lambda_A > \lambda_B$ the difference $\mathsf{P}_A - \mathsf{P}_B$ acts only on the coherence with weight $\lambda_A - \lambda_B$, giving diamond norm $\lambda_A - \lambda_B$ and $\delta(A|B) = \frac{1}{2}(\lambda_A - \lambda_B)$ (the trivial post-processing is optimal because phase coherence is monotone under CPTP maps). □

**Corollary XI.3** (Irreversible metrological cost of boost blur)**.** *For phase estimation with an equatorial probe seen through the boosted-spin channel of Theorem XI.2, the quantum Fisher information is*

$$F_Q = |\lambda_R|^2,$$

*independent of the phase. Hence boost uncertainty multiplies the attainable Fisher information by $|\lambda_R|^2$ and inflates the standard-quantum-limited variance by $|\lambda_R|^{-2}$. Because the quantum Fisher information is monotone under CPTP maps [25, 63, 64], no post-processing recovers the precision lost to the blurrier observer:*

$$\frac{F_Q[B]}{F_Q[A]} = \Big(\frac{\lambda_B}{\lambda_A}\Big)^2 \leq 1.$$

*Proof.* The probe state has Bloch vector $\mathbf{r}(\phi) = (\lambda_R \cos\phi, \lambda_R \sin\phi, 0)$, of constant length $|\lambda_R|$. For a qubit the Fisher information is $F_Q = |\partial_\phi \mathbf{r}|^2 + (\mathbf{r}\cdot\partial_\phi\mathbf{r})^2/(1-|\mathbf{r}|^2)$; the second term vanishes because $|\mathbf{r}|$ is constant, and $|\partial_\phi \mathbf{r}|^2 = |\lambda_R|^2$. Monotonicity of $F_Q$ under CPTP maps gives the ratio bound since $B$ is a dephasing of $A$. □

For the symmetric Gaussian boost law of Section XXI (particle rapidity $\zeta = 1$, a sharp observer of spread $\sigma_A = 0.6$ and a blurred observer of spread $\sigma_B \approx 0.92$) the multipliers are $\lambda_A \approx 0.877$ and $\lambda_B \approx 0.769$; the forward deficiency vanishes, the reverse deficiency is $\frac{1}{2}(\lambda_A - \lambda_B) \approx 0.054$ (confirmed by the qubit diamond-norm SDP), and the phase Fisher information drops from $F_Q[A] \approx 0.769$ to $F_Q[B] \approx 0.591$, an irrecoverable factor of about 0.77.

### C. Relativistic clocks and temporal order

A clock is a frame for time translations. A relativistic clock is additionally affected by momentum and gravitational redshift. In a weak-field or special-relativistic setting, the clock reading POVM $E_C(\mathrm{d}\tau)$ defines

$$\mathsf{P}_C(\rho) = \int U_{\bar{C}}(\tau)^\dagger \, \mathrm{Tr}_C[(E_C(\mathrm{d}\tau)^{1/2} \otimes \mathbf{1})\rho(E_C(\mathrm{d}\tau)^{1/2} \otimes \mathbf{1})] U_{\bar{C}}(\tau).$$

Finite energy bandwidth suppresses temporal Fourier modes. When temporal order observables require modes outside the support of $\widehat{\mu}_C$, temporal order is not operationally sharp in the clock perspective. If another clock $D$ resolves those modes, no deterministic $C \to D$ transformation exists. This is the channel-theoretic form of temporal-order limitations for non-ideal clocks.

## XII. RELATIVISTIC LOCALITY AND NO-SIGNALLING UNDER PERSPECTIVE CHANNELS

A relativistic QRF transformation must be completely positive, but this is not enough. It must also be local in the algebraic sense: changing an observer's quantum tetrad, clock or boost register cannot create signalling between spacelike separated subsystems. The channel formulation gives a concise condition.

Let $\mathfrak{A}(\mathcal{O}_1)$ and $\mathfrak{A}(\mathcal{O}_2)$ be commuting local algebras for spacelike separated regions. A normal Heisenberg-picture perspective map $\mathsf{P}_R^*$ is *causally local* on these regions if there exist regions $\mathcal{O}_i^R$ in the $R$-description such that

$$\mathsf{P}_R^*\big(\mathfrak{A}_R(\mathcal{O}_i^R)\big) \subseteq \mathfrak{A}(\mathcal{O}_i), \qquad i = 1, 2.$$

**Theorem XII.1** (Local QRF transformations preserve no-signalling)**.** *Let $A$ and $B$ be causally local relativistic perspectives and suppose*

$$\mathsf{P}_B = \Lambda_{A\to B}\mathsf{P}_A$$

*with $\Lambda_{A\to B}$ CPTP and normal. If the Heisenberg map $\Lambda^*_{A\to B}$ maps each $B$-local algebra into the corresponding $A$-local algebra, then spacelike commutativity and operational no-signalling are preserved under the QRF transformation:*

$$[X, Y] = 0 \quad \forall X \in \mathfrak{A}_B(\mathcal{O}_1^B),\ Y \in \mathfrak{A}_B(\mathcal{O}_2^B),$$

*and interventions represented by completely positive maps localized in $\mathcal{O}_1^B$ cannot change probabilities of effects localized in $\mathcal{O}_2^B$.*

*Proof.* By locality,

$$\begin{aligned}\mathsf{P}_B^*(X) &= \mathsf{P}_A^*\Lambda_{A\to B}^*(X) \in \mathfrak{A}(\mathcal{O}_1),\\ \mathsf{P}_B^*(Y) &= \mathsf{P}_A^*\Lambda_{A\to B}^*(Y) \in \mathfrak{A}(\mathcal{O}_2).\end{aligned}$$

The physical local algebras commute for spacelike separation, hence the pulled-back effects commute. Since all probabilities in the $B$ perspective are computed by these pulled-back effects, a localized CP intervention on one algebra leaves the marginal statistics on the commuting spacelike algebra invariant. This is the standard algebraic no-signalling argument transported through the perspective channel. □

The theorem excludes a common false solution to relativistic QRFs: one may formally write a boost-dependent map on reduced one-particle density matrices that looks covariant, but if it has no normal completely positive extension respecting local algebras, it is not a valid relativistic change of quantum reference frame. Relativistic QRF transformations are therefore constrained by three simultaneous tests: complete positivity, complete-kernel factorization, and causal locality.

## XIII. RELATION TO GAUGE THEORY AND GRAVITY

In constrained systems and general relativity, a reference frame is often a choice of material scalar fields, embedding variables, boundary/edge data or gauge conditions. Classically, changing relational reference frames is a gauge transformation between reductions. Quantum mechanically, the choice “reduce first or quantize first” can lead to inequivalent-looking descriptions. The present framework gives a common operational criterion.

Let $\mathfrak{C}$ be the constraint algebra and $\mathfrak{M}^{\mathfrak{C}}$ the physical algebra commuting with constraints. A material frame $R$ defines a gauge-fixing POVM or instrument on clock/rod fields. The corresponding perspective channel is a quantum gauge-fixing channel

$$\mathsf{P}_R : \mathsf{St}(\mathfrak{M}^{\mathfrak{C}}) \to \mathsf{St}_R.$$

Two reductions are exactly equivalent when their gauge-fixing channels have equal kernels; one reduction is a coarse-graining of another when its kernel is larger. The common invariant operator system is again the intersection of Heisenberg pullbacks.

This resolves a common confusion: a variable may be a non-observable before gauge fixing and an observable after choosing a frame because the observable is not the bare variable; it is the pullback $\mathsf{P}_R^*(A_R)$, a relational Dirac observable. Different frames yield different representatives of the same invariant exactly when their pullbacks coincide.

For quantum gravity, the result does not solve dynamics. It solves the reference-frame part: the exact mathematical object that compares quantum coordinate systems is a channel between operational reductions, and the obstruction to exact equivalence is a kernel of finite-resource or gauge-fixing information loss. In gauge theories with boundaries this viewpoint naturally interfaces with the identification of edge and soft modes as dynamical reference-frame data [40]; the quotient criterion states which such boundary-frame data are operationally redundant and which remain observable.

## XIV. DEFICIENCY GEOMETRY AND APPROXIMATE TRANSFORMATIONS

Exact kernel inclusion is a sharp criterion. Experiments and finite-resource theories often require approximate transformation. The natural measure is channel deficiency, the channel-theoretic form of the Blackwell–Le Cam comparison of statistical experiments and quantum sufficiency [26, 49–52].

**Definition XIV.1** (Reference-frame deficiency)**.** *The deficiency of $A$ relative to $B$ is*

$$\delta(B|A) = \frac{1}{2}\inf_{\Lambda\in\mathsf{CPTP}} \|\mathsf{P}_B - \Lambda\mathsf{P}_A\|_\diamond\,,$$

*where the diamond norm is taken on the physical input operator space. We write $A \succeq_\epsilon B$ when $\delta(B|A) \le \epsilon$.*

**Theorem XIV.2** (Lawvere metric of quantum perspectives)**.** *For finite-dimensional perspective channels,*

$$\delta(A|A) = 0, \qquad \delta(C|A) \le \delta(B|A) + \delta(C|B).$$

*Moreover $\delta(B|A) = 0$ if and only if $B$ is an exact CPTP degradation of $A$, i.e. $\mathsf{P}_B = \Lambda\mathsf{P}_A$ for some CPTP map $\Lambda$.*

*Proof.* The identity channel gives $\delta(A|A) = 0$. Let $\Lambda_{BA}$ and $\Lambda_{CB}$ be $\epsilon_1$- and $\epsilon_2$-optimal channels for $A \to B$ and $B \to C$. Then

$$\begin{aligned}\|\mathsf{P}_C - \Lambda_{CB}\Lambda_{BA}\mathsf{P}_A\|_\diamond &\le \|\mathsf{P}_C - \Lambda_{CB}\mathsf{P}_B\|_\diamond \\ &\quad + \|\Lambda_{CB}(\mathsf{P}_B - \Lambda_{BA}\mathsf{P}_A)\|_\diamond \\ &\le 2\epsilon_2 + 2\epsilon_1,\end{aligned}$$

because CPTP maps contract the diamond norm. Taking infima gives the triangle inequality. If $\delta(B|A) = 0$, compactness of the finite-dimensional CPTP set gives an attaining channel $\Lambda$ with zero diamond distance. The converse is immediate. ☐

The symmetric operational distance

$$\Delta(A, B) = \max\{\delta(B|A), \delta(A|B)\}$$

measures inequivalence of frames. It is zero precisely on CPTP-equivalent perspectives. In finite dimension, (XIV.1) is a semidefinite program through the Choi representation of $\Lambda$ and the diamond norm. Hence the theory is not merely conceptual: it produces a directly computable calibration distance between real clocks, gyroscopes and laboratories.

**Proposition XIV.3** (Abelian deficiency as distance to a convolution cone)**.** *For finite Abelian $G$, let $\mathsf{P}_A, \mathsf{P}_B$ be convolution perspectives with response distributions $p_A, p_B$. The covariant deficiency is*

$$\delta_{\mathrm{cov}}(B|A) = \frac{1}{2}\inf_{r\in\mathrm{Prob}(G)} \|D_{p_B} - D_{p_A * r}\|_\diamond,$$

*where $D_p$ is the corresponding Fourier-multiplier channel. In particular $\delta_{\mathrm{cov}}(B|A) = 0$ exactly when*

$$p_B = p_A * r$$

*for some probability distribution $r$, equivalently when the multiplier ratio $\widehat{p}_B/\widehat{p}_A$ has a positive-definite extension on the active Fourier sector.*

*Proof.* Random group translations are covariant post-processings, and composing one with $\mathsf{P}_A$ replaces $p_A$ by $p_A * r$; the displayed expression is the resulting distance over this family. Covariant channels are in general strictly larger than random translations, because the regular representation carries multiplicity, so this proposition only asserts the value of the random-translation (convolution-cone) deficiency. That the infimum over *all* covariant post-processings is nonetheless attained on random translations – so that the displayed quantity is the full diamond deficiency – is the content of Theorem XIV.10. The zero-deficiency condition is precisely the finite-group Bochner positive-definiteness condition of Theorem VI.2. ☐

**Proposition XIV.4** (Finite Abelian convex certificate)**.** *For a finite Abelian convolution model on $G$, the existence of a covariant concrete degradation $A \to B$ is equivalent to feasibility of the simplex equation*

$$p_B = C(p_A)r, \qquad r \in \mathrm{Prob}(G),$$

*where $C(p_A)$ is the circulant convolution matrix generated by $p_A$. The finite-dimensional certificate*

$$\Delta_{\mathrm{conv}}(B|A) = \frac{1}{2}\min_{r\in\mathrm{Prob}(G)} \|p_B - C(p_A)r\|_1$$

*is a computable lower bound on the operational deficiency restricted to tests diagonal in the group-token basis, and it vanishes iff the covariant degradation exists. Its linear-programming dual supplies an explicit separating witness when a proposed noisy-frame simulation is impossible.*

**Lemma XIV.5** (Diamond norm of a signed multiplier map)**.** *Let $G$ be a finite group acting by its left regular representation $U$ on $\mathbb{C}^{|G|}$, and for a real signed measure $\sigma \in \mathbb{R}^G$ let*

$$D_\sigma = \sum_{g\in G} \sigma(g)\ \mathrm{Ad}_{U(g)}$$

*be the associated Hermiticity-preserving multiplier map, so that $D_p - D_q = D_{p-q}$ for probability vectors $p, q$. Then*

$$\|D_\sigma\|_\diamond = \|\sigma\|_1 = \sum_{g\in G} |\sigma(g)|.$$

*Proof.* For the upper bound, each $\mathrm{Ad}_{U(g)}$ is a unitary channel with $\|\mathrm{Ad}_{U(g)}\|_\diamond = 1$; the triangle inequality for the completely bounded norm [62] then gives $\|D_\sigma\|_\diamond \le \sum_g |\sigma(g)|\,\|\mathrm{Ad}_{U(g)}\|_\diamond = \|\sigma\|_1$.

For the matching lower bound, probe with the maximally entangled state $|\Omega\rangle = |G|^{-1/2}\sum_{x\in G}|x\rangle|x\rangle$ on $\mathbb{C}^{|G|}\otimes\mathbb{C}^{|G|}$. Writing $|\Omega_g\rangle = (U(g)\otimes\mathbf{1})|\Omega\rangle$,

$$(D_\sigma\otimes\mathbf{1})\big(|\Omega\rangle\langle\Omega|\big) = \sum_{g\in G}\sigma(g)\,|\Omega_g\rangle\langle\Omega_g|.$$

In the regular representation $\mathrm{Tr}\,U(k) = |G|\,\delta_{k,e}$, so that $\langle\Omega_h|\Omega_g\rangle = |G|^{-1}\,\mathrm{Tr}\big(U(h)^\dagger U(g)\big) = \delta_{h,g}$: the vectors $|\Omega_g\rangle$ are orthonormal. The output is therefore diagonal in an orthonormal basis with eigenvalues $\sigma(g)$, and its trace norm equals $\sum_g|\sigma(g)|$. Hence $\|D_\sigma\|_\diamond \ge \big\|(D_\sigma\otimes\mathbf{1})(|\Omega\rangle\langle\Omega|)\big\|_1 = \|\sigma\|_1$. □

**Theorem XIV.6** (Symmetrization: deficiency is attained on covariant post-processings)**.** *Let $V$ be a unitary representation of a compact group $K$ on the physical Hilbert space, and suppose the perspectives are $V$-covariant, i.e. $\mathsf{P}_A, \mathsf{P}_B$ commute with $\mathrm{Ad}_{V(k)}$ for every $k\in K$. Then the unrestricted deficiency equals the $V$-covariant deficiency,*

$$\delta(B|A) = \delta^V_{\mathrm{cov}}(B|A) := \tfrac{1}{2}\inf_{\Lambda\in\mathsf{CPTP}^V}\|\mathsf{P}_B - \Lambda\,\mathsf{P}_A\|_\diamond,$$

*where $\mathsf{CPTP}^V$ denotes the channels commuting with $\mathrm{Ad}_{V(k)}$ for all $k\in K$.*

*Proof.* Covariant channels are a subset of all channels, so $\delta(B|A) \le \delta^V_{\mathrm{cov}}(B|A)$. Conversely, let $\Lambda$ be any channel and form its twirl

$$\bar\Lambda = \int_K \mathrm{Ad}_{V(k)^{-1}}\circ\Lambda\circ\mathrm{Ad}_{V(k)}\,\mathrm{d}\mu_{\mathrm{Haar}}(k),$$

which is $V$-covariant. Using covariance of $\mathsf{P}_A$ and $\mathsf{P}_B$,

$$\mathrm{Ad}_{V(k)^{-1}}\big(\mathsf{P}_B - \Lambda\mathsf{P}_A\big)\,\mathrm{Ad}_{V(k)} = \mathsf{P}_B - \big(\mathrm{Ad}_{V(k)^{-1}}\,\Lambda\,\mathrm{Ad}_{V(k)}\big)\mathsf{P}_A,$$

and unitary invariance of the diamond norm makes every summand equal in diamond norm to $\mathsf{P}_B - \Lambda\mathsf{P}_A$. Convexity of $\|\cdot\|_\diamond$ then gives $\|\mathsf{P}_B - \bar\Lambda\mathsf{P}_A\|_\diamond \le \|\mathsf{P}_B - \Lambda\mathsf{P}_A\|_\diamond$. Taking the infimum, $\delta^V_{\mathrm{cov}}(B|A)\le\delta(B|A)$. □

**Remark XIV.7** (Convolution perspectives are right-regular covariant)**.** *A left-convolution channel $D_p = \sum_g p(g)\,\mathrm{Ad}_{U(g)}$ commutes with $\mathrm{Ad}_{U(h)}$ for all $h$ only when $p$ is a class function, but it commutes with the* right*-regular representation $R(h)$ ($R(h)|x\rangle = |xh^{-1}\rangle$) for* every *$p$, because left and right translations commute. Hence Theorem XIV.6 applies with $V = R$ to any pair of convolution perspectives on any finite group, central or not, giving $\delta(B|A) = \delta^R_{\mathrm{cov}}(B|A)$; and random translations $D_r$, being right-covariant, lie in $\mathsf{CPTP}^R$.*

**Theorem XIV.8** (Exactness of the convolution LP for the random-translation cone)**.** *For convolution perspectives on a finite group realized in the regular representation, the diamond distance to the random-translation cone is computed exactly by the convolution linear program of Proposition XIV.4:*

$$\tfrac{1}{2}\inf_{r\in\mathrm{Prob}(G)}\|D_{p_B} - D_{p_A*r}\|_\diamond = \Delta_{\mathrm{conv}}(B|A) = \tfrac{1}{2}\min_{r\in\mathrm{Prob}(G)}\|p_B - C(p_A)r\|_1.$$

*Thus $\Delta_{\mathrm{conv}}$ is the genuine diamond distance to the random-translation cone over* all *probe states, with no restriction to tests diagonal in the group-token basis. Random translations are only a subset of the post-processings, so a priori $\delta(B|A)\le\Delta_{\mathrm{conv}}(B|A)$; by Remark XIV.7 the optimum may even be restricted to right-regular-covariant channels,*

$$\delta(B|A) = \delta^R_{\mathrm{cov}}(B|A) \le \Delta_{\mathrm{conv}}(B|A).$$

*In fact the inequality is an equality for* every *finite group, $\delta(B|A) = \Delta_{\mathrm{conv}}(B|A)$, proved in Theorem XIV.16 below (for Abelian $G$ this also follows from the Heisenberg–Weyl argument of Theorem XIV.10), upgrading the one-sided estimate of Proposition XIV.4 to an exact identity.*

*Proof.* Because $D_{p_B} - D_{p_A*r} = D_{p_B - p_A*r}$ and $p_A*r = C(p_A)r$, Lemma XIV.5 gives $\|D_{p_B} - D_{p_A*r}\|_\diamond = \|p_B - C(p_A)r\|_1$ for each $r$; minimizing over $r\in\mathrm{Prob}(G)$ gives the first display, the diamond distance to the random-translation cone. Random translations are valid CPTP post-processings, so $\delta\le\Delta_{\mathrm{conv}}$; the refinement $\delta = \delta^R_{\mathrm{cov}}\le\Delta_{\mathrm{conv}}$ is Remark XIV.7. The Abelian equality is Theorem XIV.10. □

**Lemma XIV.9** (Heisenberg–Weyl diamond norm equals $\ell^1$)**.** *Let $G$ be a finite Abelian group with dual $\widehat{G}$, let $U(a)$ ($a \in G$) be the translation operators and $M(\chi)$ ($\chi \in \widehat{G}$) the modulation operators $M(\chi)|x\rangle = \chi(x)|x\rangle$ on $\mathbb{C}^{|G|}$, and write $W_{a,\chi} = U(a)M(\chi)$ for the generalized Pauli operators. For a real signed array $s \in \mathbb{R}^{G\times\widehat{G}}$,*

$$\Big\| \sum_{a\in G,\ \chi\in\widehat{G}} s(a,\chi)\ \mathrm{Ad}_{W_{a,\chi}} \Big\|_\diamond = \|s\|_1 = \sum_{a,\chi} |s(a,\chi)|.$$

*Proof.* The upper bound is the triangle inequality, since each $\mathrm{Ad}_{W_{a,\chi}}$ is a unitary channel of diamond norm 1. For the lower bound, probe with the maximally entangled state $|\Omega\rangle = |G|^{-1/2}\sum_x |x\rangle|x\rangle$ and set $|\Omega_{a,\chi}\rangle = (W_{a,\chi}\otimes \mathbf{1})|\Omega\rangle$. The generalized Pauli operators are trace-orthogonal, $\mathrm{Tr}\big(W_{a,\chi}^\dagger W_{a',\chi'}\big) = |G|\,\delta_{a,a'}\delta_{\chi,\chi'}$, so $\langle\Omega_{a,\chi}|\Omega_{a',\chi'}\rangle = \delta_{a,a'}\delta_{\chi,\chi'}$ and the output $\sum_{a,\chi} s(a,\chi)\,|\Omega_{a,\chi}\rangle\langle\Omega_{a,\chi}|$ is diagonal in an orthonormal basis with eigenvalues $s(a,\chi)$; its trace norm is $\|s\|_1$. □

**Theorem XIV.10** (Heisenberg–Weyl exactness: the convolution LP is the full Abelian deficiency)**.** *For convolution perspectives $\mathsf{P}_A = D_{p_A}$, $\mathsf{P}_B = D_{p_B}$ of a finite Abelian group $G$ in the regular representation, the unrestricted diamond deficiency equals the convolution linear program exactly:*

$$\delta(B|A) = \tfrac{1}{2} \min_{r\in\mathrm{Prob}(G)} \|p_B - p_A * r\|_1 = \Delta_{\mathrm{conv}}(B|A).$$

*In particular the optimal post-processing can be taken to be a random group translation, even though the covariant channels strictly contain the random translations.*

*Proof.* Random translations are valid CPTP post-processings, so $\delta(B|A) \le \Delta_{\mathrm{conv}}(B|A)$ by Theorem XIV.8. For the reverse inequality we identify the optimal post-processing.

A convolution channel $D_p = \sum_g p(g)\,\mathrm{Ad}_{U(g)}$ commutes with every translation $\mathrm{Ad}_{U(g)}$ and, because $M(\chi)U(g)M(\chi)^\dagger = \chi(g)\,U(g)$, also with every modulation $\mathrm{Ad}_{M(\chi)}$:

$$\mathrm{Ad}_{M(\chi)}\big(D_p(X)\big) = \sum_g p(g)\,\chi(g)\overline{\chi(g)}\ \mathrm{Ad}_{U(g)}\big(\,\mathrm{Ad}_{M(\chi)}\,X\big) = D_p\big(\,\mathrm{Ad}_{M(\chi)}\,X\big).$$

Thus $\mathsf{P}_A$ and $\mathsf{P}_B$ are covariant under the entire Heisenberg–Weyl group generated by $\{U(a), M(\chi)\}$. Applying the twirl of Theorem XIV.6 successively to the translation group and to the modulation group, the optimal post-processing $\Lambda$ may be assumed to commute with $\mathrm{Ad}_{U(g)}$ and $\mathrm{Ad}_{M(\chi)}$ for all $g, \chi$.

The conjugations act on the Pauli basis by distinct characters,

$$\mathrm{Ad}_{U(g)}(W_{a,\chi}) = \overline{\chi(g)}\,W_{a,\chi}, \qquad \mathrm{Ad}_{M(\eta)}(W_{a,\chi}) = \eta(a)\,W_{a,\chi},$$

so a superoperator commuting with all of them is diagonal in $\{W_{a,\chi}\}$; being moreover completely positive and trace preserving, it is a generalized Pauli channel

$$\Lambda = \sum_{c\in G,\ \eta\in\widehat{G}} \pi(c,\eta)\ \mathrm{Ad}_{W_{c,\eta}}, \qquad \pi \in \mathrm{Prob}(G\times\widehat{G}).$$

Using $W_{c,\eta}U(g) = \eta(g)\,W_{c+g,\eta}$ (up to a phase irrelevant under Ad),

$$\Lambda\,\mathsf{P}_A = \sum_{a,\eta} \big(\pi(\cdot,\eta) *_G p_A\big)(a)\ \mathrm{Ad}_{W_{a,\eta}}, \qquad \mathsf{P}_B = \sum_a p_B(a)\ \mathrm{Ad}_{W_{a,0}}\,.$$

Both are Pauli-diagonal, so by Lemma XIV.9 the diamond distance equals the $\ell^1$ norm of the difference array, and – the minimum over Pauli channels being the unrestricted deficiency by the twirl reduction above –

$$2\,\delta(B|A) = \min_{\pi\in\mathrm{Prob}(G\times\widehat{G})} \Big(\big\|p_B - \pi(\cdot,0) *_G p_A\big\|_1 + \sum_{\eta\ne 0} \big\|\pi(\cdot,\eta) *_G p_A\big\|_1\Big).$$

Since $p_A$ is a probability vector, $\|\pi(\cdot,\eta) *_G p_A\|_1 = \sum_c \pi(c,\eta)$, and for any feasible $\pi$ the vector $r := \pi(\cdot,0) + \big(\sum_{\eta\ne0}\sum_c \pi(c,\eta)\big)\delta_e$ is a probability distribution on $G$ with $\|p_B - r *_G p_A\|_1 \le \|p_B - \pi(\cdot,0) *_G p_A\|_1 + \sum_{\eta\ne0}\sum_c \pi(c,\eta)$, i.e. no larger than the bracket. Hence the minimum is attained with all mass on $\eta = 0$, giving $2\,\delta(B|A) = \min_{r\in\mathrm{Prob}(G)} \|p_B - r *_G p_A\|_1 = 2\,\Delta_{\mathrm{conv}}(B|A)$, which is the claim. □

**Remark XIV.11** (Why multiplicity does not help, and the non-Abelian frontier)**.** *Theorem XIV.10 explains the numerical coincidence of the convolution LP and the full diamond-norm SDP: although a covariant channel may act non-trivially within the $|G|$-fold multiplicity of each Fourier mode, the additional modulation symmetry shared by the two convolution perspectives forces the optimal post-processing onto the Pauli-diagonal cone, where the off-diagonal ($\eta \neq 0$) components only add error. The argument uses Pontryagin duality and is therefore specific to Abelian groups: for non-Abelian $G$ the modulation operators are replaced by the matrix coefficients of higher-dimensional irreducibles and this particular reduction no longer applies. The equality $\delta = \Delta_{\rm conv}$ nonetheless persists for every finite group, by a different argument (an augmented entangled witness) given in Theorem XIV.16; the next proposition records the intermediate bi-regular reduction for central distributions.*

**Proposition XIV.12** (Bi-regular reduction for central perspectives)**.** *Let $G$ be an arbitrary finite group and let $\mathsf{P}_A = D_{p_A}$, $\mathsf{P}_B = D_{p_B}$ be convolution perspectives whose response distributions $p_A, p_B$ are* central *(class functions). Then:*

*(i) the deficiency is attained on* bi-regular-covariant *post-processings,*

$$\delta(B|A) = \delta^{LR}_{\rm cov}(B|A) := \tfrac{1}{2} \inf \left\{ \|\mathsf{P}_B - \Lambda \mathsf{P}_A\|_\diamond :\ \Lambda \textit{ commutes with } \mathrm{Ad}_{U(g)} \textit{ and } \mathrm{Ad}_{R(h)}\ \forall g, h \right\};$$

*(ii) the convolution LP is realized by a central kernel,*

$$\Delta_{\rm conv}(B|A) = \tfrac{1}{2} \min_{r \in \mathrm{Prob}(G)\ \mathrm{central}} \|p_B - p_A * r\|_1,$$

*so the optimal random translation may be taken central, and $\delta(B|A) \leq \Delta_{\rm conv}(B|A)$ with the degradation realizable by a central convolution.*

*Proof.* (i) Since $p_A, p_B$ are central, $D_{p_A}, D_{p_B}$ commute with the left translations $\mathrm{Ad}_{U(g)}$ – an elementary consequence of centrality, since $\sum_g p(g)\, \mathrm{Ad}_{U(hg)} = \sum_g p(g)\, \mathrm{Ad}_{U(gh)}$ iff $p(h^{-1}k) = p(kh^{-1})$ for all $h, k$ – and, by Remark XIV.7, with the right translations $\mathrm{Ad}_{R(h)}$. Apply Theorem XIV.6 first with $V = U$ to obtain a left-covariant optimizer, then with $V = R$; right-twirling preserves left covariance because left and right translations commute (the computation is that of Theorem XIV.10), so the optimizer may be taken bi-covariant.

(ii) For any $r \in \mathrm{Prob}(G)$ let $\bar{r}(g) = |G|^{-1} \sum_h r(h^{-1}gh)$ be its conjugacy average, a central probability distribution. A direct computation using centrality of $p_A$ gives $(h \cdot r) * p_A = h \cdot (r * p_A)$, where $(h \cdot q)(k) = q(h^{-1}kh)$; since $p_B$ is central, $\|p_B - (h \cdot r) * p_A\|_1 = \|p_B - r * p_A\|_1$ for every $h$, and convexity of the norm gives $\|p_B - \bar{r} * p_A\|_1 \leq \|p_B - r * p_A\|_1$. Hence the minimum over central kernels equals the minimum over all kernels, which is $2\Delta_{\rm conv}(B|A)$ by Theorem XIV.8. Random translations being valid post-processings, $\delta \leq \Delta_{\rm conv}$. ☐

**Remark XIV.13** (From reduction to proof)**.** *Proposition XIV.12 reduces the central non-Abelian case to whether a bi-regular-covariant channel can outperform a central convolution; bi-covariant channels form a proper subclass of all channels but are strictly larger than the central convolutions, so that reduction alone is not an equality. The full statement – that for* every *finite group the convolution linear program equals the diamond-norm deficiency – is proved as Theorem XIV.16 below. The route is instructive: Lemma XIV.14 and Proposition XIV.15 first show that the obvious maximally entangled certificate is provably too weak (it permits the post-processing to discard probability mass), and Theorem XIV.16 then repairs it by charging that discarded mass at the optimal-translation rate. The equality is corroborated independently by direct comparison of the full diamond-norm semidefinite program with the linear program on $S_3$, the order-eight groups $D_4$ and $Q_8$, and – reaching a three-dimensional irreducible via a commutant reduction of the program (block sizes $12, 12, 12, 36$ in place of $144$, validated against the full program on $S_3$) – the alternating group $A_4$, in every case to solver tolerance.*

**Lemma XIV.14** (The maximally entangled probe certifies only the sub-normalised program)**.** *Let $\mathsf{P}_A = D_{p_A}$, $\mathsf{P}_B = D_{p_B}$ be convolution perspectives of a finite group $G$ in the regular representation and let $\Lambda$ be any channel on $L(\ell^2(G))$. Write $|\Omega\rangle = |G|^{-1/2} \sum_x |x\rangle|x\rangle$ and $|\Omega_g\rangle = (U(g) \otimes I)|\Omega\rangle$; the family $\{|\Omega_g\rangle\}_{g \in G}$ is orthonormal and $(D_p \otimes \mathbf{1})(|\Omega\rangle\langle\Omega|) = \sum_g p(g)\, |\Omega_g\rangle\langle\Omega_g|$. With Choi state $J(\Lambda) = (\Lambda \otimes \mathbf{1})(|\Omega\rangle\langle\Omega|)$ set*

$$q_\Lambda(g) := \langle\Omega_g|J(\Lambda)|\Omega_g\rangle \geq 0, \qquad \textstyle\sum_g q_\Lambda(g) \leq 1.$$

*Then the transfer matrix $T_\Lambda(g, g') := \langle\Omega_g|(\Lambda \otimes \mathbf{1})(|\Omega_{g'}\rangle\langle\Omega_{g'}|)|\Omega_g\rangle$ equals $q_\Lambda(gg'^{-1})$, so pinching $(\Lambda D_{p_A} \otimes \mathbf{1})(|\Omega\rangle\langle\Omega|)$ onto $\{|\Omega_g\rangle\}$ gives $\sum_g (q_\Lambda * p_A)(g)\, |\Omega_g\rangle\langle\Omega_g|$. Consequently $\frac{1}{2}\|D_{p_B} - \Lambda D_{p_A}\|_\diamond \geq \frac{1}{2}\|p_B - q_\Lambda * p_A\|_1$, and minimising over $\Lambda$,*

$$\delta(B|A)\ \geq\ \Delta^{\rm sub}_{\rm conv}(B|A) := \tfrac{1}{2} \min_{f \geq 0,\ \sum_g f(g) \leq 1} \|p_B - f * p_A\|_1.$$

*Proof.* Orthonormality of $\{|\Omega_g\rangle\}$ and the image of $D_p$ follow from $\langle\Omega_g|\Omega_{g'}\rangle = \delta_{g,g'}$. Using $(A\otimes I)|\Omega\rangle = (I\otimes A^{\top})|\Omega\rangle$ and $U(a)^{\top} = U(a)^{\dagger} = U(a^{-1})$ for the real orthogonal regular representation, $|\Omega_{g'}\rangle = (I\otimes U(g'^{-1}))|\Omega\rangle$; as $\Lambda\otimes\mathbf{1}$ acts trivially on the second factor, $(\Lambda\otimes\mathbf{1})(|\Omega_{g'}\rangle\langle\Omega_{g'}|) = (I\otimes U(g'^{-1}))J(\Lambda)(I\otimes U(g'^{-1}))^{\dagger}$. With $\langle\Omega_g| = \langle\Omega|(I\otimes U(g))$ this gives $T_\Lambda(g,g') = \langle\Omega|(I\otimes U(gg'^{-1}))J(\Lambda)(I\otimes U(gg'^{-1}))^{\dagger}|\Omega\rangle = q_\Lambda(gg'^{-1})$, the last step being the case $g'=e$. Hence $\sum_{g'} T_\Lambda(g,g')p_A(g') = (q_\Lambda * p_A)(g)$. The pinching $M\mapsto\sum_g|\Omega_g\rangle\langle\Omega_g|\,M\,|\Omega_g\rangle\langle\Omega_g|$ is trace-norm contractive and sends $(D_{p_B}-\Lambda D_{p_A})\otimes\mathbf{1}$ on $|\Omega\rangle\langle\Omega|$ to $\sum_g[p_B(g)-(q_\Lambda * p_A)(g)]|\Omega_g\rangle\langle\Omega_g|$, of trace norm $\|p_B - q_\Lambda * p_A\|_1$; since $\|\Phi\|_\diamond \ge \|(\Phi\otimes\mathbf{1})(|\Omega\rangle\langle\Omega|)\|_1$, the stated bound holds. As $q_\Lambda$ is a sub-probability for every $\Lambda$, taking the infimum over $\Lambda$ yields $\delta\ge\Delta^{\mathrm{sub}}_{\mathrm{conv}}$. □

**Proposition XIV.15** (The maximally entangled certificate is strictly loose)**.** $\Delta^{\mathrm{sub}}_{\mathrm{conv}}(B|A)\le\Delta_{\mathrm{conv}}(B|A)$*, with strict inequality in general: already for* $G=\mathbb{Z}_3$ *there are* $p_A, p_B$ *with* $\Delta^{\mathrm{sub}}_{\mathrm{conv}}<\Delta_{\mathrm{conv}}$*. Hence the maximally entangled probe with a* diagonal *measurement alone cannot certify* $\delta=\Delta_{\mathrm{conv}}$*; the slack is exactly the freedom of the post-processing to discard probability mass. Theorem XIV.16 removes this slack by retaining the same probe but augmenting the measurement with a charge on the orthogonal complement of* $\mathrm{span}\{|\Omega_g\rangle\}$*, thereby proving the equality for every finite group.*

*Proof.* The inequality holds because probabilities are sub-probabilities. Strictness is a finite linear-programming phenomenon: lowering the total mass of $f$ contracts $f*p_A$ towards the origin, which strictly reduces $\|p_B - f*p_A\|_1$ whenever $p_B$ lies outside the convolution polytope $\{r*p_A : r\in\mathrm{Prob}(G)\}$ and the contraction direction has positive overlap with $p_B - f*p_A$. A direct search over $\mathbb{Z}_3$ exhibits gaps $\Delta_{\mathrm{conv}}-\Delta^{\mathrm{sub}}_{\mathrm{conv}}$ as large as $\approx 0.14$. □

The resolution is to keep the maximally entangled probe but *augment the measurement*: a charge on the orthogonal complement of $\mathrm{span}\{|\Omega_g\rangle\}$, levied at exactly the optimal-translation rate, removes the adversary's incentive to discard mass and closes the gap completely.

**Theorem XIV.16** (The convolution LP is the exact diamond-norm deficiency, for every finite group)**.** *For an arbitrary finite group* $G$ *and convolution perspectives* $\mathsf{P}_A = D_{p_A}$, $\mathsf{P}_B = D_{p_B}$ *in the regular representation,*

$$\delta(B|A) \;=\; \Delta_{\mathrm{conv}}(B|A) \;=\; \tfrac{1}{2}\min_{r\in\mathrm{Prob}(G)}\|p_B - r*p_A\|_1.$$

*Proof.* The bound $\delta\le\Delta_{\mathrm{conv}}$ is Theorem XIV.8 (random translations are admissible post-processings). For $\delta\ge\Delta_{\mathrm{conv}}$ we exhibit, for each dual vector, an entangled discrimination witness.

Fix $e\in[-1,1]^G$ and write the translate $(\delta_s * p_A)(h) = p_A(s^{-1}h)$; set $\mu_e := \max_{s\in G}\langle e,\delta_s * p_A\rangle$, so $|\mu_e|\le\|e\|_\infty\,\|\delta_s * p_A\|_1\le 1$. With $\{|\Omega_g\rangle\}$ the orthonormal family of Lemma XIV.14 and $P_\perp := I - \sum_g|\Omega_g\rangle\langle\Omega_g|$, define the Hermitian operator

$$E \;:=\; \sum_{h\in G} e(h)\,|\Omega_h\rangle\langle\Omega_h| \;+\; \mu_e\,P_\perp,$$

whose spectrum is $\{e(h)\}_h\cup\{\mu_e\}\subset[-1,1]$; thus $-I\preceq E\preceq I$. For any channel $\Lambda$, with pinched Choi diagonal $q_\Lambda$ and total captured mass $m_\Lambda = \sum_g q_\Lambda(g)\le 1$, the transfer identity $T_\Lambda(h,g) = q_\Lambda(hg^{-1})$ of Lemma XIV.14 gives, on the maximally entangled probe $|\Omega\rangle$,

$$\mathrm{Tr}\big[E\,(\Lambda D_{p_A}\otimes\mathbf{1})(|\Omega\rangle\langle\Omega|)\big] = \langle e, q_\Lambda * p_A\rangle + \mu_e\,(1-m_\Lambda) = \mu_e + \sum_{s\in G} q_\Lambda(s)\big(\langle e,\delta_s * p_A\rangle - \mu_e\big) \;\le\; \mu_e,$$

the first equality splitting $E$ into its diagonal part (contributing $\langle e, q_\Lambda * p_A\rangle$ via $\sum_g q_\Lambda(hg^{-1})p_A(g) = (q_\Lambda * p_A)(h)$) and the $P_\perp$ part (contributing $\mu_e(1-m_\Lambda)$ since each $(\Lambda\otimes\mathbf{1})(|\Omega_g\rangle\langle\Omega_g|)$ is a unit-trace state), and the inequality holding because $\langle e,\delta_s * p_A\rangle\le\mu_e$ for every $s$ while $q_\Lambda(s)\ge 0$. Since $\mathrm{Tr}[E\,(D_{p_B}\otimes\mathbf{1})(|\Omega\rangle\langle\Omega|)] = \langle e, p_B\rangle$ (the $P_\perp$ term annihilates the diagonal state $(D_{p_B}\otimes\mathbf{1})(|\Omega\rangle\langle\Omega|)$), and $\|\Phi\|_\diamond\ge\mathrm{Tr}[E\,(\Phi\otimes\mathbf{1})(|\Omega\rangle\langle\Omega|)]$ for $-I\preceq E\preceq I$,

$$\|D_{p_B} - \Lambda D_{p_A}\|_\diamond \;\ge\; \langle e, p_B\rangle - \mu_e \qquad \text{for every channel } \Lambda.$$

Taking the infimum over $\Lambda$ and then the supremum over $e\in[-1,1]^G$,

$$2\,\delta(B|A) \;\ge\; \max_{\|e\|_\infty\le 1}\Big(\langle e,p_B\rangle - \max_s\langle e,\delta_s * p_A\rangle\Big) \;=\; \min_{r\in\mathrm{Prob}(G)}\|p_B - r*p_A\|_1 \;=\; 2\,\Delta_{\mathrm{conv}}(B|A),$$

the middle equality being linear-programming duality for the total-variation distance to the convolution polytope $\{r*p_A : r\in\mathrm{Prob}(G)\} = \mathrm{conv}\{\delta_s * p_A\}_{s\in G}$. Hence $\delta\ge\Delta_{\mathrm{conv}}$, and with the reverse inequality $\delta=\Delta_{\mathrm{conv}}$. □

**Corollary XIV.17** (Frame degradation is classical)**.** *For convolution perspectives of a finite group $G$ in the regular representation the infimum defining $\delta(B|A)$ is attained at a* random translation*. If $r_\star \in \mathrm{Prob}(G)$ minimises $\|p_B - r * p_A\|_1$, then the classical post-processing $\Lambda_\star = D_{r_\star}$ satisfies*

$$\tfrac{1}{2}\big\|D_{p_B} - \Lambda_\star D_{p_A}\big\|_\diamond = \delta(B|A) = \Delta_{\mathrm{conv}}(B|A),$$

*since $D_{r_\star} D_{p_A} = D_{r_\star * p_A}$ and $\frac{1}{2}\|D_{p_B} - D_{r_\star * p_A}\|_\diamond = \frac{1}{2}\|p_B - r_\star * p_A\|_1$ by Lemma XIV.5. The optimal covariant degradation of frame $A$ toward frame $B$ is therefore implemented by sampling a single group element $g \sim r_\star$ and applying the translation $U(g)$; no channel exploiting the coherence or representation-multiplicity available on $\ell^2(G)$ achieves a smaller diamond deficiency. In particular $\delta(B|A) = 0$ precisely when $p_B = r * p_A$ for some $r \in \mathrm{Prob}(G)$, i.e. when $B$ is exactly a random-translation coarse-graining of $A$. For ideal finite quantum reference frames the resource theory of frame degradation is thus entirely classical: the quantum structure of the token enters the* value *$\delta$ only through the group convolution, never through a coherent advantage.*

**Remark XIV.18** (Scope of the exactness theorem)**.** *Theorem XIV.16 subsumes the Abelian equality of Theorem XIV.10 and the central reduction of Proposition XIV.12 as special cases, and settles affirmatively the question left open after Theorem C.3: for* every *finite group the finite convolution linear program computes the exact diamond-norm deficiency, so multiplicity-using covariant post-processings never beat random translations. The Heisenberg–Weyl argument of Theorem XIV.10 remains of independent interest because it identifies the optimal degradation as a* Pauli *(random-translation) channel explicitly, and the witness here was found precisely by repairing the deficiency of the naive maximally entangled bound diagnosed in Lemma XIV.14 and Proposition XIV.15. The construction is verified directly in `scripts/general_exactness_witness.py` on $S_3$, $D_4$, $Q_8$, $A_4$ and the Abelian $\mathbb{Z}_5$: the witness $E$ has spectrum in $[-1, 1]$, its value exceeds $2\Delta_{\mathrm{conv}}$ for every random channel sampled and meets it at the optimal translation, and the underlying identity matches to machine precision; the resulting deficiency agrees with the full diamond-norm semidefinite program wherever the latter is computable.*

**Remark XIV.19** (Regular-representation hypothesis and the continuous frontier)**.** *Two hypotheses are indispensable to Theorem XIV.16.* (i) The regular representation. *Orthonormality of $\{|\Omega_g\rangle\}$ – equivalently perfect distinguishability of the translates $U(g)$, encoded by $\mathrm{Tr}\, U(k) = |G|\,\delta_{k,e}$ – is what makes both Lemma XIV.5 and the witness work. For a non-regular finite-dimensional representation $V$ the operators $V(g)$ are in general not perfectly distinguishable, $\|D_\sigma\|_\diamond < \|\sigma\|_1$ can occur, Lemma XIV.5 fails, and the convolution linear program is then only an* upper *bound on $\delta$; the sub-normalisation gap diagnosed in Lemma XIV.14 and Proposition XIV.15 is exactly this effect seen through the maximally entangled probe. The regular representation is the* complete $G$-token *– the ideal frame whose Hilbert space resolves every group element – and it is for this canonical frame that the deficiency is computed exactly.* (ii) Finiteness. *The witness $E$ is built from the* normalisable *maximally entangled vector $|\Omega\rangle$ of the finite-dimensional regular representation. For a compact Lie group the regular representation acts on $L^2(G)$, which is infinite-dimensional, and $|\Omega\rangle$ is non-normalisable, so the construction has no direct analogue. The physically central frames – clocks ($U(1), \mathbb{R}$), gyroscopes ($SU(2)$), Lorentz boosts – are continuous, and whether $\delta = \Delta_{\mathrm{conv}}$ persists there is the natural conjecture: it is consistent with the explicit $U(1)$ computation of Theorem XI.2, a continuous Abelian instance in which the dephasing multiplier is the Thomas–Wigner characteristic function and the reverse deficiency equals the convolution value $\frac{1}{2}(\lambda_A - \lambda_B)_+$, and with finite-subgroup truncations of the integral, but we do not establish it here. The finite-group theorem should therefore be read as the exact, fully classical* core *of the resource theory, with the compact case as the principal open frontier.*

**Theorem XIV.20** (Distinguishability and metrological lower bound)**.** *For any two physical preparations $\rho, \sigma$ define*

$$D_R(\rho, \sigma) = \frac{1}{2}\|\mathsf{P}_R(\rho) - \mathsf{P}_R(\sigma)\|_1.$$

*Then*

$$\delta(B|A) \geq \frac{1}{2}\sup_{\rho,\sigma}\big(D_B(\rho,\sigma) - D_A(\rho,\sigma)\big)_+.$$

*Consequently any parameter family whose trace-distance, Chernoff or small-parameter Fisher distinguishability is strictly larger in the $B$-perspective than in the $A$-perspective supplies a quantitative no-go witness for an $A \to B$ transformation.*

*Proof.* Let $\Lambda$ be any CPTP post-processing from $A$ to the output space of $B$, and put

$$\epsilon_\Lambda = \frac{1}{2}\|\mathsf{P}_B - \Lambda \mathsf{P}_A\|_\diamond.$$

For any $\rho, \sigma$,

$$\begin{aligned} D_B(\rho,\sigma) &\le D(\Lambda \mathsf{P}_A\rho, \Lambda \mathsf{P}_A\sigma) \\ &\qquad + D(\mathsf{P}_B\rho, \Lambda \mathsf{P}_A\rho) + D(\mathsf{P}_B\sigma, \Lambda \mathsf{P}_A\sigma) \\ &\le D_A(\rho,\sigma) + 2\epsilon_\Lambda, \end{aligned}$$

using trace-norm contractivity of $\Lambda$. Rearranging and taking the supremum over $\rho, \sigma$ and then the infimum over $\Lambda$ gives (XIV.20). Chernoff and Fisher statements follow by applying the same inequality to nearby or repeated binary discrimination experiments. □

For a phase or clock frame this bound has a transparent form. Choose two states differing only by the sign of a coherence in charge mode $k$. If $\phi_R(k)$ is the frame characteristic function, then $D_R = |\phi_R(k)|$ on that binary experiment, hence

$$\delta(B|A) \ge \frac{1}{2} \max_k \big(|\phi_B(k)| - |\phi_A(k)|\big)_+.$$

Thus a noisier finite clock cannot simulate a sharper one even approximately without paying a deficiency at least equal to the lost coherent distinguishability.

**Definition XIV.21** (Perspective curvature)**.** *Choose approximate transformations $\Lambda_{A\to B}$ between several non-ideal perspectives. The curvature around an oriented triangle $A \to B \to C \to A$ is*

$$\Omega_{ABC} = \Lambda_{C\to A}\Lambda_{B\to C}\Lambda_{A\to B} - \mathbf{1}_A$$

*restricted to the operational image* $\operatorname{Ran} \mathsf{P}_A$*. Its norm*

$$\kappa(A,B,C) = \frac{1}{2}\|\Omega_{ABC}\|_\diamond$$

*is the holonomy error of the chosen calibration network.*

**Proposition XIV.22** (Flatness of exact quotient QRF theory)**.** *If all pairwise exact transformations are the quotient maps of Theorem IV.1, then every composable loop has zero curvature on the common quotient. Nonzero curvature can only arise from approximate recovery choices, incompatible kernels, or embeddings of quotient perspectives into larger non-quotient output algebras.*

*Proof.* For exact quotient maps,

$$\begin{aligned} T_{B\to C}T_{A\to B}\mathsf{P}_A(\rho) &= T_{B\to C}\mathsf{P}_B(\rho) \\ &= \mathsf{P}_C(\rho) = T_{A\to C}\mathsf{P}_A(\rho). \end{aligned}$$

Thus composition agrees with the direct quotient map on every preparable state. Around a loop with equal kernels, the composition is the identity on the quotient. If kernels are not mutually comparable, some edge is not an exact quotient map; if approximate inverses or external embeddings are chosen, the above equality no longer forces zero holonomy. □

This curvature is experimentally useful. A network of imperfect clocks and gyroscopes can be calibrated pairwise; the non-closure of a cycle is not a paradox but a quantitative witness that the chosen finite frames do not share a common exact quotient perspective.

## XV. COMPLETE BLACKWELL THEOREM FOR QRF PERSPECTIVES

The order $A \succeq B$ has an exact decision-theoretic meaning only after one specifies whether the perspective is used intrinsically as a quotient operator system or concretely as a Hilbert-space output on which arbitrary laboratory channels may act. The following hierarchy is the precise form needed to avoid a common overclaim.

**Proposition XV.1** (Comparison hierarchy)**.** *For finite-dimensional QRF perspectives the following implications hold:*

$$\exists\,\Lambda \in \mathsf{CPTP} \text{ with } \mathsf{P}_B = \Lambda \mathsf{P}_A$$
$$\Downarrow$$
$$\text{intrinsic complete-order quotient } A \to B$$
$$\Downarrow$$
$$\operatorname{Ker} \mathsf{P}_A \subseteq \operatorname{Ker} \mathsf{P}_B.$$

*The first line is equivalent to ancilla-complete Blackwell decision dominance for the chosen concrete output systems. The last line is equivalent to the existence of the unique affine quotient map. The middle line is the same factorization equipped with the matrix quotient cones. The reverse implication from the intrinsic quotient to the first line holds exactly when the concrete Choi/Arveson extension problem of Corollary IV.6 is feasible.*

*Proof.* A concrete CPTP degradation restricts to a completely positive map on every operational image and therefore gives the intrinsic quotient transformation; forgetting matrix cones gives the affine kernel condition. Conversely, kernel inclusion defines a unique affine factorization by Theorem IV.1, and the intrinsic quotient cones make it completely positive by Theorem IV.2. Extending that partially defined complete-order map to the full concrete output algebra is exactly the interpolation problem stated in Corollary IV.6. ☐

**Definition XV.2** (Ancilla-complete decision dominance)**.** *Frame $A$ Blackwell-dominates frame $B$, written $A \succeq_{\text{dec}} B$, when for every finite ensemble of physical states $\{\rho_x, p_x\}$, every ancillary system $E$, every payoff table, and every POVM/decision rule allowed after observing $(\mathsf{P}_B \otimes \mathbf{1}_E)(\rho_x)$, there exists a decision rule after observing $(\mathsf{P}_A \otimes \mathbf{1}_E)(\rho_x)$ with at least the same expected payoff.*

**Theorem XV.3** (Finite-dimensional Blackwell–QRF randomization theorem)**.** *For finite-dimensional physical sectors and fixed concrete output Hilbert spaces,*

$$A \succeq_{\text{dec}} B \quad \Longleftrightarrow \quad \exists\,\Lambda \in \mathsf{CPTP}(A \to B) \text{ such that } \mathsf{P}_B = \Lambda \mathsf{P}_A.$$

*Equivalently,*

$$A \succeq_{\text{dec}} B \quad \Longleftrightarrow \quad \delta(B|A) = 0,$$

*where $\delta$ is the diamond-norm deficiency of Definition XIV.1. Exact decision dominance therefore coincides with exact concrete post-processing, not merely with ordinary kernel inclusion unless the required complete-order extension exists.*

*Proof.* If $\mathsf{P}_B = \Lambda \mathsf{P}_A$, any $B$-decision procedure is simulated by first applying $\Lambda$ to the $A$ output and then using the same procedure. This proves decision dominance.

Conversely, suppose no CPTP $\Lambda$ satisfies $\mathsf{P}_B = \Lambda \mathsf{P}_A$. The set

$$\mathcal{C}_A = \{\Lambda \mathsf{P}_A : \Lambda \in \mathsf{CPTP}(A \to B)\}$$

is compact and convex in the finite-dimensional space of linear maps from the active physical operator system to $\mathcal{T}(\mathcal{H}_B)$. Since $\mathsf{P}_B \notin \mathcal{C}_A$, Hahn–Banach separation gives a Hermitian linear functional that is larger on $\mathsf{P}_B$ than on every element of $\mathcal{C}_A$. The finite-dimensional tester representation of such functionals realizes the separator as an ancilla-assisted ensemble, POVM and payoff. Hence there is a decision problem in which $B$ achieves a strictly larger value than every post-processing of $A$, contradicting $A \succeq_{\text{dec}} B$. Therefore a post-processing channel exists. The equivalence with $\delta(B|A) = 0$ follows because the CPTP set is compact, so the infimum in the diamond-norm definition is attained. ☐

This theorem is the QRF analogue of the Blackwell–Le Cam randomization principle. Its diagnostic content is now sharply separated from the intrinsic quotient theorem. If a proposed transformation fails at the concrete level, the failure is operationally witnessed by a finite decision problem. If it succeeds intrinsically but not concretely, the obstruction is not loss of information but the lack of a completely positive extension to the chosen oversized Hilbert-space representation.

**Corollary XV.4** (Finite decision witness for impossible concrete frame changes)**.** *If no concrete CPTP transformation $A \to B$ exists on the selected output systems, then there are finitely many physical preparations, a finite ancilla, and a finite-outcome measurement payoff for which the optimal value in the $B$ perspective is strictly larger than the optimal value obtainable from the $A$ perspective.*

*Proof.* This is exactly the separating functional constructed in the proof of Theorem XV.3, written in operational form through the duality between Hermitian forms on channels and tester/process POVM effects. ☐

## XVI. FINITE-DIMENSIONAL CERTIFICATION THEOREM

The kernel criterion is conceptually exact. In finite-dimensional experiments one also needs a certificate that can be computed from reconstructed process matrices. This section gives such a certificate.

Let $\mathsf{P}_A : \mathcal{T}(\mathcal{H}_0) \to \mathcal{T}(\mathcal{H}_A)$ and $\mathsf{P}_B : \mathcal{T}(\mathcal{H}_0) \to \mathcal{T}(\mathcal{H}_B)$ be finite-dimensional perspective channels. Choose an operator basis of $\mathcal{B}(\mathcal{H}_0)$ and write their Choi matrices as $J_A$ and $J_B$.

The complete-order quotient theorem gives the intrinsic answer. The following Choi test answers a stronger laboratory question: whether the intrinsic quotient map is representable by a physical channel acting on the full concrete Hilbert space used to encode $A$'s perspective. A concrete channel $\Lambda : \mathcal{T}(\mathcal{H}_A) \to \mathcal{T}(\mathcal{H}_B)$ realizes $A \to B$ exactly iff

$$J_\Lambda \geq 0, \qquad \mathrm{Tr}_B\, J_\Lambda = \mathbf{1}_A, \qquad J_B = (\Lambda \otimes \mathbf{1}_0)(J_A). \tag{17}$$

Thus exact QRF conversion is a semidefinite feasibility problem. If it is infeasible, the separating hyperplane theorem produces a Hermitian dual witness $Y$ such that all channels $\Lambda$ obey

$$\mathrm{Tr}[Y(\Lambda \otimes \mathbf{1}_0)(J_A)] \geq 0, \qquad \mathrm{Tr}(Y J_B) < 0,$$

after absorbing the trace-preservation constraints into the affine dual. This is an experimentally checkable no-go certificate.

**Theorem XVI.1** (Unique linear quotient and Choi witness)**.** *Assume that $\mathsf{P}_A$ is injective on the active physical operator subspace $\mathcal{V} \subseteq \mathcal{B}(\mathcal{H}_0)$ and that $\mathsf{P}_B$ is specified on the same subspace. Then there is a unique linear quotient*

$$L_{A\to B} : \mathsf{P}_A(\mathcal{V}) \to \mathsf{P}_B(\mathcal{V}), \qquad L_{A\to B}\mathsf{P}_A = \mathsf{P}_B.$$

*A deterministic physical transformation exists on this active sector iff $L_{A\to B}$ has a completely positive trace-preserving extension to $\mathcal{B}(\mathcal{H}_A)$. In the informationally complete case $\mathsf{P}_A(\mathcal{V}) = \mathcal{B}(\mathcal{H}_A)$ this reduces to*

$$J(L_{A\to B}) \geq 0, \qquad \mathrm{Tr}_B\, J(L_{A\to B}) = \mathbf{1}_A.$$

*Any negative eigenvector of $J(L_{A\to B})$ is a dual witness against exact physical implementation.*

*Proof.* Injectivity on $\mathcal{V}$ makes $L_{A\to B}$ well-defined and unique. If a CPTP $\Lambda$ implements the transformation, its restriction to $\mathsf{P}_A(\mathcal{V})$ equals $L_{A\to B}$ and is therefore a CPTP extension. Conversely any CPTP extension gives $\Lambda\mathsf{P}_A = \mathsf{P}_B$ on $\mathcal{V}$. In the informationally complete case no extension outside $\mathcal{B}(\mathcal{H}_A)$ is needed, and Choi's theorem gives the positivity criterion. A negative Choi eigenvector separates $J(L_{A\to B})$ from the cone of positive Choi matrices. ☐

**Proposition XVI.2** (Operational meaning of deficiency)**.** *For any $\epsilon > \delta(B|A)$ there exists a channel $\Lambda : A \to B$ such that, for every physical state $\rho$, every ancilla $E$, and every two-outcome measurement effect $0 \leq M \leq \mathbf{1}$ in the $B$-perspective,*

$$\left|\mathrm{Tr}\big[M\big((\mathsf{P}_B \otimes \mathbf{1}_E)(\rho) - (\Lambda\mathsf{P}_A \otimes \mathbf{1}_E)(\rho)\big)\big]\right| \leq \epsilon.$$

*Conversely, the smallest uniform bound over all entangled input tests and all binary measurements is exactly $\delta(B|A)$.*

*Proof.* The optimal binary discrimination bias between two channels is one half of their diamond-norm distance. Minimizing over $\Lambda$ gives (XIV.1). The ancilla is necessary for complete boundedness and for detecting nonclassical perspective information. ☐

**Proposition XVI.3** (Monotonicity of common information)**.** *If $A \succeq B$, i.e. $\mathsf{P}_B = \Lambda\mathsf{P}_A$ for some CPTP $\Lambda$, then*

$$\mathrm{Ran}\,\mathsf{P}_B^* \subseteq \mathrm{Ran}\,\mathsf{P}_A^*.$$

*Consequently every $B$-accessible invariant effect is already $A$-accessible. For a parametrized physical model $\rho_\theta$, any monotone distinguishability functional, including trace distance, relative entropy and quantum Fisher information whenever defined, cannot increase under $A \to B$.*

*Proof.* In the Heisenberg picture $\mathsf{P}_B^* = \mathsf{P}_A^*\Lambda^*$, proving the range inclusion. Monotonicity of the listed distinguishability functionals is the standard data-processing inequality for CPTP maps applied to $\mathsf{P}_A(\rho_\theta) \mapsto \mathsf{P}_B(\rho_\theta)$. ☐

### A. Robust finite-sample certificates

Experimental QRF comparison uses empirical reconstructions $\widehat{\mathsf{P}}_A, \widehat{\mathsf{P}}_B$ rather than exact channels. The quotient criterion remains useful because infeasibility is stable under a dual margin.

**Proposition XVI.4** (Robust no-go witness)**.** *Let*

$$\widehat{\delta}(B|A) = \frac{1}{2} \inf_{\Lambda \in \mathsf{CPTP}} \|\widehat{\mathsf{P}}_B - \Lambda \widehat{\mathsf{P}}_A\|_\diamond$$

*be the empirical deficiency obtained from reconstructed channels. If*

$$\|\widehat{\mathsf{P}}_A - \mathsf{P}_A\|_\diamond \le \varepsilon_A, \qquad \|\widehat{\mathsf{P}}_B - \mathsf{P}_B\|_\diamond \le \varepsilon_B,$$

*then*

$$\delta(B|A) \ge \widehat{\delta}(B|A) - \frac{\varepsilon_A + \varepsilon_B}{2}. \tag{18}$$

*Consequently $\widehat{\delta}(B|A) > (\varepsilon_A + \varepsilon_B)/2$ is a robust certificate that no exact concrete CPTP transformation $A \to B$ exists. Any SDP dual lower bound on $\widehat{\delta}(B|A)$ may be substituted for $\widehat{\delta}(B|A)$ in this criterion.*

*Proof.* For every channel $\Lambda$,

$$\begin{aligned} \|\mathsf{P}_B - \Lambda \mathsf{P}_A\|_\diamond \ge{}& \|\widehat{\mathsf{P}}_B - \Lambda \widehat{\mathsf{P}}_A\|_\diamond \\ &- \|\mathsf{P}_B - \widehat{\mathsf{P}}_B\|_\diamond - \|\Lambda(\mathsf{P}_A - \widehat{\mathsf{P}}_A)\|_\diamond . \end{aligned}$$

The diamond norm is contractive under post-processing by $\Lambda$, so the last term is bounded by $\varepsilon_A$. Taking the infimum over $\Lambda$ and multiplying by $1/2$ gives (18). If the right-hand side is positive, then $\delta(B|A) > 0$, hence the exact post-processing equation $\mathsf{P}_B = \Lambda \mathsf{P}_A$ is impossible. ☐

This turns the mathematical criterion into a laboratory protocol: reconstruct the source and target perspective channels by process tomography or randomized benchmarking adapted to the relational sector, solve the Choi interpolation SDP, and report either a channel with confidence radius or a dual no-go witness with a positive margin.

## XVII. STABILITY, ENTROPY BOUNDS AND CURVATURE OF IMPERFECT PERSPECTIVES

The quotient theorem is exact. The next results make it usable for noisy experiments and for the current entropy-based literature on QRFs.

**Theorem XVII.1** (Information monotonicity under exact perspective degradation)**.** *If $A \succeq B$, i.e. $\mathsf{P}_B = \Lambda \mathsf{P}_A$ for a CPTP channel $\Lambda$, then for all physical states $\rho, \sigma$ and all ancillary systems $E$,*

$$\frac{1}{2} \left\|(\mathsf{P}_B \otimes \mathbf{1}_E)(\rho - \sigma)\right\|_1 \le \frac{1}{2} \left\|(\mathsf{P}_A \otimes \mathbf{1}_E)(\rho - \sigma)\right\|_1 ,$$

*and*

$$\begin{aligned} &D((\mathsf{P}_B \otimes \mathbf{1}_E)\rho \,\|\, (\mathsf{P}_B \otimes \mathbf{1}_E)\sigma) \\ &\quad \le D((\mathsf{P}_A \otimes \mathbf{1}_E)\rho \,\|\, (\mathsf{P}_A \otimes \mathbf{1}_E)\sigma) . \end{aligned} \tag{19}$$

*Consequently no hypothesis test, Fisher-information functional obtained by local quadratic expansion of relative entropy, or entanglement monotone based only on the accessible $B$ state can increase under passage from $A$ to $B$.*

*Proof.* Both inequalities are the data-processing inequalities for trace distance and Umegaki relative entropy applied to the channel $\Lambda \otimes \mathbf{1}_E$. The final statement follows because binary discrimination advantage, local statistical distinguishability and entanglement monotones of the accessible state are monotone under CPTP post-processing. ☐

**Theorem XVII.2** (Continuity of observer-dependent entropy from deficiency)**.** *Let $\delta(B|A) \le \epsilon$ and let $\Lambda$ be a channel satisfying*

$$\frac{1}{2}\|\mathsf{P}_B - \Lambda\mathsf{P}_A\|_\diamond \le \epsilon.$$

*For every physical state $\rho$ whose $B$-perspective lives in dimension $d_B$,*

$$\frac{1}{2}\|\mathsf{P}_B(\rho) - \Lambda\mathsf{P}_A(\rho)\|_1 \le \epsilon.$$

*If $\epsilon \le 1 - 1/d_B$, then*

$$|S(\mathsf{P}_B(\rho)) - S(\Lambda\mathsf{P}_A(\rho))| \le \epsilon\log(d_B - 1) + h_2(\epsilon),$$

*where $h_2$ is the binary entropy. Thus deficiency gives a direct quantitative upper bound on how much a less informative frame can fail to reproduce the entropy assignment of a more demanding frame.*

*Proof.* The trace-norm statement is the definition of the diamond norm without using the ancillary supremum. The entropy estimate is the Audenaert–Fannes continuity bound applied to the two $d_B$-dimensional states. ☐

**Proposition XVII.3** (Curvature bound for an observer loop)**.** *Let $\Lambda_{AB}, \Lambda_{BC}, \Lambda_{CA}$ be chosen calibration channels with*

$$\begin{aligned}
\epsilon_{AB} &= \frac{1}{2}\|\mathsf{P}_B - \Lambda_{AB}\mathsf{P}_A\|_\diamond,\\
\epsilon_{BC} &= \frac{1}{2}\|\mathsf{P}_C - \Lambda_{BC}\mathsf{P}_B\|_\diamond,\\
\epsilon_{CA} &= \frac{1}{2}\|\mathsf{P}_A - \Lambda_{CA}\mathsf{P}_C\|_\diamond.
\end{aligned} \tag{20}$$

*Then the experimentally observed holonomy on the $A$-preparable sector satisfies*

$$\frac{1}{2}\left\|\Lambda_{CA}\Lambda_{BC}\Lambda_{AB}\mathsf{P}_A - \mathsf{P}_A\right\|_\diamond \le \epsilon_{AB} + \epsilon_{BC} + \epsilon_{CA}.$$

*In particular, exact quotient transformations have zero curvature on every closed loop, while nonzero holonomy lower-bounds the sum of calibration defects around the loop.*

*Proof.* Insert and subtract $\Lambda_{CA}\Lambda_{BC}\mathsf{P}_B$ and $\Lambda_{CA}\mathsf{P}_C$, then use the triangle inequality and contractivity of the diamond norm under channels. ☐

## XVIII. NORMAL AND CROSSED-PRODUCT FORM

The previous statements were written in finite dimension because that is the level at which experiments and numerical certificates are most transparent. The same quotient structure survives in algebraic QFT and gravitational applications, but the topology must be stated carefully.

Let $\mathfrak{A}_{\mathsf{Phys}}$ be a von Neumann algebra of gauge-invariant physical observables, or a crossed product algebra obtained after adjoining observer degrees of freedom. A perspective is a normal unital completely positive map

$$\Phi_R : \mathfrak{A}_R \to \mathfrak{A}_{\mathsf{Phys}},$$

with predual normal channel

$$\Phi_{R*} : \mathfrak{A}_{\mathsf{Phys}*} \to \mathfrak{A}_{R*}.$$

A normal concrete transformation from $A$ to $B$ is a normal channel $\Lambda_{A\to B}$ satisfying

$$\Phi_{B*} = \Lambda_{A\to B}\Phi_{A*}.$$

**Theorem XVIII.1** (Normal QRF factorization)**.** *A normal intrinsic transformation $A \to B$ exists on the predual quotients if and only if*

$$\operatorname{Ker}\Phi_{A*} \subseteq \operatorname{Ker}\Phi_{B*}.$$

*Equivalently, at matrix levels,*

$$\operatorname{Ker}(\mathbf{1}_n \otimes \Phi_{A*}) \subseteq \operatorname{Ker}(\mathbf{1}_n \otimes \Phi_{B*}) \qquad \forall n$$

*for the associated matrix preduals. When it exists, the transformation is unique on the normal Banach quotient and completely positive for the normal matrix quotient cones. A concrete normal channel on a selected representation exists if and only if the induced Heisenberg operator-system map admits a normal unital completely positive extension to the selected von Neumann algebra.*

*Proof.* The predual maps are bounded, hence their kernels are norm-closed. Quotienting the predual by $\operatorname{Ker}\Phi_{R*}$ gives a Banach ordered predual quotient; its dual is the weak-$*$-closed annihilator $(\operatorname{Ker}\Phi_{R*})^{\perp}$. Kernel inclusion gives the unique normal quotient map by the same factorization argument as in Theorem IV.1. Applying the construction to matrix preduals gives complete positivity with respect to the quotient cones. Passing from this intrinsic map to a concrete channel is precisely the normal unital CP extension problem in the Heisenberg picture. ☐

This is the point at which the result interfaces with gravitational entropy. Different observers may define different crossed product algebras and hence different entropy functionals. The theorem states the exact comparison criterion at the operational level: observer $A$ can reproduce observer $B$'s normal predictions precisely when $B$'s normal kernel contains $A$'s kernel and the desired concrete representation admits the corresponding normal CP extension. If not, their entropy disagreement is not a coordinate artefact; it is witnessed by a normal observable in the annihilator-defined invariant system that is not simulable from $A$.

## XIX. STINESPRING UNIQUENESS OF MICROSCOPIC FRAME IMPLEMENTATIONS

Different microscopic models of a finite clock or gyroscope can yield the same operational perspective. This is the reference-frame version of Stinespring uniqueness and its stability under small channel perturbations [57, 59]. The channel formalism explains exactly what is unique.

**Theorem XIX.1** (Operational uniqueness of frame dilations)**.** *Let $\mathsf{P}_R : \mathcal{T}(\mathcal{H}_{\text{phys}}) \to \mathcal{T}(\mathcal{H}_R)$ be a perspective channel. Any two minimal Stinespring representations*

$$\mathsf{P}_R(\rho) = \operatorname{Tr}_E\big[V\rho V^{\dagger}\big] = \operatorname{Tr}_{E'}\big[V'\rho V'^{\dagger}\big]$$

*are related by a unitary $W : E \to E'$ such that*

$$V' = (\mathbf{1}_R \otimes W)V.$$

*Consequently the operational QRF perspective determines the observer's microscopic dilation uniquely up to an inaccessible environmental unitary. Non-minimal dilations differ only by adding unused environment.*

*Proof.* This is Stinespring's uniqueness theorem applied to $\mathsf{P}_R$. Minimality means that the span of $(\langle\phi| \otimes \mathbf{1}_E)V|\psi\rangle$ is dense in the dilation environment. The equality of the two channels defines an isometry between these spans, and minimality makes it unitary. ☐

The theorem is important for the uniqueness problem. It says that the theory does not need to choose between different hardware descriptions of a non-ideal observer. What is invariant is the operator-system quotient $\operatorname{Ran}\mathsf{P}_R^*$ and the complete kernel of $\mathsf{P}_R$; all minimal microscopic models realizing the same quotient are equivalent up to an unobservable unitary on lost degrees of freedom.

## XX. WORKED MICROSCOPIC MODEL

Consider three systems: two frames $A, B$ and a system $S$, all carrying a $U(1)$ charge representation. Let the ideal relational basis be charge differences. Suppose frame $R$ has phase uncertainty with Fourier coefficients

$$c_R(k) = e^{-\sigma_R^2 k^2/2}.$$

Then

$$\mathsf{P}_R(\rho_{mn}) = e^{-\sigma_R^2(m-n)^2/2}\rho_{mn}.$$

If $\sigma_B^2 > \sigma_A^2$, then

$$\frac{c_B(k)}{c_A(k)} = e^{-(\sigma_B^2-\sigma_A^2)k^2/2}$$

is positive definite and corresponds to additional Gaussian phase noise. Therefore $A \to B$ exists as a CPTP channel:

$$T_{A\to B}(\cdot) = \int_0^{2\pi} U(\theta)(\cdot)U(\theta)^\dagger\, \nu_{BA}(\mathrm{d}\theta),$$

where, with periodic wrapping understood,

$$\nu_{BA}(\mathrm{d}\theta) = \frac{e^{-\theta^2/[2(\sigma_B^2-\sigma_A^2)]}}{\sqrt{2\pi(\sigma_B^2-\sigma_A^2)}}\,\mathrm{d}\theta.$$

with periodic wrapping understood. The reverse map would multiply by $e^{+(\sigma_B^2-\sigma_A^2)k^2/2}$ and is not positive. Thus a sharper phase reference can simulate a blurrier one, but not conversely.

If $A$ is a finite-dimensional clock with $c_A(k) = 0$ for $|k| > N$ and $B$ has nonzero $c_B(N+1)$, then no deterministic transformation $A \to B$ exists. Additional priors may choose a reconstruction, but it is not a frame transformation: it imports information not contained in the $A$-perspective.

## XXI. FINITE-MODE NUMERICAL CHECKS

The structural theorem is exact, but it is important to test its content in finite models where every map is an explicit finite matrix. The finite-mode tests consist of sixteen independent checks: a commutative cyclic-frame model, a finite-Abelian mode-support lattice, a primal–dual convolution certificate, a witness-level robust dual margin, Fourier sharpness monotones, a finite-sample statistical certificate, robust Fourier-mode calibration, a common-descendant observer-web LP, a central non-Abelian $S_3$ certificate, a noncommutative Pauli-transfer model, an error-corrected relational sector, a dynamical/resource-witness test for repeated finite clocks, observer-network fan-out/holonomy tests, a diamond-norm exactness test for the convolution LP, a general non-central $S_3$ covariant-degradation certificate, and a relativistic boosted-spin Fisher-information test.

### A. Cyclic frame on $\mathbb{Z}_{31}$

Let $G = \mathbb{Z}_N$ with $N = 31$. A translation-covariant non-ideal frame is a circulant convolution kernel $p_R$; in the charge basis the associated perspective map multiplies the $k$th mode by

$$\phi_R(k) = \sum_{g\in\mathbb{Z}_N} p_R(g)e^{2\pi i kg/N}.$$

The deterministic quotient map $A \to B$ is diagonal in Fourier variables,

$$\widehat{T}_{A\to B}(k) = \frac{\phi_B(k)}{\phi_A(k)}$$

whenever $\phi_A(k) \neq 0$. Complete positivity is equivalent, in this finite commutative sector, to positive definiteness of this ratio, or equivalently to non-negativity of the inverse discrete Fourier transform $r_{BA}$.

The numerical instance was

$$p_A(0) = 0.6, \qquad p_A(\pm 1) = 0.2,$$

and an additional blur

$$r(0) = 0.5, \qquad r(\pm 1) = 0.25, \qquad p_B = p_A * r.$$

The recovered transfer kernel

$$r_{BA} = \mathrm{IDFT}[\phi_B/\phi_A]$$

is non-negative up to roundoff and reproduces the imposed blur $r$. The formal inverse $B \to A$ exists as a linear deconvolution because no Fourier multiplier vanishes, but its kernel has large negative components and hence is not a completely positive map. This verifies the distinction between algebraic uniqueness on the quotient and physical implementability as a channel.

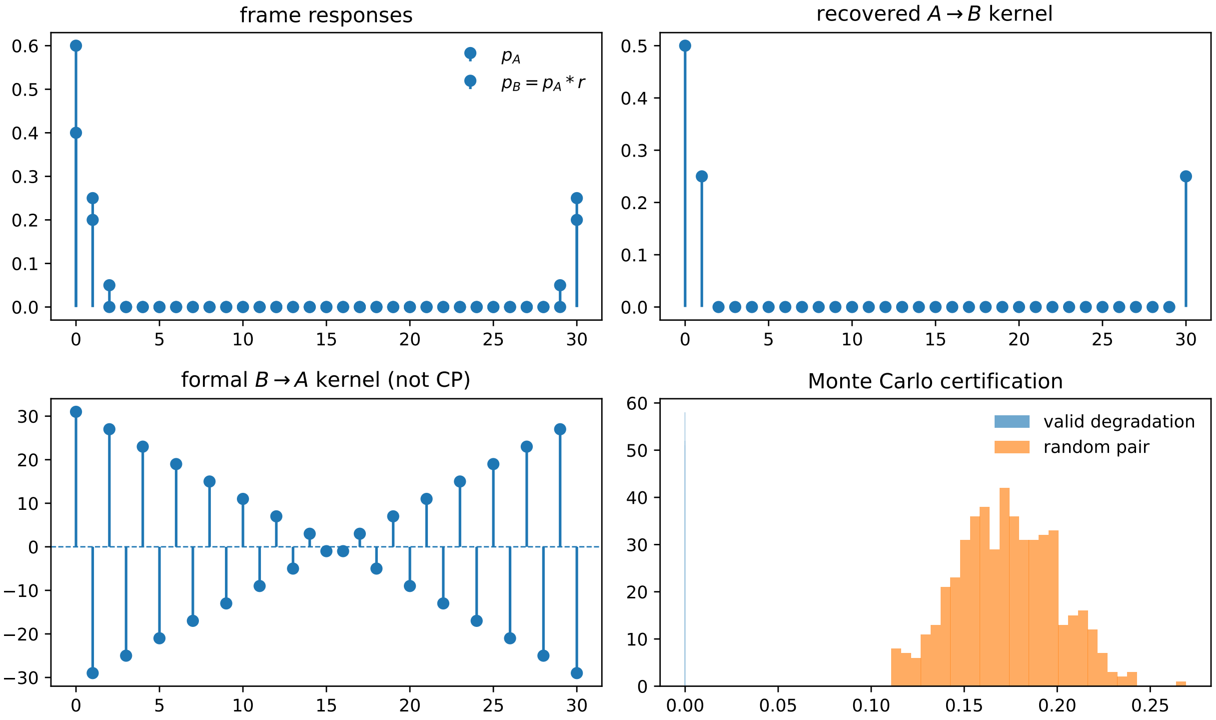

FIG. 1. Finite-mode verification on $\mathbb{Z}_{31}$. Top left: a sharper frame response $p_A$ and its degraded response $p_B = p_A * r$. Top right: the recovered $A \to B$ kernel is the imposed non-negative blur $r$. Bottom left: the formal reverse deconvolution has negative weights, so it is not a quantum channel although the linear quotient map is unique on the active image. Bottom right: in 500 random valid degradations the recovered kernel had zero error within numerical precision, whereas unrelated random pairs produced positive negative-mass witnesses.

The same finite model gives a direct implementation of Theorem VI.3 and Proposition VI.4. In the $Z_{32}$ support example the surviving mode sets satisfy

$$S_B \subset S_A, \qquad |S_A| = 16, \quad |S_B| = 8, \quad |S_A \cap S_B| = 8,$$

so $A$ determines $B$ intrinsically, while $B$ cannot determine the modes lost by its larger zero set. For the $Z_{31}$ convolution example the primal LP gives $\Delta_{\mathrm{conv}}(B|A) = 0$, whereas the reverse direction gives $\Delta_{\mathrm{conv}}(A|B) = 0.2$. The dual witness attains the same value within numerical precision and therefore supplies a certificate rather than merely a fitted inverse kernel.

The dual LP witness can also be reported as a robust finite-data margin rather than only as an optimizer for the exact or empirical LP. Theorem VI.5 gives a certificate of the form

$$\Delta_{\mathrm{conv}}(B|A) \geq \frac{1}{2}\left[\langle y, \widehat{p}_B\rangle - \max_g \langle y, \tau_g \widehat{p}_A\rangle - \varepsilon_A - \varepsilon_B\right],$$

which remains valid for the unknown true responses. The same example also illustrates the fast necessary filter supplied by Proposition VI.6: every covariant degradation reduces the nontrivial Fourier multiplier moduli and hence every weighted sharpness $M_w$.

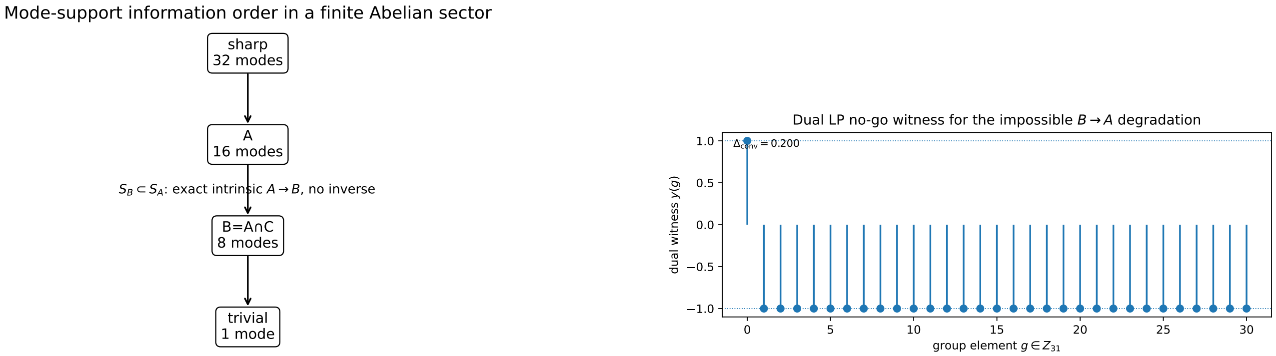

FIG. 2. Left: finite-Abelian mode-support lattice for a $Z_{32}$ clock sector. Intrinsic information flows downward by support inclusion; the common reversible core is the support intersection. Right: an optimal dual LP witness $y$ separating the sharp response $p_A$ from all covariant degradations of the noisier response $p_B$ in the impossible reverse direction.

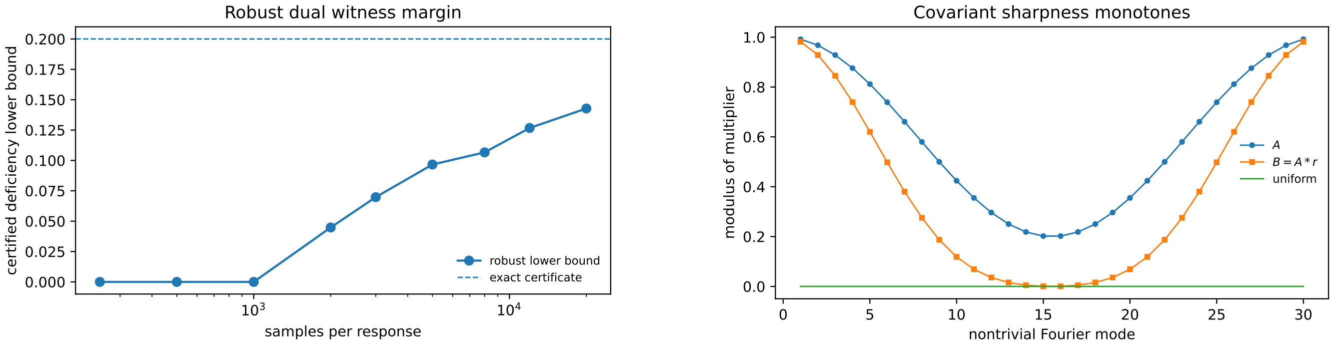

FIG. 3. Left: robust dual-witness lower bound for the impossible reverse $B \to A$ simulation after finite calibration uncertainty is subtracted from the empirical margin. Right: Fourier sharpness monotones; the degraded response $B = A * r$ has smaller nontrivial multiplier moduli than $A$, as required by any covariant post-processing.

### B. Convex deficiency certificate on $\mathbb{Z}_{31}$

The same cyclic instance was also tested through the simplex certificate of Proposition XIV.4. A linear program over probability distributions $r$ minimized

$$\|p_B - C(p_A)r\|_1.$$

TABLE I. Diagnostics for the finite cyclic model. The residual tests $\phi_B = (\phi_B/\phi_A)\phi_A$. The negative mass is $\sum_g \max\{0, -r(g)\}$ for a recovered kernel.

| diagnostic | value |
|---|---|
| $\|r_{BA} - r\|_2$ | $2.67 \times 10^{-16}$ |
| $\min_g r_{BA}(g)$ | $-3.66 \times 10^{-17}$ |
| $\min_g r_{AB}(g)$ | $-2.90 \times 10^{1}$ |
| reverse negative mass | $2.40 \times 10^{2}$ |
| valid degradations | 500/500 |
| max valid recovery error | $3.58 \times 10^{-16}$ |
| random median negative mass | $1.73 \times 10^{-1}$ |

For the physically degraded direction $A \to B$, the optimum was zero within solver tolerance. For the attempted reverse $B \to A$, the optimum was $4.0 \times 10^{-1}$, i.e. the diagonal-token lower-bound certificate gives

$$\Delta_{\mathrm{conv}}(A|B) = 2.0 \times 10^{-1}.$$

Thus the impossibility of the reverse transformation is visible already in a finite classical subtest, before using the full diamond-norm SDP.

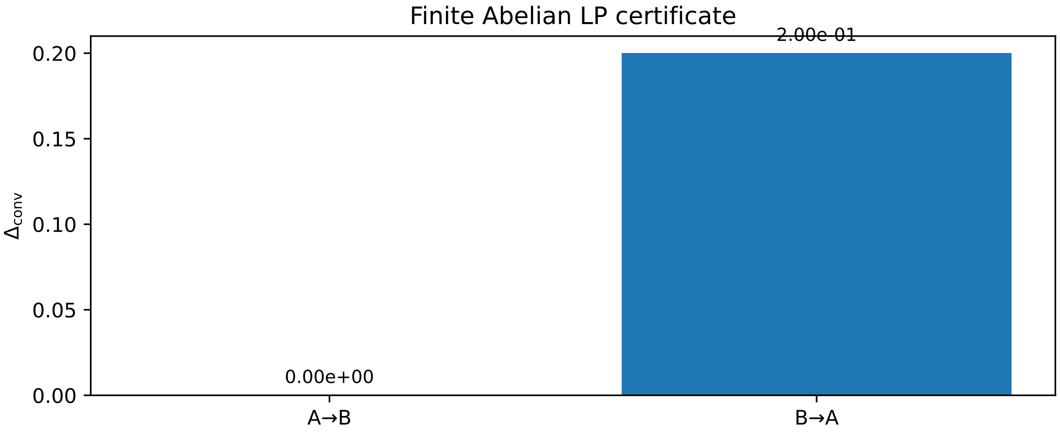

FIG. 4. Linear-programming deficiency certificate for the cyclic frame. The forward noisy-frame simulation $A \to B$ lies exactly in the convolution cone generated by $p_A$. The reverse attempt $B \to A$ has a positive $L^1$ distance from the convolution cone generated by $p_B$, producing a finite separating witness.

### C. Finite-sample no-go certification

The LP witness is stable under finite calibration statistics. For the same $Z_{31}$ reverse test $B \to A$, empirical histograms were drawn from $p_B$ and $p_A$ and inserted into the linear program of Theorem VI.8. The plotted lower curve is

$$\widehat{\Delta}_{\mathrm{conv}}(A|B) - \frac{\varepsilon_A + \varepsilon_B}{2},$$

with the finite-alphabet $\ell_1$ radius (14) at confidence 95%; the older coordinatewise Hoeffding curve is shown only as a conservative comparison. At low sample sizes the bound is intentionally inconclusive. Once the confidence radius is smaller than the LP separation margin, the same finite data certify that the reverse covariant transformation is impossible.

### D. Robust Fourier-mode calibration

The exact mode-support lattice is a theorem about calibrated response measures, not a claim that exact Fourier zeros can be inferred from finite frequencies alone. The $Z_{32}$ test used

$$p(0) = p(16) = 1/2.$$

Its true Fourier support consists of the 16 even modes. From $m = 5000$ calibration samples, Proposition VI.9 certifies all 16 nonzero even modes by the margin condition

$$|\widehat{\widehat{p}}(\ell)| > \varepsilon^{\mathrm{W}},$$

while the 16 odd modes remain statistically ambiguous rather than being overclaimed as proven zeros. This is the intended standard for experimental reporting: use Fourier intervals for certified active modes and use the LP/dual certificate for finite-data no-go statements.

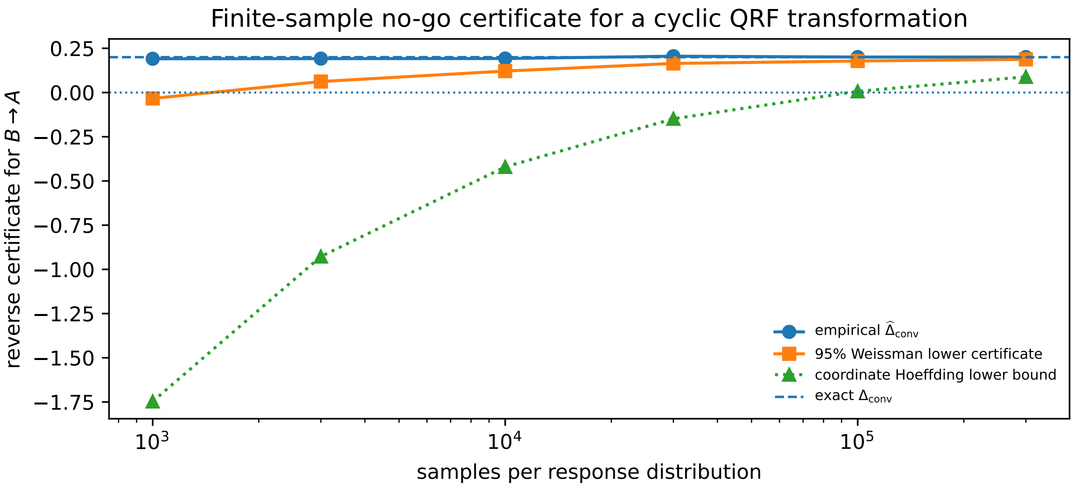

FIG. 5. Finite-sample certificate for the attempted reverse $B \to A$ transformation on $\mathbb{Z}_{31}$. The empirical LP estimate concentrates near the exact value $\Delta_{\mathrm{conv}} = 0.2$. The Weissman confidence-corrected lower bound becomes positive at a substantially smaller calibration size than the coordinatewise Hoeffding bound, giving a statistically valid no-go certificate rather than a purely model-level witness.

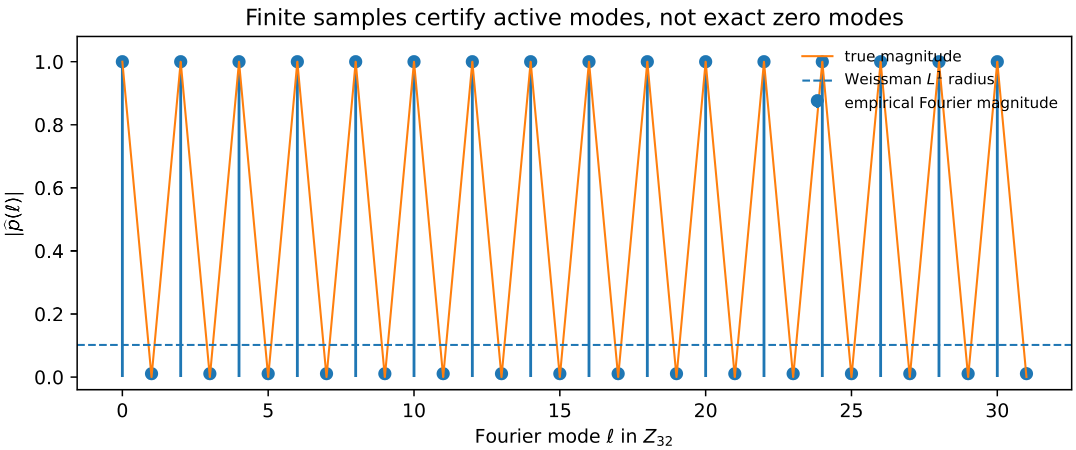

FIG. 6. Robust Fourier calibration on a $Z_{32}$ clock with exact even-mode support. The horizontal line is the Weissman $L^1$ radius, which also bounds every Fourier coefficient error. Even modes are certified active; odd modes are not certified zero from finite data alone. The figure prevents a common but serious overclaim: finite samples certify nonzero modes above margin, not exact zero modes.

### E. Common descendant for an observer web

A multi-observer experiment also needs a reproducible way to choose the part of the relational information that all calibrated frames can simulate. Proposition VI.7 turns this into a finite LP. In the $Z_{31}$ test, two incomparable token responses

$$p_A(0) = 0.62, \quad p_A(\pm 1) = 0.19, \qquad p_C(0) = 0.58, \quad p_C(\pm 2) = 0.21$$

were constrained to simulate a common descendant $q$, while maximizing $q(0)$ as a simple sharpness score. The solver found

$$q = p_A * r_A = p_C * r_C$$

with $q(0) = 0.3596$ and residuals below $10^{-15}$. This is the finite-token version of a path-consistent observer-web core: it is not necessarily equal to any input frame, but it is physically reachable from each of them by explicit stochastic post-processing.

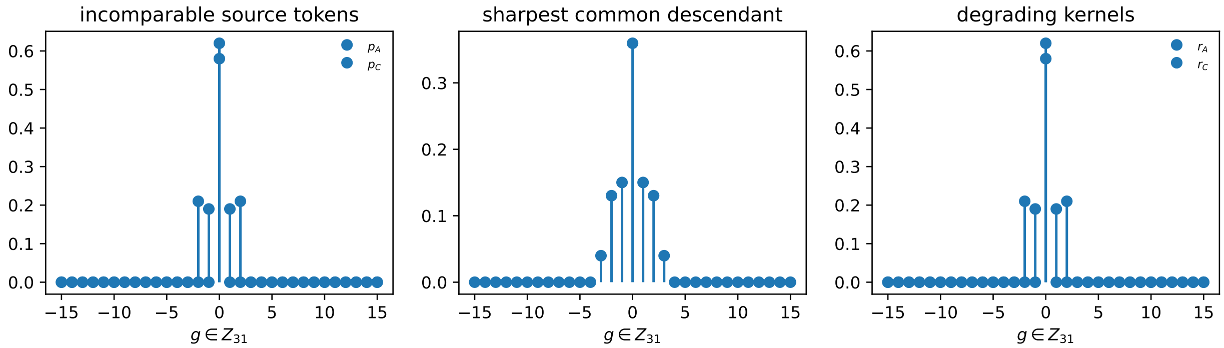

FIG. 7. Common-descendant LP for two incomparable finite cyclic frames. Left: the two source token responses. Middle: the sharpest common descendant under the identity-mass score. Right: the two degrading kernels certifying that both original observers simulate the same descendant perspective.

### F. Central non-Abelian certificate on $S_3$

The Abelian LP certificate is not the whole finite-group story: non-Abelian frames have matrix or character blocks. To verify the block criterion in a minimal non-Abelian case, the code implements central response kernels on $S_3$. The three conjugacy classes are the identity, the three transpositions and the two 3-cycles. For a central probability distribution $p$, the Fourier data are the three scalars

$$a_p(\lambda) = \frac{1}{d_\lambda} \sum_{g \in S_3} p(g) \chi_\lambda(g^{-1}), \qquad \lambda \in \{\mathbf{1}, \mathrm{sgn}, \mathrm{std}\}.$$

The tested instance uses

$$p_A = (0.62, 0.28, 0.10), \qquad r = (0.55, 0.30, 0.15)$$

as total class masses and sets $p_B = p_A * r$. The recovered forward character ratio $a_B/a_A$ reconstructs $r$ with $L_1$ error below $2 \times 10^{-15}$. The reverse character ratio reconstructs a signed class function with minimum class mass $-0.75$ and negative mass 0.986842. Thus the non-Abelian test reproduces the central message of the paper in a genuinely noncommutative group: the affine quotient can be uniquely defined on active blocks, while physical degradability is exactly positivity of the reconstructed postprocessing kernel.

### G. Maximal reversible core of two finite clocks

The quotient theorem predicts a simple spectral diagnostic for finite clocks. In the example $G = \mathbb{Z}_{32}$, frame $A$ has response distribution $(\delta_0 + \delta_{16})/2$ and keeps precisely the even Fourier modes, whereas frame $B$ has response distribution $(\delta_0 + \delta_8 + \delta_{16} + \delta_{24})/4$ and keeps precisely the modes divisible by four. Hence $S_B \subset S_A$: an exact affine transformation $A \to B$ exists and is realized by further convolution, while $B \to A$ is impossible because $B$ has erased modes that $A$ still detects. The maximal bidirectionally reversible core is not $A$ or $B$, but the quotient supported on $S_A \cap S_B = S_B$, with dimension eight at the scalar Fourier level. This is the finite-mode illustration of Theorem V.7.

### H. Qubit Pauli-transfer quotient

A second check uses qubit Pauli-covariant perspectives. A Pauli channel acts on the Bloch vector by

$$(x, y, z) \mapsto (\lambda_1 x, \lambda_2 y, \lambda_3 z).$$

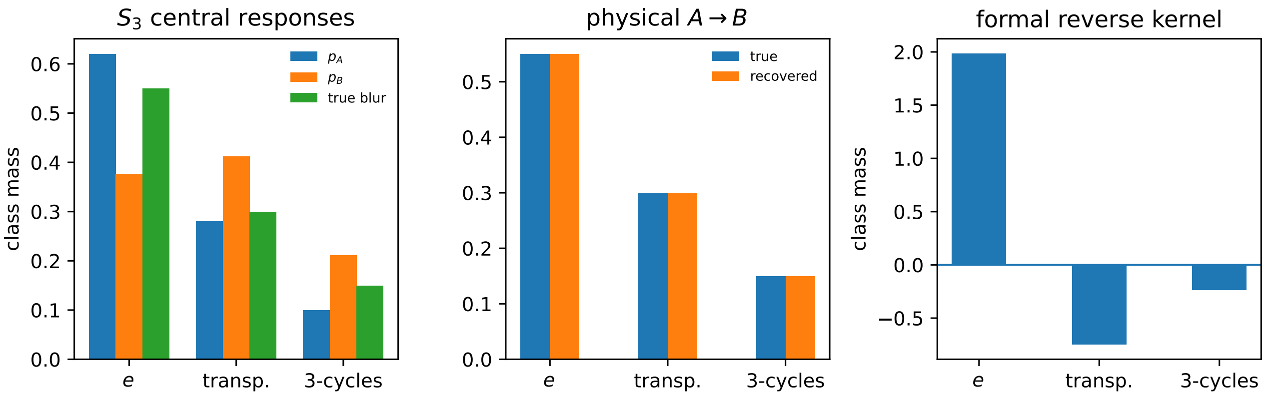

FIG. 8. Central finite non-Abelian certificate on $S_3$. Left: class-mass responses $p_A$, $p_B = p_A * r$ and the true blur $r$. Middle: the forward character-block ratio reconstructs the physical blur. Right: the formal reverse deconvolution is signed and hence not a channel.

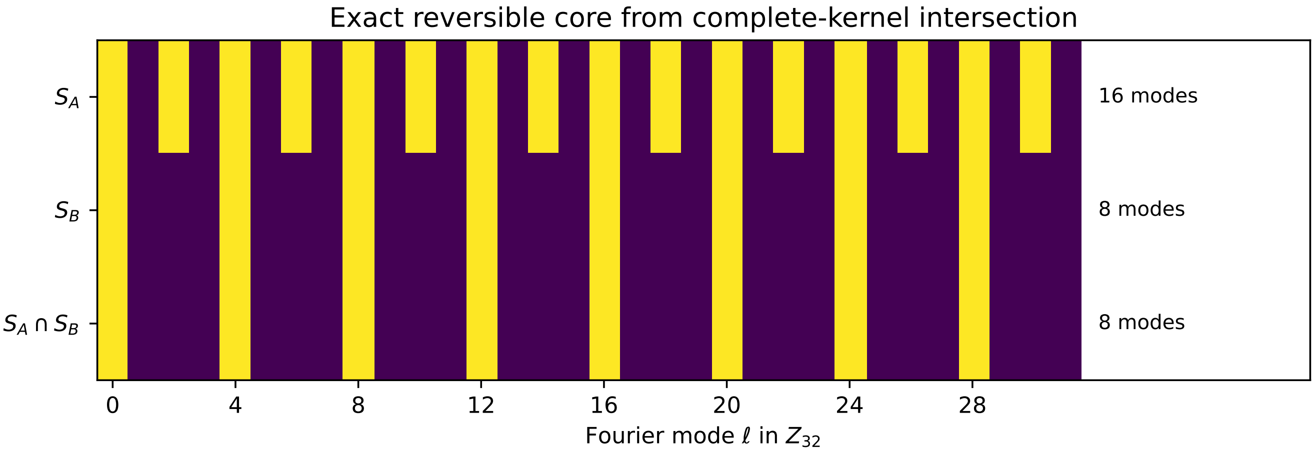

FIG. 9. Finite-clock reversible core on $\mathbb{Z}_{32}$. The first frame erases odd modes, the second erases all modes not divisible by four. The directed transformation $A \to B$ is possible, $B \to A$ is not, and the exact observer-independent reversible core is the intersection of the two surviving Fourier supports.

The unique linear quotient from $A$ to $B$ has ratios $r_i = \mu_i/\lambda_i$. It is CPTP precisely when

$$\begin{aligned} p_0 &= (1 + r_1 + r_2 + r_3)/4, \quad p_1 = (1 + r_1 - r_2 - r_3)/4, \\ p_2 &= (1 - r_1 + r_2 - r_3)/4, \quad p_3 = (1 - r_1 - r_2 + r_3)/4 \end{aligned}$$

are all non-negative. We chose

$$\begin{aligned} \lambda_A &= (0.76, 0.58, 0.50), \\ r &= (0.70, 0.55, 0.45), \\ \lambda_B &= \lambda_A \odot r. \end{aligned}$$

The recovered $A \to B$ quotient has Pauli probabilities

$$(0.675,\ 0.175,\ 0.100,\ 0.050),$$

so it is a valid channel. The formal reverse has probabilities

$$(1.617,\ -0.403,\ -0.208,\ -0.006),$$

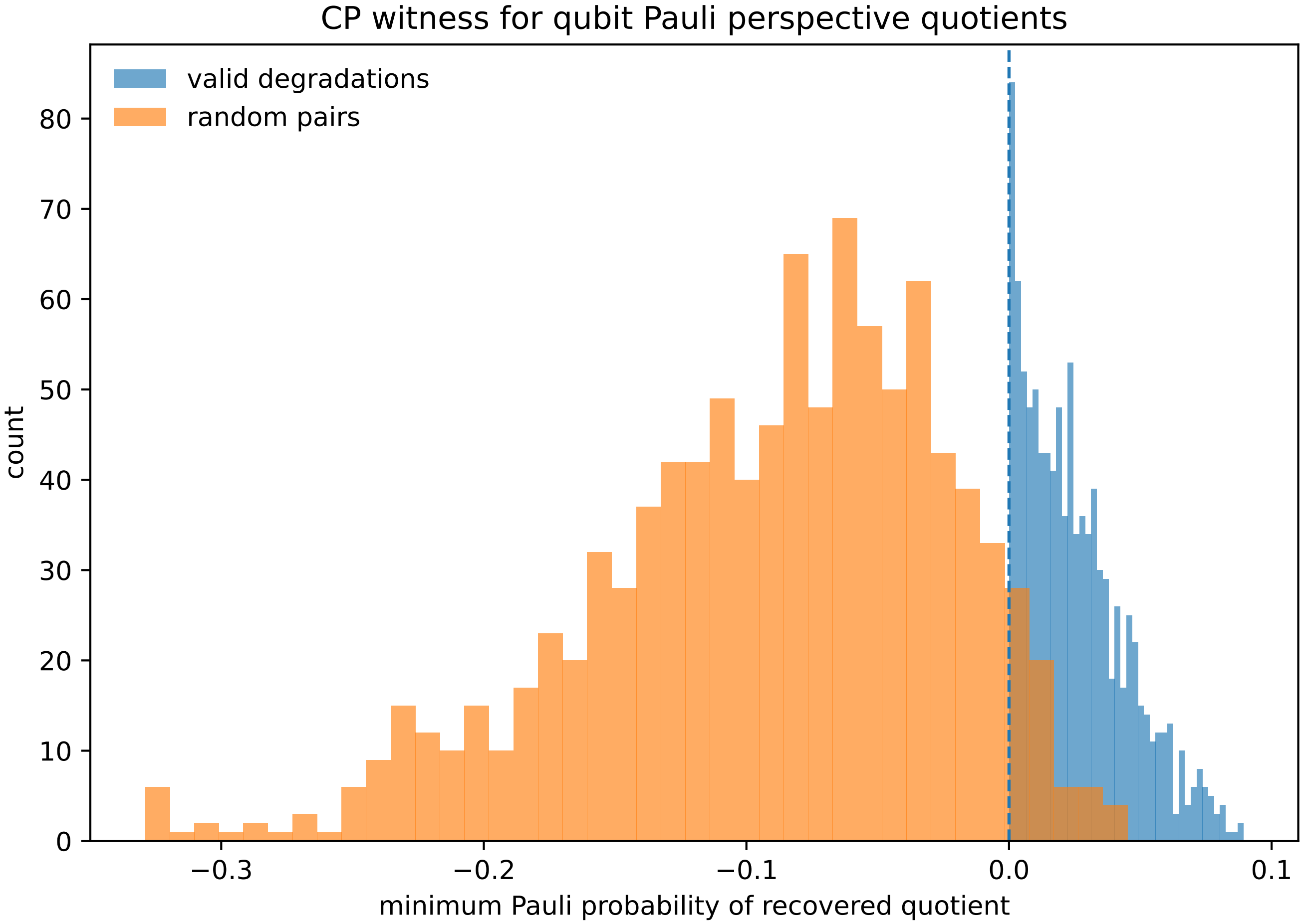

FIG. 10. Choi/Pauli positivity witness for finite-dimensional perspective quotients. The histogram shows the minimum recovered Pauli probability. Valid degradations remain inside the CP tetrahedron; generic random pairs frequently leave it, producing a negative-probability witness against exact CPTP frame transformation.

so it is not CP. This is a noncommutative finite-dimensional instance of the same kernel-and-positivity obstruction.

The two numerical tests probe complementary regimes. The cyclic model checks the Fourier-positive-definite criterion for a commutative reference variable. The Pauli model checks complete positivity of a genuinely quantum quotient. In both cases the theorem predicts not only when a transformation exists but also how to falsify its existence from finite process data.

### I. Correctable finite-resource sector

The channel theory predicts that a non-ideal frame can be globally lossy but exactly recoverable on a code sector. A minimal numerical test is the phase-token model

$$\mathcal{N}_p(\rho) = (1-p)\rho + pZ\rho Z.$$

Without encoding, the entanglement fidelity of the identity perspective is $1-p$. For the three-token relational phase code correcting one phase-token error, the uncorrectable probability is

$$p_{\mathrm{fail}} = 3p^2(1-p) + p^3 = 3p^2 - 2p^3,$$

so that

$$F_{\mathrm{e}}^{\mathrm{corr}} = 1 - 3p^2 + 2p^3.$$

The improvement is first order: frame noise is suppressed from $O(p)$ to $O(p^2)$ on the protected relational sector. This is plotted in Fig. 11.

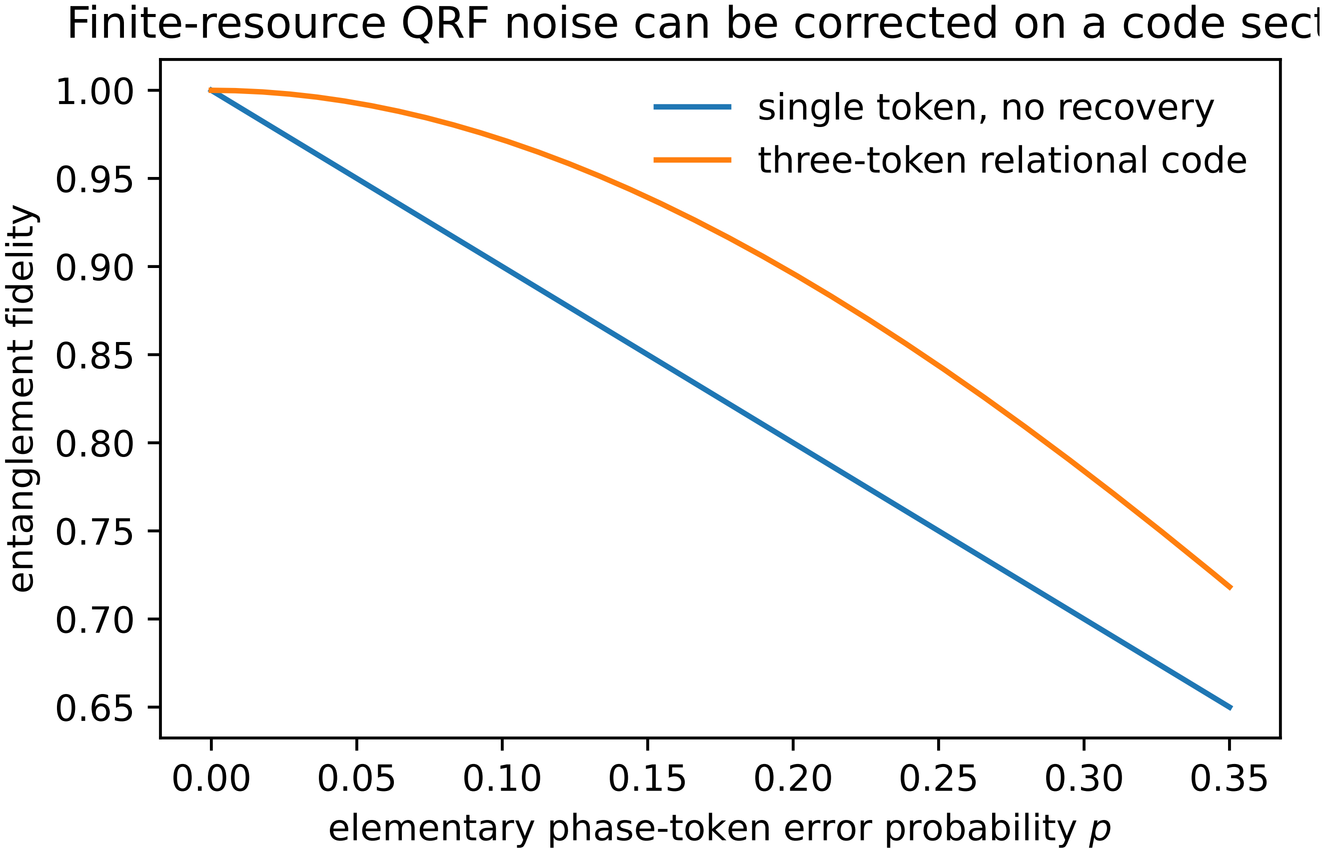

FIG. 11. Numerical check of Theorem IX.1. A raw finite-resource phase token has entanglement fidelity $1 - p$. A three-token relational code correcting a single phase-token error has fidelity $1 - 3p^2 + 2p^3$, converting first-order perspective loss into second-order loss on the recovered sector.

### J. Affine inverse versus physical inverse

The complete-order quotient theorem separates unique affine inversion from physical inversion. For a qubit phase-reference blur with transverse multiplier $\lambda = e^{-\sigma^2}$, the linear inverse has multiplier $\lambda^{-1}$. Its Choi matrix has minimum Bell-block eigenvalue

$$\lambda_{\min}^{\mathrm{Choi}} = \frac{1 - \lambda^{-1}}{2}.$$

Hence any nonzero blur $\sigma > 0$ makes the affine inverse non-CP. Figure 12 displays this witness. This illustrates why non-ideal QRF transformations cannot generally be repaired by simply inverting Fourier multipliers: the inverse exists algebraically but is not a quantum channel.

### K. Dynamical finite-clock and deficiency-witness check

The dynamical theorem is illustrated in the same $G = \mathbb{Z}_{31}$ model. The test uses a one-step source response $p_A$ and a target response $p_B = p_A * r$. The recovered one-step quotient is $r$, while the two-step process quotient for independent

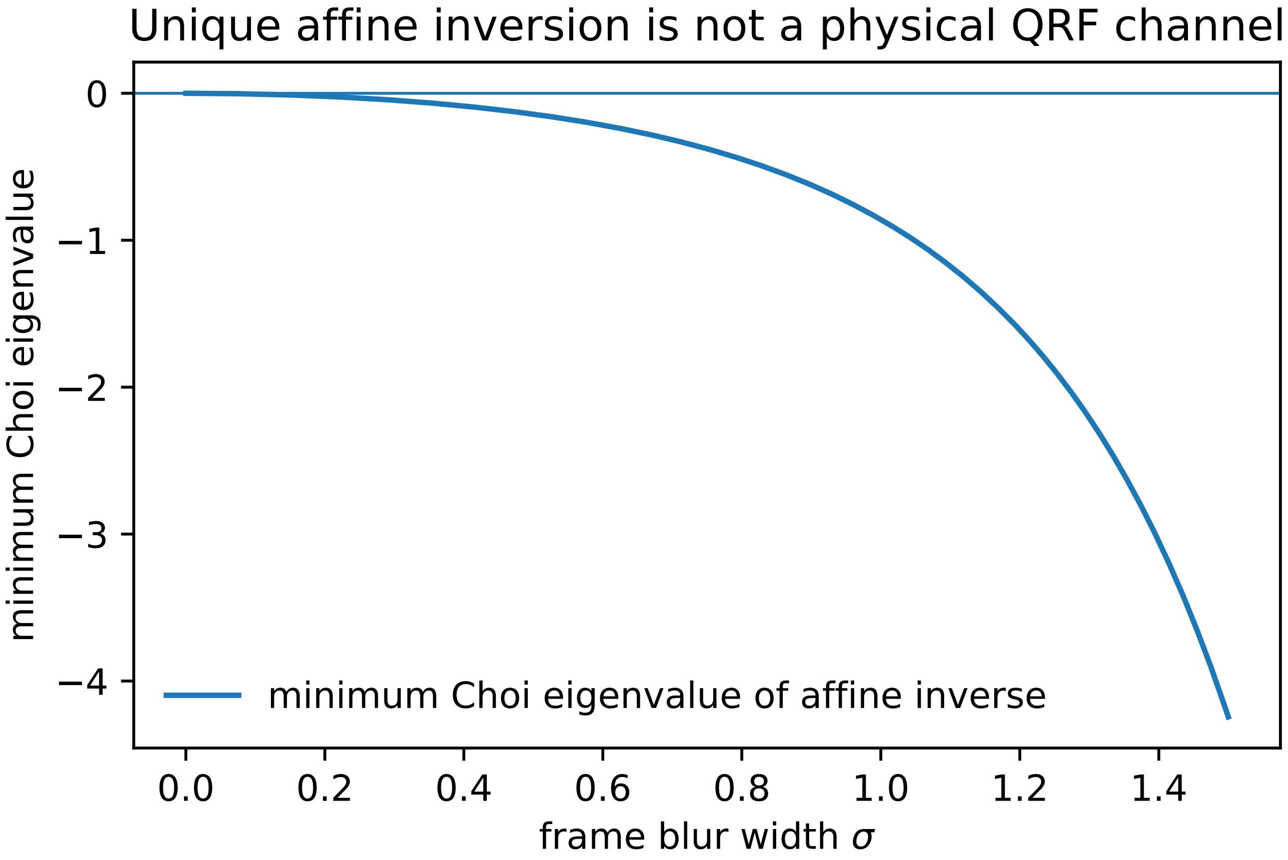

FIG. 12. Choi witness for the inverse of a blurred qubit phase-reference perspective. The affine inverse is unique whenever $\lambda \neq 0$, but its minimum Choi eigenvalue is negative for every $\sigma > 0$. Thus exact reversal of generic finite blur is not physically implementable outside a protected sector.

clock blur is $r * r$, in agreement with Proposition VIII.3. The numerical inverse of a sharper clock from a noisier source has large negative quasiprobability weights, again certifying non-CP behavior.

We also tested the lower bound (XIV). A sharper target clock preserves high Fourier coherences better than a noisy source clock. For each mode $k$, the binary pair of sign-flipped coherence states gives the experimental lower bound

$$\delta(B|A) \geq \frac{1}{2}\big(|\phi_B(k)| - |\phi_A(k)|\big)_+.$$

The maximum in the instance below is approximately 0.291. No channel post-processing the noisy perspective can beat this witness.

### L. Diamond-norm exactness of the convolution LP

This check confirms Lemma XIV.5 and Theorem XIV.8 numerically. For random signed measures on cyclic groups ($N = 3, 4, 5$) in the regular representation, the Watrous two-block semidefinite program for $\|D_\sigma\|_\diamond$ reproduces $\|\sigma\|_1$ to a maximum error of $4.27 \times 10^{-7}$, verifying the multiplier identity. For the convolution pairs on $\mathbb{Z}_5$ and $\mathbb{Z}_4$, the convolution LP and the full diamond-norm deficiency SDP coincide: forward $\Delta_{\text{conv}} = \delta = 0$ in both groups, reverse $\Delta_{\text{conv}} = \delta = 0.20$ on $\mathbb{Z}_5$ and 0.22 on $\mathbb{Z}_4$, the reverse gaps being below $4 \times 10^{-8}$. This establishes that the convolution LP is the exact diamond distance to the random-translation cone (entangled tests included, Lemma XIV.5), coinciding with the full deficiency for these Abelian instances (Theorem XIV.10), not merely a diagonal-test lower bound.

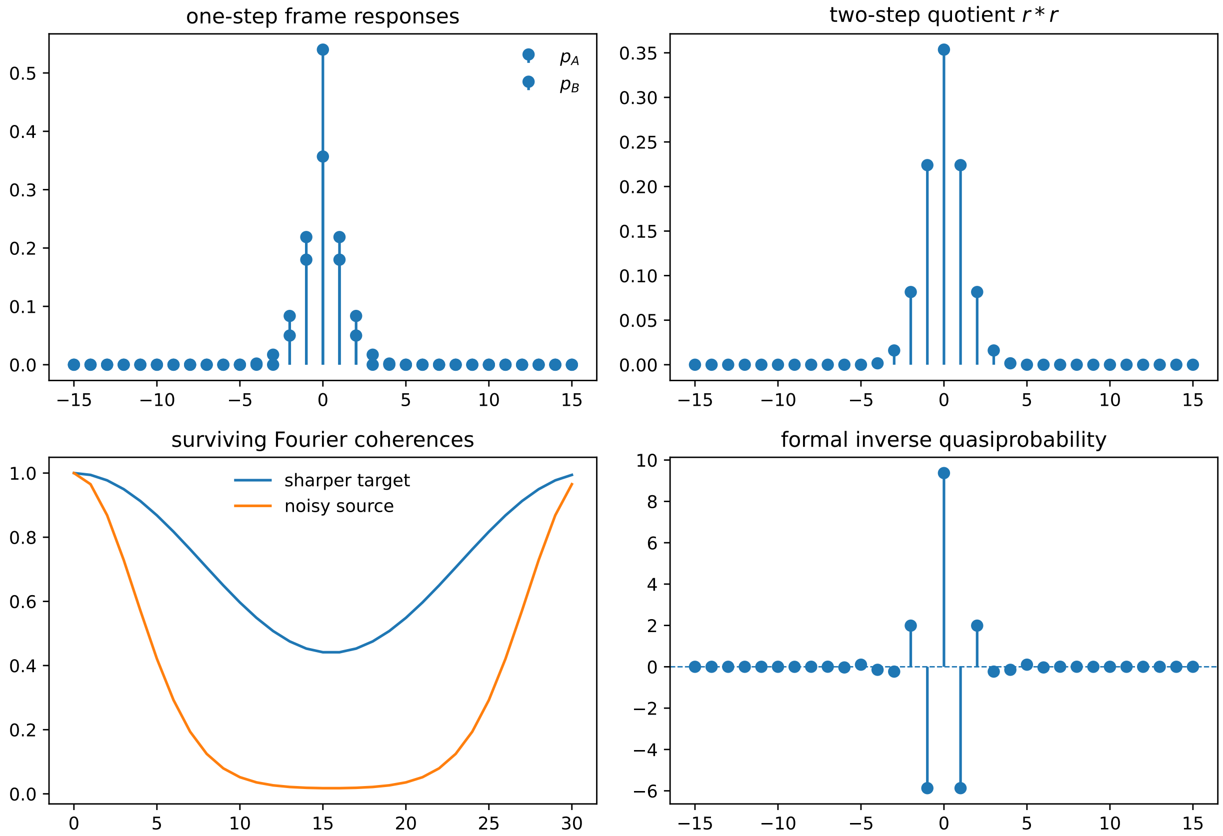

FIG. 13. Dynamical and resource-witness numerics on $\mathbb{Z}_{31}$. Top left: one-step frame responses. Top right: the process quotient kernel is the convolution power $r * r$ at two uses. Bottom left: a sharper target clock has larger surviving Fourier coherences than a noisy source clock. Bottom right: the binary distinguishability witness gives the lower bound $\delta(B|A) \geq \max_k(|\phi_B(k)| - |\phi_A(k)|)_+/2$, here about 0.291.

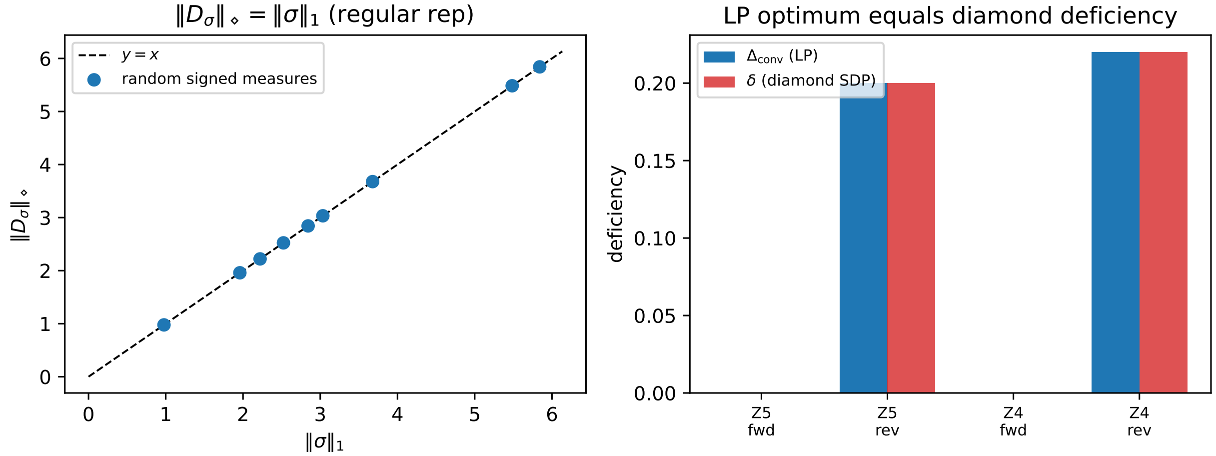

FIG. 14. Diamond exactness of the convolution LP. Left: $\|D_\sigma\|_\diamond$ versus $\|\sigma\|_1$ for random signed measures in the regular representation, lying on the diagonal (Lemma XIV.5). Right: the convolution LP value $\Delta_{\mathrm{conv}}$ equals the full diamond-norm deficiency $\delta$ on $\mathbb{Z}_5$ and $\mathbb{Z}_4$ in both directions (Theorem XIV.8).

### M. General non-central $S_3$ covariant-degradation certificate

This check exercises Theorem C.3 on a genuinely non-central pair, where the central deconvolution theorem does not apply. The source is supported on the identity and the transposition $(0\,1)$ – not a class function – and the blur kernel mixes the transposition $(1\,2)$ with a 3-cycle. The forward LP recovers the planted kernel to $\ell^1$ error $1.1 \times 10^{-16}$ with $\Delta_{\text{conv}} = 0$, while the reverse direction is infeasible with $\Delta_{\text{conv}} = \frac{1}{2}$ and a dual witness of unit $\ell^\infty$ norm. The unrestricted diamond-norm deficiency SDP confirms both values (reverse gap $1.7 \times 10^{-8}$), so on this non-Abelian example the random-translation LP already attains the full deficiency.

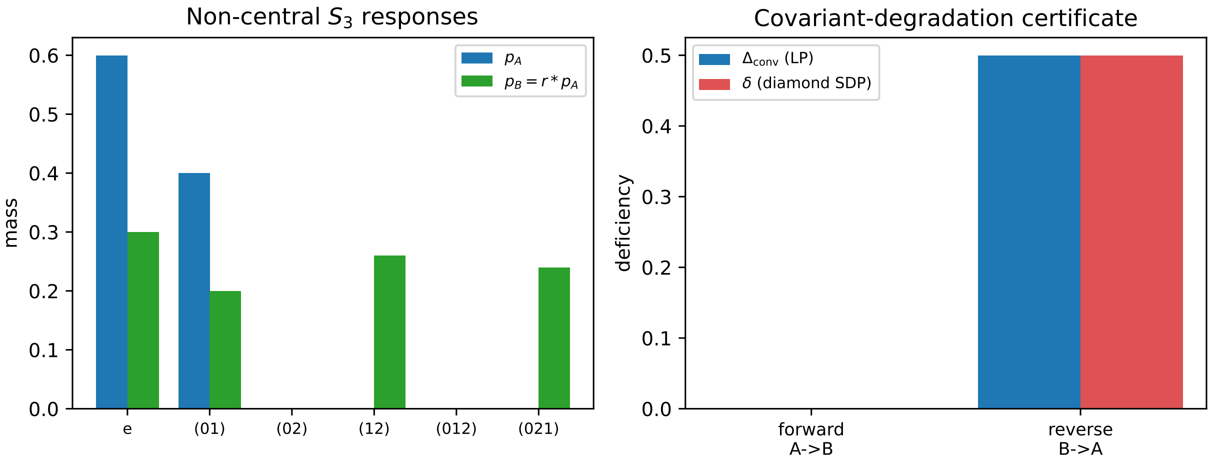

FIG. 15. Non-central $S_3$ covariant degradation. Left: source and blurred response distributions over the six group elements. Right: the convolution LP $\Delta_{\text{conv}}$ and the diamond-norm deficiency $\delta$ agree in both directions (Theorem C.3), with a separating witness certifying reverse infeasibility.

### N. Relativistic boosted-spin perspective and Fisher information

This check realizes Theorem XI.2 and Corollary XI.3. For a particle of rapidity $\zeta = 1$ seen by Gaussian-distributed perpendicular boosts, the Thomas–Wigner angle law produces coherence multipliers $\lambda_A \approx 0.877$ (sharp, $\sigma_A = 0.6$) and $\lambda_B \approx 0.769$ (blurred, $\sigma_B \approx 0.92$). The qubit diamond-norm deficiency is 0 forward and $\frac{1}{2}(\lambda_A - \lambda_B) \approx 0.054$ in reverse, matching the closed-form dephasing formula. The phase quantum Fisher information equals $\lambda^2$ to $2.2 \times 10^{-10}$, dropping from 0.769 to 0.591 – an irrecoverable metrological loss of about 23% caused purely by boost uncertainty.

## XXII. OBSERVER-WEB DESCENT AND CURVATURE

A single transformation $A \to B$ solves only the two-observer problem. In a relativistic or many-agent experiment one has a graph of partial perspectives: clocks distributed through a laboratory, gyroscopes attached to different particles, or local tetrads in a quantum spacetime region. The correct consistency question is whether these pairwise transformations glue to one global relational description.

Let $\Gamma$ be a directed graph whose vertices are frames. To each vertex $v$ assign an intrinsic operator-system perspective $\mathsf{Q}_v$. To each edge $e : v \to w$ assign a completely positive perspective map

$$\tau_e : \mathsf{Q}_v \to \mathsf{Q}_w.$$

For a path $p = e_m \cdots e_1$ write $\tau_p = \tau_{e_m} \cdots \tau_{e_1}$. For a loop $\ell$ based at $v$ define the holonomy

$$\Omega_\ell = \tau_\ell - \mathbf{1}_{\mathsf{Q}_v}.$$

**Definition XXII.1** (Descent operator system)**.** *The path-consistent global relational content of the observer web is*

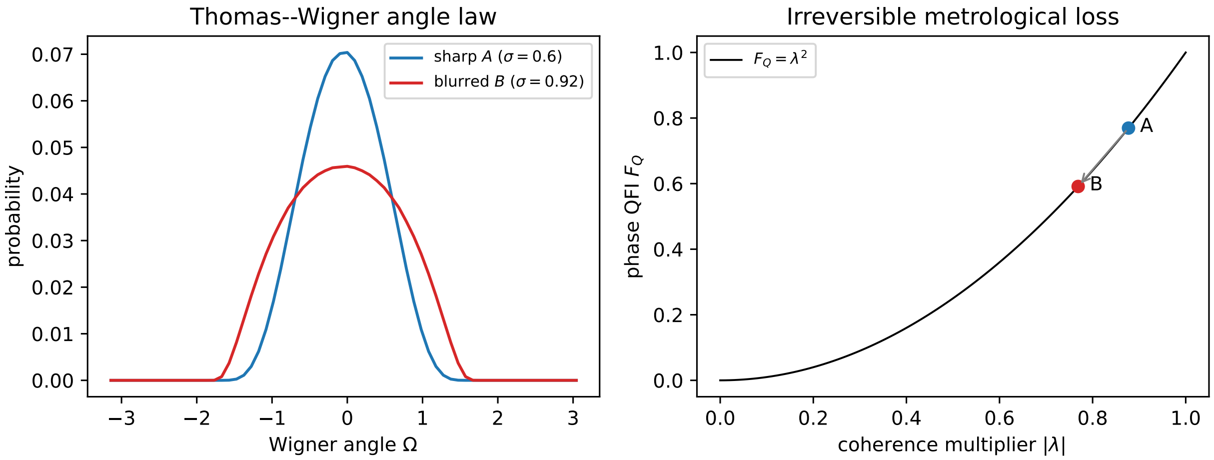

FIG. 16. Relativistic boosted-spin perspective. Left: Thomas–Wigner angle laws for a sharp and a blurred observer. Right: the phase Fisher information $F_Q = |\lambda|^2$; boost blur moves the observer down the parabola, an irreversible loss because $F_Q$ is CPTP-monotone (Corollary XI.3).

*the equalizer*

$$H^0(\Gamma, \mathsf{Q}) = \Big\{(x_v)_v \in \bigoplus_v \mathsf{Q}_v : x_w = \tau_e(x_v) \quad \text{for every } e : v \to w\Big\}.$$

*It is an operator system with matrix cones inherited from the direct sum.*

**Theorem XXII.2** (Observer-web descent theorem)**.** *Assume all edge maps are intrinsic exact QRF transformations induced by a common invariant physical theory. Then the following are equivalent on each connected component of* $\Gamma$*.*

*(i) The relational description is path independent:* $\tau_p = \tau_q$ *for any two paths with the same endpoints.*

*(ii) Every loop has zero holonomy,* $\Omega_\ell = 0$*.*

*(iii) For any base vertex* $v_0$*, the map*

$$x \longmapsto \big(\tau_{p_{v_0 \to v}} x\big)_v$$

*is a complete order isomorphism from the common path-independent quotient at* $v_0$ *onto* $H^0(\Gamma, \mathsf{Q})$*, independent of the chosen paths.*

*If loop holonomy is nonzero,* $H^0(\Gamma, \mathsf{Q})$ *is the maximal exactly glueable subsystem and the family* $\{\Omega_\ell\}$ *is the obstruction to a single global QRF perspective.*

*Proof.* Path independence immediately implies zero loop holonomy by taking one path to be a loop and the other the trivial path. Conversely, if all loop holonomies vanish, any two paths $p, q$ with the same endpoints give a loop $q^{-1}p$ whenever the reverse edges are included; in the general directed case the statement is applied in the path groupoid generated by available exact transformations. Vanishing loop holonomy gives $\tau_p = \tau_q$. The displayed map is then well-defined. It lands in the equalizer because edge composition agrees with path extension. It is completely positive at every matrix level because all $\tau_e$ are. Its inverse is projection to the base component, restricted to the equalizer. If some holonomy is nonzero, an element transported around the loop changes unless it lies in the fixed-point operator system of that holonomy; intersecting these fixed-point systems over all loops gives exactly the maximal glueable subsystem. ☐

**Proposition XXII.3** (Approximate descent bound)**.** *Let* $\ell = e_m \cdots e_1$ *be a loop and let* $\widetilde{\tau}_{e_j}$ *be implemented edge maps with*

$$\frac{1}{2}\|\widetilde{\tau}_{e_j} - \tau_{e_j}\|_\diamond \le \epsilon_j.$$

*If the ideal web is flat on the loop, then*

$$\frac{1}{2}\|\widetilde{\tau}_\ell - \mathbf{1}\|_\diamond \le \sum_{j=1}^{m} \epsilon_j$$

*for trace-norm contractive channels. Thus a measured loop curvature above this sum is not calibration error inside the assumed model; it witnesses either non-ideality not included in the edge model, a missing system/environment, or the failure of exact QRF gluing.*

*Proof.* Insert and subtract the product with one ideal edge replaced at a time. The diamond norm is submultiplicative and every channel in the product has norm one. The result is the telescoping triangle inequality. □

This section reframes the “many observers” problem as a descent problem for operator systems. Exact QRF relativity is flat. Imperfect finite-resource observers generally produce a nonzero curvature, but its fixed-point operator system remains an invariant experimentally accessible core.

## XXIII. NO-BROADCASTING OF QUANTUM PERSPECTIVES AND OBSERVER FAN-OUT

The descent theorem assumes that the vertices of the observer web are already given. A different and operationally sharper question is whether a single quantum frame can be split into two equally good frames. This is not a harmless engineering problem. For genuinely quantum perspectives it is forbidden by the same structure that forbids broadcasting noncommuting states.

Let

$$\mathsf{S}_R = \mathsf{P}_R(\mathsf{S}_{\text{phys}})$$

be the convex set of density operators that can appear in the perspective of $R$, after quotienting operationally indistinguishable physical preparations.

**Definition XXIII.1** (Exact fan-out of a perspective)**.** *A perspective $R$ is exactly broadcastable, or admits exact observer fan-out, if there exists a CPTP map*

$$\Delta : \mathcal{T}(\mathcal{H}_R) \longrightarrow \mathcal{T}(\mathcal{H}_R \otimes \mathcal{H}_R)$$

*such that for every $\sigma \in \mathsf{S}_R$,*

$$\mathrm{Tr}_2\,\Delta(\sigma) = \sigma, \qquad \mathrm{Tr}_1\,\Delta(\sigma) = \sigma.$$

*Equivalently, two downstream observers receive locally the same perspective state as the original observer.*

**Theorem XXIII.2** (No-broadcasting of noncommutative QRF perspectives)**.** *In finite dimensions a perspective state family $\mathsf{S}_R$ admits exact observer fan-out iff all its states commute pairwise:*

$$[\sigma, \tau] = 0 \qquad \forall \sigma, \tau \in \mathsf{S}_R.$$

*Consequently, a tomographically complete quantum perspective on $M_d$, $d > 1$, cannot be copied into two exact perspectives. The only exactly shareable quotient of a general QRF is its classical, jointly diagonal, commutative part.*

*Proof.* If $\Delta$ exists, then every pair $\sigma, \tau \in \mathsf{S}_R$ is broadcast by the same channel. The quantum no-broadcasting theorem of Barnum, Caves, Fuchs, Jozsa and Schumacher [27] implies that any set of states broadcast by one channel is pairwise commuting. Conversely, if all states in $\mathsf{S}_R$ commute, there is a common orthonormal decomposition in which

$$\sigma = \sum_j p_\sigma(j)|j\rangle\langle j|.$$

Measure in this common basis and prepare the perfectly correlated classical state

$$\Delta(\sigma) = \sum_j p_\sigma(j)|j\rangle\langle j| \otimes |j\rangle\langle j|.$$

Both marginals are $\sigma$. This proves the equivalence. □

**Corollary XXIII.3** (Fan-out obstruction for observer networks)**.** *Suppose a source observer $R$ is required to produce two observers $R_1, R_2$ such that each marginal observer is related to $R$ by the identity intrinsic quotient transformation. If $\mathsf{S}_R$ contains two noncommuting states, no such physical CPTP fan-out exists. Hence a many-observer QRF network cannot be built by duplicating a noncommutative perspective; it must either restrict to a commutative invariant subsystem, add noise, or encode the resulting mismatch as nonzero deficiency or loop holonomy.*

This corollary is the network-level counterpart of the complete-order quotient theorem. Complete-kernel inclusion says when one perspective can be simulated from another. No-broadcasting says that one quantum perspective cannot in general be used as two independent exact sources of the same simulation power. Thus the uniqueness result is not in conflict with the impossibility of a universal external quantum observer: the quotient map is unique when it exists, but it need not be duplicable.

**Proposition XXIII.4** (Optimal covariant qubit fan-out)**.** *For the full qubit perspective, restrict to symmetric $SU(2)$-covariant $1 \to 2$ fan-out channels. Each one-qubit marginal is necessarily a depolarizing channel*

$$D_\eta(\rho) = \eta\rho + (1-\eta)\frac{I}{2}.$$

*Complete positivity of the two-output channel implies*

$$\eta \le \frac{2}{3}.$$

*The bound is achieved by the Bužek–Hillery/Werner universal cloner. Therefore exact fan-out $\eta = 1$ is impossible, and the optimal covariant marginal error from the identity is*

$$\frac{1}{2}\|\mathbf{1} - D_{2/3}\|_\diamond = \frac{1}{4}.$$

*Proof.* Twirling any candidate fan-out over $SU(2)$ cannot decrease the average single-clone fidelity for the universal task, so an optimal symmetric fan-out can be assumed covariant. Covariance forces each marginal to be depolarizing. The optimal universal $1 \to 2$ qubit cloning fidelity is $F_{\max} = 5/6$ [28–30], equivalently $\eta = 2F_{\max} - 1 = 2/3$, and the Bužek–Hillery isometry attains it. The diamond distance formula for a qubit depolarizing channel gives

$$\|\mathbf{1} - D_\eta\|_\diamond = \frac{3}{2}(1-\eta),$$

hence the half-diamond error at $\eta = 2/3$ is $1/4$. □

The physical meaning is direct. A classical clock reading can be copied to many displays. A quantum clock or gyroscope carrying noncommuting relational information cannot. Therefore finite-resource multi-observer experiments have an unavoidable trilemma:

$$\begin{gathered}\text{classicalize} \quad \text{or} \quad \text{degrade}\\ \text{or retain observer-web curvature.}\end{gathered}$$

This is a new operational obstruction not visible in ideal unitary QRF formulas.

### A. Bochner witness beyond modulus bounds

The Fourier criterion has a second numerical trap: $|q_k| \le 1$ for every mode is necessary but not sufficient for complete positivity. On $\mathbb{Z}_{31}$ choose multipliers

$$q_0 = 1, \qquad q_{\pm 1} = -0.9, \qquad q_k = 0 \quad (k \ne 0, \pm 1).$$

Every multiplier has modulus at most one and no kernel condition is violated on the active modes. However, the inverse Fourier kernel has minimum value

$$\min_g r(g) = -2.580645 \cdot 10^{-2},$$

so the multiplier sequence is not positive definite and cannot represent a stochastic blur. The plot in Fig. 17 is therefore a direct finite witness that CP is a Bochner-positivity condition, not a scalar contraction condition.

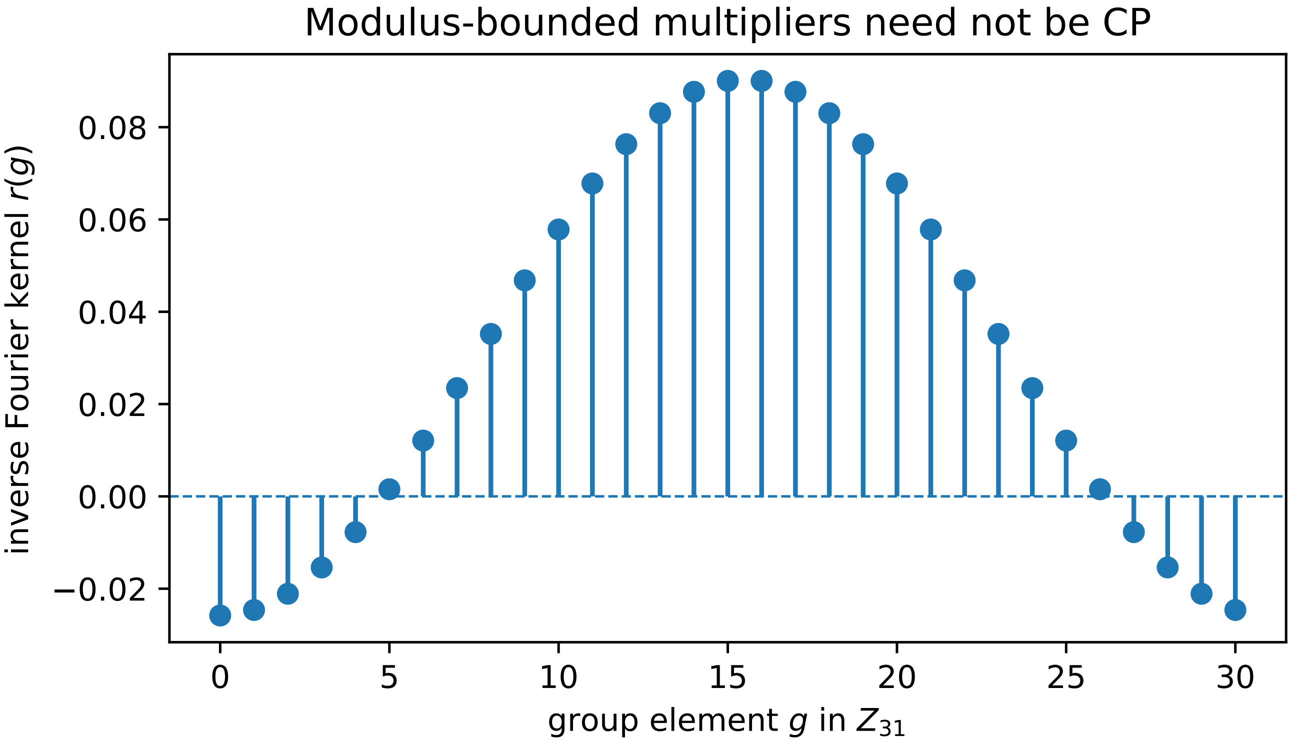

FIG. 17. Bochner-positivity witness on $\mathbb{Z}_{31}$. The multipliers satisfy $|q_k| \leq 1$, but the inverse Fourier kernel has negative entries. Thus a modewise contraction can still fail to be a completely positive QRF transformation.

### B. Observer-loop holonomy numerics

Finally we tested the descent theorem in a three-frame loop on $\mathbb{Z}_{31}$. For a flat ideal loop the three edge kernels are deterministic shifts $5, 7, -12$, whose convolution is the identity. Empirical kernels reconstructed from $M$ calibration samples per edge have loop curvature

$$h_M = \frac{1}{2}\|k_{CA}^{(M)} * k_{BC}^{(M)} * k_{AB}^{(M)} - \delta_e\|_1,$$

which decays with sample size. For a finite-resource loop with the same mean shifts but Gaussian circular width $\sigma = 1.2$ on every edge, the curvature converges instead to

$$h_\infty = 0.8080587990.$$

The plateau is not statistical error; it is intrinsic holonomy of the non-ideal observer web.

### C. No-broadcasting fan-out numerics

The final numerical check verifies Proposition XXIII.4. We implemented the Bužek–Hillery isometry

$$|0\rangle \mapsto \sqrt{\frac{2}{3}}|00\rangle|0\rangle + \sqrt{\frac{1}{6}}(|01\rangle + |10\rangle)|1\rangle,$$

$$|1\rangle \mapsto \sqrt{\frac{2}{3}}|11\rangle|1\rangle + \sqrt{\frac{1}{6}}(|01\rangle + |10\rangle)|0\rangle.$$

For 1000 Haar-random pure qubit inputs, both one-qubit marginals had Bloch vector $(2/3)r$ with maximum residual $2.3 \cdot 10^{-16}$. The right panel of Fig. 19 displays the sharp covariant boundary: $\eta > 2/3$ violates the complete-positivity fan-out witness, while $\eta = 1$ would be exact perspective duplication and is forbidden.

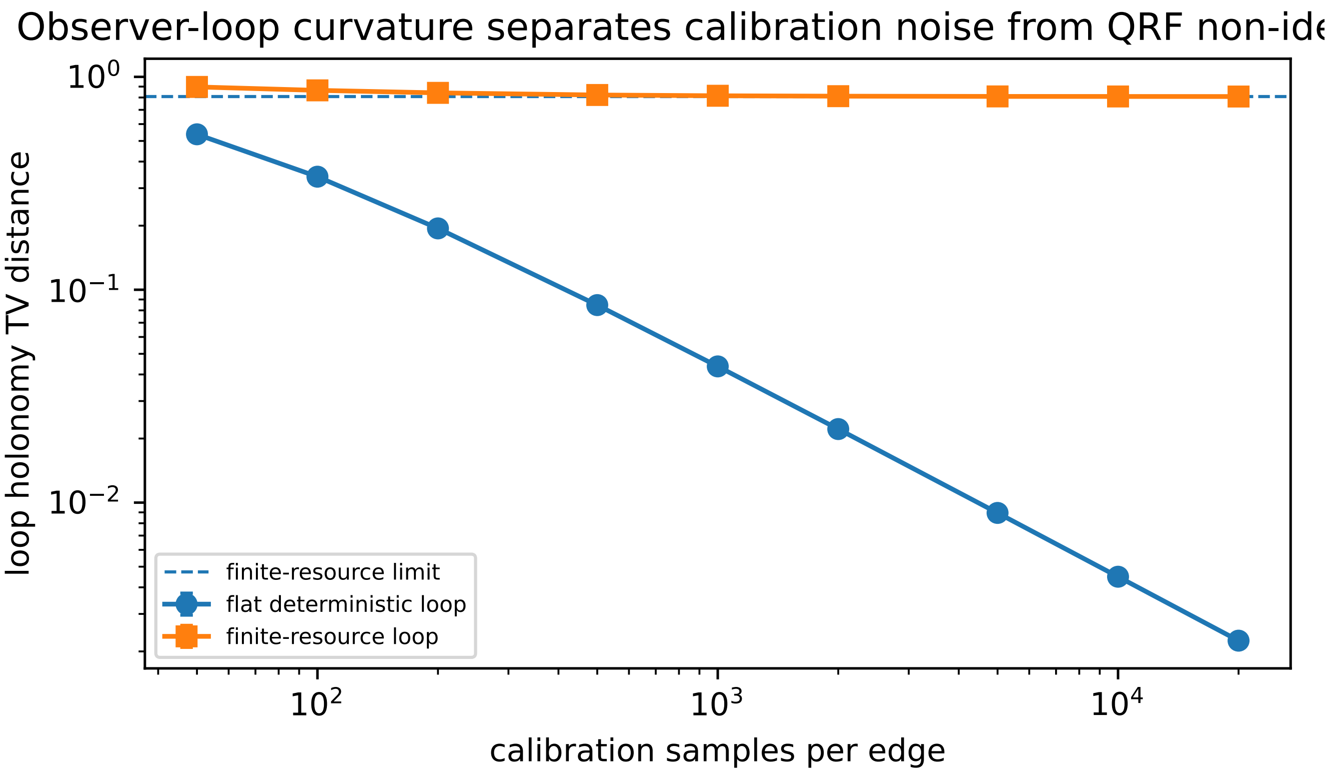

FIG. 18. Monte Carlo calibration of observer-loop curvature. The deterministic flat loop decays toward zero as calibration improves. The finite-resource loop approaches a nonzero holonomy floor, separating statistical calibration noise from genuine non-ideal QRF curvature.

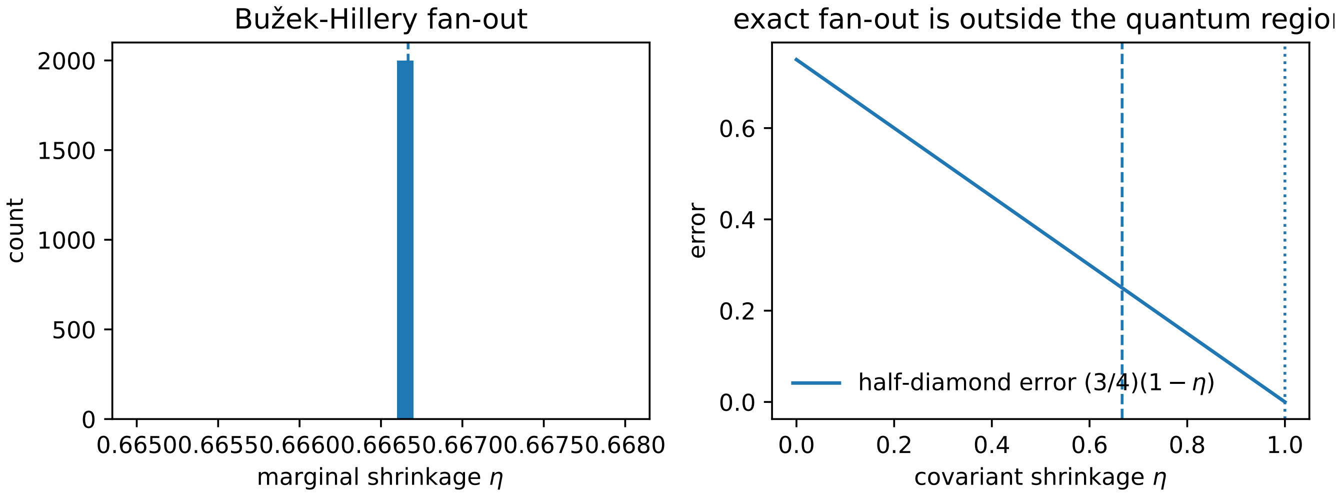

FIG. 19. Numerical no-broadcasting check for observer fan-out. Left: the Bužek–Hillery universal cloner gives marginal Bloch shrinkage $\eta = 2/3$ for random inputs up to roundoff. Right: the symmetric covariant qubit fan-out region ends at $\eta = 2/3$. Exact QRF duplication would require $\eta = 1$, which lies in the forbidden region.

## XXIV. SYNTHESIS OF THE TRANSFORMATION, INVARIANT AND UNIQUENESS PROBLEMS

Uniqueness, invariants and transformations for non-ideal, entangled and relativistic observers are three forms of the same quotient-sufficiency statement.

### 1. Unique transformations

The unique transformation is not a unitary on arbitrary representatives. It is the quotient morphism

$$T_{A\to B} : \operatorname{Ran} \mathsf{P}_A \to \operatorname{Ran} \mathsf{P}_B, \qquad T_{A\to B}(\mathsf{P}_A\rho) = \mathsf{P}_B\rho.$$

At the ordinary affine state level it exists exactly when $\operatorname{Ker} \mathsf{P}_A \subseteq \operatorname{Ker} \mathsf{P}_B$. As a physical QRF channel stable under hidden ancillas it must satisfy the complete-kernel condition

$$\operatorname{Ker}(\mathbf{1}_n \otimes \mathsf{P}_A) \subseteq \operatorname{Ker}(\mathbf{1}_n \otimes \mathsf{P}_B) \qquad \forall n.$$

For temporally extended observers the same statement holds with $\mathsf{P}_R$ replaced by the comb perspective $\mathbb{P}_R^{(m)}$. There is no further physical freedom. Any two mathematical extensions outside $\operatorname{Ran} \mathsf{P}_A$ differ only on nonpreparable or operationally irrelevant states.

### 2. Complete invariants

The complete invariant content shared by a set of frames is the common Heisenberg image

$$\mathcal{I}_{\mathcal{F}} = \bigcap_{R\in\mathcal{F}} (\operatorname{Ker} \mathsf{P}_R)^{\perp},$$

All invariant probabilities and scalar quantities are functions of the physical state restricted to this operator system. All alleged invariants outside this intersection are either frame-specific or require extra structure.

### 3. Non-ideal observers

Non-ideal observers are finite-resource channels. They lose modes. Exact reversibility is possible only on the surviving quotient. For compact Abelian frames, mode loss and degradation are classified by Fourier coefficients. The criterion is explicit:

$$\begin{gathered} \widehat{\mu}_A(\chi) = 0 \Rightarrow \widehat{\mu}_B(\chi) = 0, \\ \widehat{\mu}_B(\chi)/\widehat{\mu}_A(\chi) \text{ positive definite} \\ \text{for CPTP implementability.} \end{gathered}$$

### 4. Entangled observers

No additional postulate is required. The same perspective channel acts on entangled states. Ordinary subsystem entanglement is perspective-dependent because subsystem identifications are perspective-dependent. Invariant entanglement is algebraic entanglement across invariant subalgebras; accessible entanglement obeys data processing under non-ideal frame degradation.

### 5. Dynamical observers

Repeated clock readings and adaptive experiments are governed by the process-tensor quotient

$$\mathbb{T}^{(m)}_{A\to B}\big(\mathbb{P}^{(m)}_A W\big) = \mathbb{P}^{(m)}_B W.$$

The finite-Abelian case predicts convolution powers of the one-step blur and exponential instability of inverse perspective changes. This supplies a direct operational test: if an alleged QRF transformation works on states but fails on comb testers, it is not a valid transformation of observers.

### 6. Relativistic observers

Use $G$ as the Lorentz or Poincaré group and let the frame POVM live on the appropriate homogeneous space of origins, tetrads or boosts. The perspective channel includes Wigner rotations and superpositions of boosts. The uniqueness and invariant theorems apply on finite-energy domains and sectorwise in the Wigner decomposition. Casimirs, relational scattering probabilities and algebraic correlations are invariant; spin components, simultaneity and temporal order are relational observables whose non-ideal formulations are POVM-smeared.

### 7. Observer webs and fan-out

For many observers, pairwise consistency is not enough. The global relational description is the equalizer of the observer graph. It exists path-independently iff every loop holonomy vanishes. Nonzero holonomy is not a contradiction; it is the curvature of an imperfect observer web. Moreover, a noncommutative QRF perspective cannot be copied into two exact downstream perspectives. Exact fan-out exists only for pairwise commuting perspective state families. Hence multi-observer QRF networks cannot be reduced to cloning a single quantum frame: their shareable exact core is classical, while the genuinely quantum part must be degraded, protected by coding, or represented by deficiency and holonomy.

## XXV. PREDICTIONS AND FALSIFIABLE CONSEQUENCES

The formalism gives operational predictions.

(1) **Irreversibility threshold.** If a finite clock or gyroscope has a zero in its response Fourier transform, any experiment requiring that mode cannot be predicted from that frame perspective alone. A claimed exact QRF inverse must fail positivity or import a prior.

(2) **Reference degradation law.** If $B$ is a noisy degradation of $A$, all distinguishability measures between physical hypotheses decrease from $A$ to $B$.

(3) **Relativistic spin smearing.** A boost-uncertain observer measures spin through a Wigner-rotation-smeared POVM. The loss of spin coherence is governed by the boost response kernel.

(4) **Clock-order limitation.** Temporal-order observables requiring Fourier modes outside a clock's energy bandwidth are not transformable from that clock's perspective to a sharper clock.

(5) **Invariant algebra test.** Any proposed frame-invariant quantity must be representable in $\bigcap_R \operatorname{Ran} \mathsf{P}_R^*$. This gives a direct mathematical test for proposed invariants.

## XXVI. SCOPE, REMAINING FRONTIERS AND STANDARDS FOR EXTENSIONS

The quotient theorem gives a complete finite-dimensional comparison principle and a normal-instrument formulation under explicit closedness assumptions. The following points separate these results from extensions that require additional analytic or physical input.

(i) **Noncompact and relativistic groups.** Haar-normalized ideal frame states for noncompact groups are distributions, not normal states. The correct domain is therefore a finite-energy operational sector, a normal-weight formulation, or a rigged-Hilbert-space limit. The intrinsic criterion remains kernel inclusion; the nontrivial work is proving normality, closedness of predual kernels and existence of concrete normal extensions on the selected representation.

(ii) **Type III local algebras.** In relativistic QFT, local algebras are often type III, and QRF measurement schemes must respect that algebraic type structure [23]. The channel and invariant statements are naturally normal operator-algebraic statements, but entropy must be replaced by relative entropy, modular inclusions or crossed-product traces. The finite-dimensional von Neumann entropy bounds in Theorem XVII.2 should be read as operational finite-sector approximations to those modular quantities.

(iii) **Gauge systems and gravity.** In full quantum gravity there need not be a fixed external group $G$. The replacement is a constraint algebra, groupoid or stack of relational charts. The categorical form of the theorem survives: perspectives are completely positive functors from a perspective-neutral object to reduced operational algebras, and frame changes are unique factorisations through complete images. What remains open is the construction of the correct normal quotient category for diffeomorphism-invariant field algebras.

(iv) **Concrete extension is physical hardware.** The intrinsic quotient transformation may be completely positive while a chosen Hilbert-space realization contains unphysical complementary directions on which no canonical extension is fixed. This is not a weakness of the quotient theorem. It is the operational boundary between an informational QRF transformation and a laboratory device implementing it.

(v) **Observer networks.** Exact flat descent and exact fan-out are exceptional. Generic many-observer networks contain deficiency, holonomy and no-broadcasting obstructions. A reproducible comparison should therefore report not only a proposed QRF transformation, but also its kernel relation, complete-order quotient, concrete extension status, deficiency and fan-out loss.

(vi) **Numerical certification.** Finite experiments should publish the Choi interpolation SDP, the dual infeasibility witness when present, the confidence radii used for calibration, and the raw finite-token certificates such as $\Delta_{\mathrm{conv}}$ together with any robust dual-margin vector $y$. This makes QRF comparison falsifiable and reproducible rather than a choice of formal coordinates.

A question that might once have seemed internal to the finite theory – whether, for non-Abelian groups, covariant post-processings exploiting representation multiplicity can ever strictly outperform random translations – is settled negatively by Theorem XIV.16: for *every* finite group the convolution linear program of Theorem C.3 already equals the full diamond-norm deficiency, and the certificates of Section XXI exhibit exactly this absence of gap. The genuine frontier is therefore the passage to compact and Lie groups – clocks $U(1)$, gyroscopes $SU(2)$, Lorentz boosts – where the normalisable entangled witness of the finite regular representation has no direct analogue (Remark XIV.19); the explicit $U(1)$ boosted-spin channel of Theorem XI.2 is the first continuous instance in which the identity is seen to hold.

These frontiers specify the analytic boundary at which new physical assumptions enter. Within the stated finite-dimensional and normal-instrument regimes, uniqueness is quotient sufficiency, invariance is a common operator system, reversibility is equality of kernels on the relevant sector, and physical implementability is complete positivity stable under ancillas.

## XXVII. CONCLUSION

The central lesson is that a non-ideal quantum reference frame is not an imperfect unitary coordinate chart. It is a physical information channel from a perspective-neutral theory to the operational state space available to an observer. Once this replacement is made, the three basic problems of the subject have a common answer.

The transformation problem is a sufficiency problem. At the affine level $A$ determines $B$ precisely when $\operatorname{Ker}\mathsf{P}_A \subseteq \operatorname{Ker}\mathsf{P}_B$, and the induced map is the unique quotient morphism on the preparable image. At the physical level the same statement must be stable under arbitrary ancillas; this gives the complete-kernel criterion and the corresponding completely positive quotient map. A concrete laboratory realization on an enlarged Hilbert space is a further complete-order extension problem, not an additional freedom in the intrinsic perspective.

The invariant problem is an operator-system problem. The shared invariant effects are exactly

$$\mathcal{I}_{\mathcal{F}} = \bigcap_R (\operatorname{Ker}\mathsf{P}_R)^{\perp},$$

not necessarily a product-closed algebra. Algebraic invariants are recovered only on the multiplicative core, in fixed-point subalgebras, or in the $C^*$-envelope. The dual state-space object is the maximal common reversible quotient

$$\mathcal{Q}_{\mathcal{F}} = \mathcal{T}_{\mathrm{phys}} / \overline{\sum_R \operatorname{Ker}\mathsf{P}_R},$$

which is the largest part of the physical state data reconstructible from all observers in the chosen family.

The uniqueness problem is therefore resolved without imposing a universal unitary transformation. There is no ambiguity on the quotient where a transformation exists. Apparent non-uniqueness comes from extending a quotient map to unphysical, nonpreparable, or deliberately discarded directions. When exact transformation fails, the deficiency

$\delta(B|A)$ gives the operational distance to simulability and supplies finite witnesses through the dual Blackwell–Choi problem.

The same structure covers finite clocks, phase references, spin and Lorentz frames, temporally extended observers, entangled observers, gauge systems and crossed-product descriptions of gravitational subregions. Finite Abelian frames reduce to Fourier multipliers and convolution LPs, which a Heisenberg–Weyl symmetry argument shows compute the exact diamond-norm deficiency, with robust dual margins; finite non-Abelian frames reduce to a group-convolution linear program for any finite group, the central case giving character-block deconvolution, and this convolution program is proved to equal the full diamond-norm deficiency for *every* finite group, so multiplicity-using covariant post-processings never beat random translations; dynamical clocks reduce to convolution powers; protected relational sectors reduce to quantum error correction; observer webs reduce to descent and holonomy; and multi-observer fan-out is limited by no-broadcasting. These reductions are not separate analogies but instances of a single quotient principle: QRF transformations are completely positive factorizations between operational perspectives, and QRF invariants are the effects that survive all such factorizations.

## Appendix A: Categorical form of the theorem

Let $\mathsf{Phys}$ be the ordered vector space of physical states and let each frame $R$ define a quotient object

$$q_R : \mathsf{Phys} \to \mathsf{Phys}/\operatorname{Ker}\mathsf{P}_R.$$

Then $T_{A\to B}$ exists exactly when $q_B$ factors through $q_A$:

$$\begin{tikzcd}
\mathsf{Phys} \arrow[r, "q_A"] \arrow[dr, "q_B"'] & \operatorname{Ran}\mathsf{P}_A \arrow[d, dashed, "T_{A\to B}"] \\
& \operatorname{Ran}\mathsf{P}_B
\end{tikzcd}$$

The dashed arrow is unique by the universal property of quotients. This is the categorical reason for uniqueness.

## Appendix B: Proof of the common invariant quotient

Let $K_R = \operatorname{Ker}\mathsf{P}_R$. A scalar function $f$ on physical states is expressible as a function of every perspective when for each $R$ there exists $f_R$ with

$$f = f_R \circ \mathsf{P}_R.$$

Equivalently, $f$ is constant on every affine subspace $\rho + K_R$. Hence it is constant on the equivalence relation generated by all $K_R$, whose linear part is

$$K_{\mathcal{F}} = \sum_{R\in\mathcal{F}} K_R.$$

Thus the Schrödinger-picture common quotient is $\mathsf{V}/K_{\mathcal{F}}$. The dual annihilator is

$$K_{\mathcal{F}}^{\perp} = \bigcap_{R\in\mathcal{F}} K_R^{\perp} = \mathcal{I}_{\mathcal{F}},$$

which is exactly (6); in finite dimensions this also equals $\bigcap_R \operatorname{Ran}\mathsf{P}_R^*$.

## Appendix C: Non-Abelian finite-group extension

The Abelian Fourier classification is scalar because every irreducible representation is one-dimensional. For a finite non-Abelian group the same quotient principle remains exact, but the Fourier data are block matrices. Let

$$\mathcal{T}_{\mathrm{phys}} = \bigoplus_{\lambda\in\widehat{G}} \mathcal{T}_\lambda$$

be the Peter–Weyl decomposition of the covariant operator sector, including all multiplicity spaces. A covariant finite-resource frame acts blockwise,

$$\mathsf{P}_R|_{\mathcal{T}_\lambda} = M_R^\lambda.$$

**Proposition C.1** (Non-Abelian block quotient criterion)**.** *For finite-group covariant perspective channels, the intrinsic transformation $A \to B$ exists iff*

$$\operatorname{Ker} M_A^\lambda \subseteq \operatorname{Ker} M_B^\lambda \qquad \forall \lambda \in \widehat{G}.$$

*When it exists, the unique block map is*

$$T_{A\to B}^\lambda = M_B^\lambda (M_A^\lambda)^+ \quad \textit{on } \operatorname{Ran} M_A^\lambda,$$

*independent of the representative chosen by the pseudoinverse. A concrete covariant CPTP implementation exists iff the block family $\{T_{A\to B}^\lambda\}_\lambda$ satisfies the covariant Choi positivity and trace-preservation constraints, equivalently iff the corresponding block-interpolation SDP is feasible.*

*Proof.* The Peter–Weyl decomposition turns the intertwining equation $T\mathsf{P}_A = \mathsf{P}_B$ into independent finite-dimensional equations $T^\lambda M_A^\lambda = M_B^\lambda$. Each equation has a well-defined solution on $\operatorname{Ran} M_A^\lambda$ exactly when $\operatorname{Ker} M_A^\lambda \subseteq \operatorname{Ker} M_B^\lambda$, and then the value on $M_A^\lambda x$ is $M_B^\lambda x$. Complete positivity is not a blockwise kernel statement; it is the positivity of the Choi operator subject to the covariance intertwiners, which is precisely the finite SDP condition. ☐

There is, however, an important non-Abelian subcase where the SDP collapses to a finite character feasibility problem. This is the case of central response distributions, which includes isotropic finite gyroscopes and class-symmetric finite orientation tokens.

**Theorem C.2** (Central finite-group deconvolution certificate)**.** *Let $G$ be finite and let $p_A, p_B \in \operatorname{Prob}(G)$ be central response distributions. For each irreducible representation $\lambda \in \widehat{G}$, with character $\chi_\lambda$ and dimension $d_\lambda$, define the central Fourier scalar*

$$a_R(\lambda) = \frac{1}{d_\lambda} \sum_{g\in G} p_R(g)\chi_\lambda(g^{-1}), \qquad R \in \{A, B\}.$$

*A covariant postprocessing $p_B = p_A * r$ by a probability distribution $r \in \operatorname{Prob}(G)$ exists iff there are scalars $s_\lambda$ such that*

$$s_\lambda a_A(\lambda) = a_B(\lambda) \quad \forall \lambda \in \widehat{G}, \qquad s_{\mathbf{1}} = 1,$$

*and the inverse character expansion*

$$r_s(g) = \frac{1}{|G|} \sum_{\lambda\in\widehat{G}} d_\lambda s_\lambda \chi_\lambda(g)$$

*is non-negative for every $g \in G$. In particular, if $a_A(\lambda) \neq 0$ for all $\lambda$, the postprocessing kernel is unique and is obtained by setting $s_\lambda = a_B(\lambda)/a_A(\lambda)$.*

*Proof.* For a central distribution $p$, Schur's lemma gives $\widehat{p}(\lambda) = a_p(\lambda)\mathbf{1}_{d_\lambda}$. Hence $p_B = p_A * r$ implies, after replacing $r$ by its conjugacy average if necessary,

$$a_B(\lambda) = a_A(\lambda)a_r(\lambda).$$

The conjugacy average does not change $p_A * r$, because $p_A$ and $p_B$ are central. Thus $s_\lambda = a_r(\lambda)$ satisfies the displayed equations. Conversely, if such scalars exist and the inverse character expansion is pointwise non-negative, orthogonality of characters gives a central probability distribution $r_s$ with Fourier scalars $s_\lambda$. Its convolution with $p_A$ has the same Fourier scalars as $p_B$, hence equals $p_B$. The final statement is the zero-kernel special case. ☐

For compact Lie groups the same statement is obtained on finite-energy truncations. The full infinite-dimensional problem additionally requires domain control, normality of the induced maps and closure of the relevant predual kernels. Rotational gyroscopes, spin frames and finite angular-momentum truncations are the main physical cases where this block criterion gives a usable non-Abelian test.

**Theorem C.3** (General finite-group covariant-degradation LP)**.** *Let $G$ be an arbitrary finite group (no centrality assumption) and let $\mathsf{P}_A, \mathsf{P}_B$ be convolution perspectives with response distributions $p_A, p_B$, realized in the regular representation. A covariant random-translation degradation $A \to B$ exists iff the simplex system*

$$p_B = C_G(p_A)\, r, \qquad r \in \mathrm{Prob}(G), \qquad [C_G(p_A)]_{kg} = p_A(g^{-1}k),$$

*is feasible, where $C_G(p_A)$ is the (left) group-convolution matrix. The achievable diamond distance among random translations is the linear program*

$$\Delta_{\mathrm{conv}}(B|A) = \tfrac{1}{2} \min_{r \in \mathrm{Prob}(G)} \|p_B - C_G(p_A) r\|_1 = \tfrac{1}{2} \max_{\|y\|_\infty \le 1} \Big( \langle y, p_B \rangle - \max_{g \in G} \langle y, \tau_g p_A \rangle \Big),$$

*the second equality being its linear-programming dual, with $\tau_g$ the left-translation by $g$ and $y$ a separating witness certifying infeasibility when $\Delta_{\mathrm{conv}} > 0$. By Lemma XIV.5 the LP value coincides with the diamond distance $\frac{1}{2}\|D_{p_B} - D_{p_A * r^\star}\|_\diamond$ at the optimizer $r^\star$. The central case Theorem C.2 is the specialization in which $p_A$ is a class function, so that $C_G(p_A)$ is block-diagonalized in the character basis with scalar blocks $a_A(\lambda)\mathbf{1}_{d_\lambda}$ and the LP decouples into the scalar conditions $a_B(\lambda) = a_A(\lambda)\, a_r(\lambda)$.*

*Proof.* A $G$-covariant random translation is convolution by some $r \in \mathrm{Prob}(G)$, and composition with $\mathsf{P}_A$ sends $p_A \mapsto r * p_A = C_G(p_A) r$, which is linear in $r$; feasibility and the optimal $\ell^1$ error are therefore a linear program over the probability simplex, whose reachable set is the convex hull of the translation orbit $\{\tau_g p_A\}$. Linear-programming duality over the simplex gives the stated dual with witness bound $\|y\|_\infty \le 1$. The diamond identification is Lemma XIV.5 applied to $\sigma = p_B - C_G(p_A) r^\star$. When $p_A$ is central, $C_G(p_A)$ commutes with all translations, hence is diagonal in the character basis with the scalar blocks shown, and the LP separates modewise into the central deconvolution equations of Theorem C.2. □

Because random translations are valid CPTP post-processings (the convolution channel $D_r$ sends $\mathsf{P}_A$ to $D_{r * p_A}$), $\Delta_{\mathrm{conv}}(B|A)$ is an exactly computable upper bound on the true deficiency; by Lemma XIV.5 and the right-regular covariance of Remark XIV.7 (valid for non-central distributions, unlike left covariance),

$$\delta(B|A) = \delta^R_{\mathrm{cov}}(B|A) \le \Delta_{\mathrm{conv}}(B|A).$$

The finite-mode certificates of Section XXI include a genuinely non-central $S_3$ pair (a source supported on the identity and a single transposition, blurred by a mixed transposition/3-cycle kernel), for which the unrestricted diamond-norm deficiency SDP agrees with the LP to solver tolerance in both directions: forward $\Delta_{\mathrm{conv}} = \delta = 0$, reverse $\Delta_{\mathrm{conv}} = \delta = \frac{1}{2}$ with an explicit dual witness of unit $\ell^\infty$ norm. Whether multiplicity-using covariant channels can ever strictly outperform random translations for non-Abelian $G$ – equivalently whether the inequality above can be strict – is answered in the negative by Theorem XIV.16: for every finite group the random-translation linear program already attains the full diamond-norm deficiency, consistent with every finite case tested here.

## Appendix D: Connection with Petz recovery

When $\operatorname{Ker} \mathsf{P}_A \not\subseteq \operatorname{Ker} \mathsf{P}_B$, an exact deterministic transformation does not exist. If a prior full-rank physical state $\omega$ is supplied, one can define a Bayesian or Petz-type reconstruction

$$\mathcal{R}_{\omega,A}(X) = \omega^{1/2} \mathsf{P}_A^* \Big[ \mathsf{P}_A(\omega)^{-1/2} X \mathsf{P}_A(\omega)^{-1/2} \Big]\, \omega^{1/2}.$$

Then

$$\widetilde{T}^{(\omega)}_{A \to B} = \mathsf{P}_B \mathcal{R}_{\omega,A}$$

is a prior-dependent estimator of the $B$-perspective from the $A$-perspective. It is not a reference-frame transformation in the strict sense because it depends on $\omega$ and need not satisfy $\widetilde{T}\mathsf{P}_A = \mathsf{P}_B$ for all states. This distinction explains why many apparently different non-ideal QRF prescriptions can be useful approximations without being unique physical transformations.

## Appendix E: Numerical certification protocol

The finite examples are intended as reproducible certificates rather than illustrations drawn after the fact. The accompanying script performs the following deterministic checks with fixed random seeds where sampling is used.

(i) **Convolution degradation.** For cyclic groups it computes the exact forward blur $r$ whenever $p_B = p_A * r$, the formal reverse deconvolution, and the signed negative-mass witness when the reverse kernel is not a probability distribution.

(ii) **LP certificates.** It solves

$$\min_{r \in \mathrm{Prob}(G)} \|p_B - p_A * r\|_1$$

and the dual separation problem. Reported primal–dual gaps are therefore direct certificates of the displayed convex deficiencies.

(iii) **Robust certificates.** For empirical token histograms it applies the finite-alphabet $L_1$ radius of Theorem VI.8 and separately records witness-level margins of Theorem VI.5. Exact Fourier support is not inferred from finite samples; only modes separated from zero by the calibration radius are certified active.

(iv) **Non-Abelian central test.** The $S_3$ example uses class functions and character-block ratios. It verifies that the forward ratio reconstructs a non-negative class blur while the reverse deconvolution is signed.

(v) **Complete-positivity witnesses.** Qubit Pauli and phase-reference examples compute Choi or Pauli-simplex positivity. Negative recovered probabilities or negative Choi eigenvalues are concrete witnesses that the unique affine quotient has no exact CPTP extension.

(vi) **Observer webs and fan-out.** The descent, holonomy and broadcasting checks compute loop residuals and the covariant qubit fan-out boundary. The reported values distinguish statistical calibration error from genuine non-flat observer-web curvature and from no-broadcasting obstruction.

All numerical figures in the main text are regenerated by the script from the stated kernels and channels. The numerical file records scalar diagnostics, including LP gaps, support sizes, confidence radii, Choi minima and no-broadcasting residuals.

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